

# The Open Problems Project

edited by Erik D. Demaine      Joseph S. B. Mitchell  
Joseph O'Rourke

December 18, 2024

## 0.1 Introduction

This project originally aimed to record important open problems of interest to researchers in computational geometry and related fields. It commenced in 2001 with the publication of thirty problems in Computational Geometry Column 42 [MO01] (see Problems 1–30), and then grew to over 75 problems.

While we are no longer encouraging new problem submissions, we strongly encourage updates to existing problems, especially when those problems have been solved (completely or partially). Updates should be done in the form of a Github Pull Request; see the Github repository.

Each problem is assigned a unique number for citation purposes. Problem numbers also indicate the order in which the problems were entered. Each problem is classified as belonging to one or more categories.

The problems are also available as a single PDF file.

## 0.2 Categorized List of All Problems

Below, each category lists the problems that are classified under that category. Note that each problem may be classified under several categories.

### arrangements:

- 3-Colorability of Arrangements of Great Circles (Problem 44)

### art galleries:

- Vertex  $\pi$ -Floodlights (Problem 23)

### coloring:

- 3-Colorability of Arrangements of Great Circles (Problem 44)

### combinatorial geometry:

- $k$ -sets (Problem 7)
- Binary Space Partition Size (Problem 14)

- Chromatic Number of the Plane (Problem 57)
- Counting Polyominoes (Problem 37)
- Distances among Point Sets in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  (Problem 39)
- Extending Pseudosegment Arrangements by Subdivision (Problem 34)
- Lines Tangent to Four Unit Balls (Problem 61)
- Monochromatic Triangles (Problem 58)
- Pushing Disks Together (Problem 18)
- Rolling a Die over a Labeled Board (Problem 68)
- Slicing Axes-Parallel Rectangles (Problem 74)
- The Number of Pointed Pseudotriangulations (Problem 40)
- Thrackles (Problem 30)
- Union of Fat Objects in 3D (Problem 4)
- Vertical Decompositions in  $\mathbb{R}^d$  (Problem 19)

**convex hulls:**

- Dynamic Planar Convex Hull (Problem 12)
- Dynamic Planar Nearest Neighbors (Problem 63)
- Inplace Convex Hull of a Simple Polygonal Chain (Problem 36)
- Output-sensitive Convex Hull in  $\mathbb{R}^d$  (Problem 15)

**data structures:**

- Binary Space Partition Size (Problem 14)
- Dynamic Planar Convex Hull (Problem 12)
- Dynamic Planar Nearest Neighbors (Problem 63)
- Point Location in 3D Subdivision (Problem 13)

**Delaunay triangulations:**

- Flip Graph Connectivity in 3D (Problem 28)
- Stretch-Factor for Points in Convex Position (Problem 71)
- Voronoi Diagram of Moving Points (Problem 2)

**dissections:**

- Congruent Partitions of Polygons (Problem 73)

**folding and unfolding:**

- Edge-Unfolding Convex Polyhedra (Problem 9)
- Edge-Unfolding Polycubes (Problem 64)

- General Unfoldings of Nonconvex Polyhedra (Problem 43)
- Vertex-Unfolding Polyhedra (Problem 42)
- Volume Maximizing Convex Shape (Problem 62)

**geometric graphs:**

- Edge-Coloring Geometric Graphs (Problem 75)
- Thrackles (Problem 30)
- Yao-Yao Graph a Spanner? (Problem 70)

**graph drawing:**

- 3D Minimum-Bend Orthogonal Graph Drawings (Problem 46)
- Isoceles Planar Graph Drawing (Problem 69)
- Linear-Volume 3D Grid Drawings of Planar Graphs (Problem 51)
- Queue-Number of Planar Graphs (Problem 52)
- Smallest Universal Set of Points for Planar Graphs (Problem 45)
- Thrackles (Problem 30)

**graphs:**

- Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 53)
- Smallest Universal Set of Points for Planar Graphs (Problem 45)
- Thrackles (Problem 30)
- Traveling Salesman Problem in Solid Grid Graphs (Problem 54)

**linear programming:**

- Linear Programming: Strongly Polynomial? (Problem 8)

**lower bounds:**

- 3SUM Hard Problems (Problem 11)
- Sorting  $X + Y$  (Pairwise Sums) (Problem 41)

**meshing:**

- Hexahedral Meshing (Problem 27)
- Most Circular Partition of a Square (Problem 59)

**minimum spanning tree:**

- Bounded-Degree Minimum Euclidean Spanning Tree (Problem 48)
- Euclidean Minimum Spanning Tree (Problem 5)

**numerical computations:**

- Sum of Square Roots (Problem 33)

**optimization:**

- Bounded-Degree Minimum Euclidean Spanning Tree (Problem 48)
- Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots (Problem 35)
- Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 53)
- Packing Unit Squares in a Simple Polygon (Problem 56)
- Pallet Loading (Problem 55)
- Planar Euclidean Maximum TSP (Problem 49)
- Traveling Salesman Problem in Solid Grid Graphs (Problem 54)

**packing:**

- Most Circular Partition of a Square (Problem 59)
- Packing Unit Squares in a Simple Polygon (Problem 56)
- Pallet Loading (Problem 55)
- Rectangling a Rectangle (Problem 78)

**partitioning:**

- Congruent Partitions of Polygons (Problem 73)
- Fair Partitioning of Convex Polygons (Problem 67)

**partitioning.:**

- Rectangling a Rectangle (Problem 78)

**planar graphs:**

- Bar-Magnet Polyhedra (Problem 32)
- Isoceles Planar Graph Drawing (Problem 69)
- Pointed Spanning Trees in Triangulations (Problem 50)

**point sets:**

- $k$ -sets (Problem 7)
- Bounded-Degree Minimum Euclidean Spanning Tree (Problem 48)
- Magic Configurations (Problem 65)
- Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 53)
- Planar Euclidean Maximum TSP (Problem 49)
- Simple Polygonalizations (Problem 16)
- Smallest Universal Set of Points for Planar Graphs (Problem 45)

- Surface Reconstruction (Problem 26)
- Traveling Salesman Problem in Solid Grid Graphs (Problem 54)

**point sets.:**

- Reflexivity of Point Sets (Problem 66)

**polygons:**

- Congruent Partitions of Polygons (Problem 73)
- Fair Partitioning of Convex Polygons (Problem 67)
- Hinged Dissections (Problem 47)
- Reflexivity of Point Sets (Problem 66)
- Simple Polygonalizations (Problem 16)
- Transforming Polygons via Vertex-Centroid Moves (Problem 60)

**polyhedra:**

- 3-Colorability of Arrangements of Great Circles (Problem 44)
- Bar-Magnet Polyhedra (Problem 32)
- Edge-Unfolding Convex Polyhedra (Problem 9)
- Edge-Unfolding Polycubes (Problem 64)
- Equiprojective Polyhedra (Problem 76)
- General Unfoldings of Nonconvex Polyhedra (Problem 43)
- Hamiltonian Tetrahedralizations (Problem 29)
- Polyhedron with Regular Pentagon Faces (Problem 72)
- Vertex-Unfolding Polyhedra (Problem 42)
- Zipper Unfoldings of Convex Polyhedra (Problem 77)

**reconstruction:**

- Surface Reconstruction (Problem 26)

**robotics:**

- Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots (Problem 35)

**scheduling:**

- Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots (Problem 35)

**shortest paths:**

- Euclidean Minimum Spanning Tree (Problem 5)

- Minimum Euclidean Matching in 2D (Problem 6)
- Minimum-Link Path in 2D (Problem 22)
- Shortest Paths among Obstacles in 2D (Problem 21)

**simplification:**

- Polygonal Curve Simplification (Problem 24)
- Polyhedral Surface Approximation (Problem 25)

**spanners:**

- Stretch-Factor for Points in Convex Position (Problem 71)
- Yao-Yao Graph a Spanner? (Problem 70)

**stabbing:**

- Minimum Stabbing Spanning Tree (Problem 20)

**traveling salesman:**

- Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 53)
- Planar Euclidean Maximum TSP (Problem 49)
- Traveling Salesman Problem in Solid Grid Graphs (Problem 54)

**triangulations:**

- Compatible Triangulations (Problem 38)
- Flip Graph Connectivity in 3D (Problem 28)
- Hamiltonian Tetrahedralizations (Problem 29)
- Minimum Weight Triangulation (Problem 1)
- Pointed Spanning Trees in Triangulations (Problem 50)
- Simple Linear-Time Polygon Triangulation (Problem 10)
- The Number of Pointed Pseudotriangulations (Problem 40)

**visibility:**

- Trapping Light Rays with Segment Mirrors (Problem 31)
- Vertex  $\pi$ -Floodlights (Problem 23)
- Visibility Graph Recognition (Problem 17)

**Voronoi diagrams:**

- Dynamic Planar Nearest Neighbors (Problem 63)
- Voronoi Diagram of Lines in 3D (Problem 3)
- Voronoi Diagram of Moving Points (Problem 2)

# 1 Numerical List of All Problems

The following lists all problems sorted by number. These numbers can be used for citations and correspond to the order in which the problems were entered.

- Problem 1: Minimum Weight Triangulation
- Problem 2: Voronoi Diagram of Moving Points
- Problem 3: Voronoi Diagram of Lines in 3D
- Problem 4: Union of Fat Objects in 3D
- Problem 5: Euclidean Minimum Spanning Tree
- Problem 6: Minimum Euclidean Matching in 2D
- Problem 7:  $k$ -sets
- Problem 8: Linear Programming: Strongly Polynomial?
- Problem 9: Edge-Unfolding Convex Polyhedra
- Problem 10: Simple Linear-Time Polygon Triangulation
- Problem 11: 3SUM Hard Problems
- Problem 12: Dynamic Planar Convex Hull
- Problem 13: Point Location in 3D Subdivision
- Problem 14: Binary Space Partition Size
- Problem 15: Output-sensitive Convex Hull in  $\mathbb{R}^d$
- Problem 16: Simple Polygonalizations
- Problem 17: Visibility Graph Recognition
- Problem 18: Pushing Disks Together
- Problem 19: Vertical Decompositions in  $\mathbb{R}^d$
- Problem 20: Minimum Stabbing Spanning Tree
- Problem 21: Shortest Paths among Obstacles in 2D
- Problem 22: Minimum-Link Path in 2D
- Problem 23: Vertex  $\pi$ -Floodlights
- Problem 24: Polygonal Curve Simplification
- Problem 25: Polyhedral Surface Approximation

- Problem 26: Surface Reconstruction
- Problem 27: Hexahedral Meshing
- Problem 28: Flip Graph Connectivity in 3D
- Problem 29: Hamiltonian Tetrahedralizations
- Problem 30: Thrackles
- Problem 31: Trapping Light Rays with Segment Mirrors
- Problem 32: Bar-Magnet Polyhedra
- Problem 33: Sum of Square Roots
- Problem 34: Extending Pseudosegment Arrangements by Subdivision
- Problem 35: Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots
- Problem 36: Inplace Convex Hull of a Simple Polygonal Chain
- Problem 37: Counting Polyominoes
- Problem 38: Compatible Triangulations
- Problem 39: Distances among Point Sets in  $\mathbb{R}^2$  and  $\mathbb{R}^3$
- Problem 40: The Number of Pointed Pseudotriangulations
- Problem 41: Sorting  $X + Y$  (Pairwise Sums)
- Problem 42: Vertex-Unfolding Polyhedra
- Problem 43: General Unfoldings of Nonconvex Polyhedra
- Problem 44: 3-Colorability of Arrangements of Great Circles
- Problem 45: Smallest Universal Set of Points for Planar Graphs
- Problem 46: 3D Minimum-Bend Orthogonal Graph Drawings
- Problem 47: Hinged Dissections
- Problem 48: Bounded-Degree Minimum Euclidean Spanning Tree
- Problem 49: Planar Euclidean Maximum TSP
- Problem 50: Pointed Spanning Trees in Triangulations
- Problem 51: Linear-Volume 3D Grid Drawings of Planar Graphs
- Problem 52: Queue-Number of Planar Graphs



- Problem 53: Minimum-Turn Cycle Cover in Planar Grid Graphs
- Problem 54: Traveling Salesman Problem in Solid Grid Graphs
- Problem 55: Pallet Loading
- Problem 56: Packing Unit Squares in a Simple Polygon
- Problem 57: Chromatic Number of the Plane
- Problem 58: Monochromatic Triangles
- Problem 59: Most Circular Partition of a Square
- Problem 60: Transforming Polygons via Vertex-Centroid Moves
- Problem 61: Lines Tangent to Four Unit Balls
- Problem 62: Volume Maximizing Convex Shape
- Problem 63: Dynamic Planar Nearest Neighbors
- Problem 64: Edge-Unfolding Polycubes
- Problem 65: Magic Configurations
- Problem 66: Reflexivity of Point Sets
- Problem 67: Fair Partitioning of Convex Polygons
- Problem 68: Rolling a Die over a Labeled Board
- Problem 69: Isoceles Planar Graph Drawing
- Problem 70: Yao-Yao Graph a Spanner?
- Problem 71: Stretch-Factor for Points in Convex Position
- Problem 72: Polyhedron with Regular Pentagon Faces
- Problem 73: Congruent Partitions of Polygons
- Problem 74: Slicing Axes-Parallel Rectangles
- Problem 75: Edge-Coloring Geometric Graphs
- Problem 76: Equiprojective Polyhedra
- Problem 77: Zipper Unfoldings of Convex Polyhedra
- Problem 78: Rectangling a Rectangle

## Problem 1: Minimum Weight Triangulation

**Statement** Can a minimum weight triangulation of a planar point set be found in polynomial time? The *weight* of a triangulation is its total edge length.

**Origin** Perhaps E. L. Lloyd, 1977: [Llo77], cited in Garey and Johnson [GJ79].

**Status/Conjectures** Just solved by Wolfgang Mulzer and Günter Rote, January 2006! <http://arxiv.org/abs/cs.CG/0601002>. Entry to be updated later...

This problem is one of the few from Garey and Johnson [GJ79, p. 288] whose complexity status remains unknown.

**Partial and Related Results** The best approximation algorithms achieve a (large) constant times the optimal length [LK96]; good heuristics are known [DMM95]. If Steiner points are allowed, again constant-factor approximations are known [Epp94, CL98], but it is open to decide even if a minimum-weight Steiner triangulation exists (the minimum might be approached only in the limit).

**Appearances** [MO01]

**Categories** triangulations

**Entry Revision History** J. O'Rourke, 31 Jul. 2001; J. O'Rourke, 3 Jan. 2006.

## References

- [CL98] Siu-Wing Cheng and Kam-Hing Lee. Quadtree decomposition, Steiner triangulation, and ray shooting. In *ISAAC: 9th Internat. Sympos. Algorithms Computation*, pages 367–376, 1998.
- [DMM95] Matthew T. Dickerson, Scott A. McElfresh, and Mark H. Montague. New algorithms and empirical findings on minimum weight triangulation heuristics. In *Proc. 11th Annu. ACM Sympos. Comput. Geom.*, pages 238–247, 1995.
- [Epp94] D. Eppstein. Approximating the minimum weight Steiner triangulation. *Discrete Comput. Geom.*, 11:163–191, 1994.
- [GJ79] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, NY, 1979.
- [Llo77] Errol Lynn Lloyd. On triangulations of a set of points in the plane. In *Proc. 18th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 228–240, 1977.

- [LK96] Christos Levcopoulos and Drago Krznaric. Quasi-greedy triangulations approximating the minimum weight triangulation. In *Proc. 7th ACM-SIAM Sympos. Discrete Algorithms*, pages 392–401, 1996.
- [MO01] J. S. B. Mitchell and Joseph O’Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63–72 (2001), Issue 120.

## Problem 2: Voronoi Diagram of Moving Points

**Statement** What is the maximum number of combinatorial changes possible in a Euclidean Voronoi diagram of a set of  $n$  points each moving along a line at unit speed in two dimensions?

**Origin** Unknown (to JOR). Perhaps Michael Atallah?

**Status/Conjectures** Long conjectured to be nearly quadratic. Solved now: [Rub15]. Natan Rubin proved an upper bound of  $O(n^{2+\epsilon})$ , and a quadratic lower bound is known.

**Partial and Related Results** See [Rub15] for a review of earlier work, now superceded.

**Appearances** [MO01]

**Categories** Voronoi diagrams; Delaunay triangulations

**Entry Revision History** J. O’Rourke, 1 Aug. 2001; 19Sep2017.

## References

- [MO01] J. S. B. Mitchell and Joseph O’Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63–72 (2001), Issue 120.
- [Rub15] Natan Rubin. On kinetic Delaunay triangulations: A near-quadratic bound for unit speed motions. *Journal of the ACM*, 62(3):25, 2015.

## Problem 3: Voronoi Diagram of Lines in 3D

**Statement** What is the combinatorial complexity of the Voronoi diagram of a set of lines (or line segments) in three dimensions?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open. Conjectured to be nearly quadratic.

**Partial and Related Results** There is a gap between a lower bound of  $\Omega(n^2)$  and an upper bound that is essentially cubic [Sha94] for the Euclidean case (and yet is quadratic for polyhedral metrics [BSTY98]). A recent advance shows that the “level sets” of the Voronoi diagram of lines, given by the union of a set of cylinders, indeed has near-quadratic complexity [AS00b].

**Related Open Problems** This problem is closely related to Problem 1, because points moving in the plane with constant velocity yield straight-line trajectories in space-time.

**Appearances** [MO01]

**Categories** Voronoi diagrams

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 13 Dec. 2001.

## References

- [AS00b] Pankaj K. Agarwal and Micha Sharir. Pipes, cigars, and kreplach: The union of Minkowski sums in three dimensions. *Discrete Comput. Geom.*, 24(4):645–685, 2000.
- [BSTY98] Jean-Daniel Boissonnat, Micha Sharir, Boaz Tagansky, and Mariette Yvinec. Voronoi diagrams in higher dimensions under certain polyhedral distance functions. *Discrete Comput. Geom.*, 19(4):473–484, 1998.
- [MO01] J. S. B. Mitchell and Joseph O’Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63–72 (2001), Issue 120.
- [Sha94] Micha Sharir. Almost tight upper bounds for lower envelopes in higher dimensions. *Discrete Comput. Geom.*, 12:327–345, 1994.

## Problem 4: Union of Fat Objects in 3D

**Statement** What is the complexity of the union of “fat” objects in  $\mathbb{R}^3$ ?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open. Conjectured to be nearly quadratic.

**Partial and Related Results** The Minkowski sum of polyhedra of  $n$  vertices with the (Euclidean) unit ball has complexity  $O(n^{2+\epsilon})$  [AS99], as does the union of  $n$  congruent cubes [PSS01]. It is widely believed the same should hold true for *fat* objects, those with a bounded ratio of circumradius to inradius, as it does in  $\mathbb{R}^2$  [ES00].

**Appearances** [MO01]

**Categories** combinatorial geometry

**Entry Revision History** J. O’Rourke, 1 Aug. 2001; 1 Jan. 2003 (B. Aronov comment).

## References

- [AS99] Pankaj K. Agarwal and Micha Sharir. Pipes, cigars, and kreplach: The union of Minkowski sums in three dimensions. In *Proc. 15th Annu. ACM Sympos. Comput. Geom.*, pages 143–153, 1999.
- [ES00] A. Efrat and Micha Sharir. On the complexity of the union of fat objects in the plane. *Discrete Comput. Geom.*, 23:171–189, 2000.
- [MO01] J. S. B. Mitchell and Joseph O’Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.
- [PSS01] Janos Pach, Ido Safruit, and Micha Sharir. The union of congruent cubes in three dimensions. In *Proc. 17th Annu. ACM Sympos. Comput. Geom.*, pages 19–28, 2001.

## Problem 5: Euclidean Minimum Spanning Tree

**Statement** Can the Euclidean *minimum spanning tree* (MST) of  $n$  points in  $\mathbb{R}^d$  be computed in time close to the lower bound of  $\Omega(n \log n)$  [GKFS96]?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** Several algorithms have been developed for general graphs with arbitrary edge weights. Chazelle presented an  $O(m\alpha(m, n) \log \alpha(m, n))$ -time algorithm [Cha97], and then an  $O(m\alpha(m, n))$ -time algorithm [Cha00b], where  $\alpha(m, n)$  is the functional inverse of Ackermann’s function, and  $n$  and  $m$  are the number of vertices and edges respectively in the graph. Pettie and Ramachandran have since given an optimal algorithm for the graph setting [PR02], whose running time is an unknown function between  $\Omega(m)$  and  $O(m\alpha(m, n))$ . In particular, when  $m = \Omega(n \log n)$ ,  $\alpha(m, n) = O(1)$  and these time bounds are all linear in the number of edges,  $m$ .

But in the geometric setting, the graph is complete, so a time bound linear in the number of edges,  $m$ , is quadratic in the number of points,  $n$ . And indeed the best upper bounds for the Euclidean MST approach quadratic for large  $d$ , e.g., [CK95].

**Related Open Problems** This problem is intimately related to the bichromatic closest pair problem [AESW91].

**Appearances** [MO01]

**Categories** minimum spanning tree; shortest paths

**Entry Revision History** J. O'Rourke, 2 Aug. 2001; E. Demaine, 7 July 2002.

## References

- [AESW91] Pankaj K. Agarwal, Herbert Edelsbrunner, O. Schwarzkopf, and Emo Welzl. Euclidean minimum spanning trees and bichromatic closest pairs. *Discrete Comput. Geom.*, 6(5):407–422, 1991.
- [Cha97] Bernard Chazelle. A faster deterministic algorithm for minimum spanning trees. In *Proc. 38th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 22–31, 1997.
- [Cha00b] Bernard Chazelle. A minimum spanning tree algorithm with inverse-Ackermann type complexity. *J. ACM*, 47(6):1028–1047, November 2000.
- [CK95] P. B. Callahan and S. Rao Kosaraju. A decomposition of multidimensional point sets with applications to  $k$ -nearest-neighbors and  $n$ -body potential fields. *J. Assoc. Comput. Mach.*, 42:67–90, 1995.
- [GKFS96] Dima Grigoriev, Marek Karpinski, Friedhelm Meyer auf der Heide, and Roman Smolensky. A lower bound for randomized algebraic decision trees. In *Proc. 28th Annu. ACM Sympos. Theory Comput.*, pages 612–619, 1996.
- [MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63–72 (2001), Issue 120.
- [PR02] Seth Pettie and Vijaya Ramachandran. An optimal minimum spanning tree algorithm. *J. ACM*, 49(1):16–34, January 2002.

## Problem 6: Minimum Euclidean Matching in 2D

**Statement** What is the complexity of computing a minimum-cost Euclidean matching for  $2n$  points in the plane? The *cost* of a matching is the total length of the edges in the matching.

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** An algorithm that achieves the minimum and runs in nearly  $O(n^{2.5})$  time has long been available [Vai89]. This was improved to  $O(n^{1.5} \log^5 n)$  in [Var98]. Recently Arora showed how to achieve a  $(1 + \epsilon)$ -approximation in  $n(\log n)^{O(1/\epsilon)}$  time [Aro98], and this has been improved to  $O((n/\epsilon^3) \log^6 n)$  time [VA99].

A special case of considerable interest is bipartite matching, in which the points are red or blue and the matching connects points of different color. Here  $O(n^{2+\epsilon})$  has been achieved [AES99], and a  $(1 + \epsilon)$ -approximation can be found in  $O((n/\epsilon)^{1.5} \log^5 n)$  time [VA99].

**Appearances** [MO01]

**Categories** shortest paths

**Entry Revision History** J. O'Rourke, 2 Aug. 2001; 30 Aug. 2001; 13 Dec. 01 (thanks to M. Sharir).

## References

- [Aro98] Sanjeev Arora. Polynomial time approximation schemes for Euclidean traveling salesman and other geometric problems. *J. Assoc. Comput. Mach.*, 45(5):753–782, 1998.
- [AES99] P. K. Agarwal, A. Efrat, and Micha Sharir. Vertical decomposition of shallow levels in 3-dimensional arrangements and its applications. *SIAM J. Comput.*, 29:912–953, 1999.
- [MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63–72 (2001), Issue 120.
- [Var98] K. Varadarajan. A divide and conquer algorithm for min-cost perfect matching in the plane. In *Proc. 39th Annu. IEEE Sympos. Found. Comput. Sci.*, pages 320–329, 1998.
- [Vai89] P. M. Vaidya. Geometry helps in matching. *SIAM J. Comput.*, 18:1201–1225, 1989.
- [VA99] K. R. Varadarajan and Pankaj K. Agarwal. Approximation algorithms for bipartite and non-bipartite matching in the plane. In *Proc. 10th ACM-SIAM Sympos. Discrete Algorithms*, pages 805–814, 1999.

## Problem 7: $k$ -sets

**Statement** What is the maximum number of  $k$ -sets? (Equivalently, what is the maximum complexity of a  $k$ -level in an arrangement of hyperplanes?)

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** For a given set  $P$  of  $n$  points,  $S \subset P$  is a  $k$ -set if  $|S| = k$  and  $S = P \cap H$  for some open halfspace  $H$ . Even for points in two dimensions the problem remains open: The maximum number of  $k$ -sets as a function of  $n$  and  $k$  is known to be  $O(nk^{1/3})$  by a recent advance of Dey [Dey98], while the best lower bound is only slightly super-linear [Tot00].

**Appearances** [MO01]

**Categories** combinatorial geometry; point sets

**Entry Revision History** J. O'Rourke, 2 Aug. 2001.

## References

- [Dey98] T. K. Dey. Improved bounds on planar  $k$ -sets and related problems. *Discrete Comput. Geom.*, 19:373–382, 1998.
- [MO01] J. S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63-72 (2001), Issue 120.
- [Tot00] Géza Toth. Point sets with many  $k$ -sets. In *Proc. 16th Annu. ACM Sympos. Comput. Geom.*, pages 37–42, 2000.

## Problem 8: Linear Programming: Strongly Polynomial?

**Statement** Is linear programming strongly polynomial?

**Origin** Nimrod Megiddo [Meg82][Meg83].

**Status/Conjectures** Open.

**Partial and Related Results** It is known to be weakly polynomial, that is, polynomial in the bit complexity of the input data [Kha80, Kar84]. Subexponential time is achievable via a randomized algorithm [MSW96]. In any fixed dimension, linear programming can be solved in strongly polynomial linear time (linear in the input size), established in dimensions 2 and 3 in [Dye84] and for all dimensions in [Meg84].

**Appearances** [MO01]

**Categories** linear programming



**Entry Revision History** J. O’Rourke, 2 Aug 2001, 16 Jul 2007; E. Demaine, 12 Mar 2010.

## References

- [Dye84] M. E. Dyer. Linear time algorithms for two- and three-variable linear programs. *SIAM J. Comput.*, 13:31–45, 1984.
- [Kar84] N. Karmarkar. A new polynomial-time algorithm for linear programming. *Combinatorica*, 4:373–395, 1984.
- [Kha80] L. G. Khachiyan. Polynomial algorithm in linear programming. *U.S.S.R. Comput. Math. and Math. Phys.*, 20:53–72, 1980.
- [Meg82] N. Megiddo. Is binary encoding appropriate for the problem-language relationship? *Theoret. Comput. Sci.*, 19:337–341, 1982.
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## Problem 9: Edge-Unfolding Convex Polyhedra

**Statement** Can every convex polyhedron be cut along its edges and unfolded flat to a single, nonoverlapping, simple polygon?

**Origin** First stated in [She75], but in spirit at least goes back to Albrecht Dürer [Dür25].

**Status/Conjectures** Open. It seems to be a widespread hunch that the answer is YES.

**Partial and Related Results** The answer is known to be NO for nonconvex polyhedra even with triangular faces [BDE<sup>+</sup>03], but has been long open for convex polyhedra [She75, O’R00].

**Related Open Problems** Problem 1: Vertex-Unfolding Polyhedra.

Problem 1: General Unfolding of Nonconvex Polyhedra.

Problem 1: Edge-Unfolding Polyhedra Built from Cubes.

**Appearances** [She75, O’R00, MO01]

**Categories** folding and unfolding; polyhedra

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.

## References

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## Problem 10: Simple Linear-Time Polygon Triangulation

**Statement** Is there a deterministic, linear-time polygon triangulation algorithm significantly simpler than that of Chazelle [Cha91]?

**Origin** Implicit since Chazelle’s 1990 linear-time algorithm.

**Status/Conjectures** Open.

**Partial and Related Results** Simple randomized algorithms that are close to linear-time are known [Sei91], and a recent randomized linear-time algorithm [AGR00] avoids much of the intricacies of Chazelle’s algorithm.

**Related Open Problems** Relatedly, is there a simple linear-time algorithm for computing a shortest path in a simple polygon, without first applying a more complicated triangulation algorithm?

**Appearances** [MO01]

**Categories** triangulations

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 28 Aug. 2001.

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## Problem 11: 3SUM Hard Problems

**Statement** Can the class of *3SUM hard* problems be solved in subquadratic time? These problems can be reduced from the problem of determining whether, given three sets of integers,  $A$ ,  $B$ , and  $C$  with total size  $n$ , there are elements  $a \in A$ ,  $b \in B$ , and  $c \in C$  such that  $a + b = c$ .

**Origin** [GO95].

**Status/Conjectures** Open.

**Motivation** Many fundamental geometric problems fall in this class, e.g., computing the area of the union of  $n$  triangles.

**Partial and Related Results**  $\Omega(n^2)$  lower bounds are known for 3SUM and a few 3SUM-hard problems in restricted decision tree models of computation [ES95, Eri99a, Eri99b].

3SUM and its obvious generalizations (4SUM, 5SUM, etc.) are examples of *linear satisfiability* problems. A generic linear satisfiability problem asks, given an array of  $n$  integers, do any  $r$  of them satisfy the equation

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_r x_r = \alpha_0$$

where  $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_r$  are fixed constants. Erickson [Eri99a] proved an  $\Omega(n^{\lceil r/2 \rceil})$  lower bound for any problem of this type, in the restricted linear decision tree model. This lower bound is tight except for a logarithmic factor when  $r$  is even. Ailon and Chazelle generalized Erickson’s bound and improve it for large  $r$  or for more than  $r$  variables [AC05].

Baran et al. [BDP05] show that subquadratic algorithms for 3SUM are possible in common models of computation that allow more direct manipulation of the numbers instead of just real arithmetic, such as the word RAM. The improvement they obtain is roughly quadratic in the parallelism offered by the model; for example, with  $\lg n$ -bit words, they obtain an  $O(n^2 \left(\frac{\lg \lg n}{\lg n}\right)^2)$ -time algorithm. With this word size, the 3SUM problem becomes whether any improvement beyond polylogarithmic factors (or indeed, beyond  $\Theta(\lg^2 n)$ ) is possible.

**Appearances** [MO01]

**Categories** lower bounds

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; Jeff Erickson, 20 June 2002; E. Demaine, 7 July 2005; Raphael Clifford, 7 July 2011.

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## Problem 12: Dynamic Planar Convex Hull

**Statement** Can a planar convex hull be maintained to support both dynamic updates and queries in logarithmic time? More precisely, is there a data structure supporting insertions and deletions of points and supporting various queries about the convex hull of the current set of  $n$  points, all in  $O(\log n)$  time per operation? An *extreme-point query* asks to find the vertex of the convex hull that is extreme in a given direction. A *tangent query* asks to determine whether a given point is interior to the convex hull, and if not, to find the two tangent lines of the convex hull that passes through the given point. A *gift-wrapping query* asks to find the two vertices of the convex hull adjacent to a given vertex of the convex hull. A *line-stabbing query* asks to find the two edges of the convex hull (if any) that intersect a given line. (Note that two extreme-point queries suffice to determine whether a line intersects the convex hull, while a line-stabbing query determines where exactly the line intersects the convex hull if it does.)

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Solved (in a certain sense) by Gerth Brodal and Riko Jacob in a FOCS 2002 paper [BJ02]. See also Jacob’s PhD thesis [Jac02] for further details. Their data structure supports insertions and deletions in  $O(\log n)$  amortized time and supports extreme-point, tangent, and gift-wrapping queries in  $O(\log n)$  worst-case query bounds. It remains open whether a logarithmic bound can be achieved in the worst case, and whether logarithmic bounds can be achieved (amortized or worst case) for line-stabbing queries.

**Partial and Related Results** For 17 years, the authority on this problem was Overmars and van Leeuwen’s paper [OvL81] which describes a data structure supporting insertions and deletions in  $O(\log^2 n)$  worst-case time and all types of queries described above in  $O(\log n)$  worst-case time. Various structures achieve faster update times when either insertions or deletions are not supported [Pre79, HS92]. But the  $O(\log^2 n)$  barrier remained until Chan’s FOCS 1999 paper [Cha99], which improved the insertion and deletion time to  $O(\log^{1+\epsilon} n)$  amortized for any  $\epsilon > 0$ . The update time was further improved to  $O(\log n \log \log n)$  amortized by Brodal and Jacob [BJ00] until the problem was finally solved in optimal  $O(\log n)$  amortized

time by the same authors [BJ02, Jac02]. Both the Chan and the Brodal and Jacob structures support extreme-point, tangent, and gift-wrapping queries.

**Related Open Problems** Problem 1.

**Appearances** [MO01]

**Categories** convex hulls; data structures

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; E. Demaine, 25 Nov. 2002; 22 Aug. 2005; 24 Jan. 2006.

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## Problem 13: Point Location in 3D Subdivision

**Statement** Is there an  $O(n)$ -space data structure that supports  $O(\log n)$ -time point-location queries in a three-dimensional subdivision of  $n$  faces?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** Currently  $O(n \log n)$  space and  $O(\log^2 n)$  queries are achievable [Sno97].

**Appearances** [MO01]

**Categories** data structures

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.

## References

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## Problem 14: Binary Space Partition Size

**Statement** Is it possible to construct a *binary space partition* (BSP) for  $n$  disjoint line segments in the plane of size less than  $\Theta(n \log n)$ ?

**Origin** Paterson and Yao [PY90].

**Status/Conjectures** Solved by Csaba Tóth [Tót09, Tót11].

**Partial and Related Results** The upper bound of  $O(n \log n)$  was established by Paterson and Yao [PY90]. Tóth [Tót01b] improved the trivial  $\Omega(n)$  lower bound to  $\Omega(n \log n / \log \log n)$ . Then in 2009 he established a matching upper bound [Tót09, Tót11]. His proof is constructive and leads to a deterministic algorithm that runs in  $O(n \text{ polylog } n)$  time. As his algorithm produces BSP trees whose height might be linear in  $n$ , it remains open whether his complexity bound can be achieved while achieving  $O(\log n)$  height.

**Appearances** [MO01]

**Categories** data structures; combinatorial geometry

**Entry Revision History** J. O’Rourke, 2 Aug. 2001. Nina Amenta & JOR, 6 Jan. 2011.

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## Problem 15: Output-sensitive Convex Hull in $\mathbb{R}^d$

**Statement** What is the best output-sensitive convex hull algorithm for  $n$  points in  $\mathbb{R}^d$ ?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** The lower bound is  $\Omega(n \log f + f)$  for  $f$  facets (the output size). The best upper bound to date is  $O(n \log f + (nf)^{1-\delta} \log^{O(1)} n)$  with  $\delta = 1/(\lfloor d/2 \rfloor + 1)$  [Cha96], which is optimal for sufficiently small  $f$ .

**Appearances** [MO01]

**Categories** convex hulls

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.



## References

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## Problem 16: Simple Polygonalizations

**Statement** Can the number of simple polygonalizations of a set of  $n$  points in the plane be computed in polynomial time? A *simple polygonalization* is a simple polygon whose vertices are the points.

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** Certain special cases are known (e.g., for computing the number of monotone simple polygonalizations [ZSSM96]), but the general problem remains open. The problem is closely related to that of generating a “random” instance of a simple polygon on a given set of vertices, with each instance being generated with probability  $1/k$ , where  $k$  is the total number of simple polygonalizations. Heuristic methods are known and implemented [AH96].

See [CHUZ01] and [HMO<sup>+</sup>09] for related topics and references to relevant papers.

**Appearances** [MO01]

**Categories** polygons; point sets

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 1 Jan. 2003.

## References

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## Problem 17: Visibility Graph Recognition

**Statement** Given a visibility graph  $G$  and a Hamiltonian circuit  $C$ , determine in polynomial time whether there is a simple polygon whose vertex visibility graph is  $G$ , and whose boundary corresponds to  $C$ .

**Origin** ElGindy(?)

**Status/Conjectures** Open.

**Partial and Related Results** The problem is not even known to be in NP [O’R93], although it is for “pseudo-polygon” visibility graphs [OS97].

**Appearances** [MO01]

**Categories** visibility

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.

## References

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## Problem 18: Pushing Disks Together

**Statement** When a collection of disks are pushed closer together, so that no distance between two center points increases, can the area of their union increase?

**Origin** Kneser (1955) and Poulsen (1954).

**Status/Conjectures** Solved by K. Bezdek and R. Connelly. See their web page<sup>1</sup>. (Update as of 3 Aug. 2000.)

**Partial and Related Results** Previously only settled in the continuous-motion case [BS98], for both this and the corresponding question for intersection area decrease [Cap96]. But now both solved; see above.

**Appearances** [MO01]

**Categories** combinatorial geometry

**Entry Revision History** J. O'Rourke, 2 Aug. 2001; 3 Aug. 2003.

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## Problem 19: Vertical Decompositions in $\mathbb{R}^d$

**Statement** What is the complexity of the *vertical decomposition* of  $n$  surfaces in  $\mathbb{R}^d$ ,  $d \geq 5$ ?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** The lower bound of  $\Omega(n^d)$  was nearly achieved up to  $d = 3$  [AS00a, p. 271], but a gap remained for  $d \geq 4$ . A recent result [Kol01] covers  $d = 4$  and achieves  $O(n^{2d-4+\epsilon})$  for general  $d$ , leaving a gap for  $d \geq 5$ .

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<sup>1</sup><http://www.math.cornell.edu/~connelly/kneser.html>

**Appearances** [MO01]

**Categories** combinatorial geometry

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.

## References

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## Problem 20: Minimum Stabbing Spanning Tree

**Statement** What is the complexity of computing a spanning tree of a planar point set  $P$  having minimum stabbing number? The *stabbing number* of a tree  $T$  is the maximum number of edges of  $T$  intersected by a line.

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Solved, October 2003: the problem is NP-complete. Significant advance on approximability in 2009.

**Partial and Related Results** Fekete, Lübbecke, and Meijer [FLM04] proved strong NP-completeness of minimizing the stabbing number or axis-parallel stabbing number or crossing number or axis-parallel crossing number in a perfect matching or spanning tree. They also establish inapproximability by less than a  $6/5$  factor of minimizing the axis-parallel stabbing number in a perfect matching. They also prove strong NP-completeness of minimizing the axis-parallel crossing number in a triangulation.

The complexity of minimizing the stabbing number or crossing number in a triangulation remains open. Furthermore, it remains open whether any of these problems have constant-factor approximations. See [FLM04] for some ideas.

In the worst case, any set of  $n$  points in the plane has a spanning tree of stabbing number  $O(\sqrt{n})$  [Aga92, Cha88, Wel93] and this bound is tight. An  $O(\sqrt{n})$ -approximation follows from this result.

There has been an advance on approximations [HP09]: Har-Peled designed an algorithm that computes a spanning tree of  $n$  points  $P$  in  $\mathbb{R}^d$  whose crossing number is  $O(\min(t \log n, n^{1-1/d}))$ , where  $t$  the minimum crossing number of any spanning tree of  $P$ .

**Appearances** [MO01]

**Categories** stabbing

**Entry Revision History** J. O'Rourke, 2 Aug. 2001; E. Demaine, 16 Jan. 2004; J. O'Rourke, 26 Aug 2009.

## References

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## Problem 21: Shortest Paths among Obstacles in 2D

**Statement** Can shortest paths among  $h$  obstacles in the plane, with a total of  $n$  vertices, be found in optimal  $O(n + h \log h)$  time using  $O(n)$  space?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** The only algorithm that is linear in  $n$  in time and space is quadratic in  $h$  [KMM97];  $O(n \log n)$  time, using  $O(n \log n)$  space, is known [HS99]. In three dimensions, the Euclidean shortest path problem among general obstacles is NP-hard, but its complexity remains open for some special cases, such as when the obstacles are disjoint unit spheres or axis-aligned boxes; see [Mit00] for a survey.

**Appearances** [MO01]

**Categories** shortest paths

**Entry Revision History** J. O'Rourke, 2 Aug. 2001.

## References

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## Problem 22: Minimum-Link Path in 2D

**Statement** Can a minimum-link path among polygonal obstacles be found in subquadratic time?

**Origin** Mitchell [?].

**Status/Conjectures** Open.

**Partial and Related Results** The best algorithm known requires essentially quadratic time in the worst case [MRW92].

**Related Open Problems** What is the complexity of computing minimum-link paths in three dimensions?

**Appearances** [MO01]

**Categories** shortest paths

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.

## References

- [MO01] J. S. B. Mitchell and Joseph O’Rourke. Computational geometry column 42. *Internat. J. Comput. Geom. Appl.*, 11(5):573–582, 2001. Also in *SIGACT News* 32(3):63–72 (2001), Issue 120.
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## Problem 23: Vertex $\pi$ -Floodlights

**Statement** How many  $\pi$ -floodlights are always sufficient to illuminate any polygon of  $n$  vertices, with at most one floodlight placed at each vertex? An  $\alpha$ -floodlight is a light of aperture  $\alpha$ . (We consider here “inward-facing” floodlights, whose defining halfspace lies inside the polygon, locally in the neighborhood of the vertex. Other models of the problem allow general orientations of floodlights or restricted orientations (e.g., “edge-aligned”).)

**Origin** Jorge Urrutia, perhaps first published in [Urr00].

**Status/Conjectures** Open. Now known that the fraction of  $n$  that always suffices lies between  $5/8$  and  $2/3$ .

**Partial and Related Results** It was established in [ECOUX95] that for any  $\alpha < \pi$ , there is a polygon that cannot be illuminated with an  $\alpha$ -floodlight at each vertex. When  $\alpha = \pi$ ,  $n - 2$  lights (trivially) suffice. So it is of interest (as noted in [Urr00]) to learn whether a fraction of  $n$  lights suffice for  $\pi$ -floodlights. A (very) special case is that  $\lceil n/2 \rceil - 1$  is a tight bound for “monotone mountains” [O’R97]. Tóth established [Tót01a] that (roughly)  $(3/4)n$  suffice, in the case of general orientation floodlights (not necessarily inward-facing). A lower bound of Santos, that  $\lfloor 3(n - 1)/5 \rfloor$  inward-facing floodlights are necessary (or  $\lfloor 2(n - 2)/5 \rfloor$  generally oriented floodlights), stood for several years, but just recently (Jan. 2002) Urrutia constructed examples, based on stitching together copies of Fig. 1, that show that  $5(k + 1)/(8k + 9)$  (inward-facing) floodlights are necessary for each  $k$ , thus improving the lower bound factor from 0.6 to 0.625. Also, Speckmann and Tóth [ST01b] showed that  $\lfloor n/2 \rfloor$  vertex  $\pi$ -floodlights suffice for general orientations, while  $\lfloor (2n - c)/3 \rfloor$  suffice for inward-facing, edge-aligned orientations, where  $c$  is the number of convex vertices. In particular, this reduced the upper-bound fraction below 1.

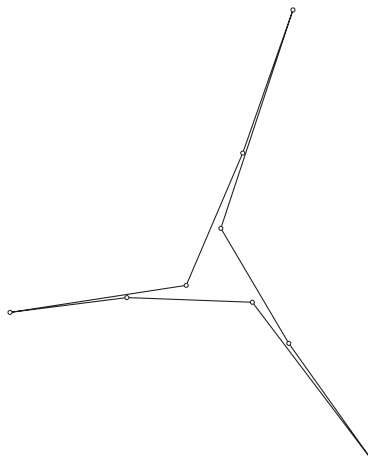


Figure 1: A polygon of 9 vertices that needs 5 vertex  $\pi$ -lights.

**Appearances** [MO01]

**Categories** art galleries; visibility

**Entry Revision History** J. Mitchell, 24 Jan 2001; J. O'Rourke, 2 Aug. 2001; 29 Aug. 2001; 23 Jan. 2002; 30 Sep. 2007.

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## Problem 24: Polygonal Curve Simplification

**Statement** Can an  $n$ -vertex polygonal curve be simplified in time nearly linear in  $n$ ?

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Partial and Related Results** The goal is to compute a simplification that uses the fewest vertices of the original curve while approximating it according to some prescribed error criterion. Quadratic-time algorithms have been known for some time [CC96] and a recent algorithm achieves time  $O(n^{4/3+\epsilon})$  for a certain error criterion [AV00]. In practice, the Douglas-Peucker algorithm is often used as a heuristic; it can be implemented to run in time  $O(n \log n)$  [HS94].

**Appearances** [MO01]

**Categories** simplification

**Entry Revision History** J. O'Rourke, 2 Aug. 2001.

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## Problem 25: Polyhedral Surface Approximation

**Statement** How efficiently can one compute a polyhedral surface that is an  $\epsilon$ -approximation of a given triangulated surface in  $\mathbb{R}^3$ ?

**Origin** Mitchell [?]

**Status/Conjectures** Open.

**Partial and Related Results** It is NP-hard to obtain the minimum-facet surface separating two nested convex polytopes [DG97], if they are allowed to have distinct structures; the complexity is open when one polytope is an offset of the other (e.g., if they arise as offsets of an underlying convex polytope that we seek to approximate). Polynomial-time approximation algorithms are known ([BG95, MS95, AS98]) for the case of nested convex polyhedra, and for separating two terrain surfaces, achieving factors within  $O(1)$  or  $O(\log n)$  of optimal. However, no polynomial-time approximation results are known for general surfaces.

**Appearances** [MO01]

**Categories** simplification

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; J. Mitchell, 14 Oct 2020.

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## Problem 26: Surface Reconstruction

**Statement** Given a sufficiently dense sample of points on a surface (technically, an  $\epsilon$ -sample), reconstruct a surface homeomorphic to the original.

**Origin** Amenta and Bern [?]

**Status/Conjectures** Open.

**Partial and Related Results** This has recently been accomplished for smooth surfaces [ACDL00], but remains open for surfaces with sharp edges and corners.

**Appearances** [MO01]

**Categories** reconstruction; point sets

**Entry Revision History** J. O’Rourke, 2 Aug. 2001.

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## Problem 27: Hexahedral Meshing

**Statement** Can the interior of every simply connected polyhedron whose surface is meshed by an even number of quadrilaterals be partitioned into a hexahedral mesh compatible with the surface meshing? [BEA<sup>+</sup>99]

**Origin** Uncertain. Scott Mitchell in [Mit96]?

**Status/Conjectures** Partially closed, Fall 2006.

**Partial and Related Results** It was known that a topological hexahedral mesh exists [Mit96, Epp96], with, in general, curved boundaries, but despite the availability of software that extends quadrilateral surface meshes to hexahedral volume meshes, it is not known if a “geometric” hexahedral mesh can be achieved, with all cells having planar faces.

A new result [CS06] settles the practical aspects of the problem, but leaves one question unresolved. This paper provides an explicit algorithm that extends a quadrilateral surface mesh to a hexahedral mesh, where all the hexahedra have straight segment edges. In a sense, these hexahedra are intermediate between the topological and geometric meshes mentioned above. The faces are not necessarily planar, but this is not a crucial aspect in applications, such a fluid dynamics simulations.

The question of whether a hexahedral mesh with planar faces exists remains open.

**Related Open Problems** See [BE01] for extension of the flipping operation described in Problem 1 to quadrilateral and hexahedral meshes.

**Appearances** [MO01]

**Categories** meshing

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 18 Feb. 2002 (thanks to D. Eppstein); J. O’Rourke, 24 Oct. 2006.

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## Problem 28: Flip Graph Connectivity in 3D

**Statement** Is the *flip graph* connected for general-position points in  $\mathbb{R}^3$ ? Given a set of  $n$  points in  $\mathbb{R}^3$ , the flip graph has a node for each tetrahedralization of the set. Two nodes are connected by an arc if there is a 2-to-3 or 3-to-2 “bistellar flip” of tetrahedra between the two simplicial complexes. In the plane, the flips correspond to convex quadrilateral diagonal switches; in  $\mathbb{R}^3$ , a 5-vertex convex polyhedron is “flipped” between two of its tetrahedralizations.

**Origin** [EPW90, Joe91]

**Status/Conjectures** Open.

**Partial and Related Results** In  $\mathbb{R}^2$  the flip graph is connected, providing a basis for algorithms to iterate toward the Delaunay triangulation. A decade ago, several [EPW90, Joe91] asked whether the same holds in  $\mathbb{R}^3$  (when no four points are coplanar), but the question remains unresolved. It is not even known whether the flip graph might contain an isolated node. Settled in the negative for points in  $\mathbb{R}^5$  by Santos [San00], by constructing polytopes with a space of triangulations not connected via bistellar flips. Settled in the negative for topological tetrahedralizations in 3D, but the constructed tetrahedralization cannot be realized geometrically [DFM04]. Settled in the positive for flip graphs of *regular triangulations* in any dimension in [PL07], based on earlier work of Gelfand, Kapranov and Zelevinsky. The result in [PL07] connects by flips that neither remove nor add vertices (i.e., 2-to-3 or 3-to-2 flips in 3D), whereas the earlier work by Gelfand et al. permits all flips (e.g., 1-to-4 and 4-to-1 flips in 3D).

**Related Open Problems** Problem 1

**Appearances** [MO01]

**Categories** triangulations; Delaunay triangulations

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 7 Dec. 2001 (thanks to F. Santos); E. Demaine, 10 Dec. 2001; J. O’Rourke, 18 Feb. 2002 (thanks to D. Eppstein); E. Demaine, 2 Aug. 2004 (thanks to M. Murphy); J. O’Rourke, 20 Aug 2006. J. O’Rourke, 22 Dec 2008 (thanks to L. Pournin).

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## Problem 29: Hamiltonian Tetrahedralizations

**Statement** Can every convex polytope in  $\mathbb{R}^3$  be partitioned into tetrahedra such that the dual graph has a Hamiltonian path?

**Origin** [AHMS96].

**Status/Conjectures** Open.

**Partial and Related Results** Every convex polygon has such a *Hamiltonian triangulation*, but not every nonconvex polygon does [AHMS96]. The existence of a Hamiltonian path permits faster rendering on a graphics screen via pipelining.

Chin, Ding, and Wang [CDW05] have shown that examples exist for which the *pulling* tetrahedralization of a convex polytope is not Hamiltonian. (A pulling tetrahedralization is obtained by joining one vertex (the apex) to all other vertices of the polytope.) It is open if the shelling tetrahedralization may be always Hamiltonian.

Escalona et al. [EFMU07] prove the conjecture up to  $n = 20$ : every points set of  $n \leq 20$  points admits a Hamiltonian Tetrahedralization. They also detail an algorithm that finds a Hamiltonian Tetrahedralization for  $n$  points by adding  $O(n)$  Steiner points. The algorithm runs in  $O(n^{3/2})$  time.

**Appearances** [MO01]

**Categories** triangulations; polyhedra

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 13 Dec. 2001; J. Mitchell, 27 Oct. 2005; J. O’Rourke, 30 Sep. 2007.

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## Problem 30: Thrackles

**Statement** What is the maximum number of edges in a thrackle? A *thrackle* is a planar drawing of a graph of  $n$  vertices by edges which are smooth curves between vertices, with the condition that each pair of edges intersect at exactly one point, and have distinct tangents there. Another phrasing is that nonincident edges cross exactly once, and no incident edges share an interior point.

**Origin** Conway, late 1960’s.

**Status/Conjectures** Open. Conway’s conjecture is that the number edges cannot exceed  $n$ .

**Partial and Related Results** The upper bound was lowered from  $O(n^{3/2})$  to  $2n - 3$  in [LPS95], and further lowered to  $(3/2)(n - 1)$  in [CN00]. The conjecture has long been known to be true for straight-line thrackles. The conjecture was extended in [CN00] to the claim that a thrackle on  $n$  vertices embedded on a surface of genus  $g$  has at most  $n + 2g$  edges. See [BMP05, Sec. 9.5] for a recent discussion and further references.

**Reward** Conway offers a reward of \$1,000 for a resolution.

**Appearances** [MO01, Weh]

**Categories** graphs; combinatorial geometry; graph drawing; geometric graphs

**Entry Revision History** J. O’Rourke, 2 Aug. 2001; 13 Dec. 2001; 18 Feb. 2002 (thanks to David Eppstein). E. Demaine, 28 May 2002 (thanks to Stephan Wehner); J. O’Rourke, 22 Sep. 2005.

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## Problem 31: Trapping Light Rays with Segment Mirrors

**Statement** Is it possible to trap all the light from one point source by a finite collection of two-sided disjoint segment mirrors? A light ray is *trapped* if it includes no point strictly exterior to the convex hull of the mirrors. The source point is disjoint from the mirrors. Although several versions of the problem are possible, it seems to make the most sense to treat the mirrors as open segments (i.e., not including their endpoints), but demand that they are disjoint as closed segments.

**Origin** O’Rourke and Petrovici [OP01]. The question seems natural enough to have been raised earlier, but no other source is known.

**Status/Conjectures** Conjecture 9 from that paper: “No collection of segment mirrors can trap all the light from one source.”

**Partial and Related Results** In [OP01] several other conjectures are formed that imply a resolution to the posed problem. The strongest—that no collection of mirrors as above can support even a single *nonperiodic* ray, i.e., one that reflects forever (so is trapped) but never rejoins its earlier path—was disproved by Ben Stephens in 2002, who designed a construction of 8 mirrors that traps a ray reflecting nonperiodically. A similar construction was discovered and described in [MSZ09], which also established that any finite number of rays can be trapped nonperiodically. Milovich [Mil04] proved that if the angles between the lines containing the mirrors are rational multiples of  $\pi$ , then all but a countable number of light rays escape. In his book on billiards, Tabachnikov says, “It is unknown whether one can construct a polygonal trap for a parallel beam of light” [Tab05, p. 116]. This is in contrast to known nonpolygonal traps for such beams.

**Related Open Problems** Pach’s “enchanted forest” of circular mirrors.

**Appearances** Presented at the Open Problem session of the *13th Canad. Conf. Comput. Geom.*, Waterloo, Ontario, Aug. 2001. Also, Oberwolfach, Jan. 2009.



**Categories** visibility

**Entry Revision History** J. O’Rourke, 28 Aug. 2001; 24 Feb. 2003; 5 Oct. 2005; 7 Sep. 2009.

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## Problem 32: Bar-Magnet Polyhedra

**Statement** Which polyhedra are bar-magnet polyhedra? For reasons detailed below, the problem can be phrased as asking which 3-connected planar graphs may have their edges directed so that the directions “alternate” around each vertex.

Let  $P$  be a polyhedron with a set of edges  $E$ . For an edge  $e \in E$ , define a *bar magnet* as a mapping of  $e$  to either  $(N, S)$  or  $(S, N)$ , which assigns the endpoints of  $e$  opposite poles of a magnet (and corresponds to directing the edge). Call a vertex  $v$  of  $P$  to be *alternating* under mappings of its edges to bar magnets if the incident edges assigns alternating magnetic poles to  $v$  in the cyclic order of those edges on the surface around  $v$ :  $(N, S, N, S, \dots)$ . Thus if  $\deg(v)$  is even, the poles alternate, and if  $\deg(v)$  is odd, at most two like poles are adjacent in the circular sequence. Finally, call a polyhedron a *bar-magnet polyhedron* if there is a bar-magnet assignment of each of its edges so that each of its vertices is alternating.

**Origin** Joseph O’Rourke, 2001. This problem is inspired by the toy “Roger’s Connection,” which provides bar magnets and steel balls to construct polyhedra. The structures are most stable when each vertex is alternating.

**Status/Conjectures** Settled by Bojan Mohar, Apr. 2004.

**Partial and Related Results** At the presentation of the problem, Therese Biedl proved that the polyhedron formed by gluing together two tetrahedra with congruent bases is not a bar-magnet polyhedron: alternation at the three degree-4 vertices of the common base forces some other edge to be directed both ways. Thus not all polyhedra are bar-magnet polyhedra. Erik Demaine proved that a polyhedron all of whose vertices have even degree is a bar-magnet polyhedron: the graph has a face 2-coloring, and the edges of the faces of color 1 can be oriented counterclockwise, which then orients each face of color 2 clockwise. He also observed that if every vertex is of degree 3, Petersen’s theorem yields a perfect matching that establishes such “simple” polyhedra are bar-magnet polyhedra.

A clean characterization was provided by Bojan Mohar, who proved [Moh04]:

**Theorem 1** *Let  $G$  be a planar graph embedded on the surface of a sphere. Define a new graph  $R$  whose nodes are the vertices of odd degree in  $G$ , with two nodes of  $R$  adjacent if they are cofacial in  $G$  (lie on a common facial walk). Then  $G$  has an NS-orientation (i.e., is a bar-magnet polyhedral skeleton) if and only if  $R$  has a perfect matching.*

A *facial walk* is a closed walk along the boundary of a face.

**Appearances** Posed by J. O’Rourke at the CCCG 2001 open-problem session [DO02].

**Categories** polyhedra; planar graphs

**Entry Revision History** J. O’Rourke, 29 Aug. 2001; 11 Oct. 2001; E. Demaine, 31 Aug. 2002; J. O’Rourke, 14 Aug. 2004; 20 Sep. 2004.

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## Problem 33: Sum of Square Roots

**Statement** What is the minimum nonzero difference between two sums of square roots of integers? More precisely, find tight upper and lower bounds on  $r(n, k)$ , the minimum positive value of

$$\left| \sum_{i=1}^k \sqrt{a_i} - \sum_{i=1}^k \sqrt{b_i} \right|$$

where  $a_i$  and  $b_i$  are integers no larger than  $n$ . Bounds should be expressed as a function of  $n$  and  $k$ . Examples:

$$r(20, 2) \approx .0002 = \sqrt{10} + \sqrt{11} - \sqrt{5} - \sqrt{18}$$

$$r(20, 3) \approx .000005 = \sqrt{5} + \sqrt{6} + \sqrt{18} - \sqrt{4} - \sqrt{12} - \sqrt{12}$$

**Origin** Posed in [O’R81]. Perhaps older in other formulations.

**Status/Conjectures** Open, although some weak bounds are known.

**Motivation** Of particular importance is whether  $\lg 1/r(n, k)$  is bounded above by a polynomial in  $k$  and  $\lg n$ . If this statement is true, then the sign of a sum of square roots of integers can be computed in polynomial time. If this statement is false, however, there still may be a polynomial-time algorithm to compute the sign.

To quote David Eppstein: “A major bottleneck in proving NP-completeness for geometric problems is a mismatch between the real-number and Turing machine models of computation: one is good for geometric algorithms but bad for reductions, and the other vice versa. Specifically, it is not known on Turing machines how to quickly compare a sum of distances (square roots of integers) with an integer or other similar sums, so even (decision versions of) easy problems such as the minimum spanning tree are not known to be in NP.”

**Partial and Related Results** Exponential upper bounds are known through root-separation bounds [BFMS00, MS00]. Specifically, [MS00, BFMS00] proves that  $-\lg r(n, k) \leq O(2^{2k} \lg n)$ . (More generally, [BFMS00, MS00] give finite algorithms to compute the sign of algebraic expressions such as sums of square roots, which are implemented and used in LEDA<sup>2</sup> and CORE<sup>3</sup> for exact geometric computation.)

John A. Anderson ([johnaa333@netzero.net](mailto:johnaa333@netzero.net)) has an unpublished proof (Aug 2003) of a similar bound:

$$r(n, k) \geq [4k^2n]^{1/2-2^{2k-2}}.$$

Cheng et al. [CMSC09] establish an upper bound on  $-\lg r(n, k)$  of  $2^{O(n/\lg n)} \lg n$ , which improves on the above bound  $O(2^{2k} \lg n)$  whenever  $n \leq ck \lg k$  for some  $c$ .

At the other extreme, Qian and Wang [QW04, QW05] show an upper bound on  $r(n, k)$  of  $O(n^{-2k+\frac{3}{2}})$ . This upper bound on  $r(n, k)$  implies a lower bound on  $\lg 1/r(n, k)$ , that is, on how many bits we need to compute from the square roots to determine the sign of the difference. In particular, it settles (positively) a question posed here by Erik Demaine (Nov. 2001):

<sup>2</sup><http://www.algorithmic-solutions.com/enleda.htm>

<sup>3</sup><http://www.cs.nyu.edu/exact/core/>

can the number of bits required to distinguish the difference from zero ever exceed the total number of bits in the input integers?

A slight variation on the problem is to ask (e.g., for  $k = 2$ ), how close can  $\sqrt{a} + \sqrt{b}$  be to an integer; Dana Angluin and Sarah Eisenstat [AE04] proved a bound of  $\Theta(1/n^{3/2})$  on this integrality gap.

Related, [Blö91] gives a polynomial-time Monte Carlo algorithm to decide whether a sum of radicals is zero.

**Appearances** [O’R81]; Usenet newsgroup sci.math<sup>4</sup> 25 Dec 90.

**Categories** numerical computations

**Entry Revision History** E. Demaine, J. O’Rourke, 19 Nov. 2001; J. O’Rourke, 3 Dec. 2001; 13 Aug. 2003; 18 Aug. 2003; 30 Aug. 2003; 7 Dec. 2003; E. Demaine, 9 Feb. 2004 (thanks to Raimund Seidel); J. O’Rourke, 10 Mar. 2004; J. Mitchell, 30 Sep. 2004; J. Mitchell, 1 Oct. 2004; J. Mitchell, 27 Oct. 2005; J. O’Rourke, 30 Dec. 2005 (thanks to Marc Glisse); J. O’Rourke, 16 May 2006; E. Demaine, 9 Sep. 2009.

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<sup>4</sup><http://www.ics.uci.edu/~eppstein/junkyard/small-dist.html>

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## Problem 34: Extending Pseudosegment Arrangements by Subdivision

**Statement** How many intersections among an arrangement of pseudosegments in the plane must be added as vertices to allow the pseudosegment arrangement to be extended to a pseudoline arrangement?

An *arrangement of pseudosegments* in the plane is a family of finite-length planar curves such that every two curves intersect in at most one point. An *arrangement of pseudolines* in the plane is a family of planar curves that extend to infinity on both ends such that every two curves intersect in at most one point. Only some pseudosegment arrangements can be *extended* to pseudoline arrangements. However, if we allow turning intersection points into vertices of the arrangement, thereby subdividing the segments, then it is always possible to make a pseudosegment arrangement extendible. The question is how many such vertices must be added in the worst-case in terms of the number  $n$  of pseudosegments.

**Origin** Perhaps [Cha00a], [AACS98], or Boris Aronov?

**Status/Conjectures** Open.

**Partial and Related Results** Chan [Cha00a] has proved an upper bound of  $O(n \log n)$ .

**Appearances** Posed by Boris Aronov during the open problem session at the Fall Workshop on Computational Geometry, Brooklyn, NY, Nov. 2–3, 2001.

**Categories** combinatorial geometry

**Entry Revision History** E. Demaine, 20 Nov. 2001.

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## Problem 35: Freeze-Tag: Optimal Strategies for Awakening a Swarm of Robots

**Statement** An optimization problem that naturally arises in the study of “swarm robotics” is to wake up a set of “asleep” robots, starting with only one “awake” robot. One robot can only awaken another when they are in the same location. As soon as a robot is awake, it may assist in waking up other robots. The goal is to compute an optimal *awakening schedule* such that all robots are awake by time  $t^*$ , for the smallest possible value of  $t^*$  (the optimal *makespan*). The  $n$  robots are initially at  $n$  points of a metric space. The problem is equivalent to finding a spanning tree with maximum out-degree two that minimizes the radius from a fixed source.

Is it NP-hard to determine an optimal awakening schedule for robots in the Euclidean (or  $L_1$ ) plane? In more general metric spaces, can one obtain an approximation algorithm with better than  $O(\log n)$  performance ratio?

**Origin** [ABF<sup>+</sup>02]

**Status/Conjectures** [ABF<sup>+</sup>02] conjecture that the freeze-tag problem is NP-hard in the Euclidean (or  $L_1$ ) plane. (They show it to be NP-complete in star metrics.)

**Motivation** What is the most efficient way to “turn on” a large swarm of robots or to distribute to them a secret or a token that requires close proximity in order to pass from one to another?

**Partial and Related Results** There are a variety of related results given in [ABF<sup>+</sup>02]. They show that the problem is NP-hard for “star metrics” (each asleep robot is at a leaf of a star graph whose spokes have various lengths). For geometric instances ( $L_p$  metrics) in fixed dimension, they give an efficient PTAS. For general metric spaces, they give an  $O(\log n)$ -approximation algorithm. They also give improved approximation methods for other special cases (star graphs, ultrametrics)

**Appearances** Posed in [ABF<sup>+</sup>02], and by Joseph Mitchell during the open problem session at the Fall Workshop on Computational Geometry, Brooklyn, NY, Nov. 2–3, 2001.

**Categories** optimization; scheduling; robotics

**Entry Revision History** E. Demaine, 20 Nov. 2001; J. Mitchell, 21 Nov. 2001.

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## Problem 36: Inplace Convex Hull of a Simple Polygonal Chain

**Statement** How much extra space is required to compute the convex hull of a simple polygonal chain or simple polygon in linear time?

More precisely, given the  $n$  points in order along the chain in an array  $A$ , the algorithm must re-arrange the points in-place in the array and output a number  $h$  so that the first  $h$  elements in the resulting array are the points on the convex hull in order. The goal is to minimize the extra storage past the array  $A$ , say to  $O(\log n)$  or ideally  $O(1)$ .

**Origin** [BIK<sup>+</sup>01]

**Status/Conjectures** Solved [BC04].

**Partial and Related Results** From the abstract of [BC04]: “we present a simple self-contained solution that uses  $O(\log n)$  space, and indicate how to improve it to  $O(1)$  space with the same techniques used for stable partition.”

**Appearances** Posed in [BIK<sup>+</sup>01], and by Hervé Brönnimann during the open problem session at the Fall Workshop on Computational Geometry, Brooklyn, NY, Nov. 2–3, 2001.

**Categories** convex hulls

**Entry Revision History** E. Demaine, 21 Nov. 2001; J. O’Rourke, 10 Mar. 2004 (thanks to Ryan Coleman).

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## Problem 37: Counting Polyominoes

**Statement** How many polyominoes on  $n$  squares are there? A *polyomino* is a connected interior-disjoint union of axis-aligned unit squares joined edge-to-edge, in other words, an edge-connected union of cells in the planar square lattice. The *order* of a polyomino is the number of unit squares forming it. The problem asks for the number of polyominoes of order  $n$ . The key constraint here is that polyominoes must be edge-connected. There are three variations on the problem, depending on whether two polyominoes are considered equivalent by factoring out just translations (fixed polyominoes), rotations and translations (chiral polyominoes), or reflections, rotations, and translations (free polyominoes).

**Origin** To quote Klarner [Kla97]: “Polyominoes have a long history, going back to the start of the 20th century, but they were popularized in the present era initially by Solomon Golomb, then by Martin Gardner in his *Scientific American* columns.”

**Status/Conjectures** Open.

**Partial and Related Results** Asymptotically, results of Klarner et al. [Kla97, Thm. 12.3.1] show that the number of fixed  $n$ -ominoes (factoring out just translations), denoted  $t(n)$ , satisfies

$$\lim_{n \rightarrow \infty} [t(n)]^{1/n} = \lambda$$

(roughly,  $t(n)$  is around  $n^\lambda$ ) for “Klarner’s constant”  $\lambda$ , with  $4.0025 < \lambda < 4.5685$ , but the precise value of  $\lambda$  remains open. The lower bound of  $> 4$  is established in [BRS16], and the upper bound in [BB15].

The exact counts have been computed for small  $n$ . See [Sloa] for the number of fixed  $n$ -ominoes for  $n \leq 28$  and for related references. The current record is  $n = 56$  by Jensen [Jen03]. See also [Epp] for related links.

**Related Open Problems** There are many related problems involving polyominoes with restricted geometric shapes (e.g., [Slob]), polyiamonds (edge-to-edge unions of unit equilateral triangles), polyhexes (edge-to-edge unions of unit regular hexagons), polyabolos (edge-to-edge unions of unit right isosceles triangles), polycubes (face-to-face unions of unit cubes), etc. All of these problems are also open.

**Appearances** [Kla97]



**Categories** combinatorial geometry

**Entry Revision History** E. Demaine & J. O’Rourke, 30Nov2001; E. Demaine, 28Aug2002; G. Barequet, 4Dec2015.

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## Problem 38: Compatible Triangulations

**Statement** Is it true that every two sets of  $n$  planar points in general position with the same number points on their convex hulls have compatible triangulations? Two triangulations are *compatible* if they have the same combinatorial structure, i.e., if their face lattices are isomorphic. For compatible triangulations  $T_1$  and  $T_2$  of point sets  $S_1$  and  $S_2$ , there is a bijection  $\phi$  between the points such that  $ijk$  is a triangle of  $T_1$  empty of points of  $S_1$  iff  $\phi(i)\phi(j)\phi(k)$  is a triangle of  $T_2$  empty of points of  $S_2$ .

**Origin** [AAK01] and [AAHK02].

**Status/Conjectures** Open. Conjectured in [AAHK02] to be true.

**Motivation** Morphing.

**Partial and Related Results** The answer to the question posed is sometimes NO for points not in general position. If the bijection between the points is given and fixed, then compatible triangulations do not always exist [Saa87]. When the bijection is not given, the conjecture is proven only for point sets with at most three points interior to the hull [AAHK02]. Compatible triangulations can always be achieved by the addition of at most a linear number of Steiner points.

**Categories** triangulations

**Entry Revision History** J. O'Rourke, 1 Jan. 2002.

## References

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## Problem 39: Distances among Point Sets in $\mathbb{R}^2$ and $\mathbb{R}^3$

**Statement** For a point set  $P$  in  $\mathbb{R}^d$ , let  $f_d(P)$  be the number of unit-distance point pairs:

$$f_d(P) = |\{(u, v) \mid u, v \in P, \|u - v\| = 1\}| ;$$

and let  $f_d(n)$  be the maximum over all sets of  $n$  points:

$$f_d(n) = \max_{|P|=n} f_d(P) .$$

Further, let  $g_d(P)$  denote the number of distinct distances induced by a set of points  $P$ :

$$g_d(P) = |\{\|u - v\| \mid u, v \in P\}| ;$$

and let  $g_d(n)$  be the minimum over all sets of  $n$  points:

$$g_d(n) = \min_{|P|=n} g_d(P) .$$

Give upper and lower bounds on  $f_d(n)$  and  $g_d(n)$ , particularly for  $d = 2$  and  $d = 3$ .

**Origin** Paul Erdős [Erd46].

**Status/Conjectures** Open.

**Partial and Related Results**  $f_2(n) = O(n^{4/3})$  [Szé97, CEG<sup>+</sup>90, SST84], and  $f_2(n) = \Omega(n^{1+c/\log \log n})$  [Erd46].  $f_3(n) = O(n^{5/3})$  and  $f_3(n) = \Omega(n^{4/3} \log \log n)$  [Erd60]. For  $g_2(n)$ , the best result is that  $g_2(n) = \Omega(n^{6/7})$  [ST01a]. Erdős conjectured that the correct answer here is  $n/\sqrt{\log n}$ ; this bound is achieved on the grid.

**Reward** Erdős offered \$500 to settle whether  $f_2(n) < cn^{1+\epsilon}$  for some  $c > 0$  and for each  $\epsilon > 0$ , and \$500 to settle whether  $g_2(n) = [1 + o(1)]cn/\sqrt{\log n}$ .

**Appearances** [CFG90, pp. 150-1].

**Categories** combinatorial geometry

**Entry Revision History** S. Venkatasubramanian, 12 Feb. 2002.

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## Problem 40: The Number of Pointed Pseudotriangulations

**Statement** For a planar point set  $S$ , is the number of pointed pseudotriangulations always at least the number of triangulations?

A *pseudotriangle* is a planar polygon with exactly three convex vertices. Each pair of convex vertices is connected by a reflex chain, which may be just one segment. (Thus, a triangle is a pseudotriangle.) A *pseudotriangulation* of a set  $S$  of  $n$  points in the plane is a partition of the convex hull of  $S$  into pseudotriangles using  $S$  as a vertex set. A minimum pseudotriangulation, or *pointed pseudotriangulation*, has the fewest possible number of edges for a given set  $S$  of points.

See [Str00, KKM<sup>+</sup>03, O’R02a] for examples, explanation of the term “pointed,” and further details.

**Origin** [RRSS01].

**Status/Conjectures** Open. Conjectured to be true, with equality only when the points of  $S$  are in convex position.

**Partial and Related Results** The conjecture has been established for all sets of at most 10 points:  $\leq 9$  by [BKPS01], and 10 by Oswin Aichholzer [personal communication, 28 Mar. 2002]. Aichholzer et al. [AAKS02] establish that the number of pointed pseudotriangulations on  $n$  points is minimized when the points are in convex position.

**Appearances** Posed by Jack Snoeyink at the CCCG 2001 open-problem session [DO02].

**Categories** triangulations; combinatorial geometry

**Entry Revision History** J. O’Rourke, 20 Mar. 2002; 28 Mar. 2002; E. Demaine, 7 Aug. 2002; 31 Aug. 2002.

## References

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## Problem 41: Sorting $X + Y$ (Pairwise Sums)

**Statement** Given two sets of numbers, each of size  $n$ , how quickly can the set of all pairwise sums be sorted? In symbols, given two sets  $X$  and  $Y$ , our goal is to sort the set

$$X + Y = \{x + y \mid x \in X, y \in Y\}.$$

**Origin** The earliest known reference is Fredman [Fre76], who attributes the problem to Elwyn Berlekamp.

**Status/Conjectures** Open.

**Motivation** This is a simple special case of the more general question of *sorting with partial information*: How many comparisons are required to sort if a partial order on the input set is already known? Hernández Barrera [Her96] and Barequet and Har-Peled [BHP01] describe several geometric problems that are “Sorting- $(X + Y)$ -hard”. Specifically, there is a subquadratic-time transformation from sorting  $X + Y$  to each of the following problems: computing the Minkowski sum of two orthogonal-convex polygons, determining whether one monotone polygon can be translated to fit inside another, determining whether one convex polygon can be rotated to fit inside another, sorting the vertices of a line arrangement, or sorting the interpoint distances between  $n$  points in  $\mathbb{R}^d$ . (Although Barequet and Har-Peled [BHP01] claim only that the problems they consider are 3SUM-hard, their proofs immediately imply this stronger result.)

Fredman also mentions an immediate application to multiplying sparse polynomials [Fre76].

**Partial and Related Results** The obvious  $O(n^2 \log n)$ -time algorithm is also the fastest known. There are  $\Omega(n^2)$  lower bounds for this problem in various restrictions of the linear decision tree model of computation [Fre76, Die89, Eri99a]. The main problem is whether the logarithmic factor can be removed.

Fredman [Fre76] proved that if a given partial order on  $m$  elements has  $L$  linear extensions, then the set can be sorted in at most  $\log_2 L + 2m$  comparisons. For the sorting  $X + Y$  problem, we have  $m = n^2$ , the Hasse diagram of the partial order is an  $n \times n$  diagonal grid, and simple arguments about hyperplane arrangements imply that  $L = O(n^{8n})$ . Thus, Fredman's algorithm can sort  $X + Y$  using only  $8n \log n + 2n^2$  comparisons; unfortunately, the algorithm needs exponential time to choose which comparisons to perform! This exponential overhead was reduced to polynomial time by Kahn and Kim [KK95] and then to  $O(n^2 \log n)$  by Lambert [Lam92] and Steiger and Streinu [SS95]. These results imply that no superquadratic lower bound is possible in the full linear decision tree model.

If the input consists of  $n$  integers between  $-M$  and  $M$ , an algorithm of Seidel based on fast Fourier transforms runs in  $O(n + M \log M)$  time [Eri99a]. The  $\Omega(n^2)$  lower bounds require exponentially large integers.

A closely related problem does have a subquadratic solution: find a minimum element of  $X + Y$ , the so-called *min-convolution* problem, posed by Jeff Erickson [DO06]. See [BCD<sup>+</sup>06] for the result and a discussion of connections to the sorting problem.

**Related Open Problems** The decision version of this problem—does the set  $X + Y$  have  $n^2$  unique elements?—is 3SUM-hard [BHP01]; see Problem 1.

**Categories** lower bounds

**Entry Revision History** E. Demaine, 6 June 2002; Jeff Erickson, 20 June 2002; J. O'Rourke, 20 Aug. 2006.

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## Problem 42: Vertex-Unfolding Polyhedra

**Statement** Consider a polyhedron with simply connected facets (no holes on a facet) and without boundary (every edge is incident to exactly two facets). Can the polyhedron be cut along potentially all of its edges, but leaving certain faces connected at vertices, and unfolded into one piece in the plane without overlap? Such an unfolding is called a *vertex-unfolding*, to distinguish from widely studied *edge-unfoldings* (see Problem 1) and *general unfoldings*. An important subproblem here is whether all convex polyhedra have vertex-unfoldings; a negative answer would also resolve Problem 1.

**Origin** [DEE<sup>+</sup>02]

**Status/Conjectures** Open.

**Partial and Related Results** All simplicial polyhedra have vertex-unfoldings [DEE<sup>+</sup>02]. These vertex-unfoldings have a special structure called a “facet path” which does not exist in general, even for convex polyhedra [DEE<sup>+</sup>02].

**Related Open Problems** Problem 1: Edge-Unfolding Convex Polyhedra.  
 Problem 1: General Unfolding of Nonconvex Polyhedra.

**Appearances** Originally posed in [DEE<sup>+</sup>02]. Posed by E. Demaine at the CCCG 2001 open-problem session [DO02].

**Categories** folding and unfolding; polyhedra

**Entry Revision History** E. Demaine, 7 Aug. 2002; 31 Aug. 2002.

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## Problem 43: General Unfoldings of Nonconvex Polyhedra

**Statement** Can every closed polyhedron be cut along its surface and unfolded into one piece in the plane without overlap? Such an unfolding is called a *general unfolding* to distinguish from *edge-unfoldings* (see Problem 1) and *vertex-unfoldings* (see Problem 1).

**Origin** Perhaps [BDE<sup>+</sup>03].

**Status/Conjectures** Open.

**Partial and Related Results** It is known that every convex polyhedron has a general unfolding. In fact, there are three general methods for unfolding convex polyhedra: the star unfolding [AO91, AAOS97], the source unfolding [MMP87], and unfolding via quasigeodesics [IOV07].

On the nonconvex side, Bern et al. [BDE<sup>+</sup>03] show a general unfolding for a nonconvex simplicial polyhedron (whose faces are all triangles) that has no edge unfolding, establishing that general unfoldings are more powerful than edge unfoldings. (This was known earlier [BDD<sup>+</sup>98] but with an example using nonconvex faces.)

It is now known that all orthogonal polyhedra (those with all edges parallel to coordinate axes) have a general unfolding [DFO07], although the resulting single piece can be exponentially thin and long. See [O’R08] for a survey of progress on orthogonal polyhedra.



**Related Open Problems** Problem 1: Edge-Unfolding Convex Polyhedra.

Problem 1: Vertex-Unfolding Polyhedra.

**Appearances** [BDE<sup>+</sup>03], [DO07, Open Prob. 22.3].

**Categories** folding and unfolding; polyhedra

**Entry Revision History** E. Demaine, 7 Aug. 2002; J. O’Rourke, 24 Jul 2008.

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## Problem 44: 3-Colorability of Arrangements of Great Circles

**Statement** Is every zonohedron face 3-colorable when viewed as a planar map?

An equivalent question, under a different guise, is the following: is the arrangement graph of great circles on the sphere always vertex 3-colorable? (The *arrangement graph* has a vertex for each intersection point, and an edge for each arc directly connecting two intersection points.) Assume that no three circles meet at a point, so that this arrangement graph is 4-regular.

**Origin** The zonohedron-face version is due to Stan Wagon, deriving from the work in [SW00]. The origin of the arrangement guise of the problem is [FHNS00].

**Status/Conjectures** Open.

**Partial and Related Results** Arrangement graphs of circles in the plane, or general circles on the sphere, can require four colors [Koe90]. The key property in this problem is that the circles must be great. All arrangement graphs of up to 11 great circles have been verified to be 3-colorable by Oswin Aichholzer (August, 2002). See [Wag02] for more details.

**Appearances** Posed by Stan Wagon at the CCCG 2002 open-problem session.

**Categories** arrangements; coloring; polyhedra

**Entry Revision History** E. Demaine & J. O'Rourke, 28 Aug. 2002.

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## Problem 45: Smallest Universal Set of Points for Planar Graphs

**Statement** How many points must be placed in the plane to support planar drawing of all planar graphs on  $n$  vertices? More precisely, call a set of points *universal* if every planar graph on  $n$  vertices can be drawn with straight-line edges and without crossings by placing the vertices on a subset of the points. What is the smallest universal set of points as a function of  $n$ ? In particular, is it  $O(n)$ ?

**Origin** Attributed to Mohar by János Pach (23 Nov. 2002). See also [CH89] for some of the history.

**Status/Conjectures** Open. Between  $\Theta(n)$  and  $\Theta(n^2)$ .

**Partial and Related Results** By definition, a universal set of points must have size at least  $n$ . Chrobak and Karloff [CH89] proved the stronger result that any universal set of points must have at least  $1.098n$  points.

On the other side, it is well-known that there are universal sets of points of size  $O(n^2)$ . In particular, every planar graph can be drawn on the  $O(n) \times O(n)$  square grid [dFPP90, Sch90]. However, any universal set of points forming a grid must have size at least  $n/3 \times n/3$  [CH89].

Stephen Kobourov asks for the smallest value of  $n$  for which a universal point set of size  $n$  does not exist. He has checked by exhaustive search that there is a universal point set of size  $n$  for all  $n \leq 14$ .

It is now known that there is a universal set of  $n$  points if one bend per edge is permitted [ELLW10].

**Appearances** Posed by Stephen Koborouv during an open-problem session at the DIMACS Workshop on Computational Geometry (12th Annual Fall Workshop on Computational Geometry), Nov. 2002.

**Categories** graphs; point sets; graph drawing

**Entry Revision History** E. Demaine, 23 Nov. 2002; 20 Sep. 2003 (thanks to Sergio Cabello); J. O'Rourke, 29 Mar 2010 (thanks to Michael Hoffmann).

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## Problem 46: 3D Minimum-Bend Orthogonal Graph Drawings

**Statement** Does every simple graph with maximum vertex degree  $\Delta \leq 6$  have a 3D orthogonal point-drawing with no more than two bends per edge? A *3D orthogonal point-drawing* of a graph maps each vertex to a unique point of the 3D cubic lattice, and maps each edge to a lattice path between the endpoints; these paths can only intersect at common endpoints. In this problem, each path must have at most two bends, that is, consist of at most three orthogonal line segments (links).

**Origin** Likely [ESW00].

**Status/Conjectures** Open.

**Partial and Related Results** Two bends would be best possible, because any drawing of  $K_5$  uses at least two bends on at least one edge. If  $\Delta \leq 5$ , two bends per edge suffice [Woo03]. Two bends also suffice for the complete multipartite 6-regular graphs  $K_7$ ,  $K_{2,2,2,2}$ ,  $K_{3,3,3}$ , and  $K_{6,6}$  [Woo00]. In general, there is a drawing with an average number of bends per edge of at most  $2 + \frac{2}{7}$  [Woo03]. Additionally, three bends per edge always suffice, even for multigraphs [ESW00, PT99, Woo01].

Two-dimensional versions of this problem have also been studied. A *2D orthogonal point-drawing* of a graph maps each vertex to a unique point of the 2D square lattice, and maps each edge to a lattice path between the endpoints; the paths are allowed to intersect at common endpoints and at proper crossings (points at which two paths meet but do not bend), but must be edge-disjoint. Every graph with maximum vertex degree  $\Delta \leq 4$  has a 2D orthogonal point-drawing with at most two bends per edge, and furthermore within a  $2n \times 2n$  rectangle of the grid [Sch95]. On the other hand, as in 3D, any drawing of  $K_5$  uses at least two bends on at least one edge [Sch95], so two bends is again best possible. For planar graphs, we can ask for 2D orthogonal point-drawings that have no (proper) crossings. In this case, again there are drawings with at most two bends per edge, unless the graph has a connected component isomorphic to the icosahedron, in which case three bends per edge is the best possible [BK98, LMS98].

**Appearances** [ESW00]. Posed by David Wood at the CCCG 2002 open-problem session [DO03b].

**Categories** graph drawing

**Entry Revision History** E. Demaine, 21 Dec. 2002; 17 July 2005.

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## Problem 47: Hinged Dissections

**Statement** Does every pair of equal-area polygons have a hinged dissection?

A *dissection* of one polygon  $A$  to another  $B$  is a partition of  $A$  into a finite number of pieces that may be reassembled to form  $B$ . A *hinged dissection* is a dissection where the pieces are hinged at vertices and the reassembling

is achieved by rotating the pieces about their hinges in the plane of the polygons.

**Origin** [DDE<sup>+</sup>03], [Fre02, p. 3].

**Status/Conjectures** Now settled: Hinged dissections exist [AAC<sup>+</sup>08]. Update to this entry soon.

**Partial and Related Results** There are two main partial results. First, any two *polyominoes* of the same area have a hinged dissection [DDE<sup>+</sup>03]. A polyomino is a polygon formed by joining unit squares at their edges; see [Kla97] and Problem 1. The polyomino result generalizes to hinged dissections of all edge-to-corresponding-edge gluings of congruent copies of any polygon. Second, any asymmetric polygon has a hinged dissection to its mirror image [Epp01]. Both of these results interpret the problem as ignoring possible intersections between the pieces as they hinge, following what Frederickson calls the “wobbly-hinged” model. This freedom may not be necessary, although this seems not to be established in the literature.

Many specific examples of hinged dissections can be found in [Fre02].

**Appearances** [O’R02b].

**Categories** polygons

**Entry Revision History** J. O’Rourke, 25 Mar 2003; J. O’Rourke, 23 Jan 2009.

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## Problem 48: Bounded-Degree Minimum Euclidean Spanning Tree

**Statement** What is the complexity of finding a bounded-degree spanning tree for a planar point set, such that the total Euclidean length  $\tau_k$  is as small as possible, subject to the constraint that no node has more than  $k = 4$  edges incident to it?

**Origin** Papadimitriou and Vazirani [PV84] conjectured the problem to be NP-hard for  $k = 4$ .

**Status/Conjectures** Solved: Proved NP-hard in [FH09].

**Motivation** Natural generalization of finding a shortest geometric Hamiltonian path; arises in network optimization

**Partial and Related Results** [PV84] proved the problem to be NP-hard for  $k = 3$ . For  $k \geq 5$ , the problem is polynomially solvable, as there always is a minimum spanning tree with no point having degree more than 5.

**Related Open Problems** Various worst-case ratios of minimum weight bounded-degree spanning trees for different degree bounds are still open, in particular comparing  $\tau_k$  to the weight  $\tau$  of a minimum spanning tree. [FKK<sup>+</sup>97] conjecture  $\tau_3/\tau \leq 1.103\dots$ ,  $\tau_4/\tau \leq 1.035\dots$  for Euclidean distances in the plane, and  $\tau_4/\tau \leq 1.25$  for Manhattan distances in the plane, and give matching lower bounds.

[KRY96] show that for Euclidean distances,  $\tau_4/\tau \leq 1.25$  and  $\tau_3/\tau \leq 1.5$  in the plane, and  $\tau_3/\tau \leq 1.66\dots$  in arbitrary dimensions.

The first two of these bounds were improved to  $\tau_4/\tau \leq 1.143$  and  $\tau_3/\tau \leq 1.402$  by [Cha03].

Now settled by NP-hard proof in [FH09].

**Categories** minimum spanning tree; optimization; point sets

**Entry Revision History** S. P. Fekete, 30 July 2003; J. O'Rourke, 29 Mar 2010 (thanks to Michael Hoffmann).

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## Problem 49: Planar Euclidean Maximum TSP

**Statement** What is the complexity of finding a tour of maximum Euclidean length for a planar point set?

**Origin** [Bar96] showed that there is a PTAS for the problem. No earlier mention is known.

**Status/Conjectures** Open.

**Motivation** How does the complexity of a natural problem depend on the geometry of distances?

**Partial and Related Results** [BJrW98] showed that a maximum length tour can be found in polynomial time for polyhedral metrics in spaces of finite dimension, i.e., for metrics for which the unit ball is a convex body with  $f$  facets. The resulting complexity is  $O(n^{f-2} \log n)$ .

[Fek99] showed that the maximum TSP can be solved in time  $O(n)$  for rectilinear distances in the plane, but is NP-hard for Euclidean distances in three-dimensional space, or on the surface of a sphere. Conjectures the case of planar Euclidean distances to be NP-hard.

More recent details and related problems can be found in [BFJ<sup>+</sup>03].

**Related Open Problems** The problem is not even known to be in NP. A polynomial algorithm would require some understanding of problem 33



(sum of square roots), at least for classes of instances arising from the computation of tour length.

Also related is the Planar Euclidean maximum scatter TSP: What is the complexity of finding a tour for a planar point set in  $\mathbb{R}^d$ , such that the Euclidean length of the shortest edge is maximized? Stated in [ACM<sup>+</sup>97], shown NP-hard in dimensions  $d \geq 3$  in [Fek99], open for  $d = 2$ . Also, no bounds on approximation are known in a geometric context; the best known approximation algorithm from [ACM<sup>+</sup>97] achieves an approximation factor of 2, but does not use geometry.

**Appearances** [Fek98], [BFJ<sup>+</sup>03]

**Categories** traveling salesman; optimization; point sets

**Entry Revision History** S.P. Fekete, 1 Aug. 2003.

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## Problem 50: Pointed Spanning Trees in Triangulations

**Statement** Does every triangulation of a set of points in the plane (in general position) contain a pointed spanning tree as a subgraph? A vertex is

pointed if one of its incident faces has an angle larger than  $\pi$  at this vertex. A spanning tree is pointed if all of its vertices are pointed.

**Origin** Oswin Aichholzer, January 2003.

**Status/Conjectures** Settled negatively, January 2004.

**Partial and Related Results** Obviously true if a triangulation contains a Hamiltonian path or a pointed pseudotriangulation as a subgraph. For both structures there exist triangulations not containing them. (See, e.g., [O'R02a] for a discussion of pseudotriangulations.) Settled negatively by Aichholzer et al. [AHK04] with a 124-point counterexample. A consequence is that there are triangulations that require  $\Omega(n)$  edge-flips to contain a pointed spanning tree, or to become Hamiltonian.

**Related Open Problems** Problem 1.

**Appearances** Posed by Oswin Aichholzer at the CCCG 2003 open-problem session, August 2003. Also posed by Bettina Speckmann as Problem 10 at the First Gremo Workshop on Open Problems in Stels, Switzerland, July 2003.

**Categories** triangulations; planar graphs

**Entry Revision History** O. Aichholzer, 13 Aug. 2003; JOR, 15 Jan. 2004.

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## Problem 51: Linear-Volume 3D Grid Drawings of Planar Graphs

**Statement** Does every  $n$ -vertex planar graph have a 3D grid drawing with  $O(n)$  volume? A *3D grid drawing* of a graph is a placement of the vertices at distinct points with integer coordinates such that the straight line segments representing the edges are pairwise non-crossing. The volume is of the bounding box.

**Origin** Felsner, Liotta, and Wismath [FLW02].

**Status/Conjectures** Open.

- Partial and Related Results**
1. [FLW02]: Every  $n$ -vertex outerplanar graph has a 3D grid drawing with  $O(n)$  volume.
  2. [DW03b]: Every  $n$ -vertex graph with bounded treewidth has a 3D grid drawing with  $O(n)$  volume.
  3. [DW04]: Every  $n$ -vertex planar graph has a 3D grid drawing with  $O(n^{3/2})$  volume.
  4. [Woo02]: Every  $n$ -vertex planar graph has an  $O(1) \times O(1) \times O(n)$  grid drawing if and only if planar graphs have  $O(1)$  queue-number. (See Problem 1 for a definition of queue-number.)

**Related Open Problems** Problem 1.

**Appearances** Above references.

**Categories** graph drawing

**Entry Revision History** D. Wood, 6 Dec. 2003; J. O'Rourke, 16 Mar. 2004.

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## Problem 52: Queue-Number of Planar Graphs

**Statement** Does every planar graph have  $O(1)$  queue-number? A *queue layout* of a graph consists of a linear order of the vertices and a partition of the edges into non-nested *queues*. Edge  $xy$  is *nested* inside edge  $vw$  if  $v < x < y < w$  in the linear order. The *queue-number* of a graph  $G$  is the minimum number of queues in a queue layout of  $G$ . This question amounts to asking whether every planar graph has a vertex ordering with a constant number of pairwise nested edges (called a rainbow).

**Origin** Heath, Leighton, and Rosenberg [HLR92, HR92].

**Status/Conjectures** Open.

- Partial and Related Results**
1. [HLR92, HR92]: Every tree has queue-number  $\leq 1$ .
  2. [HLR92, HR92]: Every outerplanar graph has queue-number  $\leq 2$ .
  3. [DW03b]: Every graph with bounded treewidth has bounded queue-number.
  4. [Woo02]: Planar graphs have  $O(1)$  queue-number if and only if every  $n$ -vertex planar graph has a  $O(1) \times O(1) \times O(n)$  grid drawing.
  5. [DW03a]: Planar graphs have  $O(1)$  queue-number if and only if Hamiltonian bipartite planar graphs have  $O(1)$  bipartite thickness. The *bipartite thickness* of a bipartite graph  $G$  is the minimum  $k$  such that  $G$  can be drawn with the vertices on each side of the bipartition along a line, with the two lines parallel, and with each edge assigned to one of  $k$  “layers” so that no two edges in the same layer cross (when drawn as straight line segments).

**Related Open Problems** Problem 1.

**Appearances** Above references.

**Categories** graph drawing

**Entry Revision History** D. Wood, 7 Dec. 2003.

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## Problem 53: Minimum-Turn Cycle Cover in Planar Grid Graphs

**Statement** What is the complexity of finding a cycle-cover of a planar grid graph that has the fewest possible  $90^\circ$  turns? (An  $180^\circ$  U-turn counts as two turns.) A *planar grid graph* is a graph whose vertices are any set of points on the planar integer lattice and whose edges connect every pair of vertices at unit distance.

**Origin** Aggarwal et al. [ACK<sup>+</sup>97] show that the more general problem of finding a cycle cover for a planar set of points that minimizes total turn angle is NP-hard. Arkin et al. [ABD<sup>+</sup>01, ABD<sup>+</sup>05] consider the problem in grid graphs, but are only able to give approximations.

**Status/Conjectures** Solved: proved NP-hard by Fekete and Krupke [FK19]

**Motivation** Minimizing turns is a natural geometric measure; understanding its algorithmic behavior is of general interest.

**Partial and Related Results** [ABD<sup>+</sup>01, ABD<sup>+</sup>05] show that the problem is polynomially solvable when restricted to *thin* grid graphs, i.e., grid graphs that do not contain an induced  $2 \times 2$  square. For this special case, the problem behaves somewhat similarly to a Chinese Postman Problem. The problem of finding a minimum-turn tour is shown to be NP-complete, even for this special case.

In 2019, Fekete and Krupke [FK19] established that finding a minimum-turn cycle cover in a planar grid graph is NP-hard, thereby providing an answer to this problem.

**Related Open Problems** Minimum-turn cycle cover in a “solid” (genus-zero) grid graph: What is the complexity of finding a minimum-turn tour for a given planar grid graph without holes?

TSP in a solid grid graph: What is the complexity of finding a minimum-length tour for a given planar grid graph without holes? (Problem 1)

**Appearances** [ABD<sup>+</sup>01, ABD<sup>+</sup>05].

**Categories** traveling salesman; optimization; point sets; graphs

**Entry Revision History** S. P. Fekete, 12 Dec. 2003; D. Krupke, 29 Nov. 2023

## References

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## Problem 54: Traveling Salesman Problem in Solid Grid Graphs

**Statement** What is the complexity of finding a shortest tour in a solid planar grid graph? A *planar grid graph* is a graph whose vertices are any set of points on the planar integer lattice and whose edges connect every pair of vertices at unit distance. Distances between nodes correspond to induced shortest-path distances in the graph, which corresponds to “Manhattan” distances. A grid graph is *solid* if it does not have any holes, i.e., its complement in the planar integer lattice is connected.

**Origin** [IPS82] show that the problem is NP-complete in general planar grid graphs.

**Status/Conjectures** Open.

## Motivation

**Partial and Related Results** [UL97] show that Hamiltonicity of a solid grid graph can be decided in polynomial time. Thus we can decide whether there is a tour of length equal to the number of vertices. In contrast, deciding Hamiltonicity is NP-hard in general planar grid graphs [IPS82].

[ABD<sup>+</sup>01] observe that finding the shortest tour is polynomially solvable when restricted to *thin* grid graphs, i.e., grid graphs that do not contain an induced  $2 \times 2$  square. This problem asks about replacing the thin restriction with the solid restriction.

**Related Open Problems** Minimum-Turn Cycle Cover in Planar Grid Graphs (Problem 1)

**Appearances** Mentioned in [ABD<sup>+</sup>01].

**Categories** traveling salesman; optimization; point sets; graphs

**Entry Revision History** S. P. Fekete, 20 Dec. 2003; E. Demaine, 16 May 2004.

## References

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## Problem 55: Pallet Loading

**Statement** What is the complexity of the pallet loading problem? Given two pairs of numbers,  $(A, B)$  and  $(a, b)$ , and a number  $n$ , decide whether  $n$  small rectangles of size  $a \times b$ , in either axis-parallel orientation, can be packed into a large rectangle of size  $A \times B$ .

This problem is not even known to be in NP, because of the compact input description, and the possibly complicated structure of a packing, if there is one.

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Motivation** Natural packing problem; first-rate example of the relevance of coding input and output.

**Partial and Related Results** Tarnowsky [Tar92] showed that the problem can be solved in time polynomial in the size of the input if we are restricted to “guillotine” patterns, i.e., arrangements of items that can be obtained by a recursive sequence of edge-to-edge cuts. This result uses some nontrivial algebraic methods.

**Related Open Problems** What is the complexity of packing a maximal number of unit squares in a simple polygon? (Problem 1)

**Appearances** [Dow87] claims the problem to be NP-hard; [Exe88] claims the problem to be in NP; but both claims are erroneous. The precise nature of the difficulty is stated in [Nel93].

**Categories** packing; optimization

**Entry Revision History** S. P. Fekete, 17 Jan. 2004.

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## Problem 56: Packing Unit Squares in a Simple Polygon

**Statement** What is the complexity of deciding whether a given number of axis-parallel unit squares can be packed into a simple polygon (without holes)?

**Origin** Unknown.

**Status/Conjectures** Solved: proved NP-hard by Abrahamsen and Stade [AS24].



**Motivation** Natural packing problem.

**Partial and Related Results** The problem is known to be NP-hard for polygons with holes [FPT81], even if the polygon is an orthogonal polygon with all coordinates being multiples of  $1/2$ . Recently this version of the problem was shown to be in NP [DEKIV09], making it NP-complete.

The problem is the decision version for two optimization problems of very different behavior. There is a PTAS for packing the maximum number of squares of fixed size [HM85]. Maximizing the size of squares such that a fixed number of squares can be packed has a lower bound on approximation of  $14/13$ , and there is a  $3/2$ -approximation [BF01].

In 2024, Abrahamsen and Stade [AS24] proved that the problem is NP-hard, even when the polygon is an orthogonal and orthogonally convex polygon with half-integer coordinates.

**Related Open Problems** What is the complexity of pallet loading? (Problem 1)

**Appearances** [BF01] conjecture the problem to be polynomially solvable.

**Categories** packing; optimization

**Entry Revision History** S. P. Fekete, 16 Jan. 2004; E. Demaine, 3 July 2009; J. Stade, 12 Dec. 2024.

## References

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## Problem 57: Chromatic Number of the Plane

**Statement** How many colors are needed to paint the plane so that no two points a unit distance apart are painted the same color? If the same question is asked of the line, the answer is 2: Coloring  $[0, 1)$  red,  $[1, 2)$  blue, etc., ensures that no two unit-separated points have the same color. One can view the question as asking for the chromatic number  $\chi(\mathbb{E}^2)$  of the infinite *unit-distance graph*  $G$ , with every point in the plane a vertex, and an edge between two vertices if they are separated by a unit distance.

**Origin** Hadwiger and Edward Nelson, 1944.

**Status/Conjectures** Open. Erdős and de Bruijn showed [EdB51] that the chromatic number of the plane is attained for some finite subgraph of  $G$ . This result led to narrowing the answer to  $4 \leq \chi(\mathbb{E}^2) \leq 7$ . For example, the lower bound of 4 is established by the “Moser graph.”

The knowledge gap for the chromatic number of (3D) space is even wider than for the plane: it is only known to satisfy  $6 \leq \chi(\mathbb{E}^3) \leq 15$ . See [Gra04a, Gra04b] for further results and references.

There is now some evidence that the chromatic number of the plane may depend on the axioms of set theory. This was first seen possible in examples constructed by Saharon Shelah and Alexander Soifer. Now Payne [Pay09] has constructed unit-distance graphs with the same property.

**Related Open Problems** Problem 1.

**Reward** Ron Graham offers \$1000 for a solution.

**Appearances** [O’R04]

**Categories** combinatorial geometry

**Entry Revision History** J. O’Rourke, 15 Aug. 2004; 6 July 2009.

## References

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- [Pay09] M. S. Payne. Unit distance graphs with ambiguous chromatic number. arXiv:0707.1177v2 [math.CO], 2009.

## Problem 58: Monochromatic Triangles

**Statement** For any (planar) triangle  $T$ , is there is a 3-coloring of the (infinite) plane with no monochromatic copy of  $T$ ? We imagine congruent copies of  $T$  moved around the plane via rigid motions, and seek a spot where  $T$  is monochromatic.  $T$  is *monochromatic* if its three vertices are painted the same color, by virtue of lying on points of the plane painted that color. Note that the coloring in the question may depend on the given triangle  $T$ .

**Origin** Ron Graham, MSRI, August 2003.

**Status/Conjectures** Open. Ron Graham conjectures that the answer is YES for all triangles  $T$ .

**Motivation** The question of the chromatic number of the Euclidean plane  $\mathbb{E}^2$  has been unresolved for over fifty years (Problem 1). This problem is an interesting, much more restricted variant, posed by Ron Graham as part of his “Geometric Ramsey Theory” investigation [Gra04a] [Gra04b] at his MSRI lectures<sup>5</sup> in August 2003.

**Partial and Related Results** See [O’R04] for further explanation.

**Related Open Problems** Problem 1.

**Reward** Ron Graham offers \$50 for a solution.

**Appearances** [O’R04]

**Categories** combinatorial geometry

**Entry Revision History** J. O’Rourke, 15 Aug. 2004.

## References

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<sup>5</sup><http://www.msri.org/publications/video/index07.html>

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- [O’R04] Joseph O’Rourke. Computational geometry column 46. *Internat. J. Comput. Geom. Appl.*, 14(6):475–478, 2004. Also in *SIGACT News*, **35**(3):42–45 (2004), Issue 132.

## Problem 59: Most Circular Partition of a Square

**Statement** What is the optimal partition of a square into convex pieces such that the circularity of the pieces is optimized? The *circularity* of a polygon is the ratio of the radius of its smallest circumscribing circle to the radius of its largest inscribed circle. Thus circular pieces have circularity near 1, and noncircular pieces have circularity greater than 1. An optimal partition minimizes the maximum ratio over all pieces in the partition.

**Origin** [DO03a]

**Status/Conjectures** Open.

**Motivation** This is a type of “fat” partition.

**Partial and Related Results** It is known from [DO03a] that the equilateral triangle requires an infinite number of pieces to achieve the optimal circularity of 1.5, and that for all regular  $k$ -gons, for  $k \geq 5$ , the one-piece partition is optimal. The square is a difficult intermediate case. It is known that the optimal ratio lies in the narrow interval [1.28868, 1.29950]. The upper bound is established by the 92-piece partition shown in Figure 2. It is conjectured in [DO03a] that, as with the equilateral triangle case, no finite partition achieves the optimal ratio, but rather optimality can be approached as closely as desired as the number of pieces goes to infinity.

**Categories** packing; meshing

**Entry Revision History** J. O’Rourke, 16 Aug. 2004.

## References

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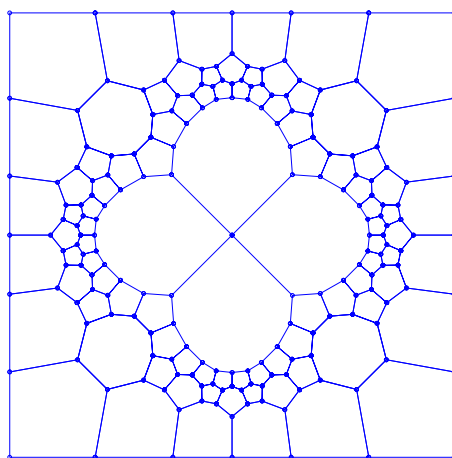


Figure 2: 92-piece partition achieving 1.29950

## Problem 60: Transforming Polygons via Vertex-Centroid Moves

**Statement** Given an arbitrary polygon, transform it by a finite sequence of “vertex-centroid” moves to a regular polygon. A *vertex-centroid move* is a translation of a vertex  $v$  along the line  $vm$ , where  $m$  is the centroid of the vertices of the polygon, i.e.,  $1/n$ -th of the sum of the vertex coordinates. Vertices may move only one at a time, but in any order and any number of times.

**Origin** Steve Gray, 2003.

**Status/Conjectures** Open.

**Partial and Related Results** Let  $v(t)$  and  $m(t)$  be the positions of the moving vertex and centroid as a function of time  $t$ , where  $t$  runs from 0 to 1 during the vertex translation. Let  $L$  be the line containing  $v(0)m(0)$ . As  $v(t)$  moves on  $L$ ,  $m(t)$  remains on  $L$ .

For  $n = 3$ , a triangle can be made equilateral in two moves. Already for  $n = 4$  the situation is less clear.

One could set many other transformational goals besides achieving regularity: scale the polygon by  $s > 0$ , rotate the polygon, etc. The notion generalizes to arbitrary dimensions.

A more difficult variant would be to use the area centroid rather than the vertex centroid, in which case  $m(t)$  does not remain on  $L$ , so that a vertex move would have the flavor of pursuit of a moving target.

**Appearances**

**Categories** polygons

**Entry Revision History** J. O'Rourke, 1 Aug. 2005; S. Gray, 15 Aug. 2005.

## Problem 61: Lines Tangent to Four Unit Balls

**Statement** Given a set of  $n$  unit-radius balls in  $\mathbb{R}^3$ , what is the number of lines that are tangent to four of the balls in the set, and miss all the others? (The balls are not necessarily disjoint.)

**Origin** [AAKS05].

**Status/Conjectures** Open, conjectured to be  $\Omega(n^3)$ .

**Motivation** The number of lines tangent to four unit balls dominates the combinatorial complexity of the space of lines that avoid all the balls. And this complexity is related to questions in visibility and in optimization.

**Partial and Related Results** In [AAKS05] it is established that the number is  $O(n^{3+\epsilon})$  for any  $\epsilon > 0$ . The best lower bound is  $\Omega(n^2)$ , which can be achieved, for example, as follows.

Place  $n/4$  balls separated along a horizontal line  $L_1$ , and another  $n/4$  along a parallel line  $L_2$  below, with each of the lower balls directly below an upper ball with their centers 1 unit apart. Thus each pair of balls overlap, their surfaces intersecting in a circle. Arrange a second set of  $n/4$  pairs of intersecting balls along lines  $L_3$  and  $L_4$ , far from  $L_1/L_2$  and with all four lines parallel, and such that all circles of sphere intersections are coplanar. Now it is easy to see that a line tangent to two circles of intersection, one from the  $L_1/L_2$  group, one from the  $L_3/L_4$  group, is tangent to four balls. And there are  $\Omega(n^2)$  such lines. (The same bound can be achieved with disjoint balls with a similar arrangement, but the analysis is slight more complex.)

The problem is also interesting if all balls are disjoint; it is not clear if disjointness affects the answer asymptotically.

**Appearances** [AAKS05].

**Categories** combinatorial geometry

**Entry Revision History** J. O'Rourke, 25 Aug. 2005.

## References

- [AAKS05] Pankaj K. Agarwal, Boris Aronov, Vladlen Koltun, and Micha Sharir. Lines avoiding unit balls in three dimensions. *Discrete Comput. Geom.*, 34:231–250, 2005.

## Problem 62: Volume Maximizing Convex Shape

**Statement** Let  $C$  be a convex piece of paper; its boundary may be a smooth curve, or a polygon. A *perimeter halving folding* is a folding of  $C$  obtained by identifying two points  $x$  and  $y$  on the boundary of  $C$  that halve the perimeter, and then folding  $C$  by “gluing”  $xy$  to  $yx$ . This always results in a unique convex shape in 3D, a polyhedron if  $C$  is a convex polygon [DO07]. What unit-area shape  $C$  achieves the maximum volume possible via a perimeter-halving folding?

**Origin** Posed by Joseph Malkevitch in 2002, in a slightly different form: for polygons, and not restricting the folding to perimeter-halving. The modifications above were suggested at CCCG’05 [DO06]. The restriction to perimeter halving eliminates some more complex foldings possible for some convex polygons, and so in that sense simplifies the problem. The extension to smooth shapes is a natural generalization. Smooth shapes only admit perimeter-halving foldings.

**Status/Conjectures** Open.

**Partial and Related Results** Even fixing the shape and finding the maximum volume perimeter halving for that shape is difficult. For a circular disk, all perimeter halvings lead to a flat doubly-covered half disk, all of volume zero. The only other shape for which the answer is known, and then only empirically, is the case of  $C$  a square [ADO03]. The resulting polyhedron of 6 vertices and 8 faces, shown in Fig. 3, achieves about 60% of the volume of a unit-area sphere.

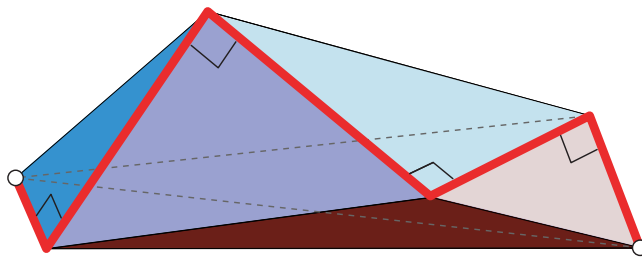


Figure 3: The maximum volume convex polyhedron foldable from a square.

**Appearances** [DO06].

**Categories** folding and unfolding

**Entry Revision History** J. O’Rourke, 26 Aug. 2005.

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## Problem 63: Dynamic Planar Nearest Neighbors

**Statement** Is there a data structure maintaining a set of  $n$  points in the plane subject to insertions, deletions, and nearest-neighbor queries in  $O(\log n)$  time? A *nearest-neighbor query* asks to find a point among the set that is nearest (in Euclidean distance) to a given a point in the plane. This problem reduces to maintaining the convex hull of a set of  $n$  points in 3D subject to insertions, deletions, and extreme-point queries.

**Origin** Uncertain, pending investigation.

**Status/Conjectures** Open.

**Motivation** This problem is the natural generalization of (1D) search trees to 2D. Standard balanced search tree data structures can maintain  $n$  points on the real line subject to insertion, deletion, and predecessor and successor queries (and thus nearest-neighbor queries) in  $O(\log n)$  time per operation. (More sophisticated data structures even attain  $O(1)$  time per update.)

**Partial and Related Results** For 14 years, the authority on this problem was Agarwal and Matoušek’s FOCS’92 paper [AM95] which describes two data structures: one supports updates in  $O(n^\epsilon)$  amortized time and queries in  $O(\log n)$  worst-case time, while the other supports updates in  $O(\log^2 n)$  amortized time queries in  $O(n^\epsilon)$  worst-case time, for any  $\epsilon > 0$ . The nearest-neighbor problem is a decomposable search problem, so when deletions are forbidden, the general techniques of Bentley and Saxe [BS80] yield an  $O(\log^2 n)$  amortized bound for updates and queries. In 2006, Chan [Cha06] obtained the first polylogarithmic data structure for 3D convex hulls and therefore 2D nearest neighbors. His data structure supports insertions in  $O(\log^3 n)$  expected amortized time, deletions in  $O(\log^6 n)$  expected amortized time, and extreme-point or nearest-neighbor queries in  $O(\log^2 n)$  worst-case time.



**Related Open Problems** Problem 1.

**Categories** convex hulls; data structures; Voronoi diagrams

**Entry Revision History** E. Demaine, 24 Jan. 2006.

## References

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## Problem 64: Edge-Unfolding Polycubes

**Statement** Is there any genus-zero orthogonal polyhedron  $P$  built by gluing together cubes face-to-face that cannot be edge-unfolded, where all cube edges on the surface of  $P$  are considered edges available for cutting? These orthogonal polyhedra are sometimes known as *polycubes*, 3D versions of 2D *polyominoes*.

**Origin** George Hart and Joseph O’Rourke, 2004.

**Status/Conjectures** Open.

**Motivation** More general problems seem even more difficult.

**Partial and Related Results** This is a special case of a more general problem, which is equally open. The goal, as in Problem 1, is to cut the surface and unfold without overlap. An *edge unfolding* only permits cutting along edges of the polyhedron. A *grid unfolding* adds extra edges to the surface by intersecting the polyhedron with planes parallel to coordinate planes through every vertex, and so is easier to edge-unfold. Easier still is the posed problem: The orthogonal polyhedron is built from cubes, and all cube edges are available for cutting. Is there any such polyhedron that cannot be edge-unfolded? Such an example would narrow the options, but it may be that every orthogonal polyhedron can be grid-unfolded. (An easy box-on-box example [BDD<sup>+</sup>98] shows that without some surface refinement [DO05], not all orthogonal polyhedra can be edge-unfolded.) The posed question is among the most specific whose answer would make progress.

Only a few narrow subclasses of orthogonal polyhedra are known to have grid-unfolding algorithms: orthotubes, orthostacks of orthogonally convex slabs, and orthogonal terrains. See [O'R08].

**Related Open Problems** Problem 1: Edge-Unfolding Convex Polyhedra.

Problem 1: Vertex-Unfolding Polyhedra.

Problem 1: General Unfolding of Nonconvex Polyhedra.

**Appearances** [DO07]

**Categories** folding and unfolding; polyhedra

**Entry Revision History** J. O'Rourke, 14 Jul 2006, 16 Jul 2007.

## References

- [BDD<sup>+</sup>98] Therese Biedl, Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Joseph O'Rourke, Mark Overmars, Steve Robbins, and Sue Whitesides. Unfolding some classes of orthogonal polyhedra. In *Proc. 10th Canad. Conf. Comput. Geom.*, pages 70–71, 1998. Full version in *Elec. Proc.*: <http://cgm.cs.mcgill.ca/cccg98/proceedings/cccg98-biedl-unfolding.ps.gz>.
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## Problem 65: Magic Configurations

**Statement** Let a finite set of points  $P$  in the plane be given, with each point assigned a positive real weight.  $P$  is called a *magic configuration* if every line determined by two or more points has the same sum of weights, i.e., the sum of the weights of the points through which each line passes is the same. The problem is to prove or disprove that there are only four essentially distinct magic configurations:

1. Points in general position, with (e.g.) every point assigned weight 1.
2. All points collinear.

3.  $n - 1$  points collinear with weight (e.g.) 1, and one point not on that line with weight  $n - 2$ .
4. The 7-point configuration shown in Fig. 4, or its projective equivalents.

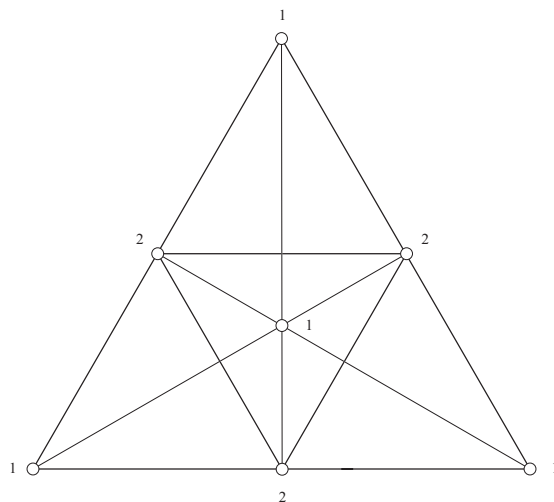


Figure 4: Edge midpoints have weight 2, while all other points have weight 1. All nine lines have sum 4.

**Origin** [Mur71]

**Status/Conjectures** Settled positively, 2007: [ABK<sup>+</sup>08]

**Motivation** The terminology “magic configuration” comes from the notion of *magic squares*, 2D matrices such that every row, column, and (optionally) diagonal sums to the same value.

**Partial and Related Results** An *ordinary line* is one that passes through exactly two points. Scaling weights of a magic configuration so that the weights on each line sum to 1, the weight of the points on ordinary lines must be  $\frac{1}{2}$  in any magic configuration other than the third example above. It is known that, for  $n \geq 3$  noncollinear points, at least  $\frac{6}{13}n$  lines must be ordinary [CS93].

Settled in [ABK<sup>+</sup>08], the journal version of a paper that originally appeared in the Proceedings of the 2007 Symposium on Computational Geometry.

**Appearances** Originally posed by U. S. R. Murty in [Mur71]. Reposed by Murty at a June 2006 celebration of V. Chvátal’s 60th birthday. Two people who heard this posing, X. Chen and P. Taslakian, brought the problem

to the conference *Discrete and Computational Geometry—Twenty Years Later* in Snowbird, June 2006. In particular, Chen posed the problem at the open-problem session.

**Categories** point sets

**Entry Revision History** J. O’Rourke, 14 Jul. 2006; E. Demaine, 15 Jul. 2006; J. O’Rourke, 16 Jul. 2008.

## References

- [ABK<sup>+</sup>08] Eyal Ackerman, Kevin Buchin, Christian Knauer, Rom Pinchasi, and Günter Rote. There are not too many magic configurations. *Discrete Comput. Geom.*, 39:3–16, 2008.
- [CS93] J. Csima and E. T. Sawyer. There exist  $6n/13$  ordinary points. *Discrete & Computational Geometry*, 9(1):187–202, December 1993.
- [Mur71] U. S. R. Murty. How many magic configurations are there?. *American Mathematical Monthly*, 78(9):1000–1002, November 1971.

## Problem 66: Reflexivity of Point Sets

**Statement** Let  $\rho(S)$  be the fewest number of reflex vertices in a polygonization of a 2D point set  $S$ , i.e., the fewest reflexivities of any simple polygon whose vertex set is  $S$ . Let  $\rho(n)$  be the maximum of  $\rho(S)$  over all sets  $S$  with  $n$  points. What is  $\rho(n)$ ?

**Origin** [AFH<sup>+</sup>03]

**Status/Conjectures** Open.

**Partial and Related Results** In [AFH<sup>+</sup>03] the authors prove that  $\lfloor n/4 \rfloor \leq \rho(n) \leq \lceil n/2 \rceil$  and conjecture that  $\rho(n) = \lfloor n/4 \rfloor$ . The upper bound was recently improved to  $\frac{5}{12}n + O(1) \approx 0.4167n$  in [AAK09].

**Related Open Problems** Problem 1: Simple Polygonalizations.

**Categories** polygons; point sets.

**Entry Revision History** J. O’Rourke, 3 Aug. 2006; 16 Jul 2008.

## References

- [AAK09] Eyal Ackerman, Oswin Aichholzer, and Balazs Keszegh. Improved upper bounds on the reflexivity of point sets. *Comput. Geom.: Theory Appl.*, 42(3), April 2009.

- [AFH<sup>+</sup>03] Esther M. Arkin, Sándor P. Fekete, Ferran Hurtado, Joseph S. B. Mitchell, Marc Noy, Vera Sacristán, and Saurabh Sethia. On the reflexivity of point sets. In B. Aronov, S. Basu, J. Pach, and M. Sharir, editors, *Discrete and Computational Geometry: The Goodman-Pollack Festschrift*, pages 139–156. Springer, 2003.

## Problem 67: Fair Partitioning of Convex Polygons

**Statement** Define a *fair partitioning* of a polygon as a partition of it into a finite number of pieces so that every piece has both the same area and the same perimeter. If all the resulting pieces are convex, call it a *fair convex partitioning*. Given any positive integer  $n$ , can any convex polygon be convex fair partitioned into  $n$  pieces?

If the answer is “Not always,” how does one decide the possibility of such a partitioning for a given polygon and a given  $n$ ? And if a fair convex partition exists for a specific polygon, how does one find a fair partitioning that minimizes the total length of the cut segments, or minimizes the sum of the perimeters of the pieces?

And finally, what could one say about higher dimensional analogs of this question?

**Origin** Posed by R. Nandakumar and N. Ramana Rao, June 2007.

**Status/Conjectures** Open. The originators tend to believe every convex polygon allows a fair convex partition into  $n$  pieces for any  $n$ . There have been recent advances in 2010: Aronov and Hubard [AH10], and independently Karasev [Kar10], have established the conjecture for any prime power,  $n = p^k$ .

**Partial and Related Results** See [NR08] for an introduction and survey and proof that the conjecture holds for  $n = 2$ , and a proof for  $n = 4$ . The conjecture has been established for  $n = 3$  in [BBS10].

There is work on partitioning convex polygons into equal area convex pieces so that every piece equally shares the boundary of the given target polygon: [ANRCU98] [AKK<sup>+</sup>98].

A proof of a weaker result—that any polygon allows fair partitioning for any  $n$  (where the pieces need not be convex) is proposed at <http://nandakumar.blogspot.com/2006/10/cutting-shapes-ii.html>.

**Categories** polygons; partitioning

**Entry Revision History** R. Nandakumar and N. Ramana Rao, 14 Jul 2007, 17 Sep 2007; J. O’Rourke, 1 Jan 2009; 23 Jan 2009; 30 Dec 2010, 16 Oct 2020.

## References

- [AH10] Boris Aronov and Alfredo Hubard. Convex equipartitions of volume and surface area. <http://arxiv.org/abs/1010.4611>, October 2010.
- [AKK<sup>+</sup>98] Jin Akiyama, A. Kaneko, M. Kano, Gisaku Nakamura, Eduardo Rivera-Campo, S. Tokunaga, and Jorge Urrutia. Radial perfect partitions of convex sets in the plane. In *Japan Conf. Discrete Comput. Geom.*, pages 1–13, 1998.
- [ANRCU98] Jin Akiyama, Gisaku Nakamura, Eduardo Rivera-Campo, and Jorge Urrutia. Perfect divisions of a cake. In *Proc. Canad. Conf. Comput. Geom.*, pages 114–115, 1998.
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## Problem 68: Rolling a Die over a Labeled Board

**Statement** Label the faces of a unit cube with numbers 1–6 as in a die. Place the cube to sit on an integer lattice grid, with one corner at the origin and sides aligned with the axes. Completely label every lattice square of a rectangular “board”  $R$ , whose corner is at the origin, with numbers in  $\{1, 2, 3, 4, 5, 6\}$ . The problem is to roll the cube over its edges so that, for each square  $s \in B$  labeled  $l$ , the cube lands on  $s$  precisely once, and when it does so, the top face of the cube has label  $l$ .

What is the computational complexity of solving an instance of this problem?

**Origin** Version posed by O’Rourke at the 2005 *Canadian Conference on Computational Geometry* [DO06], and subsequently substantially developed and embellished in [BBD<sup>+</sup>07].

**Status/Conjectures** Open.

**Motivation** This problem was inspired by van Deventer’s “Rolling block mazes” [vD04]. The paper [BBD<sup>+</sup>07] uncovered a rich history to rolling cube puzzles going back to the 1960’s, which will not be repeated here.

**Partial and Related Results** The original posed problem labeled an arbitrary connected set  $S$  of squares, rather than a rectangular board  $R$ ; the cells outside of  $S$  are *free*, and may be visited any number of times with any number on the die top. That former problem is solved in [BBD<sup>+</sup>07], which establishes that, as conjectured, the problem is NP-complete.

The posed problem has no free cells, and in fact the labels are all in a rectangular board  $R$ . This seems the most interesting specific variant, for it is left possible in [BBD<sup>+</sup>07] that, if there is a solution for  $R$ , it is “uniquely rollable.” They establish that there are boards with labeled and *blocked* (i.e., forbidden) cells for which rollable Hamiltonian cycles are not unique, but they leave open fully labeled boards. The uniquely-rollable conjecture is settled for all boards with side lengths at most 8.

**Appearances** [DO06]; see above.

**Categories** combinatorial geometry

**Entry Revision History** J. O’Rourke, 17 Jul 2007; 2 Feb 2012.

## References

- [BBD<sup>+</sup>07] Kevin Buchin, Maïke Buchin, Erik D. Demaine, Martin L. Demaine, Dania El-Khechen, Sandor Fekete, Christian Knauer, André Schulz, and Perouz Taslakian. On rolling cube puzzles. In *Proc. 19th Canad. Conf. Comput. Geom.*, pages 141–148, 2007.
- [DO06] Erik D. Demaine and Joseph O’Rourke. Open problems from CCCG 2005. In *Proc. 18th Canad. Conf. Comput. Geom.*, pages 75–80, 2006.
- [vD04] M. Oskar van Deventer. Rolling block mazes. In Barry Cipra, Erik D. Demaine, Martin L. Demaine, and Tom Rodgers, editors, *A Tribute to a Mathematician*, pages 241–250. A K Peters, November 2004.

## Problem 69: Isoceles Planar Graph Drawing

**Statement** Given a planar graph that is interior triangulated (all interior faces are triangles), is there a straight-line drawing of the graph such that each face is an isoceles triangle (i.e., it has two equal-length sides)?

The problem is worth studying both when the drawing must be planar (no crossings allowed) and when it is not.

If such drawings exist, then it is also worth studying what grid-size is needed, and whether it can be done with integer coordinates at all. If such drawings do not always exist, NP-hardness should be investigated.

**Origin** Joe Malkevitch at Graph Drawing '99.

**Status/Conjectures** Settled negatively in 2010: [Fra10].

**Partial and Related Results** If the graph is a planar 3-tree (i.e., can be obtained by starting from a triangle and repeatedly adding a vertex of degree 3 inside a face, adjacent to all other vertices in the face), then such a drawing can easily be obtained by always placing the vertex at the centroid of the face. However, this drawing will in general be non-planar. Of particular interest therefore, are planar graphs of treewidth 4 and higher.

The problem was solved negatively in [Fra10]: *Theorem*: “There exists an infinite class of maximal planar graphs that admit no isosceles planar drawing.” Frati raises the new question of whether or not every triangulation admits a possibly nonplanar isosceles drawing.

**Categories** graph drawing; planar graphs

**Entry Revision History** T. Biedl, 2 Dec. 2008; J. O’Rourke, 29 Dec. 2008; J. Malkevitch, 10 Jul 2011.

## References

[Fra10] F. Frati. A note on isosceles planar graph drawing. *Information Processing Letters*, 110(12-13):507–509, 2010.

## Problem 70: Yao-Yao Graph a Spanner?

**Statement** Is the Yao-Yao Graph a  $t$ -spanner for constant  $t$ ? A geometric graph is a  $t$ -spanner (or just a *spanner*) if, for every pair of nodes, the shortest distance between the nodes following the edges of the graph is at most  $t$  times the Euclidean distance between them. See below for the definition of the Yao-Yao graph.

**Origin** [WL02](?)

**Status/Conjectures** Open.

**Partial and Related Results** The Yao graph  $Y_k$  [Yao82] is defined as follows. At each node  $u$ , any  $k$  equally-separated rays originated at  $u$  define  $k$  cones. In each cone, choose the shortest edge  $uv$  among all edges from  $u$ , if there are any, and add a directed edge  $\vec{uv}$  to  $Y_k$ . It is known that for the undirected  $Y_k$ ,  $k = 4, k \geq 6$ ,  $Y_k$  a  $t$ -spanner (e.g., see [BDD<sup>+</sup>10]). The Yao-Yao graph  $YY_k$  [WL02] starts with the directed Yao graph, and reduces the maximum degree of nodes as follows. At each node  $u$ , all *incoming* edges from each cone are discarded, except for the shortest one  $\vec{vu}$ . And



now the result is treated as an undirected graph. Many properties of  $YY_k$  have been established, but whether or not  $YY_k$  is a  $t$ -spanner remains open.

**Categories** spanners; geometric graphs

**Entry Revision History** J. O’Rourke, 29 Dec. 2008.

## References

- [BDD<sup>+</sup>10] Prosenjit Bose, Mirela Damian, Karim Douieb, Joseph O’Rourke, Ben Seamone, Michiel Smid, and Stefanie Wurher.  $\pi/2$ -angle Yao graphs are spanners. arXiv: 1001.2913v1 [cs.CG], January 2010.
- [WL02] Yu Wang and Xiang-Yang Li. Distributed spanner with bounded degree for wireless ad hoc networks. In *IPDPS ’02: Proc. of the 16th IEEE Int. Parallel and Distributed Processing Symposium*, pages 194–201, 2002.
- [Yao82] A. C. Yao. On constructing minimum spanning trees in  $k$ -dimensional spaces and related problems. *SIAM J. Comput.*, 11(4):721–736, 1982.

## Problem 71: Stretch-Factor for Points in Convex Position

**Statement** For points  $S$  in convex position (i.e., every point is on the hull of  $S$ ), is the Delaunay triangulation of  $S$  a  $(\pi/2)$ -spanner? A geometric graph is a  $t$ -spanner (or just a *spanner*) if, for every pair of nodes, the shortest distance between the nodes following the edges of the graph is at most  $t$  times the Euclidean distance between them. The constant  $t$  is the *stretch factor* or *dilation*.

**Origin** Prosenjit Bose [DO08].

**Status/Conjectures** Now closed: false. [This entry awaiting updating.]

**Partial and Related Results** Chew conjectured that the Delaunay triangulation is a  $t$ -spanner [Che89] for some constant  $t$ . Dobkin et al. [DFS90] established this for  $t = \pi(1 + \sqrt{5})/2 \approx 5.08$ . The value of  $t$  was improved to  $2\pi/(3 \cos(\pi/6)) \approx 2.42$  by Keil and Gutwin [KG92], and further strengthened in [BM04]. Chew showed that  $t$  is  $\pi/2 \approx 1.57$  for points on a circle, providing a lower bound. “It is widely believed that, for every set of points in  $\mathbb{R}^2$ , the Delaunay triangulation is a  $(\pi/2)$ -spanner” [NS07, p. 470].

This history suggests the special case posed above.

There is a new forthcoming result: [CKX09].

**Appearances** [DO08].

**Categories** spanners; Delaunay triangulations

**Entry Revision History** J. O’Rourke, 29 Dec. 2008; 4 July 2009; 1 Apr. 2010.

## References

- [BM04] P. Bose and P. Morin. Online routing in triangulations. *SIAM J. Comput.*, 33:937–951, 2004.
- [Che89] L. P. Chew. There are planar graphs almost as good as the complete graph. *J. Comput. Syst. Sci.*, 39:205–219, 1989.
- [CKX09] Shiliang Cui, Iyad Kanj, and Ge Xia. On the dilation of Delaunay triangulations of points in convex position. In *Proc. Canad. Conf. Comp. Geom.*, pages 161–164, August 2009.
- [DFS90] D. P. Dobkin, S. J. Friedman, and K. J. Supowit. Delaunay graphs are almost as good as complete graphs. *Discrete Comput. Geom.*, 5:399–407, 1990.
- [DO08] Erik D. Demaine and Joseph O’Rourke. Open problems from CCCG 2007. In *Proc. 20th Canad. Conf. Comput. Geom.*, 2008.
- [KG92] J. M. Keil and C. A. Gutwin. Classes of graphs which approximate the complete Euclidean graph. *Discrete Comput. Geom.*, 7:13–28, 1992.
- [NS07] Giri Narasimhan and Michiel Smid. *Geometric Spanner Networks*. Cambridge University Press, 2007.

## Problem 72: Polyhedron with Regular Pentagon Faces

**Statement** Let  $M$  be a closed polyhedral surface homeomorphic to  $S^2$  which is entirely composed of equal regular pentagons. If  $M$  is immersed in 3-space, is it necessarily the boundary of a union of solid dodecahedra that are glued together at common facets?

**Origin** Richard Kenyon, first posed in 2006.

**Status/Conjectures** Open.

**Partial and Related Results** The corresponding question for equal squares has a positive answer. The question for surfaces embedded in 3-space is also interesting and open. The Kepler-Poinsot great dodecahedron has

regular pentagon faces, and is immersed, but is not homeomorphic to  $S^2$  ( $V - E + F = -6$ ).

**Appearances** Re-posed at Oberwolfach Workshop, Jan. 2009.

**Categories** polyhedra

**Entry Revision History** J. O'Rourke, 23 Jan. 2009.

## Problem 73: Congruent Partitions of Polygons

**Statement** Partition a given polygon  $P$  into  $n$  mutually congruent pieces so that the area of  $P$  not covered by the union of the pieces is as small as possible. A partition which leaves out the least area is an *optimal* congruent partition for that  $n$ . If a congruent partition is a perfect cover, leaving no area uncovered, then it is called a *perfect congruent partition*. Two polygons are *congruent* if one can be made to coincide with the other by translation, rotation, or reflection (flipping over).

**Origin** Posed by R. Nandakumar, May 2009.

**Status/Conjectures** Please see below.

**Partial and Related Results** A new introduction to the problem is now available: [Nan10a].

1. It is known that there exist quadrilaterals with no perfect congruent partition for any  $n$ : [http://domino.research.ibm.com/Comm/wwwr\\_ponder.nsf/challenges/December2003.html](http://domino.research.ibm.com/Comm/wwwr_ponder.nsf/challenges/December2003.html).
2. Deciding whether  $P$  has a perfect congruent partition appears little explored for  $n > 2$ . The case of  $n = 2$  is solved in [EKFIR08] with an  $O(n^3)$  algorithm.
3. If congruence is restricted to translation and rotation only, to what extent does the problem change?
4. Can the left-over area be upper-bounded as a function of  $P$  and  $n$ ? An attempt for  $n = 2$  is offered in [Nan10b].

**Related Open Problems** Problem 1

**Categories** polygons; partitioning; dissections

**Entry Revision History** R. Nandakumar, 13 May 2009; J. O'Rourke, 8 July 2009; 5 Jan. 2011.

## References

- [EKFIR08] Dania El-Khechen, Thomas Fevens, John Iacono, and Günter Rote. Partitioning a polygon into two mirror congruent pieces. In *Proc. 20th Canad. Conf. Comput. Geom.*, pages 131–134, August 2008.
- [Nan10b] R. Nandakumar. Cutting mutually congruent pieces from convex regions. <http://arxiv.org/abs/1012.3106>, 2010.
- [Nan10a] R. Nandakumar. ‘Congruent partitions’ of polygons—a short introduction. <http://arxiv.org/abs/1002.0122>, 2010.

## Problem 74: Slicing Axes-Parallel Rectangles

**Statement** Let us say that two rectangles in the plane are independent if both their  $x$ - and  $y$ -axis projections are disjoint. A set of rectangles is then independent if the rectangles are pairwise independent. Suppose that a collection of axes-parallel rectangles contains no independent set of size  $m$  or greater. What is the minimal number,  $f(m)$ , of horizontal and vertical lines needed to slice every rectangle in the collection?

**Origin** Vincent Vatter, Jun 2009.

**Status/Conjectures** It was known that  $f(m)$  exists and is at most exponential. An advance was made in 2010 by Werner and Lenz, who established a quadratic upper bound,  $O(m^2)$ , in [WL10]. They also uncovered a long history of the problem under other names, e.g., “ $d$ -separated interval piercing.” In fact, the result was already established by Tardos and Karolyi earlier. See the cited paper for more details.

But, as pointed out by Pablo Soberón, apparently an earlier result [Kai97, Thm. 1.4], established  $f(m) \leq 2m$ . This largely solves the problem.

**Partial and Related Results** The problem arises in the study of permutation classes, see [Vat08], where it was proved that  $f(m)$  exists and is at most exponential.

**Categories** combinatorial geometry

**Entry Revision History** V. Vatter, 24 June 2009; J.O’Rourke, 16 Mar. 2012; P. Soberón, 3 May 2012.

## References

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- [Vat08] Vincent Vatter. Small permutation classes. arXiv:0712.4006v2 [math.CO], 2008.
- [WL10] Daniel Werner and Matthias Lenz. Polynomial bounds on the rectangle slicing number. *CoRR*, abs/1004.3381, 2010. <http://arxiv.org/abs/1004.3381>.

## Problem 75: Edge-Coloring Geometric Graphs

**Statement** For a set of  $n$  points in the plane in general position, draw a straight segment between every pair of points. What is the minimum number of colors that suffice to color the edges such that no two edges that cross have the same color? (With the general position assumption, all crossings are proper crossings.)

**Origin** Ferran Hurtado: [AGH<sup>+</sup>05].

**Status/Conjectures** Open.

**Partial and Related Results** Each color class determines a plane subgraph of the complete graph. Because one can arrange for  $n/2$  pairwise crossing edges,  $n/2$  is a lower bound. The best upper bound is in [BHRCW06]:  $n - \sqrt{n/12}$ . It has been conjectured that  $(1 - \epsilon)n$  is an upper bound for some  $\epsilon > 0$ .

**Appearances** This description relies on that in the *Open Problem Garden*, written by David Wood. His posting lists several related variants.

**Categories** geometric graphs

**Entry Revision History** J. O'Rourke, 1 Apr. 2010.

## References

- [AGH<sup>+</sup>05] G. Araujo, Adrian Guitrescu, Ferran Hurtado, M. Noy, and J. Urrutia. On the chromatic number of some geometric type Kneser graphs. *Comput. Geom. Theory Appl.*, 32(1):59–69, 2005.
- [BHRCW06] Prosenjit Bose, Ferran Hurtado, Eduardo Rivera-Campo, and David R. Wood. Partitions of complete geometric graphs into plane trees. *Comput. Geom. Theory Appl.*, 34(2):116–125, 2006.

## Problem 76: Equiprojective Polyhedra

**Statement** Identify or construct all  $k$ -equiprojective polyhedra. A polyhedron  $P$  is  $k$ -equiprojective if its orthogonal projection to a plane is a  $k$ -gon in every direction not parallel to a face of  $P$ . Thus a cube is 6-equiprojective.

**Origin** Geoffrey Shephard in [She68].

**Status/Conjectures** Open.

**Partial and Related Results** A characterization is detailed in [HL08]: “A polyhedron is equiprojective iff its set of edge-face pairs can be partitioned into compensating pairs.” For term definitions, see the original paper. Building on this work, a recent paper [HHLO<sup>+</sup>10] establishes that any equiprojective polyhedron has at least one pair of parallel faces, that there is no 3- or 4-equiprojective polyhedron, and the triangular prism is the only 5-equiprojective polyhedron.

**Related Open Problems** A generalization of the problem was posted on MathOverflow, 11Feb11: [O’R11]

**Appearances** Also in [CFG90], Problem B10.

**Categories** polyhedra

**Entry Revision History** J. O’Rourke, 31 Dec. 2010; 11 Feb 2011.

## References

- [O’R11] Joseph O’Rourke. What is determined by the combinatorics of the shadows of a convex polyhedron? <http://mathoverflow.net/questions/55124/>, February 2011.
- [CFG90] H. P. Croft, K. J. Falconer, and R. K. Guy. *Unsolved Problems in Geometry*. Springer-Verlag, 1990.
- [HHLO<sup>+</sup>10] Masud Hasan, Mohammad Houssain, Alejandro Lopez-Oritz, Sabrina Nusrat, Saad Quader, and Nabila Rahman. Some new equiprojective polyhedra. <http://arxiv.org/abs/1009.2252>, 2010.
- [HL08] Masud Hasan and Anna Lubiw. Equiprojective polyhedra. *Comput. Geom. Th. Appl.*, 40(2):148–155, 2008.
- [She68] Geoffrey C. Shephard. Twenty problems on convex polyhedra—II. *Math. Gaz.*, 52:359–367, 1968.

## Problem 77: Zipper Unfoldings of Convex Polyhedra

**Statement** Does every convex polyhedron  $P$  have a zipper unfolding? A *zipper unfolding* cuts open  $P$  via a single path, necessarily a Hamiltonian path (to span all vertices), and unfolds the surface to a non-overlapping polygon in the plane. The segments of the path need not lie along edges of  $P$ .

**Origin** Posed as Open Problem 2 in [DDL<sup>+</sup>10], which introduced the term “zipper unfolding.”

**Status/Conjectures** Open.

**Partial and Related Results** With the restriction that the cuts follow edges, any  $P$  without a Hamiltonian path in its 1-skeleton has no zipper edge-unfolding, e.g., a rhombic dodecahedron. (Such polyhedra have been studied, e.g., in [Bro61].)

**Related Open Problems** Problem 1.

**Categories** polyhedra

**Entry Revision History** J. O’Rourke, 7 Feb. 2012.

## References

- [Bro61] Thomas Brown. Simple paths on convex polyhedra. *Pacific J. Math.*, 11(4):1211–1241, 1961.
- [DDL<sup>+</sup>10] Erik Demaine, Martin Demaine, Anna Lubiw, Arlo Shallit, and Jonah Shallit. Zipper unfoldings of polyhedral complexes. In *Proc. 22nd Canad. Conf. Comput. Geom.*, pages 219–222, August 2010.

## Problem 78: Rectangling a Rectangle

**Statement** Do there exist rectangles that may be partitioned into a finite number  $n$  of rectangular pieces of equal area but with all perimeters different?

**Origin** Posed by R. Nandakumar, Feb. 2012: <http://nandacumar.blogspot.in/2012/02/packing-rectangles.html>. The phrase “rectangling a rectangle” was introduced by Michael Brand at <http://brand.site.co.il/riddles/201203q.html>.

**Status** A partial solution for rectangular pieces with real edge lengths is known—a spiral layout of 7 rectangular pieces forming a larger rectangle. See Brand’s web site. The question remains open for tiling rectangles with rational edge lengths.

**Conjecture** If all edge lengths of the pieces are required to be rational, no such partition is possible (R. Nandakumar and N. Ramana Rao).

**Further questions** The question may be extended to higher dimensions  $d$  in the obvious way. The posers believe there is no solution in  $\mathbb{R}^d$  for  $d \geq 3$ .

**Categories** packing; partitioning.

**Entry Revision History** R. Nandakumar and N. Ramana Rao, Mar. 14, 2012; J. O’Rourke, 15 Mar. 2012; 25 Mar. 2012.

## **Acknowledgments**

Partially supported by NSF grants to the three editors.