



Problems on chromatic polynomials of hypergraphs

Ruixue Zhang, Fengming Dong^a

^aMathematics and Mathematics Education, National Institute of Education,
Nanyang Technological University, Singapore 637616

fengming.dong@nie.edu.sg

Abstract

Chromatic polynomials of graphs have been studied extensively for around one century. The concept of chromatic polynomial of a hypergraph is a natural extension of chromatic polynomial of a graph. It also has been studied for more than 30 years. This short article will focus on introducing some important open problems on chromatic polynomials of hypergraphs.

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1. Introduction

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ consists of two finite sets V and \mathcal{E} , where \mathcal{E} is a subset of $\{e \subseteq V : |e| \geq 2\}$. A hypergraph is called *Sperner hypergraph* when no edge is a subset of another edge.

Erdős and Hajnal [14] first extended colourings on graphs to hypergraphs. For any integer $\lambda \geq 1$, a *proper λ -colouring* of \mathcal{H} is a mapping $\phi : V \rightarrow \{1, 2, \dots, \lambda\}$ such that $|\{\phi(v) : v \in e\}| > 1$ holds for each $e \in \mathcal{E}$. Two proper λ -colourings ϕ and ψ of \mathcal{H} are regarded as *distinct* if $\phi(u) \neq \psi(u)$ for some vertex u in \mathcal{H} . Let $P(\mathcal{H}, \lambda)$ be the number of proper λ -colourings of \mathcal{H} . This function $P(\mathcal{H}, \lambda)$ is called the *chromatic polynomial* of \mathcal{H} , and it is indeed a polynomial in λ of degree $|V|$.

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We have assumed that a hypergraph \mathcal{H} has no loops. In the case that \mathcal{H} has edges e_1, e_2 with $e_1 \subseteq e_2$, $P(\mathcal{H}, \lambda) = P(\mathcal{H} - e_2, \lambda)$ holds by the definition of $P(\mathcal{H}, \lambda)$, where $\mathcal{H} - e_2$ is obtained from \mathcal{H} by removing e_2 . Hence we can assume that \mathcal{H} is Sperner in the function $P(\mathcal{H}, \lambda)$.

The graph-function $P(\mathcal{H}, \lambda)$ for a hypergraph \mathcal{H} appeared in the work of Helgason [15] in 1972, but it is unknown if it had been mentioned earlier. It has been studied extensively in the past twenty years by many researches, such as Allagan [1, 2, 3], Borowiecki and Łazuka [5, 6], Dohmen [8, 9], Drgas-Burchardt and Łazuka [13], Tomescu [19, 20, 21, 22, 23] and Walter [24]. A number of properties of chromatic polynomials of graphs related to their expressions, computations and factorizations, etc, have been extended to chromatic polynomials of hypergraphs. In this short article, we will focus on introducing some important open problems in the study of chromatic polynomials of hypergraphs which are worth for further exploring.

2. Problems on chromatic polynomials of hypergraphs

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is said to be *connected* if for any two vertices u, v in \mathcal{H} , \mathcal{H} has a sequence of edges e_1, e_2, \dots, e_k such that $u \in e_1, v \in e_k$ and $e_i \cap e_{i+1} \neq \emptyset$ for all $i = 1, 2, \dots, k - 1$. A *component* of \mathcal{H} is a connected sub-hypergraph $\mathcal{H}' = (V', \mathcal{E}')$, where $V' \subseteq V$ and $\mathcal{E}' = \{e \in \mathcal{E} : e \subseteq V'\}$, which has the property that $e \subseteq V'$ holds for all $e \in \mathcal{E}$ with $e \cap V' \neq \emptyset$.

Dohmen [8] and Tomescu [19] independently showed that $P(\mathcal{H}, \lambda)$ is a monic polynomial of degree $|V|$:

$$P(\mathcal{H}, \lambda) = \sum_{i=1}^n \left(\sum_{j=0}^m (-1)^j N(i, j) \right) \lambda^i, \tag{1}$$

where $n = |V|, m = |\mathcal{E}|$ and $N(i, j)$ is the number of spanning sub-hypergraphs (V, \mathcal{E}') , where $\mathcal{E}' \subseteq \mathcal{E}$, with i components and j edges.

Some properties on $P(\mathcal{H}, \lambda)$ can be obtained from (1). It is obvious that $P(\mathcal{H}, \lambda)$ is a monic polynomial with integral coefficients and has zero as its constant term. But the information on this polynomial from (1) or other results is far from enough to tell efficiently if a given polynomial is the chromatic polynomial of some hypergraph. It is natural to ask what polynomials are chromatic polynomials of hypergraphs.

Problem 1. Characterize chromatic polynomials of hypergraphs.

Two hypergraphs \mathcal{H}_1 and \mathcal{H}_2 are said to be *isomorphic*, denoted by $\mathcal{H}_1 \cong \mathcal{H}_2$, if there exists a bijection

$$\phi : V(\mathcal{H}_1) \rightarrow V(\mathcal{H}_2)$$

such that for any subset $e \subseteq V(\mathcal{H}_1), e \in \mathcal{E}(\mathcal{H}_1)$ if and only if $\phi(e) \in \mathcal{E}(\mathcal{H}_2)$, where $\phi(e) = \{\phi(v) : v \in e\}$, *non-isomorphic* otherwise. It is obvious that $P(\mathcal{H}_1, \lambda) = P(\mathcal{H}_2, \lambda)$ if $\mathcal{H}_1 \cong \mathcal{H}_2$. Two non-isomorphic hypergraphs may have the same chromatic polynomial. The problem of determining the family $\langle \mathcal{H} \rangle = \{\mathcal{H}' : P(\mathcal{H}, \lambda) = P(\mathcal{H}', \lambda)\}$ for a given hypergraph \mathcal{H} is very difficult. Some families $\langle \mathcal{H} \rangle$ have been confirmed in [5, 6, 19, 20, 21, 23, 24]. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is said to be *h-uniform* if $|e| = h$ holds for all $e \in \mathcal{E}$. It is unknown if all hypergraphs in $\langle \mathcal{H} \rangle$ are *h-uniform* for any given *h-uniform* hypergraph \mathcal{H} .

Problem 2. *Is it true that given any h -uniform hypergraph \mathcal{H} , all hypergraphs in $\langle \mathcal{H} \rangle$ are also h -uniform?*

A special case is on complete h -uniform hypergraphs. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is said to be *complete h -uniform* of order n , denoted by $K_n(h)$, if $|V| = n$ and $\mathcal{E} = \{e \subseteq V : |e| = h\}$.

Problem 3. *For any integers n, h with $3 \leq h \leq n$, is it true that $\langle K_n(h) \rangle = \{K_n(h)\}$?*

It is well known that for any graph G of order n , $P(G, \lambda)$ can be expressed as $\sum_{c(G) \leq i \leq n} a_i \lambda^i$, where a_i is a non-zero integer for all i with $c(G) \leq i \leq n$ and $c(G)$ is the number of components of G . In other words, $P(G, \lambda)$ has the property that its terms with non-zero coefficients are consecutive. However, by (1), the property does not hold for a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $h = \min\{|e| : e \in \mathcal{E}\} > 2$, as the coefficient of the term λ^{n-i} in the polynomial $P(\mathcal{H}, \lambda)$ is non-zero for $i = h - 1$ but is zero for all i with $1 \leq i \leq h - 2$. It is also unknown if the rest terms λ^{n-i} ($i \geq h - 1$) with non-zero coefficients are consecutive. Thus it is natural to ask what terms in the polynomial $P(\mathcal{H}, \lambda)$ have non-zero coefficients. In particular, it is interesting to know what is the minimum integer r such that the term λ^r in $P(\mathcal{H}, \lambda)$ has non-zero coefficient. By the property that $P(\mathcal{H}, \lambda) = \prod_{1 \leq i \leq k} P(\mathcal{H}_i, \lambda)$ whenever $\mathcal{H}_1, \dots, \mathcal{H}_k$ are components of \mathcal{H} , we can focus on connected hypergraphs for the above problem.

Problem 4. *Let $\mathcal{H} = (V, \mathcal{E})$ be a connected hypergraph. Which terms λ^i in the polynomial $P(\mathcal{H}, \lambda)$ have non-zero coefficients? In particular, what is the minimum integer i such that the term λ^i in the polynomial $P(\mathcal{H}, \lambda)$ has non-zero coefficient?*

It is known that the multiplicity of root ‘1’ of $P(G, \lambda)$ for a connected graph G is equal to the number of blocks of G . Is there an analogous result for hypergraphs? A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is said to be *separable* if \mathcal{H} has a vertex w and two proper subsets V_1, V_2 of vertices such that $V_1 \cap V_2 = \{w\}$, $V_1 \cup V_2 = V$ and for each $e \in \mathcal{E}$, either $w \in e$ or $e \subseteq V_i$ for some i .

Problem 5. *For a connected hypergraph $\mathcal{H} = (V, \mathcal{E})$, what is the multiplicity of root ‘1’ of $P(\mathcal{H}, \lambda)$? Is it true that the multiplicity of root ‘1’ of $P(\mathcal{H}, \lambda)$ is 1 whenever \mathcal{H} is not separable?*

It is also known that if G is a graph of order n and $P(G, \lambda)$ is the polynomial $\sum_{1 \leq i \leq n} a_i(G) \lambda^i$, then $(-1)^{n-i} a_i(G) \geq 0$ holds for all $i = 1, 2, \dots, n$. This property implies that $(-1)^n P(G, \lambda) > 0$ for all negative real numbers λ and thus $(-\infty, 0)$ is a zero-free interval for chromatic polynomials of graphs. Dohmen [9] showed that this above property holds for those hypergraphs $\mathcal{H} = (V, \mathcal{E})$ such that $|e|$ is even for each $e \in \mathcal{E}$ and each cycle in \mathcal{H} has an edge e with $|e| = 2$, where a cycle in \mathcal{H} is a sequence $(v_1, e_1, v_2, e_2, \dots, v_k, e_k)$ of distinct vertices and edges such that $v_i, v_{i+1} \in e_i$ for all $i = 1, 2, \dots, k$, where $v_{k+1} = v_1$. But this property does not hold for all hypergraphs. Observe that if $\mathcal{E} = \{V\}$, then $P(\mathcal{H}, \lambda) = \lambda^{|V|} - \lambda$, implying that $P(\mathcal{H}, \lambda)$ has negative real roots whenever $|V|$ is odd. Thus $(-\infty, 0)$ is not a zero-free interval for chromatic polynomials of hypergraphs. However, it is unknown if there is a zero-free interval (a, b) with $b \leq 0$ for chromatic polynomials of hypergraphs.

Problem 6. *Is there a zero-free interval for chromatic polynomials of hypergraphs? In particular, is there a zero-free interval (a, b) within some intervals $(-\infty, 0)$ or $(0, 1)$?*

For any graph G of order n , we have the property that G has an n -colouring and $P(G, \lambda) > 0$ holds for all real $\lambda > n - 1$. For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $|V| = n$ and $h = \min\{|e| : e \in \mathcal{E}\}$, \mathcal{H} has a $(|V| - h + 2)$ -colouring. But it is unknown if $P(\mathcal{H}, \lambda) > 0$ holds for all real $\lambda > |V| - h + 1$.

Problem 7. For a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $|V| = n$ and $h = \min\{|e| : e \in \mathcal{E}\}$, is it true that $P(\mathcal{H}, \lambda) > 0$ holds for all real $\lambda > |V| - h + 1$?

The shameful conjecture on chromatic polynomials of graphs was proposed by Bartels and Welsh [4] in 1995: for any graph G of order n , the following inequality holds:

$$\frac{P(G, n)}{P(G, n - 1)} \geq e, \tag{2}$$

where e is the base of natural logarithms.

This conjecture was finally proved by Dong [11] in 2000. Actually, Dong got a stronger result that, for any graph G of order n , the inequality $\frac{P(G, \lambda)}{P(G, \lambda - 1)} \geq \frac{\lambda^n}{(\lambda - 1)^n}$ holds for all real $\lambda \geq n$. We guess this result can be extended to hypergraphs.

Problem 8. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph with $|V| = n$. Does the following inequality

$$\frac{P(\mathcal{H}, \lambda)}{P(\mathcal{H}, \lambda - 1)} \geq \frac{\lambda^n}{(\lambda - 1)^n}$$

hold for all real $\lambda \geq n$?

Welsh in 1970s and Brenti [7] in 1992 conjectured that for any graph G , $P(G, k)^2 \geq P(G, k + 1)P(G, k - 1)$ holds for all integers $k \geq 1$. However, Seymour [18] found some graphs G with sufficiently large order such that $P(G, 6)^2 < P(G, 7)P(G, 5)$ holds. But it is still unknown if this conjecture holds for integers $k \geq n - 1$. It has been proposed in [10] that $P(G, \lambda)^2 \geq P(G, \lambda + 1)P(G, \lambda - 1)$ holds for all real numbers $\lambda \geq |V(G)| - 1$. We also believe that this conjecture also holds for hypergraphs.

Problem 9. For any hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $|V| = n$, does the following inequality

$$P(\mathcal{H}, \lambda)^2 \geq P(\mathcal{H}, \lambda - 1)P(\mathcal{H}, \lambda + 1)$$

holds for all real numbers $\lambda \geq n - 1$?

Lundow and Markström [16] conjectured that $\frac{P'(G, -1)}{P(G, -1)} < \frac{P'(K_n, -1)}{P(K_n, -1)}$ holds for any connected graph G which is not complete, where $P'(G, -1)$ is the value of $\frac{d(P(G, \lambda))}{d\lambda}$ at $\lambda = -1$. This conjecture was recently proved to be true [12]. We think this conjecture can be extended to h -uniform hypergraphs.

Problem 10. For any connected h -uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $|V| = n$ and $\mathcal{H} \not\cong K_n(h)$, does the following inequality hold

$$\frac{P'(\mathcal{H}, -1)}{P(\mathcal{H}, -1)} < \frac{P'(K_n(h), -1)}{P(K_n(h), -1)}?$$

References

- [1] J.A. Allagan, The chromatic polynomials of some linear uniform hypergraphs, *Congr. Numer.* **187** (2007), 156–160.
- [2] J.A. Allagan, Chromatic polynomials of some (m, l) -hyperwheels, *Comput. Sci. J. Moldova* **22** (2014), 64.
- [3] J.A. Allagan and S. David, Chromatic polynomials of some mixed hypergraphs, *Australas. J. Combin.* **58** (2014), 197–213.
- [4] J.E. Bartels and D.J. Welsh, The Markov chain of colourings, in *Proceedings of the Fourth Conference on Integer Programming and Combinatorial Optimisation* **920** (1995), 373–387.
- [5] M. Borowiecki and E. Łazuka, Chromatic polynomials of hypergraphs, *Discuss. Math. Graph Theory* **20** (2000), 293–301.
- [6] M. Borowiecki and E. Łazuka, On chromaticity of hypergraphs, *Discrete Math.* **307** (2007), 1418–1429.
- [7] F. Brenti, Expansions of chromatic polynomials and log-concavity, *Trans. Amer. Math. Soc.* **332** (1992), 729–756.
- [8] K. Dohmen, Chromatische polynome von graphen und hypergraphen, *Universität Düsseldorf* (1993).
- [9] K. Dohmen, A broken-circuits-theorem for hypergraphs *Arch. Math.* **64** (1995), 159–162.
- [10] F.M. Dong, K.M. Koh and K.L. Teo, *Chromatic Polynomials and Chromaticity of Graphs*, World Scientific, Singapore (2005).
- [11] F.M. Dong, Proof of a chromatic polynomial conjecture *J. Combin. Theory Ser. B* **78** (2000), 35–74.
- [12] F.M. Dong, J. Ge, H. Gong, B. Ning, Z. Ouyang and E.G. Tay, Proof of Lundow and Markström’s conjecture on chromatic polyomials via novel inequalities, *arXiv preprint arXiv:1803.08658* (2018).
- [13] E. Drgas-Burchardt and E. Łazuka, Chromatic polynomials of hypergraphs, *Applied Mathematics Letters* **20** (2007), 1250–1254.
- [14] P. Erdős and A. Hajnal, On chromatic number of graphs and set-systems, *Acta Math. Acad. Sci. Hungar.* **1966** (17), 61–99.
- [15] T. Helgason, Aspects of the theory of hypermatroids, *Hypergraph Seminar, Ohio State University 1972, Lecture Notes in Mathematics*, C. Berge and D. Ray-Chaudhuri, eds., Springer-Verlag **411** (1974), 191–213.

- [16] P.H. Lundow and K. Markström, Broken-cycle-free subgraphs and the log-concavity conjecture for chromatic polynomials, *Experiment. Math.* **15** (2006), 343–353.
- [17] R.C. Read, An introduction to chromatic polynomials, *J. Combin. Theory* **4** (1968), 52–71.
- [18] P. Seymour, Two chromatic polynomial conjectures, *J. Combin. Theory Ser. B* **70** (1997), 184–196.
- [19] I. Tomescu, Chromatic coefficients of linear uniform hypergraphs, *J. Combin. Theory Ser. B* **260** (1998), 229–235.
- [20] I. Tomescu, Sunflower hypergraphs are chromatically unique, *Discrete Math.* **285** (2004), 355–357.
- [21] I. Tomescu, On the chromaticity of sunflower hypergraphs $SH(n, p, h)$, *Discrete Math.* **307** (2007), 781–788.
- [22] I. Tomescu, Some properties of chromatic coefficients of linear uniform hypergraphs, *Graphs and Combin.* **25** (2009), 639–646.
- [23] I. Tomescu, Hypergraphs with pendant paths are not chromatically unique, *Discuss. Math. Graph Theory* **34** (2014), 23–29.
- [24] M. Walter, Some results on chromatic polynomials of hypergraphs, *Electron. J. Combin.* **16** (2009), R94.