Stability of linear density-flow hyperbolic systems under PI boundary control[☆]

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Abstract

We consider a class of density-flow systems, described by linear hyperbolic conservation laws, which can be monitored and controlled at the boundaries. These control systems are open-loop unstable and subject to unmeasured flow disturbances. We address the issue of feedback stabilization and disturbance rejection under PI boundary control. Explicit necessary and sufficient stability conditions in the frequency domain are provided.

Keywords: Hyperbolic systems, Stabilization, Proportional-Integral control

1. Introduction

We are concerned with hyperbolic systems of two linear conservation laws over a finite interval in one spatial dimension of the general form:

$$\begin{aligned} &\partial_t H + \partial_x Q = 0, \\ &\partial_t Q + \lambda_1\lambda_2\partial_x H + (\lambda_1 - \lambda_2)\partial_x Q = 0, \end{aligned} \qquad t \in [0, +\infty), \quad x \in [0, L], \tag{1}$$

where λ_1 and λ_2 are two real positive constants. In these equations $H(t, x)$ is the density and $Q(t, x)$ is the flow density of some extensive quantity of interest. Therefore, this system is called a "density-flow" system. For instance, this system may be used as a valid approximate linearised model for lossless electrical lines or for horizontal and frictionless open channels or gas pipes, see Fig.1.

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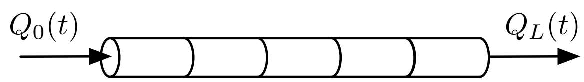

Figure 1: A density-flow system.

We are concerned with the solutions of the Cauchy problem for the system (1) under an initial condition:

$$H(0,x), Q(0,x), \qquad x \in [0,L],$$

and two boundary conditions of the form:

$$Q(t,0) = Q_0(t), \qquad Q(t,L) = Q_L(t), \qquad t \in [0,+\infty). \tag{2}$$

Any pair of constant states H^*, Q^* is a potential steady-state of the system.

The Riemann coordinates are defined around a steady-state by the following change of coordinates:

$$R_1 = Q - Q^* + \lambda_2(H - H^*),$$

$$R_2 = Q - Q^* - \lambda_1(H - H^*).$$

The inverse change of coordinates is:

$$H = H^* + \frac{R_1 - R_2}{\lambda_1 + \lambda_2},$$

$$Q = Q^* + \frac{\lambda_1 R_1 + \lambda_2 R_2}{\lambda_1 + \lambda_2}.$$

With these coordinates, the system (1) is written in characteristic form:

$$\partial_t R_1 + \lambda_1 \partial_x R_1 = 0, \qquad \partial_t R_2 - \lambda_2 \partial_x R_2 = 0. \tag{3}$$

Then, assuming a constant flow rate $Q_0(t) = Q_L(t) = Q^*$ and expressing the boundary conditions (2) in Riemann coordinates, we have:

$$R_1(t,0) = k_1 R_2(t,0), \quad R_2(t,L) = k_2 R_1(t,L),$$

with

$$k_1 = -\frac{\lambda_2}{\lambda_1}, \quad k_2 = -\frac{\lambda_1}{\lambda_2}.$$

Consequently $|k_1 k_2| = 1$ and the steady state (H^*, Q^*) is not asymptotically stable.

It is therefore relevant to study the boundary feedback stabilization of the control system (1)-(2). It is the main concern of this paper.

The PI control structure is described in Section 2. Then, in Section 3, explicit necessary and sufficient stability conditions in the frequency domain are provided. The analysis is in the continuation of previous contributions on PI control of hyperbolic systems by [9], [6], [4]. Finally in Section 4 we show how the stability analysis can be extended to (acyclic) networks of density-flow systems.

2. The PI control structure

We consider the situation where there is only one boundary control input, say $Q_0(t)$, available for feedback stabilization. The other boundary flow $Q_L(t)$ perturbs the system in an unpredictable manner. This disturbance flow cannot be measured and cannot therefore be directly compensated in the control.

We assume that, in addition to stabilization, the control objective is to regulate $H(t,x)$ at the "set point" H^* by using on-line feedback measurements of $H(t,0)$.

In such case, in order to eliminate offsets, it is useful to implement an "integral" action. A so-called "Proportional-Integral" (PI) control law may be of the following form:

$$Q_0(t) \triangleq Q_R + k_P(H^* - H(t,0)) + k_I \int_0^t (H^* - H(\tau,0))d\tau. \tag{4}$$

The first term Q_R is a constant reference value for the flow which is arbitrary and freely chosen by the designer. The second term is the proportional correction action with the tuning parameter k_P. The last term is the integral action with the tuning parameter k_I. In case of a constant (unknown) disturbance $Q_L(t) = Q^*$, the closed-loop system has a unique steady-state (H^*, Q^*). The control structure is illustrated in Fig.2.

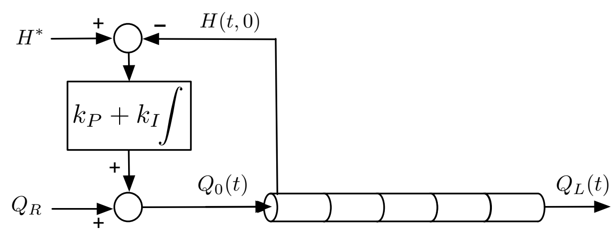

Figure 2: Block diagram of the closed-loop system with a Proportional-Integral control.

3. Stability conditions

In Riemann coordinates, the control law (4) provides a first boundary condition at $x = 0$:

$$R_1(t,0) = k_1 R_2(t,0) + k_0 Z(t), \tag{5}$$

$$\text{with } \; k_1 \triangleq \frac{k_P - \lambda_2}{k_P + \lambda_1}, \;\; k_0 \triangleq \frac{k_I}{k_P + \lambda_1} \;\text{ and}$$

$$Z(t) \triangleq \frac{Q_R - Q^*}{k_I}(\lambda_1 + \lambda_2) + \int_0^t (R_2(\tau,0) - R_1(\tau,0))d\tau.$$

The constant disturbance $Q_L(t) = Q^*$ gives the second boundary condition at $x = L$:

$$R_2(t, L) = k_2 R_1(t, L) \;\text{ with }\; k_2 = -\frac{\lambda_1}{\lambda_2}. \tag{6}$$

From (6), since $R_1(t, x)$ and $R_2(t, x)$ are constant along their respective characteristic lines, we have that

$$R_2(t + \tau, 0) = k_2 R_1(t, 0) \quad \text{ with } \quad \tau \triangleq \frac{L}{\lambda_1} + \frac{L}{\lambda_2} \tag{7}$$

and therefore that

$$\frac{dR_2(t + \tau, 0)}{dt} = k_2 \frac{dR_1(t, 0)}{dt}. \tag{8}$$

Moreover, by differentiating (5) with respect to time, the first boundary condition is rewritten as:

$$\frac{dR_1(t, 0)}{dt} = k_1 \frac{dR_2(t, 0)}{dt} + k_0\big(R_2(t, 0) - R_1(t, 0)\big). \tag{9}$$

Then, by eliminating $R_1(t, 0)$ and $dR_1(t, 0)/dt$ between (7),(8) and (9), we get that $R_2(t, 0)$ is the solution of the following delay-differential equation of neutral type:

$$\frac{dR_2(t + \tau, 0)}{dt} - k_1 k_2 \frac{dR_2(t, 0)}{dt} + k_0\Big(R_2(t + \tau, 0) - k_2 R_2(t, 0)\Big) = 0. \tag{10}$$

The Laplace transform of this equation is:

$$\Big[(e^{s\tau} - k_1 k_2)s + k_0(e^{s\tau} - k_2)\Big] R_2(s, 0) = 0. \tag{11}$$

The poles of the system are the roots of the characteristic equation

$$(e^{s\tau} - k_1 k_2)s + k_0(e^{s\tau} - k_2) = 0. \tag{12}$$

We then have the following stability Theorem. In the proof of this Theorem we use a variant of the Walton-Marshall procedure (see [8] and [7, Section 5.6]).

Theorem 1.
The poles of the system (11) have a strictly negative real part if and only if

- when $\lambda_1 \leqslant \lambda_2$ (i.e. $-1 \leqslant k_2 < 0$),

$$|k_1 k_2| < 1 \quad \text{ and } \quad 0 < k_0;$$

- when $\lambda_1 > \lambda_2$ (i.e. $k_2 < -1$),

$$|k_1 k_2| < 1 \quad \text{ and } \quad 0 < k_0 < \omega_0 \frac{k_2(1 + k_1)}{1 - k_2^2} \sin(\omega_0 \tau)$$

  where ω_0 is the smallest positive ω such that $\cos(\omega\tau) = \dfrac{1 + k_1 k_2^2}{k_2(1 + k_1)}$.

Proof. A fundamental property in the stability analysis of the neutral delay-differential system (11) is that $|k_1k_2| < 1$ is a necessary condition to have poles with a strictly negative real part i.e. $\Re(s) < 0$ (see e.g. [2] and [5]). It is also easily checked that, for every k_1 and k_2, for every $\eta > \ln(|k_1k_2|)$ and for every $C_0 > 0$, there exists $C_1 > 0$ such that

$$\Big\{|k_0| \leqslant C_0,\ |s| \geqslant C_1 \text{ and } (12)\Big\} \Rightarrow \Big\{\Re(s) \leqslant \eta\Big\}. \tag{13}$$

With the notation $s \triangleq \sigma + i\omega$, the poles satisfy the following equation:

$$\begin{aligned} k_0 &= -\frac{(e^{s\tau} - k_1k_2)s}{e^{s\tau} - k_2} \\ &= \frac{[\omega a(\sigma,\omega) - \sigma b(\sigma,\omega)] + i[\sigma a(\sigma,\omega) - \omega b(\sigma,\omega)]}{e^{2\sigma\tau} + k_2^2 - 2k_2e^{\sigma\tau}\cos\omega\tau} \end{aligned} \tag{14}$$

with

$$a(\sigma,\omega) \triangleq k_2e^{\sigma\tau}(k_1 - 1)\sin\omega\tau \ \text{ and} \tag{15}$$

$$b(\sigma,\omega) \triangleq e^{2\sigma\tau} - k_2(1+k_1)e^{\sigma\tau}\cos\omega\tau + k_1k_2^2. \tag{16}$$

Since the left-hand side of equation (14) is real, it follows that the imaginary part of the right-hand side must be zero. Therefore we are looking for the values of σ and ω such that

$$\sigma a(\sigma,\omega) = \omega b(\sigma,\omega). \tag{17}$$

Let us now consider the poles with non-positive real parts, i.e. $\sigma \leqslant 0$. If $k_0 = 0$, we see that the poles are roots of $(e^{s\tau} - k_1k_2)s = 0$. This means that there is a pole $s = 0$ at the origin and the other poles are stable if and only if $|k_1k_2| < 1$. Now for small non-zero k_0, we have:

$$(1 - k_1k_2)s + k_0(1 - k_2) \approx 0,$$

that is

$$s = -k_0\frac{1 - k_2}{1 - k_1k_2}.$$

Since $|k_1k_2| < 1$ and $k_2 = -\lambda_1/\lambda_2 < 0$, then, for small $k_0 > 0$, the pole at zero moves inside the negative half-plane while the other poles stay inside the negative half-plane.

Now, in order to analyze what happens when $k_0 > 0$ becomes larger, we consider the conditions for having poles on the imaginary axis i.e. $\sigma = 0$. Since $k_0 \neq 0$, the case $\sigma = 0, \omega = 0$ is excluded. Therefore $\sigma = 0$ implies $b = 0$ from (17), which together with (16) gives:

$$\cos(\omega\tau) = \frac{1 + k_1k_2^2}{k_2(1 + k_1)}. \tag{18}$$

In this case, it can be readily verified that if $|k_1k_2| < 1$, then

$$\lambda_1 < \lambda_2 \iff |k_2| < 1 \iff \left|\frac{1 + k_1k_2^2}{k_2(1 + k_1)}\right| > 1,$$

which implies that there is no eigenvalue on the imaginary axis (since $|\cos\omega\tau| \leqslant 1$ obviously). Then, using also (13), we can conclude, using a standard deformation argument on k_0, that, when $|k_2| < 1$ and $|k_1 k_2| < 1$, the poles remain stable for every $k_0 > 0$.

Let us now consider the case where $|k_1 k_2| < 1$ and $\lambda_1 > \lambda_2 \Leftrightarrow k_2 < -1$ (the case $\lambda_1 = \lambda_2$ is discussed later). In this case, it can be readily verified that

$$\left| \frac{1 + k_1 k_2^2}{k_2(1 + k_1)} \right| < 1.$$

Therefore, from (14) and (16) with $\sigma = 0$, there is a pair of poles $\pm i\omega$ on the imaginary axis for any positive value of ω such that:

$$\cos(\pm\omega\tau) = \frac{1 + k_1 k_2^2}{k_2(1 + k_1)} \quad \text{and} \quad \omega \sin(\omega\tau) = -\frac{k_0(k_2^2 - 1)}{k_2(1 + k_1)}. \tag{19}$$

Let ω_0 be the smallest value of ω such that (19) is satisfied. Now, if $i\omega_0$ is a pole on the imaginary axis, the corresponding value of k_0 computed from (19) $\omega = \omega_0$ is as follows:

$$k_0^* = \omega_0 \frac{k_2(1 + k_1)}{1 - k_2^2} \sin(\omega_0 \tau) > 0.$$

Then, using again (13), we can conclude, using a standard deformation argument on k_0, that the poles are stable for any k_0 such that $0 < k_0 < k_0^*$. In order to determine the motion of the pole on the imaginary axis for small variations of k_0 around k_0^*, we consider the root s of the characteristic equation as an explicit function of k_0. Then, by differentiating the characteristic equation (12), we have the following expression for the derivative of s with respect to k_0:

$$s' \triangleq \frac{ds}{dk_0} = \frac{k_2 - e^{s\tau}}{e^{s\tau}(1 + s(1 + k_0)) - k_1 k_2}. \tag{20}$$

We now evaluate this expression at $i\omega$:

$$s' = \frac{(k_2 - \cos\omega\tau) - i(\sin\omega\tau)}{[\cos\omega\tau - (1 + k_0)\omega\sin\omega\tau - k_1 k_2] + i[\sin\omega\tau + (1 + k_0)\omega\cos\omega\tau]}.$$

The real part of s' is given by

$$\Re(s') = \frac{k_2(1 + k_1)\cos(\omega\tau) - 1 - k_2(1 + k_0)\omega\sin(\omega\tau) - k_1 k_2^2}{[\cos(\omega\tau) - (1 + k_0)\omega\sin(\omega\tau) - k_1 k_2]^2 + [\sin(\omega\tau) + (1 + k_0)\omega\cos(\omega\tau)]^2}.$$

Now using (19), after some calculations, we get:

$$\Re(s') = \frac{(k_0 + k_0^2)(k_2^2 - 1)}{(1 + k_1)\Big([\cos(\omega\tau) - (1 + k_0)\omega\sin(\omega\tau) - k_1 k_2]^2 + [\sin(\omega\tau) + (1 + k_0)\omega\cos(\omega\tau)]^2\Big)}.$$

Hence, since $k_0 > 0$, $k_2 < -1$ and $|k_1| < 1$ by assumptions, $\Re(s')$ is a positive number. It follows that any pole reaching the imaginary axis from the left when

k_0 is increasing will cross the imaginary axis from left to right. This readily implies that, as soon as $k_0 > k_0^*$, there is necessarily at least one pole in the right half plane.

Let us finally consider the case where $\lambda_1 = \lambda_2$ (i.e. $k_2 = -1$). In that case, it follows directly from (18) that $\cos(\omega\tau) = -1$ and $\sin(\omega\tau) = 0$ for any pole $i\omega$ on the imaginary axis. Therefore the characteristic equation (12) reduces to

$$(k_1 - 1)i\omega = 0$$

which is impossible if $\omega \neq 0$ because the conditions $k_2 = -1$ and $|k_1 k_2| < 1$ imply that $|k_1| < 1$. Hence there is no imaginary pole when $\lambda_1 = \lambda_2$. This completes the proof of Theorem 1. $\square$

As a matter of illustration, a sketch of the root locus for fixed values of k_1 and k_2 and increasing values of k_0 from 0 to $+\infty$ is given in Fig.3.

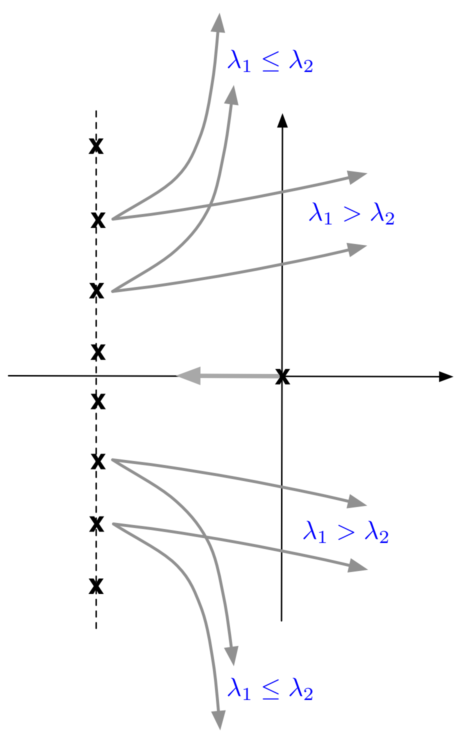

Figure 3: Sketch of the root locus for fixed values of k_1 and k_2 and increasing values of k_0 from 0 to $+\infty$.

In the previous Theorem, for the clarity of the proof, we have carried out the analysis in terms of the parameters k_0, k_1 and k_2. However, from a practical viewpoint, it is clearly more relevant and more interesting to express the stability conditions in terms of the control tuning parameters k_P and k_I. Replacing k_0, k_1 and k_2 by their expressions in function of k_P, k_I, λ_1 and λ_2 as given in (5)-(6), the conditions of Theorem 1 are translated as follows.

Theorem 2. The poles of the closed-loop system (1)-(4) have a strictly negative real part if and only if the control tuning parameters k_P, k_I are selected such that:

- when $\lambda_1 < \lambda_2$,
$$k_P > 0 \text{ and } k_I > 0 \qquad \text{or} \qquad k_P < -\frac{2\lambda_1\lambda_2}{\lambda_2 - \lambda_1} \text{ and } k_I < 0;$$
- when $\lambda_1 = \lambda_2$, $\quad k_P > 0$ and $k_I > 0$;
- when $\lambda_1 > \lambda_2$,
$$k_P > 0 \text{ and } 0 < k_I < \omega_0 \frac{(2k_P + \lambda_1 - \lambda_2)\lambda_1\lambda_2}{\lambda_1 - \lambda_2} \sin(\omega_0 \tau)$$
where ω_0 is the smallest positive ω such that
$$\cos(\omega\tau) = \frac{\lambda_2^2(k_P + \lambda_1) + \lambda_1^2(k_P - \lambda_2)}{\lambda_1\lambda_2(\lambda_2 - \lambda_1 - 2k_P)}$$

□

4. Networks of density-flow systems

In this section, we examine how the previous stability analysis can be extended to (acyclic) networks of density-flow systems. Depending on the concerned application, there are different ways of designing such networks. Here, as a matter of example, we consider a specific structure which leads to a natural generalization of the control problem addressed in the previous section. But other structures could be dealt with in a similar way (see e.g. [3], [1] for relevant related references).

The network has a compartmental structure illustrated in Fig.4. The nodes of the network are n storage compartments having the dynamics of density-flow systems (e.g. the pipes of an hydraulic network):

$$\begin{cases} \partial_t H_j + \partial_x Q_j = 0, \\ \partial_t Q_j + \lambda_j \lambda_{n+j} \partial_x H_j + (\lambda_j - \lambda_{n+j}) \partial_x Q_j = 0, \end{cases} \qquad j = 1, \ldots, n. \tag{21}$$

Without loss of generality and for simplicity, it can always be assumed that, by an appropriate scaling, all the pipes have exactly the same length L.

The directed arcs $i \to j$ of the network represent instantaneous transfer flows between the compartments. Additional input and output arcs represent interactions with the surroundings: either inflows injected from the outside into

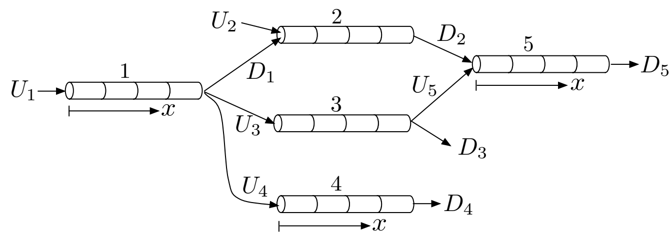

Figure 4: Physical network of density-flow systems

some compartments or outflows from some compartments to the outside. We assume that there is exactly one and only one control flow, denoted U_i, at the input of each compartment. All the other flows are assumed to be disturbances and denoted D_k $(k = 1, \dots, m)$. The set of $2n$ PDEs (21) is therefore subject to $2n$ boundary flow balance conditions of the form:

$$\begin{aligned} Q_i(t,0) &= U_i(t) + \sum_{k=1}^{m} \beta_{ik} D_k(t), \quad i = 1, \dots, n, \\ Q_i(t,L) &= \sum_{j=1}^{n} U_j(t) + \sum_{k=1}^{m} \gamma_{ki} D_k(t), \quad i = 1, \dots, n. \end{aligned} \tag{22}$$

In the summations, only the terms corresponding to actual links between adjacent compartments of the network are taken into account, i.e. the coefficients β_{ik} and γ_{ik} are equal to 1 for the existing links and 0 for the others (see Fig.4 for illustration).

With the matrix notations

$$\mathbf{H} \triangleq \begin{pmatrix} H_1 \\ \vdots \\ H_n \end{pmatrix}, \ \mathbf{Q} \triangleq \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix}, \ \mathbf{U} \triangleq \begin{pmatrix} U_1 \\ \vdots \\ U_n \end{pmatrix}, \ \mathbf{D} \triangleq \begin{pmatrix} D_1 \\ \vdots \\ D_m \end{pmatrix},$$

$$\Lambda^+ = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}, \qquad \Lambda^- = \mathrm{diag}\{\lambda_{n+1}, \dots, \lambda_{2n}\},$$

the system (21) is written

$$\begin{aligned} &\partial_t \mathbf{H} + \partial_x \mathbf{Q} = 0, \\ &\partial_t \mathbf{Q} + \Lambda^+\Lambda^- \partial_x \mathbf{H} + (\Lambda^+ - \Lambda^-)\partial_x \mathbf{Q} = 0. \end{aligned} \tag{23}$$

The boundary conditions (22) are written

$$\begin{aligned} \mathbf{Q}(t,0) &= \mathbf{U}(t) + B_0 \mathbf{D}(t), \\ \mathbf{Q}(t,L) &= A_L \mathbf{U}(t) + B_L \mathbf{D}(t), \end{aligned} \tag{24}$$

with appropriate matrices A_L, B_0, B_L. Let p denote the length of the longest path in the considered network of density-flow systems. The matrix A_L, which describes the structure of the network, has the property that

$$A_L^p = 0. \tag{25}$$

A steady state for the system (23)-(24) is a quadruple

$$\{\mathbf{H}^*, \mathbf{Q}^*, \mathbf{U}^*, \mathbf{D}^*\}$$

which satisfies the boundary conditions:

$$\mathbf{Q}^* = \mathbf{U}^* + B_0\mathbf{D}^*,$$

$$\mathbf{Q}^* = A_L\mathbf{U}^* + B_L\mathbf{D}^*.$$

The network has an infinity of positive steady states which are not asymptotically stable. In order to stabilize the network, each control input is endowed with a PI control law of the form:

$$U_i(t) \triangleq U_R + k_{Pi}(H_i^* - H_i(t,0)) + k_{Ii}\int_0^t (H_i^* - H_i(\tau,0))d\tau, \tag{26}$$

where U_R is an abitrary scaling constant, H_i^* is the set point for the i-th compartment, k_{Pi} and k_{Ii} are the control tuning parameters. In matrix form, the set of control laws (26) is written

$$\mathbf{U} = \mathbf{U}_R + K_P\Big(\mathbf{H}^* - \mathbf{H}(t,0)\Big) + K_I\int_0^t \Big(\mathbf{H}^* - \mathbf{H}(\tau,0)\Big)d\tau, \tag{27}$$

with $K_P \triangleq \text{diag}\{k_{P1}, \dots k_{Pn}\}$ and $K_I \triangleq \text{diag}\{k_{I1}, \dots k_{In}\}$.

We shall now examine how the stability analysis of Section 4 for the "single pipe" case can be generalised to the closed-loop network (23)-(24)-(27) for constant unknown disturbances $\mathbf{D}^*$. The Riemann coordinates are defined as follows:

$$\begin{cases} R_i \triangleq Q_i - Q^* + \lambda_{n+i}(H_i - H_i^*) \\ R_{n+i} \triangleq Q_i - Q^* - \lambda_i(H_i - H_i^*) \end{cases} \quad i = 1, \dots, n, .$$

Using this definition, the following equalities hold at the boundaries:

$$\begin{aligned}(\lambda_i + \lambda_{n+i})(Q_i(t,0) - Q^*) &= (\lambda_i + \lambda_{n+i})\Big[k_{P_i}(H_i^* - H_i(t,0)) + k_{Ii}Z_i(t)\Big] \\ &= \lambda_i R_i(t,0) + \lambda_{n+i}R_{n+i}(t,0) \\ &= k_{P_i}(R_{n+i}(t,0) - R_i(t,0)) + (\lambda_i + \lambda_{n+i})k_{Ii}Z_i(t),\end{aligned}$$

$$(\lambda_i + \lambda_{n+i})(Q_i(t,L) - Q^*) = \lambda_i R_i(t,L) + \lambda_{n+i}R_{n+i}(t,L),$$

with $Z_i(t)$ such that

$$\frac{dZ_i}{dt} = H_i^* - H_i(t,0) = \frac{R_{n+i}(t,0) - R_i(t,0)}{\lambda_i + \lambda_{n+i}}.$$

Since $R_i(t,x)$ and $R_{n+i}(t,x)$ are constant along their respective characteristic lines, we have that

$$R_i(t+\frac{L}{\lambda_i},L)=R_i(t,0) \text{ and } R_{n+i}(t+\frac{L}{\lambda_{n+i}},0)=R_{n+i}(t,L).$$

Then, combining appropriately these equalities, it can be shown after some computations that, in the frequency domain, the transfer function between $(Q_i(t,L)-Q^*)$ and $(Q_i(t,0)-Q^*)$ is given by:

$$G_i(s)\triangleq\frac{Q_i(s,0)-Q^*}{Q_i(s,L)-Q^*}=\frac{1}{\lambda_{n+i}}\frac{s(\lambda_i k_i-\lambda_{n+i})+c_i(\lambda_i-\lambda_{n+i})}{(e^{s\tau_i}-k_ik_{n+i})s+c_i(e^{s\tau_i}-k_{n+i})}e^{\frac{sL}{\lambda_i}},$$

with the following notations:

$$k_i\triangleq\frac{k_{Pi}-\lambda_{n+i}}{k_{Pi}+\lambda_i},\qquad k_{n+i}\triangleq-\frac{\lambda_i}{\lambda_{n+i}},$$

$$c_i\triangleq\frac{k_{Ii}}{k_{Pi}+\lambda_i},\qquad \tau_i\triangleq\frac{L}{\lambda_i}+\frac{L}{\lambda_{n+i}}.$$

It follows that the poles of the transfer function $G_i(s)$ are the roots of the characteristic equation

$$(e^{s\tau_i}-k_ik_{n+i})s+c_i(e^{s\tau_i}-k_{n+i})=0$$

which is, as expected, identical to the characteristic equation of the simple case of Section 3.

Let us now consider the closed-loop system (23)-(24)-(27) as an input-output dynamical system with input $\mathbf{D}$ and output $\mathbf{U}$. Then, by iterating equations (24) p-times and using property (25), it can be shown that the transfer matrix of the system is as follows:

$$H(s)\triangleq\sum_{i=0}^{p-1}(G(s)A_L)^i(G(s)B_L-B_0),$$

with $G(s)\triangleq\mathrm{diag}\{G_1(s),\ldots,G_n(s)\}$. It follows readily that the poles of $H(s)$ are given by the collection of the poles of the individual scalar transfer function $G_i(s)$. Consequently, the system is stable if and only if the conditions of Theorem 2 hold for each PI controller of the network.

5. Conclusion

In this paper we have addressed the issue of feedback stabilization and disturbance rejection for hyperbolic density flow systems under PI boundary control. Explicit necessary and sufficient stability conditions in the frequency domain have been provided. It has also been shown how the stability analysis can be extended to acyclic networks of density-flow systems.

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