CS364A: Problem Set #3

Due in class on Thursday, February 17, 2011

Instructions: Same as previous problem sets.

Problem 11

(a) (5 points) *Algorithmic Game Theory*, Exercise 17.2.

(b) (5 points) *Algorithmic Game Theory*, Exercise 17.3.

(c) (10 points) *Algorithmic Game Theory*, Exercise 18.2(b).

Problem 12

(25 points) *Algorithmic Game Theory*, Exercise 18.8.

Problem 13

Consider n machines and m selfish jobs (the players). Each job j has a processing time p_j and a set S_j of machines on which it can be scheduled (i.e., S_j is the strategy set of player j). Once jobs have chosen machines, the jobs on each machine are processed serially from shortest to longest. (You can assume that the p_j's are distinct.) For example, if jobs with processing times 1, 3, and 5 are scheduled on a common machine, then they will complete at times 1, 4, and 9, respectively. The following questions concern the game in which players choose machines in order to minimize their completion times.

(a) (5 points) Consider the following scheduling algorithm: (1) Sort the jobs in order from smallest to largest; (2) schedule the jobs one-at-a-time, assigning a job j to the machine of S_j with minimum load-so-far (breaking ties arbitrarily).

Prove that the (pure-strategy) Nash equilibria of the scheduling game are precisely the possible outputs of this scheduling algorithm (with the different equilibria arising from different ways of breaking ties).

[Hint: For example, if you were the smallest player, how is your personal cost affected by the others' decisions?]

(b) (9 points) For this and the next part, we consider the *makespan* objective function, defined as the time at which the final job is completed. (See also the AGT book, Section 17.2.3 and Chapter 20.) Prove that the price of anarchy in every such scheduling game, with respect to the makespan objective, is $O(\log n)$.

[Hint: You might read the proof of Theorem 20.7 in the AGT book to get an idea for the kinds of arguments that could be useful here.]

(c) (6 points) Prove that, for arbitrarily large n, there are scheduling games of the above type in which the price of anarchy is $\Omega(\log n)$.

Problem 14

In this problem we consider nonatomic selfish routing networks with one source, one sink, one unit of selfish traffic, and affine cost functions (of the form $c_e(x) = a_e x + b_e$ for $a_e, b_e \geq 0$). In parts (a)-(c), we consider the objective of the *maximum cost* incurred by a flow f:

$$\max_{P : f_P > 0} \sum_{e \in P} c_e(f_e).$$

The *price of anarchy* is then defined in the usual way, as the ratio between the maximum cost of an equilibrium flow and that of a flow with minimum-possible maximum cost. (Of course, in an equilibrium flow, all traffic incurs exactly the same cost; this is not generally true in a non-equilibrium flow.)

(a) (4 points) Prove that in a network of parallel links (each directly connecting the source to the sink), the price of anarchy with respect to the maximum cost objective is 1.

(b) (4 points) Prove that the price of anarchy with respect to the maximum cost objective can be as large as 4/3 in general networks (with affine cost functions, one source and one sink).

(c) (5 points) Prove that the price of anarchy with respect to the maximum cost objective is never larger than 4/3 (in networks with affine cost functions, one source and one sink).

[Hint: try to reduce this to facts you already know.]

(d) (7 points) A flow that minimizes the average cost of traffic generally routes some traffic on costlier paths than others. Prove that the ratio between the cost of the longest used path and that of the shortest used path in a minimum-cost flow is at most 2 (in networks with affine cost functions, one source and one sink). Prove that this bound can be achieved.

Problem 15

Recall the GSP auction from Problem 2. For this problem we'll think about bidders with known valuations and the pure-strategy Nash equilibria of the corresponding game. Note that in part (e) of Problem 2 you proved that every such game (for any k, n, valuations, and click-through rates), there is a pure-strategy Nash equilibrium with the maximum-possible surplus. That is, the price of stability of the game is always 1.

(a) (5 points) Show that even when $k = 1$ and $n = 2$, the price of anarchy of the GSP game can be arbitrarily bad.

(b) (5 points) Consider now a Nash equilibrium in which every bid b_i is at most the player's valuation v_i. Suppose that players i and j are assigned to slots with click-through rates α_h and α_ℓ, respectively, with $h < \ell$. Prove that

$$\frac{\alpha_\ell}{\alpha_h} + \frac{v_i}{v_j} \geq 1.$$

(c) (10 points) Consider again a Nash equilibrium in which every bid b_i is at most the player's valuation v_i. Prove (perhaps using (b)) that the surplus of this equilibrium is at least 50% of the maximum possible.