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Existence and Uniqueness of Solutions of a Boundary Value Problem of Fractional Order via S-Iteration

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ABSTRACT. The present paper studies the existence and uniqueness of solutions to a boundary value problem (BVP) of fractional order involving the Caputo fractional derivative and also discusses some other properties of the solutions. An example in support of all established results is given.

1. INTRODUCTION

We consider the following boundary value problem involving the Caputo fractional derivative with boundary conditions of the form:

$$\Big(D^{\alpha}_{*a}\Big)y(t) = \mathcal{F}\Big(t, y(t)\Big), \tag{1.1}$$

for $t \in I = [a, b],\ n-1 < \alpha \leq n, n \in \mathbb{N}$ with the given boundary conditions

$$y^{(j)}(a) = c_j,\ j = 0, 1, 2, \cdots, n-2;\ y^{(n-1)}(b) = c_b \tag{1.2}$$

where $\mathcal{F} : I \times X \to X$ is continuous function and $c_j\ (j = 0, 1, 2, \ldots, n-2),\ c_b$ are given elements in X.

Several researchers have introduced many iteration methods for certain classes of operators in the sense of their convergence, equivalence of convergence and rate of convergence etc. (see [2, 4, 5, 6, 10, 13, 14, 23, 24, 25, 26, 27, 28, 29, 34, 35]). The most of iterations are devoted for both analytical and numerical approaches. The $S-$ iteration method, due to simplicity and fastness, has attracted the attention and hence, it is used in this paper.

The problems of existence, uniqueness and other properties of solutions of special forms of BVP (1.1)-(1.2) and its variants have been studied by several researchers under variety of hypotheses by using different techniques (as in [1, 3, 11, 12, 15, 16, 17, 18, 19, 20, 21, 31, 32]). Recently, S. Soltuz and T. Grosan [36] have studied the special version of equation (1.1). Authors are motivated by the work of D. R. Sahu [34] and influenced by [36, 37].

The main objective of this paper is to generalize the results of the paper [37] by the use of normal $S-$iteration method which establishes the existence and uniqueness of solutions of the boundary value problem (1.1)-(1.2) and other qualitative properties of solutions.

2. PRELIMINARIES

Before proceeding to the statement of our main result, we shall setforth some preliminaries and hypotheses that will be used in our subsequent discussion.

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Let X be a Banach space with norm $\|\cdot\|$ and $I = [a, b]$ denotes an interval of the real line $\mathbb{R}$. We define $B = C^r(I, X)$ (where $r = n$ for $\alpha \in \mathbb{N}$ and $r = n-1$ for $\alpha \notin \mathbb{N}$.) as a Banach space of all r times continuously differentiable functions from I into X, endowed with the norm

$$\|y\|_B = \sup\{\|y(t)\| : y \in B\}, \quad t \in I.$$

Definition 2.1. [33] The Riemann-Liouville fractional integral (left-sided) of a function $h \in C^1[a, b]$ of order $\alpha \in \mathbf{R}_+ = (0, \infty)$ is defined by

$$I_a^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} h(s)\, ds,$$

where Γ is the Euler gamma function.

Definition 2.2. [33] Let $n-1 < \alpha \leq n,\ n \in \mathbb{N}$. Then the expression

$$D_a^\alpha h(t) = \frac{d^n}{dt^n}\left[I_a^{n-\alpha} h(t)\right],\ t \in [a, b]$$

is called the (left-sided) Riemann–Liouville derivative of h of order α whenever the expression on the right-hand side is defined.

Definition 2.3. [30] Let $h \in C^n[a, b]$ and $n-1 < \alpha \leq n,\ n \in \mathbb{N}$. Then the expression

$$\left(D_{*a}^\alpha\right) h(t) = I_a^{n-\alpha} h^{(n)}(t),\ t \in [a, b]$$

is called the (left-sided) Caputo derivative of h of order α.

Definition 2.4. [9] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that converge to a and b, respectively, and assume that there exists

$$l = \lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|}.$$

(a) If $l = 0$, then it can be said that $\{a_n\}$ converges to a faster than $\{b_n\}$ converges to b.
(b) If $0 < l < 1$, then it can be said that $\{a_n\}$ and $\{b_n\}$ have the same rate of convergence.

Suppose that for two fixed point iteration procedures $\{u_n\}$ and $\{v_n\}$, both converging to the same fixed point p, the error estimates

$$\|u_n - p\| \leq a_n,\ \forall\, n \in \mathbb{N}, \tag{2.3}$$

$$\|v_n - p\| \leq b_n,\ \forall\, n \in \mathbb{N}, \tag{2.4}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero). Then, in view of Definition 2.4, we will adopt the following concept.

Definition 2.5. [9] Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration procedures that converge to the same fixed point p and satisfy (2.3) and (2.4), respectively. If $\{a_n\}$ converges faster than $\{b_n\}$, then it can be said that $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Lemma 2.1. [22] *If the function $f = (f_1, \cdots, f_n) \in C^1[a, b]$, then the initial value problems*

$$\left(D_*^{\alpha_i}\right) y(t) = f_i(t, y_1, \cdots, y_n),\ y_i^{(k)}(0) = c_k^i,\ i = 1, 2, \cdots, n,\ k = 1, 2, \cdots, m_i$$

where $m_i < \alpha_i \leq m_i + 1$, are equivalent to Volterra integral equations:

$$y_i(t) = \sum_{k=0}^{m_i} c_k^i \frac{t^k}{k!} + I^{\alpha_i} f_i(t, y_1, \cdots, y_n),\ 1 \leq i \leq n.$$

As a consequence of the above Lemma, it is easy to observe that if $y \in B$ and $\mathcal{F} \in C^1[a,b]$, then $y(t)$ satisfies the following integral equation which is equivalent to (1.1)-(1.2):

$$y(t) = \sum_{j=0}^{n-2} \frac{c_j}{j!}\,(t-a)^j + \Big(\frac{c_b}{(n-1)!} + \frac{\mathcal{F}(a,y(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}$$
$$-\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}\mathcal{F}\Big(s,y(s)\Big)ds + \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s,y(s)\Big)ds. \tag{2.5}$$

We need the following pair of known results to establish our results.

Theorem 2.1. ([34], p.194) *Let C be a nonempty closed convex subset of a Banach space X and $T: C \to C$ a contraction operator with contractivity factor $k \in [0,1)$ and fixed point x^*. Let α_n and β_n be two real sequences in $[0,1]$ such that $\alpha \leq \alpha_n \leq 1$ and $\beta \leq \beta_n < 1$ for all $n \in \mathbb{N}$ and for some $\alpha, \beta > 0$. For given $u_1 = v_1 = w_1 \in C$, define sequences u_n, v_n and w_n in C as follows:*

S-iteration process: $\begin{cases} u_{n+1} = (1-\alpha_n)Tu_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)u_n + \beta_n Tu_n, n \in \mathbb{N}. \end{cases}$

Picard iteration: $v_{n+1} = Tv_n, n \in \mathbb{N}.$

Mann iteration process: $w_{n+1} = (1-\beta_n)w_n + \beta_n Tw_n, n \in \mathbb{N}.$

Then we have the following:

(a) $\|u_{n+1} - x^*\| \leq k^n \Big[1-(1-k)\alpha\beta\Big]^n \|u_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

(b) $\|v_{n+1} - x^*\| \leq k^n \|v_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

(c) $\|w_{n+1} - x^*\| \leq \Big[1-(1-k)\beta\Big]^n \|w_1 - x^*\|$, *for all* $n \in \mathbb{N}$.

Moreover, the S-iteration process is faster than the Picard and Mann iteration processes.

In particular, for $\alpha_n = 1,\ n \in \mathbb{N}$, the S-iteration process can be written as:

$$\begin{cases} y_0 \in C, \\ y_{n+1} = Tz_n, \\ z_n = (1-\beta_n)y_n + \beta_n Ty_n,\ n \in \mathbb{N}. \end{cases} \tag{2.6}$$

Lemma 2.2. ([36], p.4) *Let $\{\beta_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$, such that for all $n \geq n_0$ one has satisfied the inequality*

$$\beta_{n+1} \leq (1-\mu_n)\beta_n + \mu_n\gamma_n, \tag{2.7}$$

where $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0,\ \forall n \in \mathbb{N}$. Then the following inequality holds

$$0 \leq \lim\sup_{n\to\infty} \beta_n \leq \lim\sup_{n\to\infty} \gamma_n. \tag{2.8}$$

3. Existence and Uniqueness of Solutions via $S-$iteration

Now, we are able to state and prove the following main theorem which deals with the existence and uniqueness of solutions of the equation (1.1)-(1.2).

Theorem 3.2. *Assume that there exists a function $p \in C(I, \mathbb{R}_+)$ such that*

$$\Big\|\mathcal{F}\Big(t,u_1\Big) - \mathcal{F}\Big(t,v_1\Big)\Big\| \leq p(t)\Big[\Big\|u_1 - v_1\Big\|\Big]. \tag{3.9}$$

Let $\{\xi_k\}_{k=0}^{\infty}$ *be a real sequence in* $[0,1]$ *satisfying* $\sum_{k=0}^{\infty}\xi_k=\infty$. *If*

$$\Theta=\Big[\frac{(b-a)^{\alpha}p(a)}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}p(b)+{I_a}^{\alpha}p(t)\Big]<1,$$

then the BVP (1.1)-(1.2) *has a unique solution* $y\in B$ *and normal* $S-$*iterative method* (2.6) *converges to* $y\in B$ *with the following estimate:*

$$\Big\|y_{k+1}-y\Big\|_B\le\frac{\Theta^{k+1}}{e^{\left(1-\Theta\right)\sum_{i=0}^{k}\xi_i}}\Big\|y_0-y\Big\|_B. \tag{3.10}$$

Proof. Let $y(t)\in B$ and define the operator

$$\begin{aligned}(Ty)(t)=&\sum_{j=0}^{n-2}\frac{c_j}{j!}\,(t-a)^j+\Big(\frac{c_b}{(n-1)!}+\frac{\mathcal{F}(a,y(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\&-\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\mathcal{F}\Big(s,y(s)\Big)ds\\&+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\mathcal{F}\Big(s,y(s)\Big)ds,\ t\in I.\end{aligned} \tag{3.11}$$

Let $\{y_k\}_{k=0}^{\infty}$ be iterative sequence generated by normal $S-$iteration method (2.6) for the operator given in (3.11).
We will show that $y_k\to y$ as $k\to\infty$.
From (2.6), (3.11) and assumptions, we obtain

$$\begin{aligned}&\Big\|y_{k+1}(t)-y(t)\Big\|\\&=\Big\|(Tz_k)(t)-(Ty)(t)\Big\|\\&=\Big\|\sum_{j=0}^{n-2}\frac{c_j}{j!}\,(t-a)^j+\Big(\frac{c_b}{(n-1)!}+\frac{\mathcal{F}(a,z_k(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\&\quad-\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\mathcal{F}\Big(s,z_k(s)\Big)ds\\&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\mathcal{F}\Big(s,z_k(s)\Big)ds\\&\quad-\sum_{j=0}^{n-2}\frac{c_j}{j!}\,(t-a)^j-\Big(\frac{c_b}{(n-1)!}+\frac{\mathcal{F}(a,y(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\&\quad+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\mathcal{F}\Big(s,y(s)\Big)ds\\&\quad-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\mathcal{F}\Big(s,y(s)\Big)ds\Big\|\\&\le\Big(\frac{\Big\|\mathcal{F}(a,z_k(b))-\mathcal{F}(a,y(b))\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\&\quad+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\Big\|\mathcal{F}\Big(s,z_k(s)\Big)-\mathcal{F}\Big(s,y(s)\Big)\Big\|ds\end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(s,z_k(s)\Big)-\mathcal{F}\Big(s,y(s)\Big)\Big\|ds$$
$$\le \Big(\frac{p(a)\Big\|z_k(b)-y(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}$$
$$+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}p(s)\Big\|z_k(s)-y(s)\Big\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big\|z_k(s)-y(s)\Big\|ds. \tag{3.12}$$

Now, by taking supremum in the inequality (3.12), we have

$$\Big\|y_{k+1}-y\Big\|_B \le \Big(\frac{p(a)(t-a)^{n-1}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)\Big\|z_k-y\Big\|_B$$
$$+\frac{\Big\|z_k-y\Big\|_B(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}p(s)ds$$
$$+\frac{\Big\|z_k-y\Big\|_B}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)ds$$
$$\le \Big(\frac{p(a)(t-a)^{n-1}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)\Big\|z_k-y\Big\|_B$$
$$+\Big[\frac{(t-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}p(b)+{I_a}^{\alpha}p(t)\Big]\Big\|z_k-y\Big\|_B$$
$$\le \Big(\frac{p(a)(b-a)^{n-1}(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)\Big\|z_k-y\Big\|_B$$
$$+\Big[\frac{(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}p(b)+{I_a}^{\alpha}p(t)\Big]\Big\|z_k-y\Big\|_B$$
$$\le \Big[\frac{(b-a)^{\alpha}p(a)}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}p(b)+{I_a}^{\alpha}p(t)\Big]\Big\|z_k-y\Big\|_B$$
$$\le \Theta\Big\|z_k-y\Big\|_B. \tag{3.13}$$

Now, we estimate

$$\Big\|z_k(t)-y(t)\Big\| = \Big[(1-\xi_k)\|y_k(t)-y(t)\|+\xi_k\|(Ty_k)(t)-(Ty)(t)\|\Big]$$
$$\le (1-\xi_k)\|y_k(t)-y(t)\|+\xi_k\Big\{\Big(\frac{p(a)\Big\|y_k(b)-y(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}$$
$$+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}p(s)\Big\|y_k(s)-y(s)\Big\|ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big\|y_k(s)-y(s)\Big\|ds\Big\}. \tag{3.14}$$

Similarly, by taking supremum in the inequality (3.14) to get

$$\Big\|z_k-y\Big\|_B$$
$$\le \Big[1-\xi_k\Big(1-\Big[\frac{(b-a)^{\alpha}p(a)}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}p(b)+{I_a}^{\alpha}p(t)\Big]\Big)\Big]$$

$$\times \Big\| y_k - y \Big\|_B$$
$$= \Big[1 - \xi_k \Big(1 - \Theta\Big)\Big] \Big\| y_k - y \Big\|_B. \tag{3.15}$$

Therefore, using (3.15) in (3.13), we have

$$\Big\| y_{k+1} - y \Big\|_B \le \Theta \Big[1 - \xi_k \Big(1 - \Theta\Big)\Big] \Big\| y_k - y \Big\|_B. \tag{3.16}$$

Thus, by induction, we get

$$\Big\| y_{k+1} - y \Big\|_B \le \Theta^{k+1} \prod_{j=0}^{k} \Big[1 - \xi_k \Big(1 - \Theta\Big)\Big] \Big\| y_0 - y \Big\|_B. \tag{3.17}$$

Since $\xi_k \in [0,1]$ for all $k \in \mathbb{N}$, the definition of Θ yields $\xi_k \le 1 \quad \text{and} \quad \Theta < 1$

$$\Rightarrow \xi_k \Theta < \xi_k$$

$$\Rightarrow \xi_k \Big(1 - \Theta\Big) < 1, \ \forall\, k \in \mathbb{N}. \tag{3.18}$$

From the classical analysis, we know that

$$1 - x \le e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots, \ x \in [0,1].$$

Hence by utilizing this fact with (3.18) in (3.17), we obtain

$$\Big\| y_{k+1} - y \Big\|_B \le \Theta^{k+1} e^{-\big(1-\Theta\big) \sum_{j=0}^{k} \xi_j} \Big\| y_0 - y \Big\|_B$$
$$= \frac{\Theta^{k+1}}{e^{\big(1-\Theta\big) \sum_{i=0}^{k} \xi_i}} \Big\| y_0 - y \Big\|_B. \tag{3.19}$$

Thus, we have proved (3.10).

Since $\sum_{k=0}^{\infty} \xi_k = \infty$, then

$$e^{-\big(1-\Theta\big) \sum_{j=0}^{k} \xi_j} \to 0 \quad \text{as} \quad k \to \infty. \tag{3.20}$$

Hence using this, the inequality (3.19) implies $\lim_{k\to\infty} \|y_{k+1} - y\|_B = 0$ and therefore, we get $y_k \to y$ as $k \to \infty$. □

Remark 3.1. It is interesting to note that the inequality (3.19) gives the bounds in terms of known functions, which majorizes the iterations for solutions of the equation (1.1)-(1.2) for $t \in I$.

4. Continuous dependence via $S-$iteration

In this section, we shall deal with continuous dependence of solution of the problem (1.1) on the boundary data, functions involved therein and also on parameters.

4.1. **Dependence on boundary data.** Suppose $y(t)$ and $\overline{y}(t)$ are solutions of (1.1) with boundary data

$$y^{(j)}(a) = c_j,\ j = 0, 1, 2, \cdots, n-2;\ y^{(n-1)}(b) = c_b \tag{4.21}$$

and

$$\overline{y}^{(j)}(a) = d_j,\ j = 0, 1, 2, \cdots, n-2;\ \overline{y}^{(n-1)}(b) = \overline{c}_b \tag{4.22}$$

where $c_j,\ d_j\ (j = 0, 1, 2, \ldots, n-2),\ c_b,\ \overline{c}_b$ are given elements in X.
Then looking at the steps as in the proof of Theorem 3.2, we define the operator for the equation (1.1)- (4.22)

$$\begin{aligned}(\overline{T}\overline{y})(t) = \sum_{j=0}^{n-2} \frac{d_j}{j!}\,(t-a)^j &+ \Big(\frac{\overline{c}_b}{(n-1)!} + \frac{\mathcal{F}(a, \overline{y}(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\ &- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathcal{F}\Big(s, \overline{y}(s)\Big) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{y}(s)\Big) ds,\ t \in I.\end{aligned} \tag{4.23}$$

The following theorem deals with the continuous dependence of solutions of equation (1.1) on boundary data.

Theorem 4.3. *Suppose the function $\mathcal{F}$ in equation* (1.1) *satisfies the condition* (3.9)*. Consider the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in* (3.11) *and $\overline{T}$ in* (4.23)*, respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$. If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ converges to $\overline{y}$, then we have*

$$\Big\|y - \overline{y}\Big\|_B \leq \frac{3M}{\Big(1-\Theta\Big)}, \tag{4.24}$$

where

$$M = \sum_{j=0}^{n-2} \frac{\Big\|c_j - d_j\Big\|}{j!}\,(b-a)^j + \frac{\Big\|c_b - \overline{c}_b\Big\|}{(n-1)!}(b-a)^{n-1}.$$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (3.11) and $\overline{T}$ in (4.23), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$. From iteration (2.6) and equations (3.11); (4.23) and assumptions, we obtain

$$\begin{aligned}&\Big\|y_{k+1}(t) - \overline{y}_{k+1}(t)\Big\| \\ &= \Big\|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\Big\| \\ &= \Big\| \sum_{j=0}^{n-2} \frac{c_j}{j!}\,(t-a)^j + \Big(\frac{c_b}{(n-1)!} + \frac{\mathcal{F}(a, z_k(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\ &\quad - \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathcal{F}\Big(s, z_k(s)\Big) ds \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, z_k(s)\Big) ds\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=0}^{n-2} \frac{d_j}{j!}\,(t-a)^j - \Big(\frac{\overline{c}_b}{(n-1)!} + \frac{\mathcal{F}(a,\overline{z}_k(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n}\mathcal{F}\Big(s,\overline{z}_k(s)\Big)ds \\
&- \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s,\overline{z}_k(s)\Big)ds \\
\le & \sum_{j=0}^{n-2} \frac{\Big\|c_j - d_j\Big\|}{j!}\,(t-a)^j + \frac{\Big\|c_b - \overline{c}_b\Big\|}{(n-1)!}(t-a)^{n-1} \\
&+ \Big(\frac{\Big\|\mathcal{F}(a,z_k(b)) - \mathcal{F}(a,\overline{z}_k(b))\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n}\Big\|\mathcal{F}\Big(s,z_k(s)\Big) - \mathcal{F}\Big(s,\overline{z}_k(s)\Big)\Big\|ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(s,z_k(s)\Big) - \mathcal{F}\Big(s,\overline{z}_k(s)\Big)\Big\|ds \\
\le & \sum_{j=0}^{n-2} \frac{\Big\|c_j - d_j\Big\|}{j!}\,(b-a)^j + \frac{\Big\|c_b - \overline{c}_b\Big\|}{(n-1)!}(b-a)^{n-1} \\
&+ \Big(\frac{p(a)\Big\|z_k(b) - \overline{z}_k(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n}p(s)\Big\|z_k(s) - \overline{z}_k(s)\Big\|ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}p(s)\Big\|z_k(s) - \overline{z}_k(s)\Big\|ds \\
\le & M + \Big(\frac{p(a)\Big\|z_k(b) - \overline{z}_k(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n}p(s)\Big\|z_k(s) - \overline{z}_k(s)\Big\|ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}p(s)\Big\|z_k(s) - \overline{z}_k(s)\Big\|ds. \qquad (4.25)
\end{aligned}
$$

Recalling the equations (3.13) and (3.15), the above inequality becomes

$$
\Big\|y_{k+1} - \overline{y}_{k+1}\Big\|_B \le M + \Theta\Big\|z_k - \overline{z}_k\Big\|_B, \qquad (4.26)
$$

and similarly, it is seen that

$$
\Big\|z_k - \overline{z}_k\Big\|_B \le \xi_k M + \Big[1 - \xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k - \overline{y}_k\Big\|_B. \qquad (4.27)
$$

Therefore, using (4.27) in (4.26) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality becomes

$$
\begin{aligned}
\Big\|y_{k+1} - \overline{y}_{k+1}\Big\|_B &\le M + \Big\|z_k - \overline{z}_k\Big\|_B \\
&\le M + \xi_k M + \Big[1 - \xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k - \overline{y}_k\Big\|_B
\end{aligned}
$$

$$\le 2\xi_k M + \xi_k M + \Big[1 - \xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k - \overline{y}_k\Big\|_B$$
$$\le \Big[1 - \xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k - \overline{y}_k\Big\|_B + \xi_k\Big(1-\Theta\Big)\frac{3M}{\Big(1-\Theta\Big)}. \tag{4.28}$$

We denote

$$\beta_k = \Big\|y_k - \overline{y}_k\Big\|_B \ge 0,$$
$$\mu_k = \xi_k\Big(1-\Theta\Big) \in (0,1),$$
$$\gamma_k = \frac{3M}{\Big(1-\Theta\Big)} \ge 0.$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (4.28) satisfies all the conditions of Lemma 2.2 and hence we have

$$0 \le \lim\sup_{k\to\infty} \beta_k \le \lim\sup_{k\to\infty} \gamma_k$$
$$\Rightarrow 0 \le \lim\sup_{k\to\infty} \Big\|y_k - \overline{y}_k\Big\|_B \le \lim\sup_{k\to\infty} \frac{3M}{\Big(1-\Theta\Big)}$$
$$\Rightarrow 0 \le \lim\sup_{k\to\infty} \Big\|y_k - \overline{y}_k\Big\|_B \le \frac{3M}{\Big(1-\Theta\Big)}. \tag{4.29}$$

Using the assumption $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (4.29) that

$$\Big\|y - \overline{y}\Big\|_B \le \frac{3M}{\Big(1-\Theta\Big)}, \tag{4.30}$$

which shows that the dependency of solutions of BVPs (1.1)-(1.2) and (1.1)-(4.22) on given boundary data. □

4.2. **Closeness of solution via $S-$iteration.** Consider the problem (1.1)-(1.2) and the corresponding problem

$$\big(D^{\alpha}_{*a}\big)\overline{y}(t) = \overline{\mathcal{F}}\Big(t, \overline{y}(t)\Big), \tag{4.31}$$

for $t \in I = [a,b],\ n-1 < \alpha \le n, n \in \mathbb{N}$, with the given boundary conditions

$$\overline{y}^{(j)}(a) = d_j,\ j = 0,1,2,\cdots,n-2,\ \overline{y}^{(n-1)}(b) = \overline{c}_b \tag{4.32}$$

where $\overline{\mathcal{F}}$ is defined as $\mathcal{F}$ and $d_j\ (j = 0,1,2,\ldots,n-2),\ \overline{c}_b$ are given elements in X.

Then by refereeing the steps as in the proof of Theorem 3.2, we define the operator for the equation (4.31)- (4.32)

$$(\overline{T}\overline{y})(t) = \sum_{j=0}^{n-2} \frac{d_j}{j!}(t-a)^j + \Big(\frac{\overline{c}_b}{(n-1)!} + \frac{\overline{\mathcal{F}}(a,\overline{y}(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}$$
$$- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}\overline{\mathcal{F}}\Big(s,\overline{y}(s)\Big)ds$$
$$+ \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\overline{\mathcal{F}}\Big(s,\overline{y}(s)\Big)ds,\ t \in I. \tag{4.33}$$

The next theorem deals with the closeness of solutions of the problems (1.1)-(1.2) and (4.31)-(4.32).

Theorem 4.4. *Consider the sequences* $\{y_k\}_{k=0}^{\infty}$ *and* $\{\overline{y}_k\}_{k=0}^{\infty}$ *generated normal* $S-$ *iterative method associated with operators* T *in* (3.11) *and* $\overline{T}$ *in* (4.33), *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0,1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N}$. *Assume that*

(i) *all conditions of Theorem 3.2 hold, and* $y(t)$ *and* $\overline{y}(t)$ *are solutions of* (1.1)-(1.2) *and* (4.31)-(4.32) *respectively.*

(ii) *there exists non negative constant* ϵ *such that*

$$\left\|\mathcal{F}\Big(t,u_1\Big)-\overline{\mathcal{F}}\Big(t,u_1\Big)\right\| \leq \epsilon,\ \forall\, t \in I. \tag{4.34}$$

If the sequence $\{\overline{y}_k\}_{k=0}^{\infty}$ *converges to* $\overline{y}$, *then we have*

$$\left\|y-\overline{y}\right\|_B \leq \frac{3\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)}. \tag{4.35}$$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (3.11) and $\overline{T}$ in (4.33), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$. From iteration (2.6) and equations (3.11); (4.33) and hypotheses, we obtain

$$\begin{aligned}
&\Big\|y_{k+1}(t)-\overline{y}_{k+1}(t)\Big\| \\
&=\Big\|(Tz_k)(t)-(\overline{T}\overline{z}_k)(t)\Big\| \\
&=\Big\|\sum_{j=0}^{n-2}\frac{c_j}{j!}\,(t-a)^j+\Big(\frac{c_b}{(n-1)!}+\frac{\mathcal{F}(a,z_k(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&\quad-\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\mathcal{F}\Big(s,z_k(s)\Big)ds \\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\mathcal{F}\Big(s,z_k(s)\Big)ds \\
&\quad-\sum_{j=0}^{n-2}\frac{d_j}{j!}\,(t-a)^j-\Big(\frac{\overline{c}_b}{(n-1)!}+\frac{\overline{\mathcal{F}}(a,\overline{z}_k(b))(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&\quad+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\overline{\mathcal{F}}\Big(s,\overline{z}_k(s)\Big)ds \\
&\quad-\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\overline{\mathcal{F}}\Big(s,\overline{z}_k(s)\Big)ds\Big\| \\
&\leq\sum_{j=0}^{n-2}\frac{\Big\|c_j-d_j\Big\|}{j!}\,(t-a)^j+\frac{\Big\|c_b-\overline{c}_b\Big\|}{(n-1)!}(t-a)^{n-1} \\
&\quad+\Big(\frac{\Big\|\mathcal{F}(a,z_k(b))-\overline{\mathcal{F}}(a,\overline{z}_k(b))\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&\quad+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b(b-s)^{\alpha-n}\Big\|\mathcal{F}\Big(s,z_k(s)\Big)-\overline{\mathcal{F}}\Big(s,\overline{z}_k(s)\Big)\Big\|ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(s,z_k(s)\Big)-\overline{\mathcal{F}}\Big(s,\overline{z}_k(s)\Big)\Big\|ds\\
&\le \sum_{j=0}^{n-2}\frac{\Big\|c_j-d_j\Big\|}{j!}(t-a)^j+\frac{\Big\|c_b-\overline{c}_b\Big\|}{(n-1)!}(t-a)^{n-1}\\
&+\Big(\frac{\Big\|\mathcal{F}(a,\overline{z}_k(b))-\overline{\mathcal{F}}(a,\overline{z}_k(b))\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\
&+\Big(\frac{\Big\|\mathcal{F}(a,z_k(b))-\mathcal{F}(a,\overline{z}_k(b))\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\
&+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}\Big\|\mathcal{F}\Big(s,\overline{z}_k(s)\Big)-\overline{\mathcal{F}}\Big(s,\overline{z}_k(s)\Big)\Big\|ds\\
&+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}\Big\|\mathcal{F}\Big(s,z_k(s)\Big)-\mathcal{F}\Big(s,\overline{z}_k(s)\Big)\Big\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(s,\overline{z}_k(s)\Big)-\overline{\mathcal{F}}\Big(s,\overline{z}_k(s)\Big)\Big\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big\|\mathcal{F}\Big(s,z_k(s)\Big)-\mathcal{F}\Big(s,\overline{z}_k(s)\Big)\Big\|ds\\
&\le \sum_{j=0}^{n-2}\frac{\Big\|c_j-d_j\Big\|}{j!}(b-a)^j+\frac{\Big\|c_b-\overline{c}_b\Big\|}{(n-1)!}(b-a)^{n-1}\\
&+\Big(\frac{\epsilon(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}\epsilon ds\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\epsilon ds\\
&+\Big(\frac{p(a)\Big\|z_k(b)-\overline{z}_k(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\
&+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}p(s)\Big\|z_k(s)-\overline{z}_k(s)\Big\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big\|z_k(s)-\overline{z}_k(s)\Big\|ds\\
&\le M+\Big(\frac{\epsilon(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\epsilon(b-a)^{\alpha-n+1}(b-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+2)}+\frac{\epsilon(b-a)^{\alpha}}{\Gamma(\alpha+1)}\\
&+\Big(\frac{p(a)\Big\|z_k(b)-\overline{z}_k(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}\\
&+\frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)}\int_a^b (b-s)^{\alpha-n}p(s)\Big\|z_k(s)-\overline{z}_k(s)\Big\|ds\\
&+\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}p(s)\Big\|z_k(s)-\overline{z}_k(s)\Big\|ds. \qquad (4.36)
\end{aligned}
$$

Recalling the derivations obtained in equations (3.13) and (3.15), the above inequality becomes

$$\begin{aligned}\Big\|y_{k+1}-\overline{y}_{k+1}\Big\|_B \le M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\\ +\Theta\Big\|z_k-\overline{z}_k\Big\|_B, \qquad (4.37)\end{aligned}$$

and similarly, it is seen that

$$\begin{aligned}\Big\|z_k-\overline{z}_k\Big\|_B \le \xi_k\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]\\ +\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B. \qquad (4.38)\end{aligned}$$

Therefore, using (4.38) in (4.37) and using hypothesis $\Theta < 1$, and $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality becomes

$$\begin{aligned}&\Big\|y_{k+1}-\overline{y}_{k+1}\Big\|_B\\ &\le \Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]+\Big\|z_k-\overline{z}_k\Big\|_B\\ &\le \Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]\\ &\quad+\xi_k\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]\\ &\quad+\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B\\ &\le 2\xi_k\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]\\ &\quad+\xi_k\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]\\ &\quad+\Big[1-\xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B\\ &\le \Big[1-\xi_k\Big(1-\Theta\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B\\ &\quad+\xi_k\Big(1-\Theta\Big)\frac{3\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)}. \qquad (4.39)\end{aligned}$$

We denote

$$\begin{aligned}\beta_k &= \Big\|y_k-\overline{y}_k\Big\|_B \ge 0,\\ \mu_k &= \xi_k\Big(1-\Theta\Big) \in (0,1),\\ \gamma_k &= \frac{3\Big[M+\epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!\Gamma(\alpha-n+2)}+\frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)} \ge 0.\end{aligned}$$

The assumption $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$ implies $\sum_{k=0}^{\infty} \xi_k = \infty$. Now, it can be easily seen that (4.39) satisfies all the conditions of Lemma 2.2 and hence we have

$$0 \le \lim\sup_{k\to\infty} \beta_k \le \lim\sup_{k\to\infty} \gamma_k$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \Big\| y_k - \overline{y}_k \Big\|_B \leq \limsup_{k\to\infty} \frac{3\Big[M + \epsilon(b-a)^\alpha\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)} + \frac{1}{(n-1)!\Gamma(\alpha-n+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)}$$

$$\Rightarrow 0 \leq \limsup_{k\to\infty} \Big\| y_k - \overline{y}_k \Big\|_B \leq \frac{3\Big[M + \epsilon(b-a)^\alpha\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)} + \frac{1}{(n-1)!\Gamma(\alpha-n+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)}. \tag{4.40}$$

Using the assumption $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (4.40) that

$$\Big\| y - \overline{y} \Big\|_B \leq \frac{3\Big[M + \epsilon(b-a)^\alpha\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)} + \frac{1}{(n-1)!\Gamma(\alpha-n+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)}, \tag{4.41}$$

which shows that the dependency of solutions of BVP (1.1)-(1.2) on the function involved on the right hand side of the given equation. □

Remark 4.2. The inequality (4.41) relates the solutions of the problems (1.1)-(1.2) and (4.31)-(4.32) in the sense that if $\mathcal{F}$ and $\overline{\mathcal{F}}$ are close as $\epsilon \to 0$, then not only the solutions of the problems (1.1)-(1.2) and (4.31)-(4.32) are close to each other (i.e. $\|y - \overline{y}\|_B \to 0$), but also depend continuously on the functions involved therein and boundary data.

4.3. **Dependence on Parameters.** We next consider the following problems

$$\big(D^\alpha_{*a}\big)y(t) = \mathcal{F}\Big(t, y(t), \mu_1\Big), \tag{4.42}$$

for $t \in I = [a,b],\ n-1 < \alpha \leq n, n \in \mathbb{N}$ with the given boundary conditions

$$y^{(j)}(a) = c_j,\ j = 0,1,2,\cdots,n-2;\ y^{(n-1)}(b) = c_b \tag{4.43}$$

and

$$\big(D^\alpha_{*a}\big)\overline{y}(t) = \mathcal{F}\Big(t, \overline{y}(t), \mu_2\Big), \tag{4.44}$$

for $t \in I = [a,b],\ n-1 < \alpha \leq n, n \in \mathbb{N}$ with the given boundary conditions

$$\overline{y}^{(j)}(a) = d_j,\ j = 0,1,2,\cdots,n-2;\ \overline{y}^{(n-1)}(b) = \overline{c}_b \tag{4.45}$$

where $\mathcal{F} : I \times X \times \mathbb{R} \to X$ is continuous function, $c_j,\ d_j\ (j = 0,1,2,\ldots,n-2),\ c_b,\ \overline{c}_b$ are given elements in X and constants μ_1, μ_2 are real parameters.

Let $y(t),\ \overline{y}(t) \in B$ and following steps from the proof of Theorem 3.2, define the operators for the equations (4.42) and (4.44), respectively

$$\begin{aligned}(Ty)(t) = {} & \sum_{j=0}^{n-2} \frac{c_j}{j!}\,(t-a)^j + \Big(\frac{c_b}{(n-1)!} + \frac{\mathcal{F}(a, y(b), \mu_1)(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\ & - \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n}\mathcal{F}\Big(s, y(s), \mu_1\Big)ds \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\mathcal{F}\Big(s, y(s), \mu_1\Big)ds,\ t \in I. \end{aligned} \tag{4.46}$$

and

$$(\overline{T}\overline{y})(t) = \sum_{j=0}^{n-2} \frac{d_j}{j!}\,(t-a)^j + \Big(\frac{\overline{c}_b}{(n-1)!} + \frac{\mathcal{F}(a, \overline{y}(b), \mu_2)(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1}$$

$$
\begin{aligned}
&- \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathcal{F}\Big(s, \overline{y}(s), \mu_2\Big) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{y}(s), \mu_2\Big) ds, \ t \in I.
\end{aligned} \tag{4.47}
$$

The following theorem proves the continuous dependency of solutions on parameters.

Theorem 4.5. *Consider the sequences* $\{y_k\}_{k=0}^{\infty}$ *and* $\{\overline{y}_k\}_{k=0}^{\infty}$ *generated normal* $S-$ *iterative method associated with operators* T *in* (4.46) *and* $\overline{T}$ *in* (4.47), *respectively with the real sequence* $\{\xi_k\}_{k=0}^{\infty}$ *in* $[0,1]$ *satisfying* $\frac{1}{2} \leq \xi_k$ *for all* $k \in \mathbb{N}$. *Assume that*

(i) $y(t)$ *and* $\overline{y}(t)$ *are solutions of* (4.42)-(4.43) *and* (4.44)-(4.45) *respectively.*
(ii) *the function* $\mathcal{F}$ *satisfy the conditions:*

$$
\Big\|\mathcal{F}\Big(t, u_1, \mu_1\Big) - \mathcal{F}\Big(t, v_1, \mu_1\Big)\Big\| \leq \overline{p}(t)\Big\|u_1 - v_1\Big\|.
$$

and

$$
\Big\|\mathcal{F}\Big(t, u_1, \mu_1\Big) - \mathcal{F}\Big(t, u_1, \mu_2\Big)\Big\| \leq r(t)\Big|\mu_1 - \mu_2\Big|,
$$

where $\overline{p},\ r \in C(I, \mathbb{R}_+)$.

If the sequence $\{\overline{y}_n\}_{n=0}^{\infty}$ *converges to* $\overline{y}$, *then we have*

$$
\Big\|y - \overline{y}\Big\|_B \leq \frac{3\Big[M + \Big(\frac{r(a)\big|\mu_1-\mu_2\big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big) + \frac{\big|\mu_1-\mu_2\big|(b-a)^{n-1}}{(n-1)!} {I_a}^{\alpha-n+1} r(b) + \Big|\mu_1 - \mu_2\Big| {I_a}^{\alpha} r(t)\Big]}{\Big(1 - \overline{\Theta}\Big)}, \tag{4.48}
$$

where $\overline{\Theta} = \Big[\dfrac{(b-a)^{\alpha}\overline{p}(a)}{(n-2)!\Gamma(\alpha-n+2)} + \dfrac{(b-a)^{n-1}}{(n-1)!} {I_a}^{\alpha-n+1}\overline{p}(b) + {I_a}^{\alpha}\overline{p}(t)\Big] < 1.$

Proof. Suppose the sequences $\{y_k\}_{k=0}^{\infty}$ and $\{\overline{y}_k\}_{n=0}^{\infty}$ generated normal $S-$ iterative method associated with operators T in (4.46) and $\overline{T}$ in (4.47), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$. From iteration (2.6) and equations (4.46); (4.47) and hypotheses, we obtain

$$
\begin{aligned}
&\Big\|y_{k+1}(t) - \overline{y}_{k+1}(t)\Big\| \\
&= \Big\|(Tz_k)(t) - (\overline{T}\overline{z}_k)(t)\Big\| \\
&= \Big\| \sum_{j=0}^{n-2} \frac{c_j}{j!} (t-a)^j + \Big(\frac{c_b}{(n-1)!} + \frac{\mathcal{F}(a, z_k(b), \mu_1)(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&\quad - \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathcal{F}\Big(s, z_k(s), \mu_1\Big) ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, z_k(s), \mu_1\Big) ds \\
&\quad - \sum_{j=0}^{n-2} \frac{d_j}{j!} (t-a)^j - \Big(\frac{\overline{c}_b}{(n-1)!} + \frac{\mathcal{F}(a, \overline{z}_k(b), \mu_2)(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)}\Big)(t-a)^{n-1} \\
&\quad + \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \mathcal{F}\Big(s, \overline{z}_k(s), \mu_2\Big) ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \mathcal{F}\Big(s, \overline{z}_k(s), \mu_2\Big) ds \Big\| \\
&\leq \sum_{j=0}^{n-2} \frac{\Big\| c_j - d_j \Big\|}{j!} (t-a)^j + \frac{\Big\| c_b - \overline{c}_b \Big\|}{(n-1)!} (t-a)^{n-1} \\
&+ \Big(\frac{\Big\| \mathcal{F}(a, z_k(b), \mu_1) - \mathcal{F}(a, \overline{z}_k(b), \mu_2) \Big\| (b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \Big) (t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \Big\| \mathcal{F}\Big(s, z_k(s), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \mu_2\Big) \Big\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \Big\| \mathcal{F}\Big(s, z_k(s), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \mu_2\Big) \Big\| ds \\
&\leq \sum_{j=0}^{n-2} \frac{\Big\| c_j - d_j \Big\|}{j!} (t-a)^j + \frac{\Big\| c_b - \overline{c}_b \Big\|}{(n-1)!} (t-a)^{n-1} \\
&+ \Big(\frac{\Big\| \mathcal{F}(a, \overline{z}_k(b), \mu_1) - \mathcal{F}(a, \overline{z}_k(b), \mu_2) \Big\| (b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \Big) (t-a)^{n-1} \\
&+ \Big(\frac{\Big\| \mathcal{F}(a, z_k(b), \mu_1) - \mathcal{F}(a, \overline{z}_k(b), \mu_1) \Big\| (b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \Big) (t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \Big\| \mathcal{F}\Big(s, \overline{z}_k(s), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \mu_2\Big) \Big\| ds \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \Big\| \mathcal{F}\Big(s, z_k(s), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \mu_1\Big) \Big\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \Big\| \mathcal{F}\Big(s, \overline{z}_k(s), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \mu_2\Big) \Big\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \Big\| \mathcal{F}\Big(s, z_k(s), \mu_1\Big) - \mathcal{F}\Big(s, \overline{z}_k(s), \mu_1\Big) \Big\| ds \\
&\leq \sum_{j=0}^{n-2} \frac{\Big\| c_j - d_j \Big\|}{j!} (b-a)^j + \frac{\Big\| c_b - \overline{c}_b \Big\|}{(n-1)!} (b-a)^{n-1} \\
&+ \Big(\frac{r(a) \Big| \mu_1 - \mu_2 \Big| (b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \Big) (t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} r(s) \Big| \mu_1 - \mu_2 \Big| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} r(s) \Big| \mu_1 - \mu_2 \Big| ds \\
&+ \Big(\frac{\overline{p}(a) \Big\| z_k(b) - \overline{z}_k(b) \Big\| (b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \Big) (t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \overline{p}(s) \Big\| z_k(s) - \overline{z}_k(s) \Big\| ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \Big\| z_k(s) - \overline{z}_k(s) \Big\| ds \\
&\leq M + \Big(\frac{r(a)\Big|\mu_1 - \mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \Big) \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} r(s) \Big|\mu_1 - \mu_2\Big| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} r(s) \Big|\mu_1 - \mu_2\Big| ds \\
&+ \Big(\frac{\overline{p}(a)\Big\| z_k(b) - \overline{z}_k(b)\Big\|(b-a)^{\alpha-n+1}}{(n-2)!\Gamma(\alpha-n+2)} \Big)(t-a)^{n-1} \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \overline{p}(s) \Big\| z_k(s) - \overline{z}_k(s) \Big\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \Big\| z_k(s) - \overline{z}_k(s) \Big\| ds \\
&\leq M + \Big(\frac{r(a)\Big|\mu_1 - \mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \Big) + \frac{\Big|\mu_1 - \mu_2\Big|(b-a)^{n-1}}{(n-1)!} {I_a}^{\alpha-n+1} r(b) + \Big|\mu_1 - \mu_2\Big| {I_a}^{\alpha} r(t) \\
&+ \Big(\frac{\overline{p}(a)\Big\| z_k(b) - \overline{z}_k(b)\Big\|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \Big) \\
&+ \frac{(t-a)^{n-1}}{(n-1)!\Gamma(\alpha-n+1)} \int_a^b (b-s)^{\alpha-n} \overline{p}(s) \Big\| z_k(s) - \overline{z}_k(s) \Big\| ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \overline{p}(s) \Big\| z_k(s) - \overline{z}_k(s) \Big\| ds. \qquad (4.49)
\end{aligned}
$$

Recalling the derivations obtained in equations (3.13) and (3.15), the above inequality becomes

$$
\begin{aligned}
\Big\| y_{k+1} - \overline{y}_{k+1} \Big\|_B &\leq M + \Big(\frac{r(a)\Big|\mu_1 - \mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \Big) + \frac{\Big|\mu_1 - \mu_2\Big|(b-a)^{n-1}}{(n-1)!} {I_a}^{\alpha-n+1} r(b) \\
&+ \Big|\mu_1 - \mu_2\Big| {I_a}^{\alpha} r(t) + \overline{\Theta} \Big\| z_k - \overline{z}_k \Big\|_B, \qquad (4.50)
\end{aligned}
$$

and similarly, it is seen that

$$
\begin{aligned}
\Big\| z_k - \overline{z}_k \Big\|_B &\leq \xi_k \Big[M + \Big(\frac{r(a)\Big|\mu_1 - \mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \Big) + \frac{\Big|\mu_1 - \mu_2\Big|(b-a)^{n-1}}{(n-1)!} {I_a}^{\alpha-n+1} r(b) \\
&+ \Big|\mu_1 - \mu_2\Big| {I_a}^{\alpha} r(t) \Big] + \Big[1 - \xi_k \Big(1 - \overline{\Theta} \Big) \Big] \Big\| y_k - \overline{y}_k \big\|_B. \qquad (4.51)
\end{aligned}
$$

Therefore, using (4.51) in (4.50) and using hypothesis $\overline{\Theta} < 1$, and $\frac{1}{2} \leq \xi_k$ for all $k \in \mathbb{N}$, the resulting inequality becomes

$$
\begin{aligned}
&\Big\| y_{k+1} - \overline{y}_{k+1} \Big\|_B \\
&\leq \Big[M + \Big(\frac{r(a)\Big|\mu_1 - \mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)} \Big) + \frac{\Big|\mu_1 - \mu_2\Big|(b-a)^{n-1}}{(n-1)!} {I_a}^{\alpha-n+1} r(b) + \Big|\mu_1 - \mu_2\Big| {I_a}^{\alpha} r(t) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\|z_k-\overline{z}_k\Big\|_B\\
&\le\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]\\
&\quad+\xi_k\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]\\
&\quad+\Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B\\
&\le 2\xi_k\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]\\
&\quad+\xi_k\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]\\
&\quad+\Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B\\
&\le\Big[1-\xi_k\Big(1-\overline{\Theta}\Big)\Big]\Big\|y_k-\overline{y}_k\Big\|_B\\
&\quad+\xi_k\Big(1-\overline{\Theta}\Big)\frac{3\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}.
\end{aligned}
\tag{4.52}
$$

We denote

$$
\begin{aligned}
&\beta_k=\Big\|y_k-\overline{y}_k\Big\|_B\ge 0,\\
&\mu_k=\xi_k\Big(1-\overline{\Theta}\Big)\in(0,1),\\
&\gamma_k=\frac{3\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}\ge 0.
\end{aligned}
$$

The assumption $\frac{1}{2}\le\xi_k$ for all $k\in\mathbb{N}$ implies $\sum_{k=0}^{\infty}\xi_k=\infty$. Now, it can be easily seen that (4.52) satisfies all the conditions of Lemma 2.2 and hence we have

$$
\begin{aligned}
&0\le\limsup_{k\to\infty}\beta_k\le\limsup_{k\to\infty}\gamma_k\\
&\Rightarrow 0\le\limsup_{k\to\infty}\Big\|y_k-\overline{y}_k\Big\|_B\\
&\qquad\le\limsup_{k\to\infty}\frac{3\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}\\
&\Rightarrow 0\le\limsup_{k\to\infty}\Big\|y_k-\overline{y}_k\Big\|_B
\end{aligned}
$$

$$\leq \frac{3\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}. \tag{4.53}$$

Using the assumption $\lim_{k\to\infty} y_k = y$, $\lim_{k\to\infty} \overline{y}_k = \overline{y}$, we get from (4.53) that

$$\Big\|y-\overline{y}\Big\|_B \leq \frac{3\Big[M+\Big(\frac{r(a)\Big|\mu_1-\mu_2\Big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big)+\frac{\Big|\mu_1-\mu_2\Big|(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}r(b)+\Big|\mu_1-\mu_2\Big|{I_a}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)}, \tag{4.54}$$

which shows the dependence of solutions of the problem (1.1)-(1.2) is on the parameters μ_1 and μ_2. □

Remark 4.3. The result (4.54) deals with the property of a solution called "dependence of solutions on parameters". Here the parameters are scalars and also note that the boundary conditions do not involve parameters. The dependence on parameters is an important aspect in various physical problems.

5. Example

We consider the following problem:

$$\big(D_*^{\alpha}\big)y(t) = \frac{3t}{5}\Big[\frac{t-\sin(y(t))}{2}\Big],\ t\in[0,1],\ n-1<\alpha\leq n,\ n\in\mathbb{N} \tag{5.55}$$

with the given boundary conditions

$$y^{(j)}(0)=0,\ j=0,1,2,\cdots,n-2,\ y^{(n-1)}(1)=1. \tag{5.56}$$

Comparing this equation with the equation (1.1), we get

$$\mathcal{F}\in C(I\times\mathbb{R},\mathbb{R}),\ \text{with}\ \mathcal{F}(t,y(t)) = \frac{3t}{5}\Big[\frac{t-\sin(y(t))}{2}\Big].$$

Now, we have

$$\begin{aligned}\Big|\mathcal{F}(t,y(t))-\mathcal{F}(t,\overline{y}(t))\Big| &\leq \Big|\frac{3t}{5}\Big|\Big|\frac{t-\sin(y(t)))}{2}-\frac{t-\sin(\overline{y}(t))}{2}\Big|\\ &\leq \frac{3t}{10}\Big|\sin(y(t))-\sin(\overline{y}(t))\Big|\\ &\leq \frac{3t}{10}\Big|y(t)-\overline{y}(t)\Big|,\end{aligned} \tag{5.57}$$

where $p(t)=\dfrac{3t}{10}$.

5.1. Existence and Uniqueness of Solutions.

Therefore, we the estimate

$$\begin{aligned}\Theta &= \Big[\frac{(b-a)^{\alpha}p(a)}{(n-2)!\Gamma(\alpha-n+2)}+\frac{(b-a)^{n-1}}{(n-1)!}{I_a}^{\alpha-n+1}p(b)+{I_a}^{\alpha}p(t)\Big]\\ &= \Big[\frac{p(0)}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!}I^{\alpha-n+1}p(1)+I^{\alpha}p(t)\Big]\\ &= \Big[\frac{0}{(n-2)!\Gamma(\alpha-n+2)}+\frac{1}{(n-1)!}I^{\alpha-n+1}p(1)+I^{\alpha}p(t)\Big]\ (p(0)=0)\end{aligned}$$

$$= \frac{3}{10}\Big[\frac{1}{(n-1)!\Gamma(\alpha-n+1)}\int_0^1 (1-s)^{\alpha-n} s ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s ds\Big]$$

$$\leq \frac{3}{10}\Big[\frac{1}{(n-1)!\Gamma(\alpha-n+3)} + \frac{1}{\Gamma(\alpha+2)}\Big] \; (t \leq 1). \tag{5.58}$$

If the quantity $\frac{3}{10}\Big[\frac{1}{(n-1)!\Gamma(\alpha-n+3)} + \frac{1}{\Gamma(\alpha+2)}\Big] < 1$, then $\Theta < 1$. In particular, we choose $\alpha = \frac{5}{2}$, then we have $n = [\alpha] + 1 = [\frac{5}{2}] + 1 = 2 + 1 = 3$ and

$$\begin{aligned}\Theta &\leq \frac{3}{10}\Big[\frac{1}{(3-1)!\Gamma(\frac{5}{2}-3+3)} + \frac{1}{\Gamma(\frac{5}{2}+2)}\Big] \\ &= \frac{3}{10}\Big[\frac{1}{2\Gamma(\frac{5}{2})} + \frac{1}{\Gamma(\frac{9}{2})}\Big] \\ &= \frac{3}{5\sqrt{\pi}}\Big[\frac{1}{3} + \frac{8}{105}\Big] \\ &= \frac{43}{175\sqrt{\pi}} \\ &\simeq 0.1387 \\ &< 1.\end{aligned}$$

We define the operator $T : B \to B$ for the given problem by

$$\begin{aligned}(Ty)(t) = \frac{t^2}{2} - \frac{t^2}{2}\frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-s)^{-\frac{1}{2}} \frac{3s}{5}\Big[\frac{s - \sin(y(s))}{2}\Big] ds \\ + \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}} \frac{3s}{5}\Big[\frac{s - \sin(y(s))}{2}\Big] ds, \; t \in I.\end{aligned} \tag{5.59}$$

Since all conditions of Theorem 3.2 are satisfied and so by its conclusion, the sequence $\{y_k\}$ associated with the normal $S-$iterative method (2.6) for the operator T in (5.59) converges to a unique solution $y \in B$. This convergence under S-iteration process is faster than the Picard, Mann and Ishikawa iteration processes.
Now, we will discuss the simplicity and fastness of the S- iteration method. By refereing [7, 8, 27, 34], the definitions of $a_k,\ b_k$, c_k and d_k under S-iteration, Picard iteration, Mann iteration and Ishikawa iteration are given, respectively:

(a) $a_k = \nu^k \Big[1 - (1-\nu)\alpha\beta\Big]^k \|u_1 - x^*\|$,
(b) $b_k = \nu^k \|v_1 - x^*\|$,
(c) $c_k = \Big[1 - (1-\nu)\beta\Big]^k \|w_1 - x^*\|$,
(d) $d_k = \Big[1 - (1-\nu)^2\beta\Big]^k \|x_1 - x^*\|$,

where $\nu \in [0, 1)$ is contracting factor. For given $u_1 = v_1 = w_1 = x_1 \in \mathbb{R}$, the convergence of sequences $\{a_k\}$, $\{b_k\}$, $\{c_k\}$ and $\{d_k\}$ depend only the factors $\Theta_1 = \nu^k\Big[1 - (1-\nu)\alpha\beta\Big]^k$, $\Theta_2 = \nu^k$, $\Theta_3 = \Big[1 - (1-\nu)\beta\Big]^k$ and $\Theta_4 = \Big[1 - (1-\nu)^2\beta\Big]^k$ respectively. Therefore, the following comparison table shows the values of the factors $\Theta_1,\ \Theta_2$, Θ_3 and Θ_4 under respective iteration processes for the numerical example discussed in this paper with $\nu = \Theta = 0.138629441$ and $\xi_n = \beta_n = \frac{1}{2}$:

Iteration (k)	S-iteration (Θ_1)	P-iteration (Θ_2)	M-iteration (Θ_3)	I-iteration (Θ_4)
1	0.078923781	0.138629441	0.569314720	0.629020380
2	0.006228963	0.019218122	0.324119251	0.395666639
3	0.000491613	0.002664197	0.184525861	0.248882379
4	0.000038800	0.000369336	0.105053289	0.156552089
5	0.000003062	0.000051201	0.059808384	0.098474454
6	0.000000242	0.000007098	0.034049793	0.061942439
7	0.000000019	0.000000984	0.019385049	0.038963056
8	0.000000002	0.000000136	0.011036193	0.024508557
9	0.000000000	0.000000019	0.006283067	0.015416382
10	0.000000000	0.000000003	0.003577043	0.009697218

Hence, observing the above table and Definitions 2.4, 2.5, it is easy to see that $\lim_{k\to\infty}\frac{a_k}{b_k}=0$, $\lim_{k\to\infty}\frac{a_k}{c_k}=0$ and $\lim_{k\to\infty}\frac{a_k}{d_k}=0$. Therefore, we conclude that the S-iteration process is faster than the Picard, Mann and Ishikawa iteration processes.

5.2. **Error Estimate.** Further, we also have for any $y_0\in B$

$$\begin{aligned}\|y_{k+1}-y\|_B &\le \frac{\Theta^{k+1}}{e^{\left(1-\Theta\right)\sum_{i=0}^{k}\xi_i}}\|y_0-y\|_B\\ &\le \frac{\left[\frac{43}{175\sqrt{\pi}}\right]^{k+1}}{e^{\left[1-\frac{43}{175\sqrt{\pi}}\right]\sum_{i=0}^{k}\xi_i}}\|y_0-y\|_B\\ &\le \frac{\left(\frac{43}{175\sqrt{\pi}}\right)^{k+1}}{e^{\left(1-\frac{43}{175\sqrt{\pi}}\right)\sum_{i=0}^{k}\frac{1}{1+i}}}\|y_0-y\|_B, \end{aligned} \tag{5.60}$$

where we have chosen $\xi_i=\frac{1}{1+i}\in[0,1]$. The estimate obtained in (5.60) is called a bound for the error (due to truncation of computation at the $k-$th iteration).

5.3. **Continuous Dependence.** One can check easily the continuous dependence of solutions of equations (1.1) on boundary data. Indeed, for $c_0=c_1=d_0=d_1=0,\ c_b=1,\ \overline{c_b}=\frac{1}{2}$, we have

$$\begin{aligned}\|y-\overline{y}\|_B &\le \frac{3M}{\Big(1-\Theta\Big)}\\ &\le \frac{3\Big[\sum_{j=0}^{n-2}\frac{\left\|c_j-d_j\right\|}{j!}(b-a)^j+\frac{\left\|c_b-\bar{c}_b\right\|}{(n-1)!}(b-a)^{n-1}\Big]}{\Big(1-\Theta\Big)}\\ &\le \frac{3\frac{\left[1-\frac{1}{2}\right]}{2}}{\left(1-\frac{43}{175\sqrt{\pi}}\right)}\\ &\le \frac{3}{3.4454}\\ &\simeq 0.8707. \end{aligned} \tag{5.61}$$

5.4. **Closeness of Solutions.** Next, we consider the perturbed equation:

$$\big(D_*^{\alpha}\big)\overline{y}(t) = \frac{3t}{5}\Big[\frac{t-\sin(\overline{y}(t))}{2}\Big] - t + \frac{1}{7},\ t \in [0,1], \tag{5.62}$$

with the given boundary conditions

$$\overline{y}(0) = 0,\ \overline{y}'(0) = 0,\ \overline{y}''(1) = \frac{1}{2}. \tag{5.63}$$

Similarly, comparing it with the equation (4.31), we have

$$\overline{\mathcal{F}}(t, \overline{y}(t)) = \frac{3t}{5}\Big[\frac{t-\sin(\overline{y}(t))}{2}\Big] - t + \frac{1}{7}.$$

One can easily define the mapping $\overline{T} : B \to B$ by

$$\begin{aligned}(\overline{T}\overline{y})(t) &= \frac{t^2}{4} + \frac{2}{7}\frac{t^2}{\sqrt{\pi}} - \frac{t^2}{2}\frac{1}{\Gamma(\frac{1}{2})}\int_0^1 (1-s)^{-\frac{1}{2}}\Big\{\frac{3s}{5}\Big[\frac{s-\sin(\overline{y}(s))}{2}\Big] - s + \frac{1}{7}\Big\}ds \\ &+ \frac{1}{\Gamma(\frac{5}{2})}\int_0^t (t-s)^{\frac{3}{2}}\Big\{\frac{3s}{5}\Big[\frac{s-\sin(\overline{y}(s))}{2}\Big] - s + \frac{1}{7}\Big\}ds,\ t \in I.\end{aligned} \tag{5.64}$$

In perturbed equation, all conditions of Theorem 3.2 are also satisfied and so by its conclusion, the sequence $\{\overline{y}_k\}$ associated with the normal $S-$iterative method (2.6) for the operator $\overline{T}$ in (5.64) converges to a unique solution $\overline{y} \in B$.
Now, one can easily prove the estimate:

$$|\mathcal{F}(t, y(t)) - \overline{\mathcal{F}}(t, y(t))| \le \frac{8}{7} = \epsilon. \tag{5.65}$$

Consider the sequences $\{y_k\}_{k=0}^{\infty}$ with $y_k \to y$ as $k \to \infty$ and $\{\overline{y}_k\}_{k=0}^{\infty}$ with $\overline{y}_k \to \overline{y}$ as $k \to \infty$ generated normal $S-$ iterative method associated with operators T in (5.59)and $\overline{T}$ in (5.64), respectively with the real sequence $\{\xi_k\}_{k=0}^{\infty}$ in $[0,1]$ satisfying $\frac{1}{2} \le \xi_k$ for all $k \in \mathbb{N}$. Then we have from Theorem 4.3 that for $M = \frac{1}{4},\ a = 0,\ b = 1,\ \epsilon = \frac{8}{7}$

$$\begin{aligned}\|y - \overline{y}\|_B &\le \frac{3\Big[M + \epsilon(b-a)^{\alpha}\Big(\frac{1}{(n-2)!\Gamma(\alpha-n+2)} + \frac{1}{(n-1)!\Gamma(\alpha-n+2)} + \frac{1}{\Gamma(\alpha+1)}\Big)\Big]}{\Big(1-\Theta\Big)} \\ &\le \frac{3\Big[\frac{1}{4} + \frac{8}{7}\Big(\frac{1}{\Gamma(\frac{3}{2})} + \frac{1}{2\Gamma(\frac{3}{2})} + \frac{1}{\Gamma(\frac{7}{2})}\Big)\Big]}{\Big(1 - \frac{43}{175\sqrt{\pi}}\Big)} \\ &\le \frac{7.5848}{\Big(1 - \frac{43}{175\sqrt{\pi}}\Big)} \\ &\le \frac{7.5848}{0.8613} \\ &\simeq 8.8062.\end{aligned} \tag{5.66}$$

This shows the closeness and dependency of solutions on functions which are involved therein.

5.5. **Dependence on Parameters.** Finally, we shall prove the dependency of solutions on real parameters.
We consider the following integral equations involving real parameters $\mu_1,\ \mu_2$:

$$D_*^{\frac{5}{2}}y(t) = \frac{3t}{5}\Big[\frac{t-\sin(y(t))}{2} + \mu_1\Big],\ t \in [0,1], \tag{5.67}$$

with the given boundary conditions

$$y(0) = 0,\ y'(0) = 0,\ y''(1) = 1 \tag{5.68}$$

and

$$D_*^{\frac{5}{2}}\overline{y}(t) = \frac{3t}{5}\Big[\frac{t-\sin(\overline{y}(t))}{2} + \mu_2\Big],\ t \in [0,1], \tag{5.69}$$

with the given boundary conditions

$$\overline{y}(0) = 0,\ \overline{y}'(0) = 0,\ \overline{y}''(1) = \frac{1}{2}. \tag{5.70}$$

Based on the above discussion, one can observe that $p(t) = \overline{p}(t) = \frac{3t}{10},\ r(t) = \frac{3t}{5}$ and therefore, we have $\Theta = \overline{\Theta}$. Hence by making similar arguments and from Theorem 4.5, one can have ($M = \frac{1}{4},\ a = 0,\ b = 1,\ p(t) = \overline{p}(t) = \frac{3t}{10},\ r(t) = \frac{3t}{5}$)

$$\begin{aligned}
\|y-\overline{y}\|_B &\leq \frac{3\Big[M + \Big(\frac{r(a)\big|\mu_1-\mu_2\big|(b-a)^{\alpha}}{(n-2)!\Gamma(\alpha-n+2)}\Big) + \frac{\big|\mu_1-\mu_2\big|(b-a)^{n-1}}{(n-1)!}I_a{}^{\alpha-n+1}r(b) + \Big|\mu_1-\mu_2\Big|I_a{}^{\alpha}r(t)\Big]}{\Big(1-\overline{\Theta}\Big)} \\
&\leq \frac{3\Big[\frac{1}{4} + \Big(\frac{r(0)\big|\mu_1-\mu_2\big|}{\Gamma(\frac{3}{2})}\Big) + \frac{\big|\mu_1-\mu_2\big|}{2}I^{\frac{1}{2}}r(1) + \Big|\mu_1-\mu_2\Big|I^{\frac{5}{2}}\frac{3t}{5}\Big]}{\Big(1-\frac{43}{175\sqrt{\pi}}\Big)} \\
&\leq \frac{3\Big[\frac{1}{4} + \frac{86}{175\sqrt{\pi}}\Big|\mu_1-\mu_2\Big|\Big]}{\Big(1-\frac{43}{175\sqrt{\pi}}\Big)}.
\end{aligned} \tag{5.71}$$

In particular, if we choose $\mu_1 = 1,\ \mu_2 = \frac{1}{2}$, then we have from (5.71) that

$$\begin{aligned}
\|y-\overline{y}\|_B &\leq \frac{3\Big[\frac{1}{4} + \frac{86}{175\sqrt{\pi}}\Big|1-\frac{1}{2}\Big|\Big]}{\Big(1-\frac{43}{175\sqrt{\pi}}\Big)} \\
&\leq \frac{1.1659}{0.8613} \\
&\simeq 1.3537.
\end{aligned} \tag{5.72}$$

This proves the dependence of solutions is on both boundary data and real parameters.

6. Conclusions

First, we proved existence and uniqueness of the solution of the BVP (1.1)-(1.2) using S-iterative method. Next, we discussed various properties of solutions like continuous dependence on the boundary data, closeness of solutions, dependence of solutions on parameters and functions involved therein. Finally, we gave suitable example which illustrate all proved results along with the comparison table showing that S-iteration method is faster than Picard, Mann and Ishikawa iteration processes.

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