

Convergence Rate of an Inertial Extragradient Method for Strongly Pseudomonotone Equilibrium Problems in Hilbert Spaces

Duong Viet Thong* Phan Tu Vuong†

Abstract

In this work, we establish the R -linear convergence rate of the inertial extragradient method for solving strongly pseudo-monotone equilibrium problems with a new self adaptive step-size. The linear convergence rate of the proposed methods is obtained without the prior knowledge of the Lipschitz-type constants of the bifunction. We also discuss the application of the obtained results to variational inequality problems involving strongly pseudomonotone and Lipschitz continuous mapping.

Keywords: Equilibrium problem; inertial extragradient method; strongly pseudomonotone bifunction; R -linear rate

AMS class: 65Y05; 65K15; 68W10; 47H05; 47J25

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H . Let $f : H \times H \rightarrow \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem for the bifunction f on C , denoted by $EP(f, C)$, is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0, \forall y \in C.$$

Equilibrium problem is also called the Ky Fan inequality due to his contribution to this field [6]. Mathematically, $EP(f, C)$ is a generalization of many mathematical models including variational inequality problems, optimization problems and fixed point problems, see [1, 11, 12, 19, 24, 30]. EPs have been considered by numerous scholars in recent years, e.g., see [3, 7, 8, 13, 14, 18, 22, 25, 26, 27, 28, 29] and the references therein. Some notable methods for EPs have been proposed such as: proximal point methods (PPM) [17], auxiliary problem principle methods [15] and gap function methods [16].

The PPM is often applied to solve monotone EPs and it based on a regularized equilibrium problem which is strongly monotone and so that the unique solution is found easily. The auxiliary problem principle was proposed in [5], which was also called the proximal-like method. Its convergence was further investigated in [26] under different assumptions that equilibrium bifunction is pseudomonotone and satisfies a Lipschitz-type condition. The methods in [5, 26] are also called extragradient methods (EGM) due to the results of Korpelevich [10]. Under some suitable conditions imposed on parameters and bifunctions, solution approximation sequences generated by the

*Applied Analysis Research Group, Faculty of Mathematics and Statistics, Ton Duc Thang University, Ho Chi Minh City, Vietnam. Email: duongvietthong@tdtu.edu.vn

†Mathematical Sciences School, University of Southampton, UK. Email: t.v.phan@soton.ac.uk

extragradient method are proved to be convergent to some solution of $EP(f, C)$. In recent years, the extragradient methods have received a great attention by many authors, see, e.g., [2, 22, 27, 28]. The advantage of the extragradient method [26] is that the sub-problems are easier to solve than the PPM sub-problems. Moreover, it can be applied to more general class of pseudomonotone bifunctions.

The main drawback of EGM is that the chosen step-size depends on the Lipschitz-type constants of the bifunctions [4, 13, 14, 17]. This requirement can make some restrictions in applications because the Lipschitz-type constants are often unknown or difficult to estimate. In this work, we propose a new inertial extragradient method for solving strongly pseudo-monotone equilibrium problem and prove its linear convergence. An extragradient method with inertial effect for solving EPs can be founded in [27] but no convergence rate was obtained. It is worth pointing out that the proposed algorithm uses a new step-size rule which does not require the knowledge of the Lipschitz-type constants of the bifunction as required in [27]. As a consequence, we obtain convergence rate analysis for a modified extragradient method for solving variational inequality problems in Hilbert spaces.

The paper is organized as follows: In Section 2, we collect some definitions and preliminary results for further use. Section 3 presents the new algorithm and the convergence analysis. Finally, we discuss the applications to variational inequalities in Section 4, following with some concluding remarks.

2 Preliminaries

Let C be a nonempty closed convex subset of H . We begin with some concepts of monotonicity of a bifunction [1, 19]. A bifunction $f : H \times H \rightarrow \mathbb{R}$ is said to be:

(i) strongly monotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) + f(y, x) \leq -\gamma \|x - y\|^2, \quad \forall x, y \in C.$$

(ii) strongly pseudomonotone on C if there exists a constant $\gamma > 0$ such that

$$f(x, y) \geq 0 \implies f(y, x) \leq -\gamma \|x - y\|^2, \quad \forall x, y \in C.$$

(iii) satisfied Lipschitz-type condition on C if there exist two positive constants c_1, c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \|x - y\|^2 - c_2 \|y - z\|^2, \quad \forall x, y, z \in C.$$

From the definitions above, it is obvious that (i) \implies (ii).

The normal cone N_C to C at a point $x \in C$ is defined by $N_C(x) = \{w \in H : \langle w, x - y \rangle \geq 0, \forall y \in C\}$. For every $x \in H$, the metric projection $P_C x$ of x onto C is defined by $P_C x = \arg \min \{\|y - x\| : y \in C\}$. Since C is nonempty, closed and convex, $P_C x$ exists and is unique.

For each $x, z \in H$, by $\partial_2 f(z, x)$, we denote the subdifferential of convex function $f(z, \cdot)$ at x , i.e.,

$$\partial_2 f(z, x) := \{u \in H : f(z, y) \geq f(z, x) + \langle u, y - x \rangle, \forall y \in H\}.$$

In particular,

$$\partial_2 f(z, z) = \{u \in H : f(z, y) \geq \langle u, y - z \rangle, \forall y \in H\}.$$

For proving the convergence of the new algorithm, we need the following basic lemma.

Lemma 2.1. [23, Proposition 3.61] *Let C be a nonempty closed convex subset of H and $g : H \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function on H . Assume either that g is continuous at some point of C , or that there is an interior point of C where g is finite. Then, x^* is a solution to the following convex problem $\min \{g(x) : x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^* .*

3 Convergence Analysis

Now, we are in a position to present a modified version of inertial extragradient method in [5, 26] for solving equilibrium problems.

Algorithm 3.1.

Initialization. *Let $u_0, u_1 \in H$, $\lambda_1 > 0$, $\rho \in [0, 1)$, $\mu \in (0, 1)$. Let $\{\tau_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.*

Step 1. *Given the current iterates u_{n-1} and u_n ($n \geq 1$), compute*

$$\begin{cases} t_n = u_n + \rho(u_n - u_{n-1}), \\ v_n = \operatorname{argmin}_{y \in C} \{ \lambda_n f(t_n, y) + \frac{1}{2} \|y - t_n\|^2 \}. \end{cases}$$

If $v_n = t_n$ the stop and v_n is a solution. Otherwise, go to Step 2.

Step 2. *Compute*

$$u_{n+1} = \operatorname{argmin}_{y \in C} \{ \lambda_n f(v_n, y) + \frac{1}{2} \|y - t_n\|^2 \},$$

and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu}{2} \frac{\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2}{f(t_n, u_{n+1}) - f(t_n, v_n) - f(v_n, u_{n+1})}, \lambda_n + \tau_n \right\} \\ \lambda_n + \tau_n \text{ otherwise.} \end{cases} \quad (1)$$

Set $n := n + 1$ and return to Step 1.

Remark 3.1. *The adaptive step sizes $\{\lambda_n\}$ is chosen as in (1) is allowed to increase from iteration to iteration. This means that the adaptive step-size rule in Algorithm 3.1 is different to the other adaptive step-size rules studied in the literature [7, 4, 20, 26, 27].*

In order to establish the convergence of Algorithm 3.1, we assume that bifunction $f : H \times H \rightarrow \mathbb{R}$ satisfies the following conditions.

Condition 1

- (A1) f is γ -strongly pseudomonotone on C .
- (A2) f satisfies Lipschitz-type condition on H with two constants c_1 and c_2 .
- (A3) $f(x, \cdot)$ is convex and lower semicontinuous on H for every fixed $x \in H$.
- (A4) Either $\operatorname{int}C \neq \emptyset$ or $f(x, \cdot)$ is continuous at some point in C for every $x \in H$.

Remark 3.2. *From the conditions (A1) and (A2) we get $f(x, x) = 0$ for all $x \in C$. It is also known that under Condition 1, the problem $EP(f, C)$ has unique solution [21].*

Next, we will establish the convergence rate of Algorithm 3.1. We start with the following lemmas which play an important role in proving the convergence of the proposed algorithm.

Lemma 3.1. ([31]) *Let $\{\lambda_n\}$ be a sequence generated by Algorithm 3.1. Then $\lim_{n \rightarrow \infty} \lambda_n = \lambda \in \left[\min\left\{\frac{\mu}{2 \max\{c_1, c_2\}}, \lambda_1\right\}, \lambda_1 + \tau\right]$, where $\tau = \sum_{n=1}^{\infty} \tau_n$.*

Lemma 3.2. *For any $\lambda > 0$ and $x \in C$, let*

$$z = \operatorname{argmin}_{y \in C} \left\{ \lambda f(x, y) + \frac{1}{2} \|y - x\|^2 \right\}, \quad (2)$$

then

$$\lambda (f(x, y) - f(x, z)) \geq \langle x - z, y - z \rangle \quad \forall y \in C.$$

Proof: Since z is the unique solution of the strongly convex minimization problem (2). The optimality condition (Lemma 2.1) implies that there exists $s \in \partial_2 f(x, z)$ such that

$$0 \in \lambda s + z - x + N_C(z),$$

where $N_C(z)$ denotes the normal cone to C at z . Hence, by definition of this cone, we obtain that

$$\langle x - z - \lambda s, y - z \rangle \leq 0 \quad \forall y \in C. \quad (3)$$

On the other hand, since $s \in \partial_2 f(x, z)$, we have

$$f(x, y) - f(x, z) \geq \langle s, y - z \rangle \quad \forall y \in C. \quad (4)$$

Combining (3) and (4), we obtain

$$\lambda (f(x, y) - f(x, z)) \geq \langle \lambda s, y - z \rangle \geq \langle x - z, y - z \rangle \quad \forall y \in C.$$

Lemma 3.3. *Let C be a nonempty closed convex subset of H and $f : H \times H \rightarrow \mathbb{R}$ be a bifunction satisfying Condition 1. Let u be the unique solution of $EP(f, C)$. Then the following inequality holds*

$$\begin{aligned} \|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|t_n - v_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|u_{n+1} - v_n\|^2 \\ &\quad - 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned} \quad (5)$$

Proof: From

$$u_{n+1} = \operatorname{argmin}_{y \in C} \left\{ \lambda_n f(v_n, y) + \frac{1}{2} \|y - t_n\|^2 \right\},$$

by Lemma 3.2, we get

$$\lambda_n (f(v_n, y) - f(v_n, u_{n+1})) \geq \langle t_n - u_{n+1}, y - u_{n+1} \rangle \quad \forall y \in C.$$

Substituting $y := u \in C$, we obtain

$$\lambda_n (f(v_n, u) - f(v_n, u_{n+1})) \geq \langle t_n - u_{n+1}, u - u_{n+1} \rangle. \quad (6)$$

Since u is the unique solution of $EP(f, C)$ and $v_n \in C$, we have $f(u, v_n) \geq 0$. By the strong pseudomonotonicity assumption of f , we obtain $f(v_n, u) \leq -\gamma \|v_n - u\|^2$. It implies from (6) that

$$\begin{aligned} -\lambda_n f(v_n, u_{n+1}) &\geq \langle t_n - u_{n+1}, u - u_{n+1} \rangle - \lambda_n f(v_n, u) \\ &\geq \langle t_n - u_{n+1}, u - u_{n+1} \rangle + \lambda_n \gamma \|v_n - u\|^2. \end{aligned} \quad (7)$$

Again, since

$$v_n = \operatorname{argmin}_{y \in C} \left\{ \lambda_n f(t_n, y) + \frac{1}{2} \|y - t_n\|^2 \right\},$$

Lemma 3.2 implies

$$\lambda_n (f(t_n, u_{n+1}) - f(t_n, v_n)) \geq \langle t_n - v_n, u_{n+1} - v_n \rangle. \quad (8)$$

Adding (7) and (8) we get

$$\begin{aligned} & 2\lambda_n (f(t_n, u_{n+1}) - f(t_n, v_n) - f(v_n, u_{n+1})) \\ & \geq 2\langle t_n - v_n, u_{n+1} - v_n \rangle + 2\langle t_n - u_{n+1}, u - u_{n+1} \rangle + 2\lambda_n \gamma \|v_n - u\|^2 \\ & = (\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 - \|u_{n+1} - t_n\|^2) + \\ & \quad + (\|t_n - u_{n+1}\|^2 + \|u_{n+1} - u\|^2 - \|t_n - u\|^2) + 2\lambda_n \gamma \|v_n - u\|^2 \\ & = \|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 + \|u_{n+1} - u\|^2 - \|t_n - u\|^2 + 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} \|u_{n+1} - u\|^2 & \leq \|t_n - u\|^2 - \|t_n - v_n\|^2 - \|u_{n+1} - v_n\|^2 + 2\lambda_n (f(t_n, u_{n+1}) \\ & \quad - f(t_n, v_n) - f(v_n, u_{n+1})) - 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned} \quad (9)$$

On the other hand, from the definition of the sequence λ_n we get

$$2(f(t_n, u_{n+1}) - f(t_n, v_n) - f(v_n, u_{n+1})) \leq \frac{\mu}{\lambda_{n+1}} \left(\|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2 \right). \quad (10)$$

Substituting (9) into (10) we obtain

$$\begin{aligned} \|u_{n+1} - u\|^2 & \leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \|t_n - v_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) \|u_{n+1} - v_n\|^2 \\ & \quad - 2\lambda_n \gamma \|v_n - u\|^2. \end{aligned}$$

In the following theorem we will show that the sequence $\{u_n\}$ generated by Algorithm 3.1 converges strongly to the unique solution u with a R -linear rate.

Theorem 3.1. *Let C be a nonempty closed convex subset of H . Let $f : H \times H \rightarrow \mathbb{R}$ be a bifunction satisfying Condition 1 and be γ -strongly pseudomonotone on C . Let $\theta \in (0, 1)$ be arbitrary and ρ be a real number such that*

$$0 \leq \rho \leq \frac{w\epsilon}{w\epsilon + 2w + \epsilon}, \quad (11)$$

where $w := 1 - \min \left\{ \frac{(1 - \mu)\theta}{2}, \gamma\lambda \right\}$ and $\epsilon := \frac{1}{2}(1 - \mu)(1 - \theta)\theta$. Then the sequence $\{u_n\}$ is generated by Algorithm 3.1 converges in norm to the unique solution u of the problem $EP(f, C)$ with a R -linear rate.

Proof: First, we show that there exists $N_1 \in \mathbb{N}$ such that

$$\|u_{n+1} - u\|^2 \leq \omega \|t_n - u\|^2 - \epsilon \|u_{n+1} - t_n\|^2 \quad \forall n \geq N_1. \quad (12)$$

Indeed, since $\lim_{n \rightarrow \infty} \lambda_n = \lambda > 0$, there exists $N > 0$ such that

$$\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}} \right) > 0 \quad \forall n \geq N.$$

Thanks to (5) and $\theta \in (0, 1)$, we have for all $n \geq N$ that

$$\begin{aligned}
\|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \|v_n - t_n\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \|u_{n+1} - v_n\|^2 \\
&\quad - 2\lambda_n \gamma \|v_n - u\|^2 \\
&= \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta \|v_n - t_n\|^2 \\
&\quad - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \left[\|v_n - t_n\|^2 + \|u_{n+1} - v_n\|^2 \right] - 2\lambda_n \gamma \|v_n - u\|^2 \\
&\leq \|t_n - u\|^2 - \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta \|v_n - t_n\|^2 - \frac{1}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) \|u_{n+1} - t_n\|^2 \\
&\quad - 2\lambda_n \gamma \|v_n - u\|^2, \tag{13}
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality in the last estimation. Moreover, we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) &= \frac{1}{2} (1 - \mu) (1 - \theta) \geq \frac{1}{2} (1 - \mu) (1 - \theta) \theta, \\
\lim_{n \rightarrow \infty} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta &= (1 - \mu) \theta \geq 2 \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\}, \\
\lim_{n \rightarrow \infty} \lambda_n \gamma &= \lambda \gamma \geq \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\}.
\end{aligned}$$

Using the definition of the limit there exists $N_1 \in \mathbb{N}$ and $N_1 \geq N$, such that for all $n \geq N_1$

$$\begin{aligned}
\frac{1}{2} \left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) (1 - \theta) &\geq \frac{1}{2} (1 - \mu) (1 - \theta) \theta, \\
\left(1 - \mu \frac{\lambda_n}{\lambda_{n+1}}\right) \theta &\geq 2 \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\},
\end{aligned}$$

and

$$\lambda_n \gamma \geq \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\}.$$

Using (13) we obtain for all $n \geq N_1$ that

$$\begin{aligned}
\|u_{n+1} - u\|^2 &\leq \|t_n - u\|^2 - 2 \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\} \|v_n - t_n\|^2 - \frac{1}{2} (1 - \mu) (1 - \theta) \theta \|u_{n+1} - t_n\|^2 \\
&\quad - 2 \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\} \|v_n - u\|^2 \\
&= \|t_n - u\|^2 - \frac{1}{2} (1 - \mu) (1 - \theta) \theta \|u_{n+1} - t_n\|^2 \\
&\quad - 2 \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\} (\|v_n - t_n\|^2 + \|v_n - u\|^2) \\
&\leq \|t_n - u\|^2 - \frac{1}{2} (1 - \mu) (1 - \theta) \theta \|u_{n+1} - t_n\|^2 - \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\} \|t_n - u\|^2 \\
&= \left(1 - \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\}\right) \|t_n - u\|^2 - \frac{1}{2} (1 - \mu) (1 - \theta) \theta \|u_{n+1} - t_n\|^2 \\
&\leq \left(1 - \min \left\{ \frac{(1 - \mu) \theta}{2}, \gamma \lambda \right\}\right) \|t_n - u\|^2 - \frac{1}{2} (1 - \mu) (1 - \theta) \theta \|u_{n+1} - t_n\|^2 \\
&= \omega \|t_n - u\|^2 - \epsilon \|u_{n+1} - t_n\|^2.
\end{aligned}$$

Next, we show that the sequence $\{u_n\}$ converges strongly to the unique solution u of the problem EP(f,C). Indeed, we have

$$\begin{aligned}\|t_n - u\|^2 &= \|(1 + \rho)(u_n - u) - \rho(u_{n-1} - u)\|^2 \\ &= (1 + \rho)\|u_n - u\|^2 - \rho\|u_{n-1} - u\|^2 + \rho(1 + \rho)\|u_n - u_{n-1}\|^2\end{aligned}$$

and

$$\begin{aligned}\|u_{n+1} - t_n\|^2 &= \|u_{n+1} - u_n - \rho(u_n - u_{n-1})\|^2 \\ &= \|u_{n+1} - u_n\|^2 + \rho^2\|u_n - u_{n-1}\|^2 - 2\rho\langle u_{n+1} - u_n, u_n - u_{n-1} \rangle \\ &\geq \|u_{n+1} - u_n\|^2 + \rho^2\|u_n - u_{n-1}\|^2 - 2\rho\|u_{n+1} - u_n\|\|u_n - u_{n-1}\| \\ &\geq \|u_{n+1} - u_n\|^2 + \rho^2\|u_n - u_{n-1}\|^2 - \rho\|u_{n+1} - u_n\|^2 - \rho\|u_n - u_{n-1}\|^2 \\ &= (1 - \rho)\|u_{n+1} - u_n\|^2 - \rho(1 - \rho)\|u_n - u_{n-1}\|^2.\end{aligned}$$

Combining these inequalities with (12) we obtain

$$\begin{aligned}\|u_{n+1} - u\|^2 &\leq \omega(1 + \rho)\|u_n - u\|^2 - \omega\rho\|u_{n-1} - u\|^2 + \omega\rho(1 + \rho)\|u_n - u_{n-1}\|^2 \\ &\quad - \epsilon(1 - \rho)\|u_{n+1} - u_n\|^2 + \epsilon\rho(1 - \rho)\|u_n - u_{n-1}\|^2 \quad \forall n \geq N_1,\end{aligned}$$

or equivalently

$$\begin{aligned}\|u_{n+1} - u\|^2 - \omega\rho\|u_n - u\|^2 + \epsilon(1 - \rho)\|u_{n+1} - u_n\|^2 \\ \leq \omega [\|u_n - u\|^2 - \rho\|u_{n-1} - u\|^2 + \epsilon(1 - \rho)\|u_n - u_{n-1}\|^2] \\ - (\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho))\|u_n - u_{n-1}\|^2 \quad \forall n \geq N_1.\end{aligned}$$

Setting

$$\Sigma_n := \|u_n - u\|^2 - \rho\|u_{n-1} - u\|^2 + \epsilon(1 - \rho)\|u_n - u_{n-1}\|^2,$$

since $\omega \in (0, 1)$, we can write

$$\begin{aligned}\Sigma_{n+1} &\leq \|u_{n+1} - u\|^2 - \omega\rho\|u_n - u\|^2 + \epsilon(1 - \rho)\|u_{n+1} - u_n\|^2 \\ &\leq \omega\Sigma_n - (\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho))\|u_n - u_{n-1}\|^2 \quad \forall n \geq N_1.\end{aligned}$$

Now, using (11) we show that

$$\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho) \geq 0.$$

Indeed, from (11) we get $\rho \in [0, 1)$, thus we obtain $1 + \rho \leq 2$ and $\rho(1 - \rho) \leq \rho$, hence

$$\begin{aligned}\omega\epsilon(1 - \rho) - \omega\rho(1 + \rho) - \epsilon\rho(1 - \rho) &\geq \omega\epsilon(1 - \rho) - 2\omega\rho - \epsilon\rho \\ &= \omega\epsilon - \rho(\omega\epsilon + 2\omega + \epsilon) \geq 0.\end{aligned}$$

Therefore

$$\Sigma_{n+1} \leq \omega\Sigma_n \quad \forall n \geq N_1.$$

Next, we show that $\Sigma_n \geq 0$ for all n . Indeed, from (11) we get

$$\rho \leq \frac{\omega\epsilon}{\omega\epsilon + 2\omega + \epsilon} \leq \frac{\omega\epsilon}{\omega\epsilon + 2\omega} = \frac{\epsilon}{2 + \epsilon},$$

this implies that $\rho \leq \frac{\epsilon(1-\rho)}{2}$, using this fact, we obtain

$$\begin{aligned}\Sigma_n &= (1 - \epsilon(1 - \rho))\|u_n - u\|^2 + \epsilon(1 - \rho) (\|u_n - u\|^2 + \|u_n - u_{n-1}\|^2) - \rho\|u_{n-1} - u\|^2 \\ &\geq (1 - \epsilon(1 - \rho))\|u_n - u\|^2 + \frac{\epsilon(1 - \rho)}{2}\|u_{n-1} - u\|^2 - \rho\|u_{n-1} - u\|^2 \\ &\geq (1 - \epsilon(1 - \rho))\|u_n - u\|^2 \geq 0.\end{aligned}$$

Hence

$$\begin{aligned}\Sigma_{n+1} &\leq \omega\Sigma_n \leq \dots \leq \omega^{n-N_1+1}\Sigma_{N_1}. \\ \|u_n - u\|^2 &\leq \frac{\Sigma_{N_1}}{\omega^{N_1-1}}\omega^n,\end{aligned}$$

which means that $\{u_n\}$ converges R -linearly to u .

Remark 3.3. *Using the similar technique in [27, 31], one can obtain the weak convergence of Algorithm 3.1 under conditions: f is pseudomonotone on C ; $f(\cdot, y)$ is weakly upper semicontinuous on C , Conditions (A2), (A3), (A4) are satisfied and the solution set $EP(f, C) \neq \emptyset$. Hence, we omit the proof here.*

4 Application to Variational Inequalities

In this Section, we discuss the applications of the main result obtained in Section 3 for solving variational inequality problems in Hilbert spaces. Let $f(x, y) = \langle Fx, y - x \rangle \quad \forall x, y \in C$, where $F : H \rightarrow H$ is a continuous mapping. Then the equilibrium problems becomes the variational inequality problem, i.e., find $x^* \in C$ such that

$$\langle Fx^*, y - x^* \rangle \geq 0 \quad \forall y \in C. \quad (14)$$

The solution set of (14) is denoted by $Sol(F, C)$. Moreover, we have

$$v_n = \operatorname{argmin}_{y \in C} \{ \lambda_n f(t_n, y) + \frac{1}{2} \|y - t_n\|^2 \} = P_C(t_n - \lambda_n F(t_n)).$$

We recall that the mapping F is δ -strongly pseudomonotone on C if there exists a constant $\delta > 0$ such that

$$\langle Fx, y - x \rangle \geq 0 \implies \langle Fy, y - x \rangle \geq \delta \|x - y\|^2 \quad \forall x, y \in C.$$

If F is Lipschitz-continuous and strongly pseudomonotone, then the conditions (A1)-(A4) hold for f with $c_1 = c_2 = \frac{L}{2}$ (see, e.g. [26]). Note that, under these assumption, $Sol(F, C)$ is nonempty and singleton [9]. For solving variational inequality (14), we propose the following algorithm.

Algorithm 4.1.

Initialization. *Let $u_0, u_1 \in H$, $\lambda_1 > 0$, $\rho \in [0, 1)$, $\mu \in (0, 1)$. Let $\{\tau_n\}$ be a nonnegative real numbers sequence such that $\sum_{n=1}^{\infty} \tau_n < +\infty$.*

Step 1. *Given the current iterates u_{n-1} and u_n ($n \geq 1$), compute*

$$\begin{cases} t_n = u_n + \rho(u_n - u_{n-1}), \\ v_n = P_C(t_n - \lambda_n F t_n). \end{cases}$$

If $v_n = t_n$ or $Ft_n = 0$ then the stop and t_n is a solution of VI (14). Otherwise, go to Step 2.

Step 2. Compute

$$u_{n+1} = P_C(t_n - \lambda_n Fv_n), \quad (15)$$

and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|t_n - v_n\|^2 + \|u_{n+1} - v_n\|^2}{2 \langle Fv_n - Ft_n, u_{n+1} - v_n \rangle}, \lambda_n + \tau_n \right\} \\ \quad \text{if } \langle Fv_n - Ft_n, u_{n+1} - v_n \rangle > 0; \\ \lambda_n + \tau_n \text{ otherwise.} \end{cases}$$

Set $n := n + 1$ and return to Step 1.

The following theorem is a direct consequence of Theorem 3.1.

Theorem 4.1. Assume that $F : H \rightarrow H$ is L -Lipschitz continuous on H and δ -strongly pseudomonotone on C . Let $\theta \in (0, 1)$ be arbitrary and ρ be a real number such that

$$0 \leq \rho \leq \frac{w\epsilon}{w\epsilon + 2w + \epsilon},$$

where $w := 1 - \min \left\{ \frac{(1 - \mu)\theta}{2}, \gamma\lambda \right\}$ and $\epsilon := \frac{1}{2}(1 - \mu)(1 - \theta)\theta$. Then the sequence $\{x_n\}$ is generated by Algorithm 4.1 converges in norm to the unique solution x^* of the problem $\text{Sol}(F, C)$ with a R -linear rate.

Remark 4.1. As a consequence of Remark 3.3 we can also obtain the weak convergence of Algorithm 4.1 under the following condition conditions: F is pseudomonotone and L -Lipschitz continuous on C ; F is sequentially weakly continuous on H and the solution set $\text{Sol}(F, C) \neq \emptyset$. Moreover, the second projection in (15) can be replaced by an explicit projection onto a half-space as in the subgradient extragradient method [2].

5 Conclusions

The paper presented a linear convergence analysis of the inertial extragradient method for approximating solutions of equilibrium problems in Hilbert spaces under the strongly pseudomonotone and Lipschitz assumptions imposed on equilibrium bifunctions. Application of our main result for solving variational inequality problems in Hilbert spaces is also investigated.

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