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Three new models for preference voting and aggregation

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Preference voting and aggregation require the determination of the weights associated with different ranking places. This paper proposes three new models to assess the weights. Two of them are linear programming (LP) models which determine a common set of weights for all the candidates considered and the other is a nonlinear programming (NLP) model that determines the most favourable weights for each candidate. The proposed models are examined with two numerical examples and it is shown that the proposed models cannot only choose a winner, but also give a full ranking of all the candidates.

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Introduction

In preferential voting systems, each voter selects m candidates from among n candidates ($n \geqslant m$) and ranks them from the most to the least preferred. Each candidate may receive some votes in different ranking places. The total score of each candidate is the weighted sum of the votes he/she receives in different places. The winner is the one with the biggest total score. So, the key issue of the preference aggregation in a preferential voting system is how to determine the weights associated with different ranking places.

Borda–Kendall (BK) method (Cook and Kress, 1990; Wang *et al*, 2005) is perhaps the most widely used procedure for determining the weights. By the BK method, the first place is given a weight or mark of m, the second place is given a weight or mark of $m-1$, followed by $m-2, \ldots, 2$ and the last place is given a weight or mark of one. Because of its computational simplicity, the BK method is very popular. But the determination of the weights is somewhat subjective.

To avoid the subjectivity in determining the weights, Cook and Kress (1990) suggest using data envelopment analysis (DEA) to determine the most favourable weights for each candidate. Different candidates utilize different sets of weights to calculate their total scores, which are referred to as the best relative total scores and are all restricted to be less than or equal to one. The candidate with the biggest relative total score of one is said to be DEA efficient and may be considered as a winner. This approach proves to be effective, but very often leads to more than one candidate to be DEA efficient.

To choose a winner from among the DEA-efficient candidates, Cook and Kress (1990) suggest maximizing the gap between the weights so that only one candidate is left DEA efficient. This has been found equivalent to imposing a common set of weights on all the candidates and equivalent to the BK method in a specific discrimination intensity function. Green *et al* (1996) suggest using the cross-efficiency evaluation technique in DEA to choose the winner. Noguchi *et al* (2002) also utilize cross-efficiency evaluation technique to select the winner, but present a strong ordering constraint condition on the weights. Hashimoto (1997) proposes the use of the DEA exclusion model (ie super-efficiency model) to identify the winner. Obata and Ishii (2003) suggest excluding non-DEA-efficient candidates and using normalized weights to discriminate the DEA-efficient candidates. Their method is subsequently extended to rank non-DEA-efficient candidates by Foroughi and Tamiz (2005) (see also Foroughi *et al*, 2005). It is found that although the DEA approaches mentioned above do not require predetermining the weights subjectively, they still need to choose a discrimination intensity function and/or discriminating intensity factor subjectively. It is also found that the winner may be unstable and affected by the choice of discrimination intensity function and discriminating intensity factor.

To avoid the instability in choosing a winner, we propose three new models in this paper to determine the weights of ranking places. The proposed models do not require

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predetermining any discrimination intensity function and discriminating intensity factor subjectively and each of them can lead to a stable full ranking for all the candidates considered. This will be illustrated with two numerical examples.

The rest of the paper is organized as follows. In the next section, we develop the models for preference aggregation to assess the weights associated with different ranking places. We then examine two numerical examples using the proposed models to illustrate their applications and show their capabilities of identifying the winner and producing a stable full ranking for all the candidates considered. Finally, we conclude the paper.

The models

Let w_j be the relative importance weight attached to the jth ranking place ($j=1, \ldots, m$) and v_{ij} be the vote of candidate i being ranked in the jth place. The total score of each candidate is defined as

$$Z_i = \sum_{j=1}^{m} v_{ij} w_j, \qquad i = 1, \ldots, n \tag{1}$$

which is a linear function of the relative importance weights. Once the weights are given or determined, candidates can be ranked in terms of their total scores.

To determine the relative importance weights, Cook and Kress (1990) suggest the following DEA model, which determines the most favourable weights for each candidate:

$$\begin{aligned} \text{Maximize} \quad & Z_i = \sum_{j=1}^{m} v_{ij} w_j \\ \text{s.t.} \quad & \sum_{j=1}^{m} v_{ij} w_j \leqslant 1, \qquad i = 1, \ldots, n \\ & w_j - w_{j+1} \geqslant d(j, \varepsilon), \qquad j = 1, \ldots, m-1 \\ & w_m \geqslant d(m, \varepsilon) \end{aligned} \tag{2}$$

where $d(., \varepsilon)$ is referred to as a discrimination intensity function that is nonnegative and monotonically increasing in a nonnegative discriminating intensity factor ε and satisfies $d(., 0) = 0$. It has been found that the choice of the discrimination intensity functional $d(., \varepsilon)$ and the discriminating intensity factor ε has significant impacts on the winner. For example, Cook and Kress (1990) investigate three special cases of the discrimination intensity function $d(., \varepsilon)$: $d(j, \varepsilon) = \varepsilon$, $d(j, \varepsilon) = \varepsilon/j$ and $d(j, \varepsilon) = \varepsilon/j!$. Each of them leads to a different winner. Noguchi *et al* (2002) examine the six special cases of the discriminating intensity factor ε: $\varepsilon = 0$, 0.01, 0.05, 0.055, 0.06 and 0.07. These cases also result in different winners.

To avoid the difficulties in determining the discrimination intensity function $d(., \varepsilon)$ and the discriminating intensity factor ε, Noguchi *et al* (2002) suggest a strong ordering DEA model, which is shown below:

$$\begin{aligned} \text{Maximize} \quad & Z_i = \sum_{j=1}^{m} v_{ij} w_j \\ \text{s.t.} \quad & \sum_{j=1}^{m} v_{ij} w_j \leqslant 1, \qquad i = 1, \ldots, n \\ & w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m \\ & w_m \geqslant \varepsilon = \frac{2}{Nm(m+1)} \end{aligned} \tag{3}$$

where N is the number of voters. In our view, the strong ordering constraint $w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m$ makes sense because it satisfies $w_1 > w_2 > \cdots > w_m$ and $w_1 - w_2 > w_2 - w_3 > \cdots > w_{m-1} - w_m > 0$. It also makes the choice of the discrimination intensity function $d(., \varepsilon)$ unnecessary. So, this strong ordering constraint will be adopted in the new models to be developed. However, it is found that the choice of the discriminating intensity factor ε is somewhat arbitrary and there is no evidence to support ε to take the value of $2/Nm(m+1)$. In effect, ε can take any value within the interval $[0, 1/Nm]$. In addition, to determine the value of ε in model (3), the number of voters is required to be known, but this is not always the case (see Example 2 for an instance).

In what follows, we present our new models, which do not require predetermining any parameters because the new models usually produce only one best candidate and there is no need to make any further choice with the help of any parameters. The new models are given as follows:

LP-1:

$$\begin{aligned} \text{Maximize} \quad & \alpha \\ \text{s.t.} \quad & Z_i = \sum_{j=1}^{m} v_{ij} w_j \geqslant \alpha, \qquad i = 1, \ldots, n \\ & w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m \geqslant 0 \\ & \sum_{j=1}^{m} w_j = 1 \end{aligned} \tag{4}$$

LP-2:

$$\begin{aligned} \text{Maximize} \quad & \alpha \\ \text{s.t.} \quad & \alpha \leqslant Z_i = \sum_{j=1}^{m} v_{ij} w_j \leqslant 1, \qquad i = 1, \ldots, n \\ & w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m \geqslant 0 \end{aligned} \tag{5}$$

NLP-1:

$$\begin{aligned} \text{Maximize} \quad & Z_i = \sum_{j=1}^{m} v_{ij} w_j \\ \text{s.t.} \quad & w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m \geqslant 0 \\ & \sum_{j=1}^{m} w_j^2 = 1 \end{aligned} \tag{6}$$

LP-1 and LP-2 are two linear programming models. Both of them maximize the minimum of the total scores of the n

candidates and determine a common set of weights for all the candidates. The differences between the two models lie in that LP-1 requires the weights to be summed to one, while LP-2 does not, and that LP-2 requires the total score of each candidate to be equal to or less than one, while LP-1 has no such a requirement. Once the weights are determined, the total score of each candidate can be computed by Equation (1) and the winner can be selected.

NLP-1 is a nonlinear programming model, which determines the most favourable weights within the feasible region

$$\Omega = \left\{ W = (w_1, \ldots, w_m) \,|\, w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m \geqslant 0, \sum_{j=1}^{m} w_j^2 = 1 \right\}$$

for each candidate. The NLP-1 is a variant of the following multiple attribute decision making model developed by Wang and Fu (1993):

$$\begin{aligned} \text{Maximize} \quad & Z_i = \sum_{j=1}^{m} z_{ij} w_j \\ \text{s.t.} \quad & \sum_{j=1}^{m} w_j^2 = 1 \\ & w_j \geqslant 0, \qquad j = 1, \ldots, m \end{aligned} \tag{7}$$

where z_{ij} is the normalized attribute value of the ith decision alternative with respect to the jth attribute and w_j is the relative importance weight of the jth attribute. The analytical solution to model (7) is found to be

$$w_j^* = z_{ij} \Big/ \sqrt{\sum\nolimits_{j=1}^{m} z_{ij}^2}, \qquad j = 1, \ldots, m \tag{8}$$

However, due to the presence of the strong ordering constraint $w_1 \geqslant 2w_2 \geqslant \cdots \geqslant m w_m \geqslant 0$, the NLP-1 cannot usually be solved analytically, but can be solved using Microsoft Excel Solver or the LINGO software package very easily.

Numerical examples

In this section, we examine two numerical examples using the proposed models to illustrate their applications and show their capabilities of choosing the winner and ranking candidates. Models, linear and nonlinear, are all implemented in Microsoft Excel worksheets and are solved using MS-Excel Solver.

Example 1 Consider the example investigated by Cook and Kress (1990), in which 20 voters are asked to rank four out of six candidates A–F on a ballot. The votes each candidate receives are shown in Table 1.

Table 1 Votes received by six candidates

Candidate	*First place*	*Second place*	*Third place*	*Fourth place*
A	3	3	4	3
B	4	5	5	2
C	6	2	3	2
D	6	2	2	6
E	0	4	3	4
F	1	4	3	3

For this example, $n = 6$, $m = 4$ and $N = 20$. If we set $\varepsilon = 2/Nm(m+1)$ and $4/Nm(m+1)$, respectively, and solve Noguchi *et al*'s model (3), then three and two DEA-efficient candidates are, respectively, identified. This shows the choice of ε does have impact on the results and should be chosen very carefully. However, if the new models are employed to solve the example, then there is no need to choose any parameter. By solving LP-1, we get

$$\alpha^* = 1.92, \quad w_1^* = 0.48, \quad w_2^* = 0.24, \quad w_3^* = 0.16 \quad \text{and} \quad w_4^* = 0.12$$

Solving LP-2, we have

$$\alpha^* = 0.4364, \quad w_1^* = 0.1091, \quad w_2^* = 0.0545, \quad w_3^* = 0.0364 \quad \text{and} \quad w_4^* = 0.0273$$

By solving NLP-1 for each candidate, we obtain the following two sets of weights for different candidates:

$$w_1^* = 0.8381, \quad w_2^* = 0.4191, \quad w_3^* = 0.2794 \quad \text{and} \quad w_1^* = 0.2095 \quad \text{for candidates A, B, D, E, and F}$$

$$w_1^* = 0.8422, \quad w_2^* = 0.4142, \quad w_3^* = 0.2761 \quad \text{and} \quad w_1^* = 0.2071 \quad \text{for candidate C}$$

Table 2 Scores and rankings of the six candidates by different models

	LP-1		*LP-2*		*NLP-1*		*Borda–Kendall*	
Candidate	*Score*	*Rank*	*Score*	*Rank*	*Score*	*Rank*	*Score*	*Rank*
A	3.16	4	0.7182	4	5.52	4	32	4
B	4.16	2	0.9455	2	7.26	2	43	1
C	4.08	3	0.9273	3	7.12	3	38	3
D	4.40	1	1	1	7.68	1	40	2
E	1.92	6	0.4364	6	3.35	6	22	6
F	2.28	5	0.5182	5	3.98	5	25	5

The rankings of the six candidates produced by the three new models are shown in Table 2, from which it is clear that the three new models all lead to the same ranking, that is, $D \succ B \succ C \succ A \succ F \succ E$, where the symbol '$\succ$' means 'be preferred or superior to'. So, candidate D is the winner. Such a conclusion is consistent with Cook and Kress's

Table 3 Votes received by seven candidates

Candidate	*First rank*	*Second rank*
A	32	10
B	28	20
C	13	36
D	20	27
E	27	19
F	30	8
G	0	30

Table 4 Scores and rankings of the seven candidates by different models

	LP-1		*LP-2*		*NLP-1*		*Borda–Kendall*	
Candidate	*Score*	*Rank*	*Score*	*Rank*	*Score*	*Rank*	*Score*	*Rank*
A	24.67	2	0.9737	2	33.53	2	74	2
B	25.33	1	1	1	33.99	1	76	1
C	20.67	6	0.8158	6	27.73	6	62	6
D	22.33	5	0.8816	5	27.96	5	67	5
E	24.33	3	0.9605	3	32.65	3	73	3
F	22.67	4	0.8947	4	31.05	4	68	4
G	10.00	7	0.3947	7	13.42	7	30	7

recommendation by setting $d(j, \varepsilon) = \varepsilon/j$ and maximizing ε, but is slightly different from the conclusion that B is the winner drawn by the BK method. This is mainly because our models and Cook and Kress's model put more emphasis upon the first ranking place than the BK method. In fact, the total of weights used by the BK method is $4 + 3 + 2 + 1 = 10$, and the relative weight assigned to the first ranking place is $4/10 = 0.4$, which is smaller than $w_1^* = 0.48$ obtained by LP-1, LP-2 and NLP-1. The three models all produce the same normalized relative weights for this example. The advantage of our models over the BK method is that our models do not need to specify the relative weights, which are determined automatically by the models, and more emphasis is put upon the first ranking place. This is useful in preference voting.

Example 2 Consider the example investigated by Obata and Ishii (2003) and Foroughi and Tamiz (2005), in which seven candidates A–G are ranked. Table 3 shows the votes each candidate receives in the first two places.

For this example, $n = 7$ and $m = 2$, but the exact number of voters is not known. So, the value of the discriminating intensity factor ε will be difficult to set by Noguchi *et al*'s approach. However, there exists no such difficulty with the new models proposed in this paper. By solving LP-1, we have

$$\alpha^* = 10, \quad w_1^* = 2/3 \quad \text{and} \quad w_2^* = 1/3$$

By solving LP-2, we obtain

$$\alpha^* = 0.3947, \quad w_1^* = 0.0263 \quad \text{and} \quad w_2^* = 0.0132$$

By solving NLP-1 for the seven candidates, respectively, we get the following three sets of weights for different candidates:

$$w_1^* = 0.9545 \quad \text{and} \quad w_2^* = 0.2983 \quad \text{for candidate A}$$

$$w_1^* = 0.8944 \quad \text{and} \quad w_2^* = 0.4472$$

$$\text{for candidate B, C, D, E and G}$$

$$w_1^* = 0.9662 \quad \text{and} \quad w_2^* = 0.2577 \quad \text{for candidate F}$$

The rankings of the seven candidates generated by the three new models and the BK method are shown in Table 4. As can be seen from Table 4, the three new models and the BK method all lead to the same ranking: $B \succ A \succ E \succ F \succ D \succ C \succ G$. So, candidate B is the winner.

Conclusions

How to determine the weights associated with different ranking places is an important issue of preference voting and aggregation. In this paper, we have proposed three new models, LP-1, LP-2 and NLP-1, to assess the weights of different ranking places. The proposed models do not require predetermining any discrimination intensity function and discriminating intensity factor subjectively and each of them can lead to a stable full ranking for the candidates considered. The proposed models can be implemented on MS Excel worksheets and can be solved using MS-Excel Solver very easily. Two numerical examples are provided and have been examined using the proposed models. It has been shown that the proposed models have very strong capabilities of choosing the winner and ranking the candidates. It is expected that the proposed models can play an important role in preferential voting and preferential election decision making in the future.

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