

The Nonnegative l_0 Norm Minimization under Generalized Z -matrix Measurement ^{*}

Ziyan Luo,[†] Linxia Qin,[‡] Lingchen Kong[§] Naihua Xiu,[¶]

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Abstract

In this paper, we consider the l_0 norm minimization problem with linear equation and nonnegativity constraints. By introducing the concept of generalized Z -matrix for a rectangular matrix, we show that this l_0 norm minimization with such a kind of measurement matrices and nonnegative observations can be exactly solved via the corresponding l_p ($0 < p \leq 1$) norm minimization. Moreover, the lower bound of sample number is allowed to be k for recovering the unique k -sparse solution of the underlying l_0 norm minimization. A practical application in communications is presented which satisfies the generalized Z -matrix recovery condition.

Key words. Nonnegative l_0 norm minimization, generalized Z -matrix, k -sparse solution, sample number

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1 Introduction

The l_0 norm minimization problem is a core model in the compressed sensing (CS) which was initiated by Donoho [10], Candés, Romberg and Tao [6, 7] with the involved essential idea—recovering some original n -dimensional but sparse signal/image from linear measurement with dimension far fewer than n . CS has attracted much attention and obtained rapid

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[†]Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, P.R. China; (starkeynature@hotmail.com).

[‡]Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, P.R. China; (lxqin.echo@163.com).

[§]Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, P.R. China; (konglchen@126.com).

[¶]Department of Mathematics, School of Science, Beijing Jiaotong University, Beijing 100044, P.R. China; (nhxiu@bjtu.edu.cn).

developments in recent years [2, 8] owing to its wide applications in signal processing, communications, machine learning, medicine science, sensor location and many other fields. However, in some specific applications, the unknown signal\image may inherently possess some prior information such as the nonnegativity. Such a kind of signals\images is extensively encountered in communications, DNA microarrays, spectroscopy, tomography, network monitoring, and hidden Markov models [14, 18, 20, 21, 25]. This scenario drives us to study the following l_0 norm minimization with linear equation and nonnegativity constraints (also called *sparse nonnegative recovery*)

$$(L_0) \quad \begin{array}{ll} \min & \|x\|_0 \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{array}$$

where $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $m \leq n$ and $b \in \mathbb{R}^m$. Here $\|x\|_0$ denotes the number of the nonzero entries of x , and a vector x is called k -sparse if its l_0 norm is no more than k .

Mathematically, problem (L_0) is actually to find the sparsest nonnegative solutions for an underdetermined system of linear equations, which is NP-hard in general due to the combinatorial search [17, 22]. One popular method is to substitute the l_0 norm by the l_p ($0 < p \leq 1$) norm. This leads to the following relaxation—the l_p norm minimization problem with linear equation and nonnegative constraints

$$(L_p) \quad \begin{array}{ll} \min & \|x\|_p \\ \text{s.t.} & Ax = b, \\ & x \geq 0. \end{array}$$

As in the CS setting, a fundamental question arises: under what conditions the sparse nonnegative signal\image can be exactly recovered via the above relaxation (L_p) with a promising compression ratio (i.e., n/m)? Specifically, how to design the measure matrix A

- (1) to guarantee the equivalence of problems (L_0) and (L_p) ;
- (2) to decrease the lower bound of sample number for the improvement of the compression ratio.

To achieve the above equivalence, many researchers have done a series of investigations from different perspectives, see, e.g., [3, 12, 13, 19, 20, 23, 26, 27] and the references therein. The first characterization of this equivalence, introduced by Donoho and Tanner in [12] from a geometric point of view, is that the polytope AT is outwardly k -neighborly with T being the standard simplex in \mathbb{R}^n . In [12] Donoho and Tanner also certified that the random polytope generated by a Gaussian matrix offers this property with an overwhelming probability. Zhang [27] analyzed the algebraic structure of the measurement matrix and built up the equivalence under the condition that the null space of A is strictly half k -balance. Recently, by employing the optimality condition of (L_1) , Juditsky, Karzan and Nemirovski [19] developed some verifiable k -semigood conditions for the equivalence based on some prior information on signs of the original image, which includes the nonnegative constraints as a

special case. More recently, by developing the well-known restricted isometry property (RIP) condition, Qin, Xiu, Kong and Li [23] proposed a nonnegative RIP of the measurement matrix A to get the desired equivalence.

Some more special case was also analyzed for the desired equivalence. It is trivial that if the feasible set of (L_0) is a singleton, the unique feasible solution is definitely the unique sparse solution which can be achieved by minimizing any objective function over this constraint set. Based on this observation, some recovery conditions were tailored for the uniqueness of feasible solutions to problem (L_0) . Toward this direction, Bruckstein, Elad and Zibulevsky [3] gave a sufficient condition that A has a row-span intersecting the positive orthant. Later, this condition was shown to be also necessary by Xu and Tang [26]. By exploiting the geometrical structure features of the feasible set, Donoho and Tanner [13] derived that the desired uniqueness holds if and only if polytope $A\mathbb{R}_+^n$ and \mathbb{R}_+^n have the same number of k -faces. Another equivalent condition for the uniqueness property was proposed by Khajehnejad, Dimakis, Xu and Hassibi [20] in terms of the support size of vectors in the null space of A .

In this paper, by employing Z -matrices and the least element theory in linear complementarity problems (for details, see [1, 4]), together with the relation among problems (L_0) , (L_p) and the following multi-objective programming

$$(L_{multi}) \quad \begin{array}{ll} \min & x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \end{array}$$

we present that if the observation vector b is nonnegative and the measurement matrix is a generalized Z -matrix (see Section 2 for details), problems (L_0) and (L_p) share the common unique solution, which is exactly the unique least element solution of (L_{multi}) . In comparison to the existing recovery conditions, our condition on the measurement matrix A is quite easy to check, only at the price of restricting the observations to be nonnegative.

To improve the lower bound of sample number, there are also plenty of meaningful explorations, such as the works in [5, 6, 9] to name a few. For the k -sparse recovery problem with no prior information, the existing best lower bound for stable recovery (with noise corruption) is $\mathcal{O}(k \log(n/k))$ which is shown to be tight in [9]. Intuitively, it is quite possible to improve this bound when the original signal has some favorable features besides sparsity, such as the nonnegativity. This has also been pointed out by Donoho and Tanner in [12] as “there are substantial quantitative improvements on the breakdown points when nonnegativity constraints are present”. Up to date, only a few discussions are addressed on this lower bound for the sparse nonnegative recovery [12, 26, 27], and to our best knowledge, the corresponding best lower bound is $2k$ as stated by Donoho and Tanner in [12] and Zhang in [27]. In this paper, under the same conditions for the aforementioned equivalence, the exact recovery of a nonnegative k -sparse signal/image can be pursued with no less than k linear measurements. This lower bound k obviously improves the compression ratio greatly.

The organization of this paper is as follows. The concept of generalized Z -matrix for

a rectangular matrix is introduced and the feasibility of problem (L_0) with such a kind of measurement matrices is discussed in Section 2. The equivalence of problems (L_0) and (L_p) , together the lower bound of sample number, is presented under generalized Z -matrix measurement by using (L_{multi}) as a bridge in Section 3. A practical application in communications is exhibited in Section 4 which satisfies our proposed recovery conditions.

2 Generalized Z -matrix

In this section, we introduce the concept of generalized Z -matrix and explore some of its interesting properties. The feasibility of problem (L_0) with such generalized Z -matrix as the measurement matrix is also studied. We begin with recalling the definition of Z -matrix.

Definition 2.1 ([16]) *A matrix $M \in \mathbb{R}^{n \times n}$ is called a Z -matrix if all its off-diagonal entries are nonpositive.*

Z -matrices appear in various fields and have extensive applications in differential equations, dynamical systems, optimization, economics, etc., see [1] for details. In this paper, we will generalize this important concept from square matrices to rectangular matrices in the following fashion.

Definition 2.2 *Let $A \in \mathbb{R}^{m \times n}$ with $m \leq n$. We say that A is a generalized Z -matrix if $[A^T \ 0]^T \in \mathbb{R}^{n \times n}$ is a Z -matrix.*

We use $\mathbb{Z}_{m \times n}$ to denote the set of all generalized Z -matrices in $\mathbb{R}^{m \times n}$ ($m \leq n$). The generalized Z -matrix we defined here is different from the one introduced in [24] for block matrices. However, we still use “generalized” since it possesses these properties: (a) the generalized Z -matrix turns to be the Z -matrix when $m = n$; (b) the submatrix A_m , formed by the first m columns of A is a Z -matrix.

An interesting property of the generalized Z -matrix is presented as follows by observation.

Lemma 2.3 *Let $A \in \mathbb{Z}_{m \times n}$, and $A_I \in \mathbb{R}^{|I| \times n}$ be its submatrix formed by those rows with index set I . Then $\bar{A}_I := [A_{II} \ A_{I\bar{I}}] \in \mathbb{Z}_{|I| \times n}$, where $A_{II} \in \mathbb{R}^{|I| \times |I|}$ is the submatrix of A_I formed by those columns with index set I and \bar{I} is the complement set of I with respect to $\{1, \dots, n\}$.*

Another interesting but essential property of the generalized Z -matrix is related to the least element theory which is analogous to that of the standard Z -matrix.

Proposition 2.4 *Let $A \in \mathbb{Z}_{m \times n}$ and $b \in \mathbb{R}^m$. If $\tilde{\mathcal{F}} := \{x \in \mathbb{R}^n \mid Ax \geq b, x \geq 0\} \neq \emptyset$, then $\tilde{\mathcal{F}}$ has a least element u which satisfies $u_i(Au - b)_i = 0$ for all $i = 1, \dots, m$. Moreover, if $\tilde{\mathcal{F}}_u := \{x \in \mathbb{R}^n \mid Ax \geq b, 0 \leq x \leq u\} \neq \emptyset$ for some $u \geq 0$, then $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}_u$ share the same least element.*

The remaining of this section is devoted to the feasibility of problem (L_0) for the generalized Z -matrix case. For simplicity, we denote the corresponding feasible set by

$$\mathcal{F} := \{x \mid Ax = b, x \geq 0\}.$$

Proposition 2.5 *Let $A \in \mathbb{Z}_{m \times n}$ and $b \in \mathbb{R}_+^m$. The following two systems are equivalent:*

- (a) $Ax = b, x \geq 0$;
- (b) $x \geq 0, Ax - b \geq 0, x_i(Ax - b)_i = 0, \forall i = 1, \dots, m$.

Proof. For any solution x^* of system (a), it is trivial that x^* is a solution to (b). Let \bar{x} be any solution of system (b). To get the converse part, it suffices to show that $(A\bar{x} - b)_i = 0$ for any $i \in \{1, \dots, m\}$ with $\bar{x}_i = 0$. By the observation that

$$(A\bar{x} - b)_i = A_{ii}\bar{x}_i + \sum_{j=1, j \neq i}^n A_{ij}\bar{x}_j - b_i \leq 0,$$

together with $A\bar{x} - b \geq 0$, the desired result follows. \square

The above proposition provides a necessary and sufficient condition for the feasibility of problem (L_0) . Below, we will give a sufficient condition and a necessary condition for this feasibility. The involved P -matrix means that all the principal sub-determinants of a matrix are positive, see [4] for details.

Proposition 2.6 *Let $A \in \mathbb{Z}_{m \times n}$ and $b \in \mathbb{R}_+^m$.*

- (a) *If the submatrix A_m , formed by the first m columns of A , is a P -matrix, then $\mathcal{F} \neq \emptyset$;*
- (b) *If $\mathcal{F} \neq \emptyset$, then for any $i \in \{1, \dots, m\}$ with $b_i > 0$, we have $A_{ii} > 0$.*

Proof. For (a), let $A = [A_m, A_{n-m}]$ with $A_m \in \mathbb{R}^{m \times m}$ being a P -matrix and $A_{n-m} \in \mathbb{R}^{m \times (n-m)}$ being the remaining part of A . Note that A_m is a Z -matrix. By employing the equivalence of P -matrix and inverse-positiveness for the Z -matrix A_m [1], we can obtain that A_m^{-1} exists and all its entries are nonnegative. Combining with $b \geq 0$, we have $A_m^{-1}b \geq 0$, which further implies that $x^* := ((A_m^{-1}b)^\top \ 0)^\top \in \mathcal{F}$.

To get (b), for any $b_i > 0$, we have $(Ax^* - b)_i = A_{ii}x_i^* + \sum_{j=1, j \neq i}^n A_{ij}x_j^* - b_i = 0$ which implies that $A_{ii} > 0$. Henceforth, the desired assertion holds. \square

It is worth mentioning that under the condition $\mathcal{F} \neq \emptyset$, A_{ii} could be negative when $b_i = 0$. A simple example follows for the illustration.

Example 2.7 *Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Evidently $A \in \mathbb{Z}_{2 \times 3}$ and $b \in \mathbb{R}_+^2$. By direct calculation, the feasible set $\mathcal{F} = \{(1, 0, t)^\top \mid t \in \mathbb{R}_+\} \neq \emptyset$.*

3 Main Results

In this section, we explore the relation of problems (L_0) , (L_p) ($0 < p \leq 1$) and (L_{multi}) for the generalized Z -matrix case. For the sake of simplicity, we use S_0 and S_p to respectively denote the optimal solution sets of problems (L_0) and (L_p) , and S_{multi} to denote the set of all Pareto points of the multi-objective programming problem (L_{multi}) . Here a Pareto solution u of problem (L_{multi}) means that there exists no feasible solution y such that $y \leq u$ and $y_j < u_j$ for at least one index $j \in \{1, \dots, n\}$. Before stating the main theorem of this section, we give the relation of S_0 , S_p and S_{multi} for the general case as a start.

Proposition 3.1 *Let $A \in \mathbb{R}^{m \times n}$ ($m \leq n$) and $b \in \mathbb{R}^m$. We have $S_0 \cup S_p \subseteq S_{multi}$.*

Proof. When $\mathcal{F} = \emptyset$, the assertion holds trivially since $S_0 = S_p = S_{multi} = \emptyset$. For the case $\mathcal{F} \neq \emptyset$, it is equivalent to show that $S_0 \subseteq S_{multi}$ and $S_p \subseteq S_{multi}$. For any $x^* \in S_0$, assume on the contrary that $x^* \notin S_{multi}$. Then by the definition of Pareto point, there exists some $y \in \mathcal{F}$ such that $y \leq x^*$ and $y_j < x_j^*$ for some $j \in \{1, \dots, n\}$. Let $\beta := I(x^*) = \{i : x_i^* > 0\}$ and $\bar{\beta} := \{1, \dots, n\} \setminus \beta$. From the nonnegativity of y , we can easily get $y_{\bar{\beta}} := (y_i)_{i \in \bar{\beta}} = 0$. Noticing that x^* is one of the sparsest solution in \mathcal{F} , we immediately have $y_{\beta} := (y_i)_{i \in \beta} > 0$. Thus,

$$x^* - y \in \mathbb{R}_+^n, \quad \emptyset \neq I(x^* - y) \subseteq \beta.$$

Let $\delta := \min\{\frac{x_i^*}{x_i^* - y_i}\}$ and $z := x^* - \delta(x^* - y)$. It follows that

$$z \geq 0, \quad Az = b, \quad \|z\|_0 \leq \|x^*\|_0 - 1,$$

which contradicts to $x^* \in S_0$. Henceforth, $x^* \in S_{multi}$.

To get the assertion $S_p \subseteq S_{multi}$, it suffices to show that for any $x^* \in S_p$, there exists no $y \in \mathcal{F}$ such that $y \leq x^*$ and $y_j < x_j^*$ for at least one index $j \in \{1, \dots, n\}$. While this is trivial from the fact that $\|x^*\|_p = \left(\sum_{i=1}^n (x_i^*)^p\right)^{\frac{1}{p}}$ is the minimum of $\left(\sum_{i=1}^n u_i^p\right)^{\frac{1}{p}}$ for any $u \in \mathcal{F}$ with $p \in (0, 1]$. This completes the proof. \square

Generally, S_0 , S_p and S_{multi} are not equivalent to each other, as the following example shows.

Example 3.2 *Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 8 & 4 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$. By direct calculation, we have*

$$S_0 = \{(0, 0, 8)^\top\}, \quad S_1 = \left\{\left(\frac{2}{3}, \frac{2}{3}, 0\right)^\top\right\}, \quad S_{\min} = \left\{\left(\frac{2t}{3}, \frac{2t}{3}, 8 - 8t\right)^\top : 0 \leq t \leq 1\right\}.$$

The equivalence of the above three solution sets are attainable by employing the concept of generalized Z -matrix as introduced in Section 2. This is established in the following main theorem.

Theorem 3.3 Let $A \in \mathbb{Z}_{m \times n}$ and $b \in \mathbb{R}_+^m$ such that $\mathcal{F} \neq \emptyset$. We have

- (i) problems (L_0) , (L_p) ($p \in (0, 1]$) and (L_{multi}) share the same unique solution x^* ;
- (ii) $\|b\|_0 \leq \|x^*\|_0 \leq m$.

Proof. By employing Propositions 2.4 and 2.5, we can derive that \mathcal{F} has a unique least element, says x^* . Evidently, x^* is exactly the unique solution of problem (L_{multi}) by definition. Note that problems (L_0) and (L_p) are solvable under the assumption that $\mathcal{F} \neq \emptyset$. It further implies that $S_0 = S_p = S_{multi} = \{x^*\}$ by invoking Proposition 3.1. This arrives at the assertion in (i).

For (ii), observe that if $b_i > 0$, we have $0 < (Ax^*)_i = A_{ii}x_i^* + \sum_{j=1, j \neq i}^n A_{ij}x_j^*$. This implies that $x_i^* > 0$ by the condition that A is a generalized Z -matrix. Henceforth, $\|b\|_0 \leq \|x^*\|_0$. To get the other inequality, we take $p = 1$. In this case, problem (L_p) turns out to be a linear program. Utilizing (i), we know that this linear program has the unique solution x^* , which further implies that x^* is an extreme point of \mathcal{F} by the property in linear program theory which says at least one optimal solution of a linear program should be an extreme point of the corresponding feasible set. This immediately leads to $\|x^*\|_0 \leq m$. This completes the whole proof. \square

Remark 3.4 In [15], Fung and Mangasarian have proven that for a bounded system of linear equalities and inequalities, problem (L_0) is completely equivalent to problem (L_p) for a sufficiently small $p \in (0, 1)$. Here, by imposing the generalized Z -matrix measurement and the nonnegativity of the observation vector, this equivalence can be achieved for any $p \in (0, 1]$, as shown in Theorem 3.3.

A corollary follows immediately from Proposition 2.6 and Theorem 3.3 for the P -matrix case.

Corollary 3.5 Let $A \in \mathbb{Z}_{m \times n}$, $b \in \mathbb{R}_+^m$, and $A_m \in \mathbb{R}^{m \times m}$ be a P -matrix, where A_m is the submatrix formed by the first m columns of A . Then $((A_m^{-1}b)^\top 0)^\top$ is the unique solution of problem (L_0) .

Proof. According to the proof of Proposition 2.6, we know that $((A_m^{-1}b)^\top 0)^\top$ is actually a feasible solution of problem (L_0) . It suffices to show that this feasible solution is the unique solution of S_{multi} from Theorem 3.3 (i). Assume on the contrary that there exists some feasible solution $\hat{x} = (\hat{x}_m^\top \hat{x}_{n-m}^\top)^\top \neq x^*$ with $\hat{x} \leq x^*$ and $\hat{x} \neq x^*$. Obviously, $\hat{x}_{n-m} = 0$ and $A\hat{x} = A_m\hat{x}_m = b = A_mx_m^*$. By the invertibility of A_m , we obtain that $\hat{x}_m = x_m^*$ and hence $\hat{x} = x^*$. This contradicts to the assumption $\hat{x} \neq x^*$. Therefore, x^* is the unique solution of problem (P_0) . This completes the proof. \square

4 An Application

In this section, we will briefly introduce an application in communications, more specifically, in the joint power and admission control problem for single-input single-output interference channel. It is known from [21] that this problem can be regarded as two stages, with the first stage to get a maximum number of links, and the second one to achieve a minimum total transmission power to support that maximum number of links at their specified signal to interference plus noise ratio (SINR) targets. The involved subproblem of the second stage can be actually reformulated as

$$\begin{aligned} \min \quad & \bar{p}^\top q \\ \text{s.t.} \quad & C_I q \geq c_I \\ & 0 \leq q \leq e \end{aligned} \tag{4.1}$$

where the variable $q \in \mathbb{R}^K$ is the normalized power allocation vector with K the total number of all possible links in the whole communication system, $p \in \mathbb{R}^K$ is the prescribed upper bound of power vector for all possible links, $I \subseteq \{1, \dots, K\}$ is the set of links with the maximum total number, $c \in \mathbb{R}^K > 0$ is the normalized noise vector and $c_I = \{c_i\}_{i \in I}$, C is the normalized channel matrix whose entries have the form

$$C_{ij} = \begin{cases} 1, & \text{if } i = j; \\ -\frac{\gamma_i g_{ij} \bar{p}_j}{g_{ii} \bar{p}_i}, & \text{otherwise.} \end{cases}$$

Here $\gamma_i > 0$ is the specified requirement of the SINR for link i , $g_{ij} > 0$ is the channel gain from transmitter j to receiver i . And $C_I \in \mathbb{R}^{|I| \times K}$ is the corresponding submatrix formed by those rows of C with the row index set I . By choosing suitable subset I , problem (4.1) is always feasible and hence solvable in practical cases.

Let \bar{I} be complement set of I in $\{1, \dots, K\}$. We can rewritten $C_I q \geq c_I$ as

$$C_{II} q_I + C_{I\bar{I}} q_{\bar{I}} \geq c_I.$$

By setting $\bar{A} := [C_{II} \ C_{I\bar{I}}]$, $x := [q_I^\top \ q_{\bar{I}}^\top]^\top$, $\tilde{p} := [\bar{p}_I^\top \ \bar{p}_{\bar{I}}^\top]^\top$ and $\bar{b} := c_I$, problem (4.1) can be expressed as

$$\begin{aligned} \min \quad & \tilde{p}^\top x \\ \text{s.t.} \quad & \bar{A} x \geq \bar{b} \\ & 0 \leq x \leq e. \end{aligned} \tag{4.2}$$

Utilizing Lemma 2.3, we know that \bar{A} is a generalized Z -matrix. Combining with Proposition 2.4 and the fact $\tilde{p} > 0$, the above problem is equivalent to

$$\begin{aligned} \min \quad & \tilde{p}^\top x \\ \text{s.t.} \quad & \bar{A} x \geq \bar{b} \\ & x \geq 0, \end{aligned} \tag{4.3}$$

which can be equivalently transformed to

$$\begin{aligned} \min \quad & \|x\|_p \\ \text{s.t.} \quad & \bar{A} x = \bar{b} \\ & x \geq 0, \end{aligned} \tag{4.4}$$

with $p \in (0, 1]$ by invoking Propositions 2.4 and 2.5. Furthermore, Theorem 3.3 tells us that problem (4.4) is also equivalent to the following l_0 minimization problem

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & \bar{A}x = \bar{b} \\ & x \geq 0, \end{aligned} \tag{4.5}$$

and the multi-objective programming problem

$$\begin{aligned} \min \quad & x \\ \text{s.t.} \quad & \bar{A}x = \bar{b} \\ & x \geq 0. \end{aligned} \tag{4.6}$$

This immediately gives a theoretical explanation of the convention that to pursue the minimum total transmission power for all links is consistent to aim for the minimum power for every link which is widely applied in practice.

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