A lattice-theoretic interpretation of independence of frames

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Outline

1. The theory of evidence
2. Independence of frames
3. Notion of independence
4. The lattice of frames
5. Independence on lattices and independence of frames
6. Conclusions and perspectives

The theory of evidence

- formulated as a theory of **subjective probability**;
- mathematical description of how a body of evidence affects one's belief;
- contains standard probability as special case;
- knowledge state represented by **belief functions** instead of finite probabilities;
- Bayes' rule is replaced by more general **Dempster's rule**;

Belief and probability measures

- probability distribution: $p : \Theta \rightarrow [0,1]$ s.t.

$$p(\emptyset) = 0, \sum_{x \in \Theta} p(x) = 1, p(x) \geq 0 \ \forall x \in \Theta$$

- probability measure $p(A) = \sum_{x \in A} p(x)$
- **Basic belief assignment** $m : 2^{\Theta} \rightarrow [0,1]$ such that

$$m(\emptyset) = 0, \sum_{A \subseteq \Theta} m(A) = 1, m(A) \geq 0 \ \forall A \subseteq \Theta$$

- **belief function** $b : 2^{\Theta} \rightarrow [0,1]$: $b(A) = \sum_{B \subseteq A} m(B)$

Dempster's combination

Definition

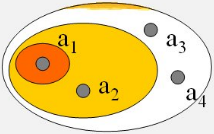
The *orthogonal sum* or *Dempster's sum* of two b.f.s b_1, b_2 on Θ is a new belief function $b_1 \oplus b_2$ on Θ with b.p.a.

$$m_{b_1 \oplus b_2}(A) = \frac{\sum_{B \cap C = A} m_{b_1}(B) m_{b_2}(C)}{\sum_{B \cap C \neq \emptyset} m_{b_1}(B) m_{b_2}(C)}.$$

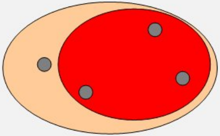
- when the denominator of the above equation is zero the two b.f.s are said to be *non-combinable*

Example

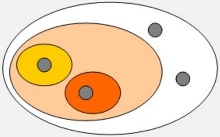
- $m_1(a_1) = 0.7$, $m_1(a_1, a_2) = 0.3$;



- $m_2(\Theta) = 0.1$, $m_2(a_2, a_3, a_4) = 0.9$;



- $m_1 \oplus m_2(a_1) = 0.19$,
  $m_1 \oplus m_2(a_2) = 0.73$,
  $m_1 \oplus m_2(a_1, a_2) = 0.08$.



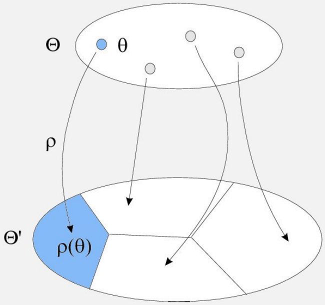
Refining

between pairs of finite domains or frames

Definition

A map $\rho : 2^{\Theta} \to 2^{\Theta'}$ is a **refining** when it maps Θ to a disjoint partition of Θ'

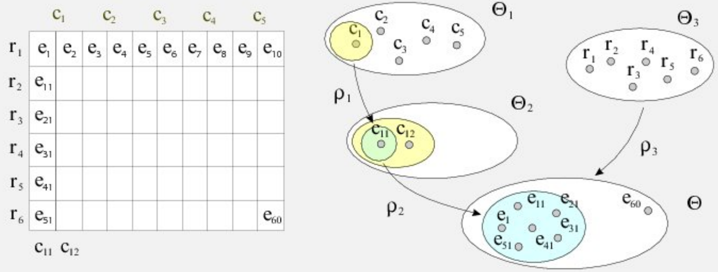
- Θ' **refinement** of Θ
- Θ **coarsening** of Θ'



Family of compatible frames

Definition

In a **family of compatible frames** each finite collection of frames admits a common refinement (amongst other things)



- **minimal refinement** -¿ smallest such common refinement

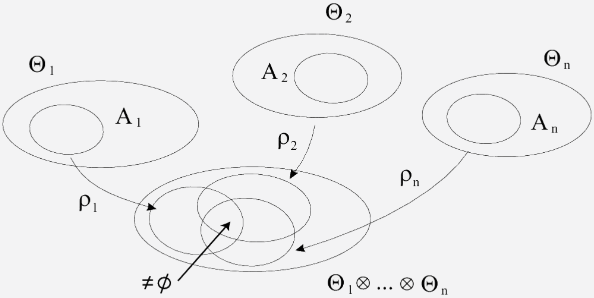
Independence of frames

as Boolean sub-algebras

Definition

$\Theta_1, ..., \Theta_n$ are **independent** [Shafer'76] ($\mathcal{IF}$) if

$$\rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \quad \forall \emptyset \neq A_i \subset \Theta_i$$



- comes from independence as Boolean algebras

Independence of frames and Dempster's rule

Proposition

$\Theta_1, ..., \Theta_n$ *are* ***independent*** *iff* ***all*** *the possible collections of b.f.s* $b_1, ..., b_n$ *on* $\Theta_1, ..., \Theta_n$ *are* ***combinable*** *on their minimal refinement* $\Theta_1 \otimes \cdots \otimes \Theta_n$

- independence of sources is then **equivalent** to independence of frames!
- in this form can be analyzed from an algebraic point of view

Independence in algebra and uncertainty theory

- similarity between $\mathcal{IF}$ and independence of vector subspaces

$$\begin{array}{c} \rho_1(A_1) \cap \cdots \cap \rho_n(A_n) \neq \emptyset, \quad \forall A_i \subseteq \Theta_i \\ \Theta_1 \otimes \cdots \otimes \Theta_n = \Theta_1 \times \cdots \times \Theta_n \\ \equiv \\ v_1 + \cdots + v_n \neq 0, \quad \forall v_i \in V_i \\ span\{V_1, ..., V_n\} = V_1 \times \cdots \times V_n. \end{array}$$

- obtained from each other under:

$$v_i \leftrightarrow A_i, \quad V_i \leftrightarrow 2^{\Theta_i}, \quad + \leftrightarrow \cap, \quad 0 \leftrightarrow \emptyset, \quad \otimes \leftrightarrow span.$$

- they share the algebraic structure of **semi-modular lattice**
- lattice independence / (Boolean) independence of sources?

Lattices

- *partially ordered set*: a set endowed with a relation $\leq$ s.t.
  1. $x \leq x$;
  2. if $x \leq y$ and $y \leq x$ then $x = y$;
  3. if $x \leq y$ and $y \leq z$ then $x \leq z$
- *least upper bound* $x \vee y$ is the smallest element bigger than both x and y
- *greatest lower bound* $x \wedge y$ is the biggest element smaller than both x and y
- **lattice** L is a poset in which each *pair* of elements admits both inf and sup

The lattice of frames

- in a family of frames we can define the following order

$$\Theta_1 \leq \Theta_2 \Leftrightarrow \exists \rho : \Theta_2 \rightarrow 2^{\Theta_1} \textit{ refining}$$

- i.e. Θ_1 smaller then Θ_2 iff Θ_1 is a refinement of Θ_2

Proposition

Both $(\mathcal{F}, \leq)$ and $(\mathcal{F}, \leq^*)$ where $\mathcal{F}$ is a family of compatible frames are lattices.

The lattice of frames is semimodular

- x "covers" y ($x \succ y$) if $x \geq y$ and there is no intermediate element in the chain linking them

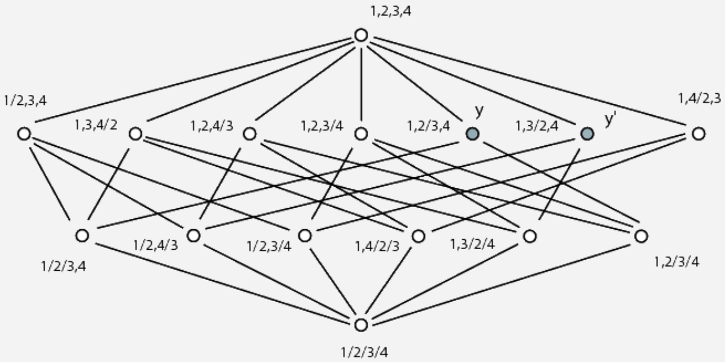
Definition

A lattice L is **upper semi-modular** if for each pair x, y of elements of L, $x \succ x \wedge y$ implies $x \vee y \succ y$.
A lattice L is **lower semi-modular** if for each pair x, y of elements of L, $x \vee y \succ y$ implies $x \succ x \wedge y$.

Theorem

$(\mathcal{F}, \leq)$ is an upper semi-modular lattice; $(\mathcal{F}, \leq^*)$ is a lower semi-modular lattice.

Finite lattice of frames



- both $y = \{1, 2/3, 4\}$ and $y' = \{1, 3/2, 4\}$ cover $y \wedge^* y' = \{1, 2, 3, 4\}$ but $y \vee^* y' = \{1/2/3/4\}$ does not cover them
- $(P(\Theta), \leq^*)$ is not upper semi-modular but lower semi-modular.

A lattice-theoretic interpretation

... of independence of sources?

- reinterpret the analogy between subspaces of a vector space V and elements of a family of compatible frames
- **both are lattices**: according to the chosen order relation we get an upper $L(\Theta)$ or lower $L^*(\Theta)$ semi-modular lattice

lattice	$L(V)$	$L^*(\Theta)$	$L(\Theta)$
initial element **0**	$\{0\}$	$\mathbf{0}_{\mathcal{F}}$	Θ
sup $l_1 \vee l_2$	$span(V_1, V_2)$	$\Theta_1 \otimes \Theta_2$	$\Theta_1 \oplus \Theta_2$
inf $l_1 \wedge l_2$	$V_1 \cap V_2$	$\Theta_1 \oplus \Theta_2$	$\Theta_1 \otimes \Theta_2$
order relation $l_1 \leq l_2$	$V_1 \subseteq V_2$	Θ_1 coars. of Θ_2	Θ_1 refin. of Θ_2
height $h(l_1)$	$\dim(V_1)$	$\|\Theta_1\| - 1$	$\|\Theta\| - \|\Theta_1\|$

Independence on lattices

from vectors ..

- abstract independence can be defined on the elements of a semi-modular lattice
- $v_1, ..., v_n$ are **linearly independent** iff

$$\sum_i \alpha_i v_i = 0 \vdash \alpha_i = 0 \quad \forall i$$

- **equivalent conditions** are

$$\mathcal{I}_1: \quad v_j \not\subset span(v_i, i \neq j) \quad \forall j = 1, ..., n;$$

$$\mathcal{I}_2: \quad v_j \cap span(v_1, ..., v_{j-1}) = 0 \quad \forall j = 2, ..., n;$$

$$\mathcal{I}_3: \quad \dim(span(v_1, ..., v_n)) = n.$$

Independence on lattices

... to the general case

- generalize these relations to collections $\{l_1, ..., l_n\}$ of non-zero elements of any semi-modular lattice with initial element $\mathbf{0}$

$$\mathcal{I}_1 : \quad l_j \not\leq \bigvee_{i \neq j} l_i \quad \forall j = 1, ..., n;$$

$$\mathcal{I}_2 : \quad l_j \wedge \bigvee_{i < j} l_i = \mathbf{0} \quad \forall j = 2, ..., n;$$

$$\mathcal{I}_3 : \quad h(\bigvee_i l_i) = \sum_i h(l_i).$$

Lattice-theoretic independence on the lattice of frames

- how we write those relations for frames?

$$\Theta_1, ..., \Theta_n \ \mathcal{I}_1^* \quad \Leftrightarrow \quad \Theta_j \oplus \bigotimes_{i \neq j} \Theta_i \neq \Theta_j \qquad \forall j = 1, ..., n$$

$$\Theta_1, ..., \Theta_n \ \mathcal{I}_2^* \quad \Leftrightarrow \quad \Theta_j \oplus \bigotimes_{i=1}^{j-1} \Theta_i = \mathbf{0}_{\mathcal{F}} \qquad \forall j = 2, ..., n$$

$$\Theta_1, ..., \Theta_n \ \mathcal{I}_3^* \quad \Leftrightarrow \quad \left| \bigotimes_{i=1}^{n} \Theta_i \right| - 1 = \sum_{i=1}^{n} (|\Theta_i| - 1)$$

- $\mathcal{I}_3^* \equiv$ the dimension of the probability polytope for the minimal refinement is the sum of the dimensions of the polytopes associated with the individual frames

Evidential independence ...

... is a stronger condition then the first two forms

Theorem

$\Theta_1, ..., \Theta_n \, \mathcal{IF}$ *and* $\Theta_j \neq \mathbf{0}_{\mathcal{F}} \; \forall j$ *then* $\Theta_1, ..., \Theta_n \, \mathcal{I}_1^*$.

Theorem

$\Theta_1, ..., \Theta_n \, \mathcal{IF} \vdash \Theta_1, ..., \Theta_n \, \mathcal{I}_2^*$.

- unless some frame is unitary, $\mathcal{IF} \vdash \mathcal{I}_1^* \wedge \mathcal{I}_2^*$
- the opposite implication does not hold

but also evidential independence ...

... is opposed to the third form

- independence of frames is **incompatible** with $\mathcal{I}_3^*$ ...

Theorem

If $\Theta_1, ..., \Theta_n \, \mathcal{IF}$, $n > 2$ then $\Theta_1, ..., \Theta_n \neg \mathcal{I}_3^*$. If $\Theta_1, \Theta_2 \, \mathcal{IF}$ then $\Theta_1, \Theta_2 \, \mathcal{I}_3^*$ iff $\exists \Theta_i = \mathbf{0}_{\mathcal{F}}$ $i \in \{1, 2\}$.

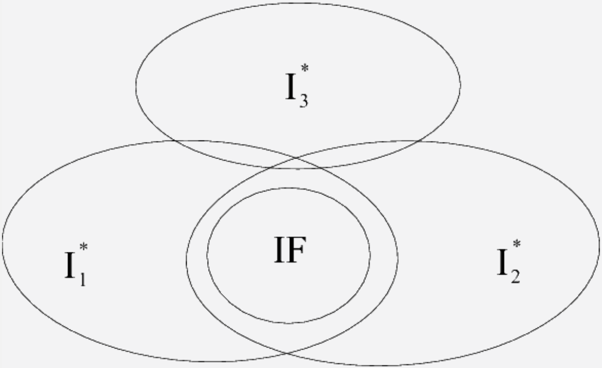
Theorem

$\Theta_1, \Theta_2 \in A^*$ are $\mathcal{IF}$ iff $\Theta_1, \Theta_2 \neg \mathcal{I}_3^*$.

- ... and they are the **negation** of each other for pairs of frames of size $n - 1$

Evidential and lattice-theoretic independence

... in a nutshell



What we learned ...

- the fact that families of frames and projective geometries are the same kind of lattices apparently explains very well the analogy between independence in uncertainty theory and algebra
- however, independence of sources is **not a form of lattice-theoretic independence**
- but it is **strictly related**, as:
  - it is stronger than some of its incarnations,
  - opposed to the remaining one

... and what is still to learn

- can independence of sources explained algebraically?
- natural to conjecture: is related to classical **matroidal** independence?
- probably the answer is no
- need for a **more general definition** of independence which encompasses both
- general analysis relation between Boolean and matroidal/lattice independence