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# NUMERICAL VERIFICATION OF WEAK SOLUTIONS OF THE CROCCO TYPICAL BOUNDARY PROBLEM USING AN IMPLICIT SECOND ORDER DIFFERENCE SCHEME

*M.R. Petrichenko, E.V. Kotov*

Peter the Great St. Petersburg Polytechnic University, St. Petersburg, Russian Federation

To verify the solution of a typical Crocco boundary problem, a numerical experiment has been performed using an implicit second-order difference scheme. The computational experiment showed uniform convergence in the $0 \le x \le 1$ interval for the numerical approximation of the solution to a weak solution with a small interval discrete sampling (of the order of $N$ = 104 nodes). It was shown that a numerical solution approximated a weak solution of the typical Crocco limit problem, except for the right end of the integration interval. The solution of the Crocco boundary problem could be continued to the left of the point $x = x_0$ while preserving the continuity and smoothness of the solution at this point. The point $x = 1$ represents the natural upper bound of the solution domain.

**Keywords:** Crocco's typical boundary problem, implicit difference scheme, weak solution, homotopy

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# ЧИСЛЕННАЯ ВЕРИФИКАЦИЯ СЛАБЫХ РЕШЕНИЙ ТИПИЧНОЙ ПРЕДЕЛЬНОЙ ЗАДАЧИ КРОККО С ПОМОЩЬЮ НЕЯВНОЙ РАЗНОСТНОЙ СХЕМЫ ВТОРОГО ПОРЯДКА

*М.Р. Петриченко, Е.В. Котов*

Санкт-Петербургский политехнический университет Петра Великого,

Санкт-Петербург, Российская Федерация

Для верификации решения типичной предельной задачи Крокко проведен численный эксперимент с использованием неявной разностной схемы второго порядка. Вычислительный эксперимент показал равномерную на промежутке $0 \le x \le 1$ сходимость численной аппроксимации решения к слабому решению при небольшой плотности дискретизации промежутка (порядка $N = 10^4$ узлов). Показано, что численное решение аппроксимирует слабое решение типичной предельной задачи Крокко, кроме правого конца промежутка интегрирования – точки $x = 1$. Решение предельной задачи Крокко может быть продолжено левее точки $x$=0 с сохранением непрерывности и гладкости решения в этой точке. Точка $x = 1$ представляет естественную верхнюю границу области определения решения.

**Ключевые слова:** типичная предельная задача Крокко, неявная разностная схема, слабое решение, гомотопия

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## Introduction

It is known that the typical Crocco boundary problem is stated as follows [1]:

$$yy'' + \gamma x = 0, D(y) = (x : 0 \le x_0 < x < 1);$$
$$\mathrm{Im}(y) = (y : y_0 > y > 0); \qquad (1)$$
$$y'(x_0) = y(1) = 0,$$

where $y_0 := y(x_0) > 0$.

In the classic case of a typical boundary problem,

$$\gamma = 1/2,\ x_0 = 0,\ y_0 := y(0).$$

This study deals with this particular classical case.

We can prove that two-point boundary conditions (1) are equivalent to the Cauchy condition:

$$y(0) - a = y'(0) = 0.$$

Let $a = 0$. Then $y(x) = \pm\sqrt{2/3(-x)^3}$ is the solution of a homogeneous one-point problem for the Crocco equation on the negative semi-axis $x < 0$.

In hydrodynamic problems, $y(x)$ is the dimensionless friction factor, $x$ is the dimensionless longitudinal component of velocity in the boundary layer on a plate in plane flow in the longitudinal direction.

Then $y(0) = a$ is the shear stress on the wall (Blasius constant) [2]. In seepage theory, $x$ is the dimensionless depth of seepage flow through a scalar (homogeneous and isotropic) porous medium, $y$ is the Crocco potential, defined as

$$y(x) = \int_x^1 s dx', \quad y(1) = y'(0) = 0,$$

where $s$ is the longitudinal coordinate measured along the seepage flow.

The constant $y_0 = y(x_0)$ in seepage problems is proportional to seepage flow in the outflow face [3].

Steady-state solutions for free surface seepage in a scalar medium are found in terms of analytic theory of ordinary differential equations [3]. Modern results of such solutions are given in [4–6].

The following statements are true for a typical Crocco problem (1).

1. The Crocco equation has two solution branches: positive $y_+(x)$ and negative $y_-(x)$. The negative branch is defined as the solution to a boundary problem:

$$2y_- y_-'' + \gamma x = 0, \quad D(y_-) = (x : x_0 < x < 1),$$
$$y_-'(0) = y(1) = 0, \quad \mathrm{Im}(y_-) = (y_- : -y_0 > y_- > 0);$$

with $y_+(x) + y_-(x) = 0,\ \forall x \in (0,1)$.

The proof is trivial.

Below we consider only the positive branch of the solution of the Crocco equation, i.e., $y(x) := y_+(x)$.

2. The solution of a typical Crocco boundary problem (1) has the following properties:

$$y'(x) < 0, \quad y''(x) < 0;$$
$$y'(x) \xrightarrow[x \to 1-0]{} -\infty,$$

so $y_0 = a > y(x)$, $0 < x < 1$.

To prove Statement 2, we formally reduce the order of the Crocco equation, reformulating it as an integral equation:

$$2y' = -\int_0^x \frac{tdt}{y(t)} \to y' \le 0, \quad 0 \le x < 1.$$

The integral on the right-hand side can be calculated using the Bonnet mean value theorem. We obtain:

$$2yy' = -1/2x^2(1 - \theta^2), \qquad (2)$$

where $\theta$ is a regular fraction, $0 < \theta < 1$.

Now we need only to pass to the limit for $x \to 1-0$, Q.E.D.

The solution of Eq. (2), such that the value of $y(1)$ is zero, $y(1) = 0$, has the following form:

$$y^2(x,\theta) = 1/6(1-\theta^2)(1-x^3). \qquad (3)$$

Solution (3) continuously depends on the magnitude of the fraction $\theta$. Its mean over $\theta$ is the so-called weak solution of the typical Crocco boundary problem, interpreted as the distribution over $\theta$ with a distribution density $y(x; \theta)$ [7].

In view of expression (3), the weak solution of the typical Crocco boundary problem is:

$$y(x) = 1/3\sqrt{1-x^3}, \qquad (4)$$

and then $y_0 = y(0) = 1/3$, which is a good rational approximation for the Blasius constant. The exact value of the Blasius constant was calculated in Varin's study [8]. It can be seen from formula (4) that the weak solution can be continued to negative values of $x$ while preserving the solution smooth and continuous at the point $x = 0$.

The solution of the typical Crocco boundary problem is related to the solution of a nonlinear integral equation:

$$y(x) = (1/2)\left\{\int_0^1 \frac{(1-s)sds}{y(s)} - \int_0^x \frac{(x-s)sds}{y(s)}\right\}, \quad (5)$$

which gives the following expression for the Blasius constant:

$$y_0 := y(0) = (1/2)\int_0^1 \frac{(1-s)sds}{y(s)}.$$

The solution of Eq. (5) can also be obtained in the form of a Lagrange series [9]. It was proved that the convergence radius of the Lagrange number is less than unity and the series diverges at $x \to 1-0$.

An alternative solution in the form of a Lagrange series is forming an iterative process:

$$y_k(x) = (1/2)\left\{\int_0^1 \frac{(1-s)sds}{y_{k-1}(s)} - \int_0^x \frac{(x-s)}{y_{k-1}(}\right.$$

$$k = 1(1)\infty,$$

where the subscript $k$ indicates the iteration number.

The values of the Blasius constant obtained during the iterative process are found from the sequence

$$y_k(0) = (1/2)\int_0^1 \frac{(1-s)sds}{y_{k-1}(s)}.$$

We successively find the following values for different $k$:

$$k = 1: y_0(x) = y_0 = \sqrt{1/12} = 0.2887;$$

$$k = 2: y_1(x)\cdot y_0 = (1/12)(1-x^3),$$

$$y_1(x) = (1-x^3)/\sqrt{12},\ y_1(0) = 1/\sqrt{12};$$

$$k = 3:$$

$$y_2(x) = \sqrt{3}\left(\int_0^1 \frac{(1-s)sds}{1-s^3} - \int_0^x \frac{(x-s)sds}{1-s^2}\right) -$$

$$-\sqrt{3}\left\{\begin{array}{l} \ln\sqrt{3}\,\dfrac{x+2}{3}\ln\sqrt{1+x+x^2} + \\ +(1/3)(1-x)\ln\left(\dfrac{1}{1-x}\right) - \dfrac{\pi}{6\sqrt{3}} + \\ +\dfrac{1}{\sqrt{3}}\left(\operatorname{arctg}\dfrac{2x+1}{\sqrt{3}} - \dfrac{\pi}{6}\right) \end{array}\right\},$$

$$y_2(0) = \sqrt{3}\left(\ln\sqrt{3} - \frac{\pi}{6\sqrt{3}}\right)$$

and so on.

Accordingly, the first three iterated values of the Blasius constant form a sequence

$$y_0(0) = 1/\sqrt{12} = 0.2887...,$$

$$y_1(0) = 0.2887...,$$

$$y_3(0) = 0.4278...,$$

and, on average, $y(0)$ for the first three iterations lies in the range

$$0.3299 < y(0) < 0.3344.$$

The iterative process leads to trivial and lengthy calculations, which is already clear at the third iteration. Evidently, any iterated solution has all the basic properties of the solution to boundary problem (1):

$$\forall x \in (0,1),\ \forall k = 1(1)\infty,$$

$$y_k'(x) < 0,\ y_k''(x) < 0,$$

$$y'(x) \xrightarrow[x\to 1-0]{} -\infty.$$

The iterative process is inconvenient as the expressions for the iterated solutions are cumbersome and there is no proof for the convergence of the process. Both of these obstacles can be avoided by using the difference approximation of boundary problem (1).

Interest in numerical solutions to the Blasius equation appeared immediately after he published his study in 1908 [2], due to general disappointment in the integration method using power series (see [8] and its preprint detailing the history of the issue). Modern studies [10, 11, 13–21] mainly consist of attempts to improve the convergence of predictor-corrector methods for solving ordinary differential equations of the boundary layer. Ref. [22] is an exception, developing Kaplun's method interpreted in terms of homotopy mappings of the integration interval on a compact set. In the case of boundary problem (1), the mappings are compact.

Let linear homotopy

$$F(t,x): ((0 < t < 1) \times (0 < x < 1)) \to (0, a)$$

represent the solution to boundary problem (1).

Then $F(0,x)$ represents the solution in the neighborhood of the point $x = 0$, and $F(1,x)$ in the neighborhood of the point $x = 1$. For example, for a weak solution,

$$F(0,x) = (1/3)\left(1 - x^3/2 - x^6/8\right),$$

$$F(1,x) = \left(1/\sqrt{3}\right)\sqrt{1-x}.$$

A linear homotopy mapping has the form:

$$y(x) = F(t,x) = (1-t)F(0,x) + tF(1,x) =$$

$$= (1-t)/3\left(1 - x^3/3 - x^6/8\right) + t\sqrt{\frac{1-x}{3}}.$$

A weak solution also represents some homotopy with the parameter $\theta \in (0,1)$. Indeed,

$$y^2(x,\theta) = (1/6)\left(1-\theta^2\right)\left(1-x^3\right),$$

$$y^2(x,1) = 0,\ y^2(x,0) = (1/6)\left(1-x^3\right).$$

Finally, [12] reintroduces the method of power expansions. However, its results coincide with the data given in [8] on flat series, as well as preprints of this study in Keldysh Institute Preprints, published earlier.

The computational domain in the numerical solution of problem (1) on the interval $x \in (0,1)$ consists of $N$ segments with a constant step $h = 1/N$ ($x_j = jh$, $j = 0, 1, \ldots, N$). We use a second-order difference scheme for discretizing Eq. (1)

$$\frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} + \gamma\frac{x_j}{y_j} = 0. \qquad (6)$$

Equality (6) is a discrete equivalent of the exact equality

$$y'' = -\gamma\frac{x}{y}.$$

This expression is linear with respect to the component $y_{j+1}$, and therefore, if the components $y_{j-1}$, $y_j$ (where $j = 1(1)$) of the vector $\mathbf{y}$ are known, a linear system of algebraic equations is obtained to calculate $y_{j+1}$.

The boundary conditions in problem (1) take the following form upon discretization:

$$\frac{3y_0 - 4y_1 + y_2}{2h} = 0,\ \ y_N = 0. \qquad (7)$$

If the differences in equalities (6), (7) are denoted by

$$\begin{cases} f_0 = 3y_0 - 4y_1 + y_2, \\ f_j = y_{j-1} - 2y_j + y_{j+1} + \gamma h^2\dfrac{x_j}{y_j}, \\ f_N = y_N, \end{cases} \qquad (8)$$

then problem (6)–(8) can be written in the form equivalent to a linear algebraic system

$$\mathbf{F(y)} = 0,$$

where $\mathbf{F}$, $\mathbf{y}$ are vectors taking the form

$$F = [f_0 f_1 \ldots f_N]^T,$$

$$y = [y_0 y_1 \ldots y_N]^T.$$

The resulting nonlinear system is solved by the Newton iterative method:

$$y^{(k+1)} = y^{(k)} + \Delta y^{(k)},$$

where $\Delta y^{(k)}$ is the residual vector,

$$\Delta y^{(k)} = [\Delta y_0^{(k)}\ \Delta y_1^{(k)} \ldots\ \Delta y_N^{(k)}]^T.$$

It is obtained as a solution of the linearized matrix equation with the Jacobi matrix $\mathbf{J}_F(\mathbf{y})$ of order $N + 1$:

$$J_F(y^{(k)})\Delta y^{(k)} = -F(y^{(k)}), \qquad (9)$$

$$J_F(\mathbf{y}^{(k)}) = \frac{\partial(f_0, \ldots, f_N)}{\partial(y_0, \ldots, y_N)}. \qquad (10)$$

It is assumed that the matrix $\mathbf{J}_F(\mathbf{y})$ is well-conditioned. Then system (10) is correct and uniquely solvable:

$$\Delta\mathbf{y}^{(k)} = -\mathbf{J}_F^{-1}\left(\mathbf{y}^{(k)}\right)\mathbf{F}\left(\mathbf{y}^{(k)}\right).$$

Substituting equality (8) into Eq. (9), we obtain, in view of equality (10), the following expressions:

$$\begin{aligned} 3\Delta y_0^{(k)} - 4\Delta y_1^{(k)} + \Delta y_2^{(k)} &= -f_0^{(k)}, \\ f_0^{(k)} &= 3y_0^{(k)} - 4\Delta y_1^{(k)} + y_2^{(k)}, \end{aligned} \qquad (11)$$

$$f_j^{(k)} = y_{j-1}^{(k)} - 2y_j^{(k)} + y_{j+1}^{(k)} + \gamma h^2\frac{x_j}{y_j^{(k)}}, \quad (12)$$

$$a_j \Delta y_{j-1}^{(k)} + b_j \Delta y_j^{(k)} + c_j \Delta y_{j+1}^{(k)} = -f_j^{(k)}, \quad (13)$$

$$a_j = 1,\ b_j = -2 - \gamma h^2 \frac{x_j}{\left(y_j^{(k)}\right)^2},\ c_j = 1, \quad (14)$$

$$\Delta y_N^{(k)} = -y_N^{(k)}.$$

Evidently, system of equations (11)–(14) contains three unknowns in each of the equations and is similar to a tridiagonal system. The first and the last equations in such systems usually contain only two unknowns. However, the first equation in this system contains three unknowns: $\Delta y_0^{(k)}$, $\Delta y_1^{(k)}$, $\Delta y_2^{(k)}$.

To eliminate the unknown $\Delta y_0^{(k)}$, Eq. (11) can be represented as follows:

$$\Delta y_0^{(k)} = \frac{1}{3}\left[4\Delta y_1^{(k)} - \Delta y_2^{(k)} - f_0^{(k)}\right]. \quad (15)$$

Next, substituting expressions (13) and (14) into Eq. (15) with $j = 1$, we obtain the expression:

$$\hat{b}_1 \Delta y_1^{(k)} + \hat{c}_1 \Delta y_2^{(k)} = -\hat{f}_1^{(k)}, \quad (16)$$

where

$$\hat{b}_1 = b_1 + 4/3a_1,$$
$$\hat{c}_1 = c_1 - 1/3a_1, \quad (17)$$
$$\hat{f}_1^{(k)} = f_1^{(k)} - 1/3 f_0^{(k)}.$$

The matrix of system of equations (11), (15), (16) is tridiagonal. This system can be solved by sweeping with respect to the indices $j$:

$$\Delta y_j^{(k)} = p_j - q_j \Delta y_{j+1}^{(k)}. \quad (18)$$

It follows from equality (16) that

$$p_1 = -\hat{f}_1^{(k)} / \hat{b}_1, q_1 = \hat{c}_1 / \hat{b}_1. \quad (19)$$

It follows from Eqs. (15), (19) that

$$a_j \left(p_{j-1} - q_{j-1} \Delta y_j^{(k)}\right) + b_j \Delta y_j^{(k)} + $$
$$+ c_j \Delta y_{j+1}^{(k)} + f_j^{(k)} = 0. \quad (20)$$

In view of the boundary condition $y_N = 0$, we obtain the following equalities for all $k$:

$$y_N^{(k)} = \Delta y_N^{(k)} = 0.$$

After calculating $p_j$ and $q_j$ for $j = 1, 2, \ldots, N - 1$ using expressions (18) and (19), we can calculate $\Delta y_j^{(k)}$ for $j = N - 1, N - 2, \ldots, 0$ using expression (18).

Calculations continue until a predetermined accuracy ε is reached:

$$\left\|\Delta \mathbf{y}^{(k)}\right\| \le \varepsilon,$$

where ‖*‖ denotes, for example, sup that is the norm of the residual vector or any equivalent norm of the matrix.

Fig. 1 shows the numerical solution of problem (4), (5) on the interval $x \in [0, 1]$ for $\gamma = 1$ with a different number of steps $N$ for $\varepsilon = 10^{-6}$. The fiber bundle of numerical solutions is small on the scale of the figure even when changing the number $N$ of the nodes into which the integration interval $0 < x < 1$ is divided by 4(!) orders, $10^2 \le N \le 10^6$. The following expression is considered as the initial approximation:

$$y_0 = (1/2)\ (1 - x^2)^S.$$

The bold solid line in Fig. 1 corresponds to a weak solution (4) with the Blasius constant of 0.4714 (the exact value is 0.4696).

Table 1 lists the Blasius constants $y(0)$, calculated with $\gamma = 1$ and different numbers of steps $N$, and the values obtained by other authors [12–16].

It follows from the data given in Table that the first three exact significant digits of the Blasius constant can be calculated with a small number of nodes, with $N > 10{,}000$. The derivative of the numerical solution at the right endpoint of the integration interval, i.e., at $x = 1 - 0$, is bounded from below and no numerical solution curve has a vertical tangent (see Fig. 1). It is to be expected that the values of numerical derivatives should be bounded, since one-sided differences are used.

To extend the solution of problem (1) to the domain $x < 0$, a second-order difference scheme (6) is used with the following boundary conditions:

$$y(0) - \tilde{y}_0 = y'(0) = 0, \quad (21)$$

where $\tilde{y}_0$ is the value of $y(0)$ from the solution obtained on the interval $x \in [0, 1]$, i.e., the Blasius constant of the numerical solution.

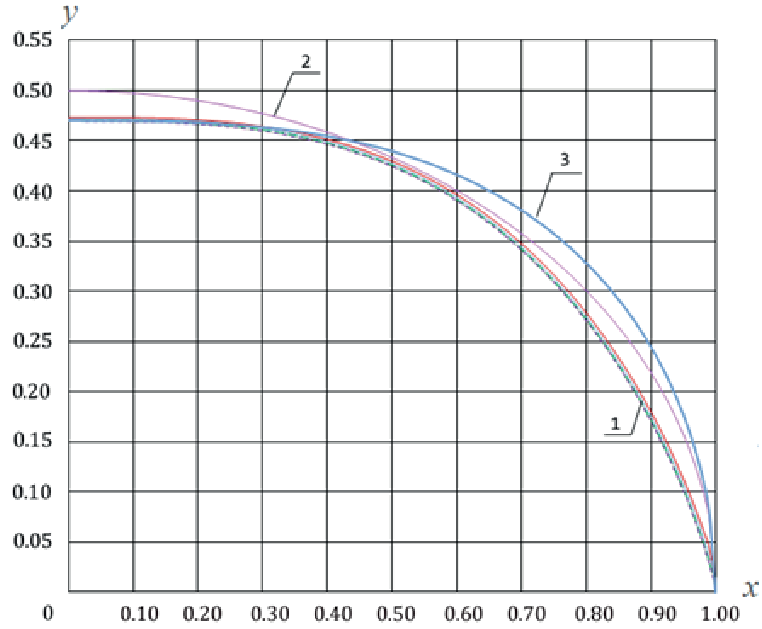


Fig. 1. Numerical solution of Crocco problem on interval $x \in [0, 1]$ with $\gamma = 1$, with a different number of steps $N$: 100, 1000, 10 000, 100 000, $10^6$ (fiber bundle of lines *1*); line *2* is the initial approximation $y_0 = 1/2$, line *3* is weak solution (4) with the Blasius constant $y_0 = 0.4714$

Table

**Calculated values of Blasius constant $y(0)$ with varying parameters and number of partitions of integration interval**

| Source | Number of steps $N$ | Value of $y(0)$ | |
|---|---|---|---|
| | | $\gamma = 0.5$ | $\gamma = 1.0$ |
| This paper | 100 | 0.339566 | 0.472865 |
| | 1000 | 0.335198 | 0.471984 |
| | 10,000 | 0.332051 | 0.470430 |
| | 100,000 | 0.332053 | 0.469855 |
| | 1,000,000 | 0.332053 | 0.469676 |
| [13] | – | 0.332057 | 0.469600 |
| [14] | | 0.3320573362 | 0.4695999889 |
| [15] | | 0.332057 | 0.469599 |

Upon discretization, boundary conditions (21) take the form

$$y_0 - \tilde{y}_0 = (y_0 - y_{-1}) / h = 0,$$

and it follows then that $y_0 = \tilde{y}_0 = y_{-1}$.

Therefore,

$$y_j = 2y_{j+1} - y_{j+2} - \gamma h^2 x_{j+1} / y_{j+1},$$
$$j = -2, -3, \ldots, -M,$$

where $M$ is the number of calculation steps in the region $x < 0$ (a natural number).

Fig. 2 shows a positive numerical solution of boundary problem (1), extended to the negative semi-axis. The extended solution is preserved continuous and smooth at the point of contact $x = -0$.

Extension of the positive and negative branches of the weak solution to the negative semi-axis has the following form:

$$y(x) = \pm a\sqrt{1-(-x)^3},\ a = 1/\left(3\sqrt{\gamma}\right)$$

Evidently, if $-x >> 1$, a weak solution has an order that coincides with the order of the exact solution of boundary problem (1):

$$y(x) \sim (-x)^{3/2}.$$

## Conclusions

The study we have carried out allows us to draw the following conclusions.

1. The weak solution of the Crocco problem has all the properties of an exact solution: there is a zero derivative at $x = 0$, an unbounded derivative at $x = 1$, the solution can be extended to the negative semi-axis $x < 0$ while preserving continuity and smoothness at $x = 0$.

2. The values of the Blasius constant that we have obtained for the weak solution were: $y(0) = 1/3$ with $\gamma = 1/2$ and $y(0) = 0.4714$ with $\gamma = 1$; the approximate value of the Blasius constant differs from the exact value

$$(y(0) = 0.332059,\ \gamma = 1/2 \text{ and}$$
$$y(0) = 0.4696,\ \gamma = 1)$$

by less than 0.4%.

3. The numerical experiment revealed that the numerical approximation of the solution uniformly converges on the interval $0 \le x \le 1$ to a weak solution with a small discretization of the interval (of the order of $N = 10^4$ nodes).

4. The derivative of the numerical solution is bounded from below at the right endpoint of the integration interval, $x = 1 - 0$, and the numerical solution curve does not have a vertical tangent. It is to be expected that the values of numerical derivatives should be bounded, since one-sided differences are used.

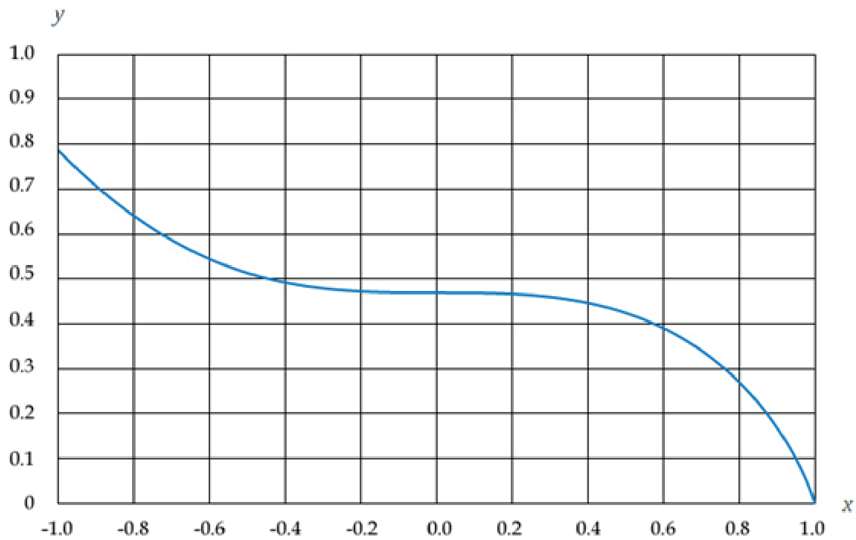


Fig. 2. Solution of Crocco problem on interval $x \in [-1, 1]$ with $\gamma = 1$

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## THE AUTHORS

**PETRICHENKO Mikhail R.**
*Peter the Great St. Petersburg Polytechnic University*
29 Politechnicheskaya St., St. Petersburg, 195251, Russian Federation
fonpetrich@mail.ru

**KOTOV Eugeniy V.**
*Peter the Great St. Petersburg Polytechnic University*
29 Politechnicheskaya St., St. Petersburg, 195251, Russian Federation
ekotov.cfd@gmail.com

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## СВЕДЕНИЯ ОБ АВТОРАХ

**ПЕТРИЧЕНКО Михаил Романович** – *доктор технических наук, заведующий кафедрой гидравлики и прочности Санкт-Петербургского политехнического университета Петра Великого.*
195251, Российская Федерация, г. Санкт-Петербург, Политехническая ул., 29
fonpetrich@mail.ru

**КОТОВ Евгений Владимирович** – *ассистент кафедры гидравлики и прочности Санкт-Петербургского политехнического университета Петра Великого.*
195251, Российская Федерация, г. Санкт-Петербург, Политехническая ул., 29
ekotov.cfd@gmail.com