On a Glimm – Effros dichotomy theorem for Souslin relations in generic universes

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Abstract

We prove that if every real belongs to a set generic extension of the constructible universe then every $\mathbf{\Sigma}^1_1$ equivalence E on reals either admits a $\mathbf{\Delta}_1$ reduction to the equality on the set $2^{<\omega_1}$ of all countable binary sequences, or continuously embeds E_0, the Vitali equivalence.

The proofs are based on a topology generated by OD sets.

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Introduction

This paper presents a proof of the following theorem:

Theorem 1 [1] *Let* E *be a* $\boldsymbol{\Sigma}^1_1$ *equivalence on reals. Assume that*

(†) *each real belongs to a "virtual" generic extension* [2] *of the constructible universe* L.

Then at least one [3] *of the following two statements hold:*

(I) E *admits a* $\boldsymbol{\Delta}^{\mathrm{HC}}_1$ *reduction* [4] *to the equality on the set* $2^{<\omega_1}$ *of all countable binary sequences.*

(II) $\mathsf{E}_0 \sqsubseteq \mathsf{E}$ *continuously.*

Remarks on the theorem

By a *"virtual" generic extension* L we mean a set generic extension, say, $\mathrm{L}[G]$, which is not necessarily an inner class in the basic universe V (in other words, $G \in \mathrm{V}$ is not assumed). [5]

Notice that the assumption (†) of the theorem follows e. g. from the hypothesis that the universe is a set generic extension of L. In fact the theorem remains true in a weaker assumption that each real x belongs to a "virtual" generic extension of $\mathrm{L}[z_0]$ for one and the same real z_0 which does not depend on x.

We refer the reader to Harrington, Kechris, and Louveau [2] on matters of the early history of "Glimm – Effros" theorems — those of type: *each equivalence of certain class either admits a reduction to equality or embeds* E_0 — and relevant problems in probability and the measure theory. (However Section 1 contains the basic notation.)

The modern history of the topic began in the paper [2] where it is proved that each Borel equivalence on reals either admits a Borel reduction to the equality on reals or embeds E_0. The proof is based on an advanced tool in descriptive set theory, the *Gandy – Harrington topology* on reals, generated by Σ^1_1 sets.

Hjorth and Kechris [4] found that the case of $\boldsymbol{\Sigma}^1_1$ relations is much more complicated. Some examples have shown that one cannot find a reasonable "Glimm – Effros" result

[1] It follows from an e-mail discussion between G. Hjorth and the author in May – July 1995 that G. Hjorth may have proved equal or similar theorem independently.

[2] By a generic extension of some M we always mean a set generic extension via a forcing notion $P \in M$. Here the extensions could be different for different reals.

[3] If all reals are constructible from one of them then the statements are compatible.

[4] By $\boldsymbol{\Delta}^{\mathrm{HC}}_1$ we denote the class of all subsets of HC (the family of all hereditarily countable sets) which are Δ_1 in HC by formulas which may contain reals and countable ordinals as parameters.

[5] The assumption that a set $S \subseteq \mathrm{Ord}$ belongs to a "virtual" set generic extension of L can be adequately formalized as follows: *there exists a Boolean valued extension of* $\mathrm{L}[S]$ *in which it is true that the universe is a set generic extension of the constructible universe*, see Lemma 5 below.

for $\mathbf{\Sigma}^1_1$ relations simply taking a nonBorel reduction in (I) or discontinuous embedding in (II); it seemms that the equality on *reals* rather than countable binary sequences in (I) does not match completely the nature of $\mathbf{\Sigma}^1_1$ relations.

Hjort and Kechris [4] suggested the adequate approach: one has to take $2^{<\omega_1}$ as the domain of the equality in (I). (This approach is referred to as the *Ulm – type classification* in [4], in connection with a classification theorem of Ulm in algebra.) On this way they proved that the dichotomy (I) vs. (II) holds for each $\mathbf{\Sigma}^1_1$ equivalence relation on reals, in the assumption of the "sharps" hypothesis (and the latter can be dropped provided the $\mathbf{\Sigma}^1_1$ relation occasionally has only Borel equivalence classes).

Theorem 1 of this paper establishes the same result (not paying attention on the possible compatibility of (I) and (II)) in the completely different than sharps assumption: each real belongs to a generic extension of L. Of course it is the principal problem (we may refer to the list of open problems in [4]) to eliminate the "forcing" assumption and prove the result in **ZFC**.

One faces much more problems in higher projective classes. In fact there exists a sort of upper bound for "Glimm – Effros" theorems in **ZFC**. Indeed, in a nonwellfounded (of "length" $\omega_1 \times \mathbb{Z}$, i. e. ω_1 successive copies of the integers) iterated Sacks extension [6] of L the Σ^1_2 equivalence

$$x \mathsf{E} y \quad \text{iff} \quad \mathrm{L}[x] = \mathrm{L}[y]$$

neither continuously embeds E_0 nor admits a real–ordinal definable reduction to the equality on $\mathcal{P}(\kappa)$ for a cardinal κ.

Thus the interest can be paid on classes $\mathbf{\Pi}^1_1$, $\mathbf{\Delta}^1_2$, $\mathbf{\Pi}^1_2$. One may expect that $\mathbf{\Delta}^1_2$ relations admit a theorem similar to Theorem 1. [7]

More complicated relations can be investigated in strong extensions of **ZFC** or in special models. Hjorth [3] proved that in the assumption of **AD** and $\mathrm{V} = \mathrm{L}[\text{reals}]$ every equivalence on reals either admits a reduction (here obviously a real–ordinal definable reduction) to the equality on a set 2^κ, $\kappa \in \mathrm{Ord}$, or continuously embeds E_0. Kanovei [6] proved even a stronger result (reduction to the equality on $2^{<\omega_1}$) in Solovay model for $\mathbf{ZF} + \mathbf{DC}$.

The organization of the proof

Theorem 1 is the main result of this paper. The proof is arranged as follows.

First of all, we shall consider only the case when E is a lightface Σ^1_1 relation; if in fact E is $\Sigma^1_1(z)$ in some $z \in \mathcal{N}$ then this z simply enters the reasoning in a uniform way, not influenting substantially any of the arguments.

The splitting point between the statements (I) and (II) of Theorem 1 is determined in Section 1. It occurs that we have (I) in the assumption that

[6] See Groszek [1] or Kanovei [7] on matters of nonwellfounded Sacks iterations.

[7] G. Hjorth informed the author that he had partial results in this domain.

($\ddagger$) each real x belongs to a "virtual" λ-collapsing generic extension of L (for some ordinal λ) in which E is closed in a topolody generated by OD sets on the set $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ of all reals λ-weak over L. (We say that $x \in \mathcal{D}$ is λ-*weak over* L iff it belongs to a α-collapsing extension of L for some $\alpha < \lambda$.)

On the opposite side, we have (II) provided the assumption ($\ddagger$) fails.

Both sides of the proof depend on properties of reals in collapsing extensions close to those of Solovay model. The facts we need are reviewed in Section 2.

Section 3 proves assertion (I) of Theorem 1 assuming ($\ddagger$). The principal idea has a semblance of the corresponding parts in [2] and especially [4] [8] : in the assumption of ($\ddagger$), each λ-weak over L real in the relevant "virtual" λ-collapsing extension belongs to a set (one and the same for all E-equivalent reals) which admits a characterization in terms of an element of $2^{<\omega_1}$. An absoluteness argument allows to extend this fact to the universe of Theorem 1.

Sections 4 and 5 prove (II) of Theorem 1 in the assumption that ($\ddagger$) *fails* (but ($\dagger$) still holds, as Theorem 1 assumes). In fact is this case E is *not* closed on the set $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ in a "virtual" λ-collapsing extension of L for some λ. This suffices to see that E embeds E_0 continuously in the "virtual" universe; moreover, E embeds E_0 in a certain special sense which can be expressed by a Σ^1_2 formula (unlike the existence of an embedding in general which needs Σ^1_3). We conclude that E embeds E_0 in the universe of Theorem 1 as well by Shoenfield.

The construction of the embedding of E_0 into E follows the principal idea of Harrington, Kechris, and Louveau [2], yet associated with another topology and arranged in a different way. (In particular we do not play the strong Choquet game to define the necessary sequence of open sets.)

Important remark

It will be more convenient to consider $\mathcal{D} = 2^\omega$, the *Cantor space*, rather than $\mathcal{N} = \omega^\omega$, as the basic Polish space for which Theorem 1 is being proved.

[8] Yet we use a technique different from the approach of [4], completely avoiding any use of recursion theory.

1 Approach to the proof of the main theorem

First of all, we shall prove only the "lightface" case of the theorem, so that E will be supposed to be a Σ^1_1 equivalence on reals. The case when E is $\Sigma^1_1[z]$ for a real z does not differ much: the z uniformly enters the reasoning.

By "reals" we shall understand points of the *Cantor set* $\mathcal{D} = 2^\omega$ rather than the *Baire space* $\mathcal{N} = \omega^\omega$; this choice is implied by some technical reasons.

The purpose of this section is to describe how the two cases of Theorem 1 will appear. This needs to recall some definitions.

1–A Collapsing extensions

Let α be an ordinal. Then $\alpha^{<\omega}$ is the forcing to collapse α down to ω. If $G \subseteq \alpha^{<\omega}$ is $\alpha^{<\omega}$-generic over a transitive model M (M is a set or a class) then $f = \bigcup G$ is a function from ω onto α, so that α is countable in $M[G] = M[f]$. Functions $f : \omega \longrightarrow \alpha$ obtained this way will be called $\alpha^{<\omega}$*-generic over* M .

By λ*-collapse universe hypothesis*, λ-CUH in brief, we shall mean the following assumption: $\mathrm{V} = \mathrm{L}[f_0]$ for a $\lambda^{<\omega}$-generic over L collapse function $f_0 \in \lambda^\omega$.

By the assumption of Theorem 1, each real z belongs to a "virtual" $\lambda^{<\omega}$-generic extension of L, the constructible universe, for some ordinal λ. Such an extension satisfies λ-CUH.

Remark 2 The extension is not necessarily supposed to be an inner class in the universe of Theorem 1, see Introduction. □

A set is λ*-weak over* M (λ an ordinal in a model M) iff it belongs to a "virtual" $\alpha^{<\omega}$-generic extension of M for some $\alpha < \lambda$. We define

$$\mathsf{Weak}_\lambda(M) = \{x : x \text{ is } \lambda\text{-weak over } M\}\,.$$

In the assumption λ-CUH, reals in $\mathsf{Weak}_\lambda(\mathrm{L})$ behave approximately like all reals in Solovay model.

1–B The OD topology

In **ZFC**, Let $\mathcal{T}$ be the topology generated on a given set X (for instance, $X = \mathcal{D} = 2^\omega$, the Cantor set) by all OD subsets of X. $\mathcal{T}^2$ is the product of two copies of $\mathcal{T}$, a topology on $\mathcal{D}^2$.

This topology plays the same role in our consideration as the Gandy – Harrington topology in the proof of the classical Glimm – Effros theorem (for Borel relations) in Harrington, Kechris, and Louveau [2]. In particular, it has similar (although not completely similar: some special Σ^1_1-details vanish) properties.

We define $\overline{\mathsf{E}}$ to be the $\mathcal{T}^2$-closure of E in $\mathcal{D}^2$. Thus $x \,\overline{\not\mathsf{E}}\, y$ iff there exist OD sets X and Y containing resp. x and y and such that $x' \not\mathsf{E}\, y'$ for all $x' \in X$, $y' \in Y$. Obviously X and Y can be chosen as E-invariant (simply replace them by their E-saturations), and then Y can be replaced by the complement of X, so that

$$x \,\overline{\mathsf{E}}\, y \;\longleftrightarrow\; \forall\, X\, [\, X \text{ is OD } \& \ X \text{ is } \mathsf{E}\text{-invariant} \;\longrightarrow\; (x \in X \;\longleftrightarrow\; y \in X)\,]\,.$$

Therefore $\overline{\mathsf{E}}$ is an OD equivalence on $\mathcal{D}$.

1–C The cases

In [2], the two cases are determined by the equality $\mathsf{E} = \overline{\mathsf{E}}$: if it holds that E admits a Borel reduction on $\Delta(\mathcal{D})$, otherwise E embeds E_0. Here the splitting condition is a little bit more complicated. First of all, we have to consider the equality in different universes. Second, the essential domain of the equivalence is now a proper subset of $\mathcal{D}$, the set of all weak reals.

Case 1. For each real z, there exist an ordinal λ and a "virtual" $\lambda^{<\omega}$-generic extension V of the constructible universe L containing z such that the following is true in V : E coincides with $\overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ and x is λ-weak over L.

(Notice that, for a Σ^1_1 binary relation E, the assertion that E is an equivalence is Π^1_2, therefore absolute for all models with the same ordinals, in particular for L and all generic extensions of L.)

Case 2. Not Case 1.

Theorem 3 *Suppose that each real belongs to a "virtual" generic extension of* L. *Then, for the given* Σ^1_1 *equivalence relation* E, *we have*

- *assertion* (I) *of Theorem 1 in Case 1,* *and*
- *assertion* (II) *of Theorem 1 in Case 2.*

This is how Theorem 1 well be proved.

2 On collapsing extensions

In this section, we fix a limit constructible cardinal λ. The purpose is to establish some properties of λ-collapsing generic extensions (= the universe under the hypothesis λ-CUH). It will be shown that weak ponts (introduced in Section 1) behave approximately like all reals in Solovay model.

2–A Basic properties

We recall that a set S is *λ-weak over M* iff S belongs to an $\alpha^{<\omega}$-generic extension of the model M for some $\alpha < \lambda$.

The hypothesis λ-CUH (the one which postulates that the universe is a λ-generic extension of L) will be assumed during the reasoning, but we shall not mind to specify λ-CUH in all formulations of theorems.

Proposition 4 *Assume λ-CUH. Let $S \subseteq \mathrm{Ord}$ be λ-weak over L. Then*

1. *The universe V is a $\lambda^{<\omega}$-generic extension of $\mathrm{L}[S]$.*
2. *If Φ is a sentence containing only sets in $\mathrm{L}[S]$ as parameters then Λ (the empty sequence) decides Φ in the sense of $\lambda^{<\omega}$ as a forcing notion over $\mathrm{L}[S]$.*
3. *If a set $X \subseteq \mathrm{L}[S]$ is $\mathrm{OD}[S]$ then $X \in \mathrm{L}[S]$.*

($\mathrm{OD}[S]$ = *S–ordinal definable*, that is, definable by an $\in$-formula having S and ordinals as parameters.) The proof (a copy of the proof of Theorem 4.1 in Solovay [10]) is based on several lemmas, including the following crucial lemma:

Lemma 5 *Suppose that $P \in \mathrm{L}$ is a p.o. set, and a set $G \subseteq P$ is P-generic over L. Let $S \in \mathrm{L}[G]$, $S \subseteq \mathrm{Ord}$. Then there exists a set $\Sigma \subseteq P$, $\Sigma \in \mathrm{L}[S]$ such that $G \subseteq \Sigma$ and G is Σ-generic over $\mathrm{L}[S]$.*

Proof of the lemma. We extract the result from the proof of Lemma 4.4 in [10].

We argue in $\mathrm{L}[S]$.

Let $\underline{S}$ be the name for S in the language of the forcing P.

Define a sequence of sets $A_\alpha \subseteq P$ $(\alpha \in \mathrm{Ord})$ by induction on α.

(A1) $p \in A_0$ iff either $\sigma \in S$ but p forces (in L and in the sense of P as the notion of forcing) $\sigma \notin \underline{S}$, or $\sigma \notin S$ but p forces $\sigma \in \underline{S}$ — for some $\sigma \in \mathrm{Ord}$.

(A2) $p \in A_{\alpha+1}$ iff there exists a dense set $D \subseteq P$, $D \in \mathrm{L}$ such that every $q \in D$ satisfying $p \leq q$ (means: q is stronger than p) belongs to A_α.

(A3) If α is a limit ordinal then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

The following properties of these sets are easily verifiable (see Solovay [10]): first, if $p \in A_\alpha$ and $p \le q \in P$ then $q \in A_\alpha$, second, if $\beta < \alpha$ then $A_\beta \subseteq A_\alpha$.

Since each A_α is a subset of P, it follows that $A_\delta = A_{\delta+1}$ for some ordinal δ. We put $\Sigma = P \setminus A_\delta$. Thus Σ intends to be the set of all conditions $p \in P$ which do not force something about $\underline{S}$ which contradicts the factual information about S.

We prove, following [10], that Σ is as required. This yields a pair of auxiliary facts.

(Σ1) $G \subseteq \Sigma$.

Indeed assume on the contrary that $G \cap A_\gamma \ne \emptyset$ for some γ. Let γ be the least such an ordinal. Clearly γ is not limit and $\gamma \ne 0$; let $\gamma = \alpha + 1$. Let $p \in A_\gamma \cap G$. Since G is generic, definition (A2) implies $G \cap A_\alpha \ne \emptyset$, contradiction.

(Σ2) *If* $D \in \mathrm{L}$ *is a dense subset of* P *then* $D \cap \Sigma$ *is a dense subset of* Σ.

This is easy: if $p \in \Sigma$ then $p \notin A_{\delta+1}$; hence by (A2) there exists $q \in D \setminus A_\delta$, $q \ge p$.

We prove that G is Σ-generic over $\mathrm{L}[S]$. Let $D \in \mathrm{L}[S]$ be a dense subset of Σ; we have to check that $D \cap G \ne \emptyset$. Suppose that $D \cap G = \emptyset$, and get a contradiction.

Since $D \in \mathrm{L}[S]$, there exists an $\in$-formula $\Phi(x, y)$ containing only ordinals as parameters and such that $\Phi(S, y)$ holds in $\mathrm{L}[S]$ iff $y = D$.

Let $\Psi(G')$ be the conjunction of the following formulas:

(1) $S' = \underline{S}[G']$ (the interpretation of the "term" $\underline{S}$ via G') is a set of ordinals, and there exists unique $D' \in \mathrm{L}[S']$ such that $\Phi(S', D')$ holds in $\mathrm{L}[S']$;

(2) D' is a dense subset of Σ' where Σ' is the set obtained by applying our definition of Σ within $\mathrm{L}[S']$;

(3) $D' \cap G' = \emptyset$.

Then $\Psi(G)$ is true in $\mathrm{L}[G]$ by our assumptions. Let $p \in G$ force Ψ over L. Then $p \in \Sigma$ by (Σ1). By the density there exists $q \in D$ with $p \le q$. We can consider a Σ-generic over $\mathrm{L}[S]$ set $G' \subseteq \Sigma$ containing q. Then G' is also P-generic over L by (Σ1). We observe that $\underline{S}[G'] = S$ because $G' \subseteq \Sigma$. It follows that D' and Σ' (as is the description of Ψ) coinside with resp. D and Σ. In particular $q \in D' \cap G'$, a contradiction because p forces (3). □

Proof of the proposition. *Item 1.* Lemma 5 (for $P = \lambda^{<\omega}$) implies that the universe is a Σ-generic extension of $\mathrm{L}[S]$ for a certain tree $\Sigma \subseteq \lambda^{<\omega}$, $\Sigma \in \mathrm{L}[S]$. Notice that λ is a cardinal in $\mathrm{L}[S]$ because S is α-weak over L where $\alpha < \lambda$; on the other hand, λ is countable in the universe by λ-CUH. It follows that there exists a condition $u \in G$ such that the set of all λ-branching points of Σ is cofinal over u in Σ. In other words, the set $\{v \in \Sigma : u \subseteq v\}$ includes in $\mathrm{L}[S]$ a cofinal subset order isomorphic to $\lambda^{<\omega}$.

Items 2 and 3. It suffices to refer to item 1 and argue as in the proofs of Lemma 3.5 and Corollary 3.5 in [10] for $\mathrm{L}[S]$ as the initial model. □

2–B Coding of reals and sets of reals in the model

We let $\mathbb{F}_\alpha(M)$ be the set of all $\alpha^{<\omega}$-generic over M functions $f \in \alpha^\omega$.

The following definitions intend to introduce a useful coding system for reals (i. e. points of $\mathcal{D} = 2^\omega$ in this research) and sets of reals in collapsing extensions.

Let $\alpha \in \mathrm{Ord}$. By Term_α we denote the set of all indexed sets $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle$ – the "terms" – such that $t_n \subseteq \alpha^{<\omega}$ for each n. We put $\mathrm{Term}_{<\lambda} = \bigcup_{\alpha<\lambda} \mathrm{Term}_\alpha$.

"Terms" $t \in \mathrm{Term}_\alpha$ are used to code functions $C : \alpha^\omega \longrightarrow \mathcal{D} = 2^\omega$; namely, for every $f \in \alpha^\omega$ we define $x = \mathrm{C}_t(f) \in \mathcal{D}$ by: $x(n) = 1$ iff $f\,|\,m \in t_n$ for some m.

Assume that $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle \in \mathrm{Term}_\alpha$, $u \in \alpha^{<\omega}$, M arbitrary. We introduce the sets $\mathsf{X}_{tu}(M) = \{\mathrm{C}_t(f) : u \subset f \in \mathbb{F}_\alpha(M)\}$ and $\mathsf{X}_t(M) = \mathsf{X}_{t\Lambda}(M) = \mathrm{C}_t\text{"}\mathbb{F}_\alpha(M)$.

Proposition 6 *Assume* λ-CUH. *Let* $S \subseteq \mathrm{Ord}$ *be* λ-*weak over* L. *Then:*

1. *If* $\alpha < \lambda$, $F \subseteq \mathbb{F}_\alpha(\mathrm{L}[S])$ *is* $\mathrm{OD}[S]$, *and* $f \in F$, *then there exists* $m \in \omega$ *such that each* $f' \in \mathbb{F}_\alpha(\mathrm{L}[S])$ *satisfying* $f'\,|\,m = f\,|\,m$ *belongs to* F.
2. *For each real* $x \in \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L}[S])$, *there exist* $\alpha < \lambda$, $t \in \mathrm{Term}_\alpha \cap \mathrm{L}[S]$, *and* $f \in \mathbb{F}_\alpha(\mathrm{L}[S])$ *such that* $x = \mathrm{C}_t(f)$.
3. *Each* $\mathrm{OD}[S]$ *set* $X \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L}[S])$ *is a union of sets of the form* $\mathsf{X}_t(\mathrm{L}[S])$, *where* $t \in \mathrm{Term}_{<\lambda} \cap \mathrm{L}[S]$.
4. *Suppose that* $t \in \mathrm{Term}_\alpha \cap \mathrm{L}[S]$, $\alpha < \lambda$, *and* $u \in \alpha^{<\omega}$. *Then every* $\mathrm{OD}[S]$ *set* $X \subseteq \mathsf{X}_{tu}(\mathrm{L}[S])$ *is a union of sets of the form* $\mathsf{X}_{tv}(\mathrm{L}[S])$, *where* $u \subseteq v \in \alpha^{<\omega}$.

Proof *Item 1.* We observe that $F = \{f' \in \alpha^\omega : \Phi(S, f')\}$ for an $\in$-formula Φ. Let $\Psi(S, f')$ denote the formula: "Λ $\lambda^{<\omega}$-forces $\Phi(S, f')$ over the universe", so that

$$F = \{f' \in \alpha^\omega : \Psi(S, f') \text{ is true in } \mathrm{L}[S, f']\}.$$

by Proposition 4 (items 1 and 2). Therefore, since $f \in F \subseteq \mathbb{F}_\alpha[S]$, there exists $m \in \omega$ such that the restriction $u = f\,|\,m$ $\alpha^{<\omega}$-forces $\Psi(S, \hat{f})$ over $\mathrm{L}[S]$ where $\hat{f}$ is the name of the α-collapsing function.

Item 2. By the choice if x, this real belongs to a $\alpha^{<\omega}$-generic extension of $\mathrm{L}[S]$. Thus $x \in \mathrm{L}[S, f]$ where $f \in \mathbb{F}_\alpha(\mathrm{L}[S])$. Let $\hat{x}$ be the name of x. It suffices to define $t_n = \{u \in \alpha^{<\omega} : u \text{ forces } \hat{x}(n) = 1\}$ and take $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle$.

Item 3. Consider a real $x \in X$. We use item 2 to obtain $\alpha < \lambda$, $f \in \mathbb{F}_\alpha(\mathrm{L}[S])$, and $t \in \mathrm{Term}_\alpha \cap \mathrm{L}[S]$ such that $x = \mathrm{C}_t(f)$. Then we apply item 1 to the $\mathrm{OD}[S]$ set $F = \{f' \in \mathbb{F}_\alpha[S] : \mathrm{C}_t(f') \in X\}$ and the f defined above. This results in a condition $u = f\,|\,m \in \lambda^{<\omega}$ ($m \in \omega$) such that $x \in \mathsf{X}_{tu}[S] \subseteq X$. Finally the set $\mathsf{X}_{tu}[S]$ is equal to $\mathsf{X}_{t'}[S]$ for some other $t' \in \mathrm{Term}_\alpha \cap \mathrm{L}[S]$.

Item 4. Similar to the previous item. □

3 The case of closed relations: classifiable points

In this section, we prove the "case 1" of Theorem 3. Thus let E be a Σ^1_1 equivalence relation.

3–A Classifiable points

First of all, we introduce the notion of an E-classifiable point.

As usual, HC denotes the set of all hereditarily countable sets. Σ_1^{HC} will denote the collection of all subsets of HC definable in HC by a parameter-free Σ_1 formula. The class Π_1^{HC} is understood the same way, and $\Delta_1^{\mathrm{HC}} = \Sigma_1^{\mathrm{HC}} \cap \Pi_1^{\mathrm{HC}}$.

Let us fix a constructible Δ_1^{HC} enumeration $\mathrm{Term} \cap \mathrm{L} = \{\tau[\xi] : \xi < \omega_1\}$ such that each $t \in \mathrm{Term} \cap \mathrm{L}$ has uncountably many numbers $\xi < \omega_1$ satisfying $t = \tau[\xi]$. The following lemma gives a more special characterization for $\overline{\mathsf{E}}$, the $\mathcal{T}^2$-closure of E, based on this enumeration.

Lemma 7 *Assume λ-CUH. Let $x, y \in \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$. Then $x \,\overline{\mathsf{E}}\, y$ if and only if for each $\xi < \omega_1$ we have $x \in [\mathsf{X}_{\tau[\xi]}(\mathrm{L}_\xi)]_{\mathsf{E}} \longleftrightarrow y \in [\mathsf{X}_{\tau[\xi]}(\mathrm{L}_\xi)]_{\mathsf{E}}$.*

Proof The "only if" part follows from the fact that the sets $\mathsf{X}_{\tau[\xi]}(\mathrm{L}_\gamma)$ are OD. Let us prove the "if" direction. Assume that $x \,\overline{\not\mathsf{E}}\, y$. There exists an OD set X such that $x \in [X]_{\mathsf{E}}$ but $y \notin [X]_{\mathsf{E}}$. By Proposition 6, we obtain $x \in \mathsf{X}_t(\mathrm{L}) \subseteq [X]_{\mathsf{E}}$, where $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle \in \mathrm{Term}_\alpha \cap \mathrm{L}$, $\alpha < \lambda$. Since λ is a limit cardinal in L, there exists a constructible cardinal γ, $\alpha < \gamma < \lambda$, such that $\mathbb{F}_\alpha(\mathrm{L}) = \mathbb{F}_\alpha(\mathrm{L}_\gamma)$. Then $t' = \langle \gamma, \langle t_n : n \in \omega \rangle \rangle$ is $\tau[\xi]$ for some ξ, $\gamma \le \xi < \omega_1$. Then $\mathsf{X}_t(\mathrm{L}) = \mathsf{X}_{\tau[\xi]}(\mathrm{L}_\xi)$. □

For each $x \in \mathcal{D}$, we define $\varphi_x \in 2^{\omega_1}$ as follows: $\varphi_x(\xi) = 1$ iff $x \in [\mathsf{X}_{\tau[\xi]}(\mathrm{L}_\xi)]_{\mathsf{E}}$.

Definition 8 We introduce the notion of a E-classifiable point. We let T be the set of all triples $\langle x, \psi, t \rangle$ such that $x \in \mathcal{D}$, $\psi \in 2^{<\omega_1}$, $t \in \mathrm{Term}_\alpha \cap \mathrm{L}_\gamma[\psi]$, where $\alpha < \gamma = \mathrm{dom}\,\psi < \omega_1$, and the following conditions (a) through (d) are satisfied.

(a) $\mathrm{L}_\gamma[\psi]$ models $\mathbf{ZFC}^-$ (minus the Power Set axiom) so that ψ can occur as an extra class parameter in Replacement and Separation.

(b) It is true in $\mathrm{L}_\gamma[\psi]$ that $\langle \Lambda, \Lambda \rangle$ forces $\mathrm{C}_t(\hat{f}) \,\mathsf{E}\, \mathrm{C}_t(\hat{g})$ in the sense of $\alpha^{<\omega} \times \alpha^{<\omega}$ as the forcing, where $\hat{f}$ and $\hat{g}$ are names for the generic functions in α^ω.

(c) For each $\xi < \gamma$, $\psi(\xi) = 1$ iff $x \in [\mathsf{X}_{\tau[\xi]}(\mathrm{L}_\xi)]_{\mathsf{E}}$ — so that $\psi = \varphi_x \,|\, \gamma$.

(d) x belongs to $[\mathsf{X}_t(\mathrm{L}_\gamma[\psi])]_{\mathsf{E}}$.

A point $x \in \mathcal{D}$ is E-*classifiable* iff there exist ψ and t such that $\langle x, \psi, t \rangle \in T$. □

The author learned from Hjorth and Kechris [4] the idea of forcing over countable models to get a Δ_1 reduction function, the key idea of this definition.

Lemma 9 T_{E} *is a* Δ_1^{HC} *set* (provided E is Σ_1^1).

Proof Notice that conditions (a) and (b) in Definition 8 are Δ_1^{HC} because they reflect truth within $\mathrm{L}_\gamma[\psi]$ and the enumeration $\tau[\xi]$ was chosen in Δ_1^{HC}.

Condition (d) is obviously Σ_1^{HC} (provided E is at least Σ_2^1), so it remains to convert it also to a Π_1^{HC} form. Notice that in the assumption of (a) and (b), the set $X = \mathsf{X}_t(\mathrm{L}_\gamma[\psi])$ consists of pairwise E-equivalent points.

(Indeed, consider a pair of $\alpha^{<\omega}$-generic over $\mathrm{L}_\gamma[\psi]$ functions $f, g \in \alpha^\omega$ (not necessarily a *generic pair*). Let $h \in \alpha^\omega$ be an $\alpha^{<\omega}$-generic over both $\mathrm{L}_\gamma[\psi, f]$ and $\mathrm{L}_\gamma[\psi, g]$ function. Then, by (b), $\mathrm{C}_t(h) \mathrel{\mathsf{E}} \mathrm{C}_t(f)$ holds in $\mathrm{L}_\gamma[\psi, f, h]$, therefore in the universe by Shoenfield. Similarly, $\mathrm{C}_t(h) \mathrel{\mathsf{E}} \mathrm{C}_t(g)$. It follows that $\mathrm{C}_t(f) \mathrel{\mathsf{E}} \mathrm{C}_t(g)$, as required.)

Therefore (d) is equivalent to the formula $\forall\, y \in \mathsf{X}_t(\mathrm{L}_\gamma[\psi])\ (x \mathrel{\mathsf{E}} y)$ because $\mathsf{X}_t(\mathrm{L}_\gamma[\psi])$ is not empty. This is clearly Π_1^{HC} provided E is at least Π_2^1.

Let us consider (c). The right–hand side of the equivalence "iff" in (c) is Σ_1^1 with inserted Δ_1^{HC} functions, therefore Δ_1^{HC}. It follows that (c) itself is Δ_1^{HC}. [9] □

3–B The classification theorem

The following lemma will allow to define a Δ_1^{HC} reduction of the given Σ_1^1 equivalence relation E to the equality on $2^{<\omega_1}$.

Lemma 10 *In the assumption of Case 1 of Subsection 1–C, each point* $x \in \mathcal{D}$ *is* E-*classifiable.*

Proof Let $x \in \mathcal{D}$. By the assumption of Case 1, there exist an ordinal λ and a "virtual" $\lambda^{<\omega}$-generic extension V of the constructible universe L containing x such that E coincides with $\overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ in V and x is λ-weak over L in V.

Thus we have the two universes, V and the universe of the lemma, with one and the same class of ordinals. Since by Lemma 9 "being E-classifiable" is a Σ_1^{HC}, therefore Σ_2^1 notion, it suffices to prove that x is E-classifiable in the "virtual" universe V.

We observe that λ-CUH is true in V.

We argue in V.

Notice that $\varphi = \varphi_x$ is λ-weak over L: indeed $\varphi \in \mathrm{L}[x]$ by Proposition 4 since φ is $\mathrm{OD}[x]$. It follows that $[x]_{\mathsf{E}}$ is $\mathrm{OD}[\varphi]$ by Lemma 7, because $\mathsf{E} = \overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$. Therefore by Proposition 6, $x \in \mathsf{X}_t(\mathrm{L}[\varphi]) \subseteq [x]_{\mathsf{E}}$ for some $t \in \mathrm{Term}_\alpha \cap \mathrm{L}[\varphi]$, $\alpha < \lambda$.

[9] Here we do not see how to weaken the assumption that E is Σ_1^1; even if the relation is Π_1^1, (c) becomes Δ_2^{HC}.

The model $\mathrm{L}_{\omega_1}[\varphi]$ has an elementary submodel $\mathrm{L}_\gamma[\psi]$, where $\gamma < \omega_1$ and $\psi = \varphi\,|\,\gamma$, containing t and α. We prove that $\langle x, \psi, t\rangle \in T_{\mathsf{E}}$. Since conditions (a) and (c) of Definition 8 obviously hold for $\mathrm{L}_\gamma[\psi]$, let us check requirements (b) and (d).

We check (b). Indeed otherwise there exist conditions $u, v \in \alpha^{<\omega}$ such that $\langle u, v\rangle$ forces $\mathrm{C}_t(\hat f) \not\mathsf{E}\, \mathrm{C}_t(\hat g)$ in $\mathrm{L}_\gamma[\psi]$ in the sense of $\alpha^{<\omega}{\times}\alpha^{<\omega}$ as the notion of forcing. Then $\langle u, v\rangle$ also forces $\mathrm{C}_t(\hat f) \not\mathsf{E}\, \mathrm{C}_t(\hat g)$ in $\mathrm{L}_{\omega_1}[\varphi]$. Let us consider an $\alpha^{<\omega}\times\alpha^{<\omega}$-generic over $\mathrm{L}[\varphi]$ pair $\langle f, g\rangle \in \alpha^\omega \times \alpha^\omega$ such that $u \subset f$ and $v \subset g$. Then both $y = \mathrm{C}_t(f)$ and $z = \mathrm{C}_t(g)$ belong to $\mathsf{X}_t(\mathrm{L}[\varphi])$, so $y \,\mathsf{E}\, z$ because $\mathsf{X}_t(\mathrm{L}[\varphi]) \subseteq [x]_{\mathsf{E}}$.

On the other hand, $y \,\mathsf{E}\, z$ is *false* in $\mathrm{L}_{\omega_1}[\varphi, f, g]$, that is, in $\mathrm{L}[\varphi, f, g]$, by the forcing property of $\langle u, v\rangle$. Therefore we have $x \not\mathsf{E}\, y$ (in the "virtual" universe V) by Shoenfield, contradiction.

We check (d). Take any $\alpha^{<\omega}$-generic over $\mathrm{L}[\varphi]$ function $f \in \alpha^\omega$. Then $y = \mathrm{C}_t(f)$ belongs to $\mathsf{X}_t(\mathrm{L}[\varphi])$, hence $y \,\mathsf{E}\, x$. On the other hand, f is generic over $\mathrm{L}_\gamma[\psi]$.

Thus $\langle x, \psi, t\rangle \in T_{\mathsf{E}}$. This means that x is E-classifiable, as required. □

Definition 11 Let $x \in \mathcal{D}$. It follows from Lemma 10 that there exists the least ordinal $\gamma = \gamma_x < \omega_1$ such that $T_{\mathsf{E}}(x, \varphi_x\,|\,\gamma, t)$ for some t. We put $\psi_x = \varphi_x\,|\,\gamma$ and let t_x denote the least, in the sense of the $\mathrm{OD}[\psi_x]$ wellordering of $\mathrm{L}_\gamma[\psi_x]$, "term" $t \in \mathrm{Term}[\psi_x] \cap \mathrm{L}_\gamma[\psi_x]$ which satisfies $T_{\mathsf{E}}(x, \psi_x, t)$. We put $U(x) = \langle \psi_x, t_x\rangle$. □

Lemma 12 *If each $x \in \mathcal{D}$ is E-classifiable then the map U is a Δ^{HC}_1 reduction of E to equality.*

Proof First of all, U is Δ^{HC}_1 by Lemma 9.

If $x \,\mathsf{E}\, y$ then $U(x) = U(y)$ because Definition 8 is E-invariant for x.

Let us prove the converse. Assume that $U(x) = U(y)$, that is, in particular, $\psi_x = \psi_y = \psi \in 2^{<\omega}$ and $t_x = t_y = t \in \mathrm{Term}_\alpha[\psi] \cap \mathrm{L}_\gamma[\psi]$, where $\alpha < \gamma = \mathrm{dom}\,\psi < \omega_1$.

By (d) we have $\mathrm{C}_t(f) \,\mathsf{E}\, x$ and $\mathrm{C}_t(g) \,\mathsf{E}\, y$ for some $\alpha^{<\omega}$-generic over $\mathrm{L}_\gamma[\psi]$ functions $f, g \in \alpha^\omega$. However $\mathrm{C}_t(f) \,\mathsf{E}\, \mathrm{C}_t(g)$ (see the proof of Lemma 9). □

Corollary 13 [The classification theorem]
In the assumption of Case 1 of Subsection 1–C, E admits a Δ^{HC}_1 reduction to the equality on $2^{<\omega_1}$.

Proof The range of the function U can be covered by a subset $R \subseteq \mathrm{HC}$ (all pairs $\langle \psi, t\rangle$ such that ...) which admits a $1-1$ Δ^{HC}_1 correspondence with $2^{<\omega_1}$. □

This completes the proof of the "case 1" part of Theorem 3.

4 OD forcing

This section starts the proof of the "Case 2" part of Theorem 3. At the beginning, we reduce the problem to a more elementary form.

4–A Explanation

Thus let us suppose that each real x belongs to a "virtual" generic extension of L, but the assumption of Case 1 in Subsection 1–C fails.

This means the following. There exists a real $z \in \mathcal{D}$ such that for every ordinal λ and a "virtual" $\lambda^{<\omega}$-generic extension V of the constructible universe L containing z, the following is true in V : if z is λ-weak over L then E *does not* coincide with $\overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$.

We know indeed that z belongs to a "virtual" generic extension of L. Therefore there exists a limit constructible cardinal λ such that z belongs to a $\lambda^{<\omega}$-generic extension V of L and z is weak in V. (Simply take λ sufficiently large.)

Let us fix λ and V . As the condensed matter of this reasoning, we obtain

- V is a "virtual" $\lambda^{<\omega}$-generic extension of L, λ is a limit cardinal in L, and $\mathsf{E} \underset{\neq}{\subsetneq} \overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ in V .

This is the description of the starting position of the proof of the "Case 2" part of Theorem 3. The aim is to see that in this case E continuously embeds E_0 in the universe of Theorem 3.

The general plan will be first to prove that E continuously embeds E_0 *in the auxiliary "virtual" universe* V, and second, to get the result in the universe of Theorem 3 by Shoenfield.

After a short examination, one can see a problem in this plan: the existence of a continuous embedding E_0 into E is in fact a Σ^1_3 statement:

$$\exists \text{ continuous } 1-1\ U : \mathcal{D} \longrightarrow \mathcal{D}\ \forall\, x,\, y \in \mathcal{D} \left[\begin{array}{lcll} x \mathrel{\mathsf{E}_0} y & \longrightarrow & U(x) \mathrel{\mathsf{E}} U(y), & \text{and} \\ x \mathrel{\not{\mathsf{E}}_0} y & \longrightarrow & U(x) \mathrel{\not{\mathsf{E}}} U(y) & \end{array} \right]$$

The lower implication in the square brackets is Π^1_1, which would match the total Σ^1_2, but the upper one is Σ^1_1, so that the total result is Σ^1_3, worse than one needs for Shoenfield.

4–B Special embeddings and proof of the "Case 2" part of Theorem 3

To overcome this obstacle, we strengthen the upper implication to convert it to a Π^1_1 (actually Δ^1_1) statement. We recall that the Σ^1_1 set $\mathsf{E} \subseteq \mathcal{D}^2$ admits a partition $\mathsf{E} = \bigcup_{\alpha<\omega_1} \mathsf{E}_\alpha$ onto Borel sets E_α – the *constituents*, uniquely defined as soon as we have fixed a Π^0_1 set $F \subseteq \mathcal{D}^2 \times \mathcal{N}$ which projects onto E.

Definition 14 A $1-1$ function $\phi : \mathcal{D} \longrightarrow \mathcal{D}$ is a *special embedding* of E_0 into E iff

(1) there exists an ordinal $\alpha < \omega_1$ such that $\langle \phi(0^k{}^\wedge 0^\wedge z), \phi(0^k{}^\wedge 1^\wedge z)\rangle \in \mathsf{E}_\alpha$ for all $z \in \mathcal{D}$ and $k \in \omega$, and

(2) for all $x, y \in \mathcal{D}$, if $x \not\mathsf{E}_0\, y$ then $\phi(x) \not\mathsf{E}\, \phi(y)$. □

(0^k is the sequence of k zeros.) First of all, let us see that a special embedding is an embedding in the usual sense. We have to prove that $x\,\mathsf{E}_0\,y$ implies $\phi(x)\,\mathsf{E}\,\phi(y)$. We say that a pair of points $x, y \in \mathcal{D}$ is a *neighbouring pair* iff there exist $k \in \omega$ and $z \in \mathcal{D}$ such that $x = 0^k{}^\wedge 0^\wedge z$ and $y = 1^k{}^\wedge 1^\wedge z$ or vice versa. Obviously a neighbouring pair is E_0-equivalent. Conversely, if $x\,\mathsf{E}_0\,y$ then x and y can be connected by a finite chain of neighbouring pairs in $\mathcal{D}$. Therefore condition (1) actually suffices to guarantee that $x\,\mathsf{E}_0\,y \longrightarrow \phi(x)\,\mathsf{E}\,\phi(y)$.

Obviously the existence of a *special* embedding of E_0 into E is a Σ^1_2 property. Thus, by Shoenfield, to complete the proof of the "Case 2" part of Theorem 3, it suffices to prove the following theorem (and apply it in the auxiliary "virtual" universe V).

Theorem 15 *Assume* λ-CUH. *Let* E *be a* Σ^1_1 *relation and* $\mathsf{E} \subsetneq \overline{\mathsf{E}}$ *on* $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$. *Then* E_0 *admits a special continuous embedding into* E.

This theorem is being proved in this and the next section. During the course of the proof, we assume λ-CUH and fix a Σ^1_1 equivalence E satisfying $\mathsf{E} \subsetneq \overline{\mathsf{E}}$ on the set $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ (although the last assumption will not be used at the beginning).

In this section, we consider important interactions between E and $\overline{\mathsf{E}}$. The next section defines the required embedding. This will complete the proof of theorems 15 and 3, and Theorem 1 – the main theorem.

4–C OD topology and the forcing

We recall that $\mathcal{T}$ be the topology generated by all OD sets.

A set X will be called *$\mathcal{T}$-separable* if the OD power set $\mathcal{P}^{\mathrm{OD}}(X) = \mathcal{P}(X) \cap \mathrm{OD}$ has only countably many different OD subsets.

Lemma 16 *Assume* λ-CUH. *Let* $\alpha < \lambda$ *and* $t \in \mathrm{Term}_\alpha \cap \mathrm{L}$. *Each set* $X = \mathsf{X}_t(\mathrm{L})$ *satisfying* $X \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ *is* $\mathcal{T}$*-separable.*

Proof By Proposition 6 every OD subset of X is uniquely determined by an OD subset of $\alpha^{<\omega}$. Since each OD set $S \subseteq \alpha^{<\omega}$ is constructible, we obtain an OD map $h : \alpha^+$ onto $\mathcal{P}^{\mathrm{OD}}(X)$, where α^+ is the least cardinal in L bigger than α. Therefore $\mathcal{P}^{\mathrm{OD}}(X)$ has $\leq \alpha^{++}$-many OD subsets. It remains to notice that $\alpha^{++} < \lambda$ because λ is a limit cardinal in L, but λ is countable in the universe. □

Let $\mathbb{X} = \{X \subseteq \mathcal{D} : X$ is OD and nonempty$\}$.

Let us consider $\mathbb{X}$ as a forcing notion (smaller sets are stronger conditions) for generic extensions of L. Of course formally $\mathbb{X} \notin$ L, but $\mathbb{X}$ is OD order isomorphic to a partially ordered set in L . (Indeed it is known that there exists an OD map ℓ : ordinals onto the class of all OD sets. Since $\mathbb{X}$ itself is OD, $\mathbb{X}$ is a 1–1 image of an OD set $\mathbb{X}'$ of ordinals via ℓ. By Proposition 4 both $\mathbb{X}'$ and the ℓ-preimage of the order on $\mathbb{X}$ belong to L .)

It also is true that a set $G \subseteq \mathbb{X}$ is $\mathbb{X}$-generic over L iff it nonempty intersects every dense OD subset of $\mathbb{X}$.

Corollary 17 *Assume* λ-CUH. *If a set* $X \in \mathbb{X}$ *satisfies* $X \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ *then there exists a* $\mathbb{X}$*-generic over* L *set* $G \subseteq \mathbb{X}$ *containing* X *.*

Proof We can suppose, by Proposition 6, that $X = \mathsf{X}_t(\mathrm{L})$ where $t \in \mathrm{Term}_\alpha \cap \mathrm{L}$ and $\alpha < \lambda$. Now apply Lemma 16. □

Lemma 18 *Assume* λ-CUH. *If* $G \subseteq \mathbb{X}$ *is a generic over* L *set containing the set* $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ *then the intersection* $\bigcap G$ *is a singleton* $\{a\} = \{a_G\}$ *.*

Proof Assume that this is not the case. Let $\mathbb{X}' \in \mathrm{L}$ be a constructible p. o. set order isomorphic $\mathbb{X}$ via an OD function $\ell : \mathbb{X}'$ onto $\mathbb{X}$. Then $G' = \ell^{-1}(G)$ is $\mathbb{X}'$-generic over L. We assert that the statement that $\bigcap G$ is not a singleton can be converted to a sentence relativized to $\mathrm{L}[G']$.

(Indeed, it follows from the reasoning in the proof of Lemma 16 that $\mathrm{L}[G']$ is in fact a P-generic extension of L for a certain set $P \in \mathrm{L}$, $P \subseteq \mathbb{X}'$ of a cardinality $\alpha < \lambda$ in L. The next L-cardinal α^+ is $< \lambda$ since λ is a limit cardinal in L. Therefore G' belongs to a $\alpha^{+<\omega}$-generic extension of L, so G' is weak. Then by Proposition 4 the universe $\mathrm{V} = \mathrm{L}[f_0]$ is a $\lambda^{<\omega}$-generic extension of $\mathrm{L}[G']$. This is enough to convert any statement about G' in V – like the statement: $\bigcap \ell\,"G'$ is not a singleton – to a sentence relativized to $\mathrm{L}[G']$.)

Then there exists $X \in \mathbb{X}$, $X \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$, such that $\bigcap G$ is not a singleton for *every* generic over L set $G \subseteq \mathbb{X}$ containing X. We can assume that $X = \mathsf{X}_t(\mathrm{L})$, where $t \in \mathrm{Term}_\alpha \cap \mathrm{L}$, $\alpha < \lambda$. Then X is $\mathcal{T}$-separable; let $\{\mathcal{X}_n : n \in \omega\}$ be an enumeration of all OD dense subsets of $\mathcal{P}^{\mathrm{OD}}(X)$. Using Proposition 4 (item 1), we obtain an increasing $\alpha^{<\omega}$-generic over L sequence $u_0 \subseteq u_1 \subseteq u_2 \subseteq \ldots$ of $u_n \in \alpha^{<\omega}$ such that $X_n = \mathsf{X}_{t\,u_n}(\mathrm{L}) \in \mathcal{X}_n$. Obviously this gives an $\mathbb{X}$-generic over L set $G \subseteq \mathbb{X}$ containing X and all X_n .

Now let $f = \bigcup_{n\in\omega} u_n$; $f \in \alpha^\omega$ and f is $\alpha^{<\omega}$-generic over L. Then $x = \mathrm{C}_t(f) \in X_n$ for all n, so $x \in \bigcap G$. Since $\bigcap G$ obviously cannot contain more than one point, it is a singleton, so we get a contradiction with the choice of X . □

Reals a_G will be called OD-*generic over* L .

4–D The product forcing

We recall that E is assumed to be a Σ^1_1 equivalence on $\mathcal{D}$; $\overline{\mathsf{E}}$ is the closure of E in the topology $\mathcal{T}^2$ (the product of two copies of $\mathcal{T}$).

For a set $P \subseteq \mathcal{D}^2$, we put $\mathrm{pr}_1 P = \{x : \exists y\, P(x,y)\}$ and $\mathrm{pr}_2 P = \{y : \exists x\, P(x,y)\}$. Notice that if P is OD, so are $\mathrm{pr}_1 P$ and $\mathrm{pr}_2 P$.

The classical reasoning in Harrington, Kechris, and Louveau [2] plays on interactions between E and $\overline{\mathsf{E}}$. In the forcing setting, we have to fix a restriction by $\overline{\mathsf{E}}$ directly in the definition of the product forcing. Thus we consider

$$\mathbb{P} = \mathbb{P}(\overline{\mathsf{E}}) = \{P \subseteq \overline{\mathsf{E}} : P \text{ is OD and nonempty and } P = (\mathrm{pr}_1 P \times \mathrm{pr}_2 P) \cap \overline{\mathsf{E}}\}$$

as a forcing notion. As above for $\mathbb{X}$, the fact that formally $\mathbb{P}$ does not belong to L does not cause essential problems.

The following assertion connects $\mathbb{P}$ and $\mathbb{X}$.

Assertion 19 *Assume* λ-CUH. *Then*

1. *If* $P \in \mathbb{P}$ *then* $\mathrm{pr}_1 P$ *and* $\mathrm{pr}_2 P$ *belong to* $\mathbb{X}$.
2. *If* $X, Y \in \mathbb{X}$ *and* $P = (X \times Y) \cap \overline{\mathsf{E}} \neq \emptyset$ *then* $P \in \mathbb{P}$.
3. *If* $P \in \mathbb{P}$, $X \in \mathbb{X}$, $X \subseteq \mathrm{pr}_1 P$, *then there exists* $Q \in \mathbb{P}$, $Q \subseteq P$, *such that* $X = \mathrm{pr}_1 Q$. *Similarly for* pr_2.

Proof Set $Q = \{\langle x,y\rangle \in P : x \in X \;\&\; y \,\overline{\mathsf{E}}\, x\}$ in item 3. □

A set $P \in \mathbb{P}$ is $\mathbb{P}$-*separable* if the set $\mathbb{P}_{\subseteq P} = \{Q \in \mathbb{P} : Q \subseteq P\}$ has only countably many different OD subsets.

Lemma 20 *Assume* λ-CUH. *Let* $t, t' \in \mathrm{Term}_{<\lambda} \cap \mathrm{L}$. *Suppose that the sets* $X = \mathsf{X}_t(\mathrm{L})$ *and* $Y = \mathsf{X}_{t'}(\mathrm{L})$ *satisfy* $X \cup Y \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$, *and finally that* $P = (X \times Y) \cap \overline{\mathsf{E}}$ *is nonempty. Then* $P \in \mathbb{P}$ *and* P *is* $\mathbb{P}$-*separable.*

Proof $P \in \mathbb{P}$ by Assertion 19. A proof of the $\mathbb{P}$-separability can be obtained by a minor modification of the proof of Lemma 16. □

Lemma 21 *Assume* λ-CUH. *Let* $G \subseteq \mathbb{P}$ *be a* $\mathbb{P}$-*generic over* L *set containing* $(\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L}))^2 \cap \overline{\mathsf{E}}$. *Then the intersection* $\bigcap G$ *contains a single point* $\langle a,b\rangle$ *where* a *and* b *are* OD-*generic over* L *and* $a \,\overline{\mathsf{E}}\, b$.

Proof By Assertion 19, both $G_1 = \{\mathrm{pr}_1 P : P \in G\}$ and $G_2 = \{\mathrm{pr}_1 P : P \in G\}$ are OD-generic over L subsets of $\mathbb{X}$, so that there exist unique OD-generic over L points $a = a_{G_1}$ and $b = a_{G_2}$. It remains to show that $\langle a,b\rangle \in \overline{\mathsf{E}}$.

Suppose not. There exists an E-invariant OD set A such that we have $x \in A$ and $y \in B = \mathcal{D} \setminus A$. Then $A \in G_1$ and $B \in G_2$ by the genericity. There exists a condition $P \in G$ such that $\mathrm{pr}_1 P \subseteq A$ and $\mathrm{pr}_2 B \subseteq B$, therefore $P \subseteq (A \times B) \cap \overline{\mathsf{E}} = \emptyset$, which is impossible. □

Pairs $\langle a, b \rangle$ as in Lemma 21 will be called $\mathbb{P}$-*generic* and denoted by $\langle a_G, b_G \rangle$.

For sets X and Y and a binary relation R, let us write $X \,\mathsf{R}\, Y$ if and only if $\forall\, x \in X \;\exists\, y \in Y \; (x \,\mathsf{R}\, y)$ and $\forall\, y \in Y \;\exists\, x \in X \; (x \,\mathsf{R}\, y)$.

Lemma 22 *Assume* λ-CUH. *Let* $P_0 \in \mathbb{P}$, $P_0 \subseteq (\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L}))^2$, *points* $a, a' \in X_0 = \mathrm{pr}_1 P_0$ *be* OD-*generic over* L, *and* $a \,\overline{\mathsf{E}}\, a'$. *There exists a point* b *such that both* $\langle a, b \rangle$ *and* $\langle a', b \rangle$ *belong to* P_0 *and are* $\mathbb{P}$-*generic pairs.*

Proof By Lemma 20 and Proposition 6 there exists a $\mathbb{P}$-separable set $P_1 \subseteq P_0$ such that $a \in X_1 = \mathrm{pr}_1 P_1$. We put $Y_1 = \mathrm{pr}_2 P_1$; then $X_1 \,\overline{\mathsf{E}}\, Y_1$, and $P_1 = (X_1 \times Y_1) \cap \overline{\mathsf{E}}$.

We let $P' = \{\langle x, y \rangle \in P_0 : y \in Y_1\}$. Then $P' \in \mathbb{P}$ and $P_1 \subseteq P' \subseteq P_0$. Furthermore $a' \in X' = \mathrm{pr}_1 P'$. (Indeed, since $a \in X_1$ and $X_1 \,\overline{\mathsf{E}}\, Y_1$, there exists $y \in Y_1$ such that $a \,\overline{\mathsf{E}}\, y$; then $a' \,\overline{\mathsf{E}}\, y$ as well because $a \,\overline{\mathsf{E}}\, a'$, therefore $\langle a', y \rangle \in P'$.) By Lemma 20 and Proposition 6 there exists a $\mathbb{P}$-separable set $P'_1 \subseteq P'$ such that $a' \in X'_1 = \mathrm{pr}_1 P'_1$. Then $Y'_1 = \mathrm{pr}_2 P'_1 \subseteq Y_1$.

It follows from the choice of P and P' that $\mathbb{P}$ admits only countably many different dense OD sets below P_1 and below P'_1. Let $\{\mathcal{P}_n : n \in \omega\}$ and $\{\mathcal{P}'_n : n \in \omega\}$ be enumerations of both families of dense sets. We define sets $P_n, P'_n \in \mathbb{P}$ $(n \in \omega)$ satisfying the following conditions:

(i) $a \in X_n = \mathrm{pr}_1 P_n$ and $a' \in X'_n = \mathrm{pr}_1 P'_n$;

(ii) $Y'_n = \mathrm{pr}_2 P'_n \subseteq Y_n = \mathrm{pr}_2 P_n$ and $Y_{n+1} \subseteq Y'_n$;

(iii) $P_{n+1} \subseteq P_n$, $P'_{n+1} \subseteq P'_n$, $P_n \in \mathcal{P}_{n-2}$, and $P'_n \in \mathcal{P}'_{n-2}$.

By (iii) both sequences $\{P_n : n \in \omega\}$ and $\{P'_n : n \in \omega\}$ are $\mathbb{P}$-generic over L, so by Lemma 21 they result in two generic pairs, $\langle a, b \rangle \in P_0$ and $\langle a', b \rangle \in P_0$, having the first terms equal to a and a' by (i) and second terms equal to each other by (ii). Thus it suffices to conduct the construction of P_n and P'_n.

The construction goes on by induction on n.

Assume that P_n and P'_n have been defined. We define P_{n+1}. By (ii) and Assertion 19, the set $P = (X_n \times Y'_n) \cap \overline{\mathsf{E}} \subseteq P_n$ belongs to $\mathbb{P}$ and $a \in X = \mathrm{pr}_1 P$. (Indeed, $\langle a, b \rangle \in P$, where b satisfies $\langle a', b \rangle \in P'_n$, because $a \,\overline{\mathsf{E}}\, a'$.) However $\mathcal{P}_{n-1}$ is dense in $\mathbb{P}$ below $P \subseteq P_0$; therefore $\mathrm{pr}_1 \mathcal{P}_{n-1} = \{\mathrm{pr}_1 P' : P' \in \mathcal{P}_{n-1}\}$ is dense in $\mathbb{X}$ below $X = \mathrm{pr}_1 P$. Since a is generic, we have $a \in \mathrm{pr}_1 P'$ for some $P' \in \mathcal{P}_{n-1}$, $P' \subseteq P$. It remains to put $P_{n+1} = P'$, and then $X_{n+1} = \mathrm{pr}_1 P_{n+1}$ and $Y_{n+1} = \mathrm{pr}_2 P_{n+1}$.

After this, to define P'_{n+1} we let $P = (X'_n \times Y_{n+1}) \cap \overline{\mathsf{E}}$, etc. □

4–E The key set

We recall that λ-CUH is assumed, E is a Σ^1_1 equivalence on $\mathcal{D}$, and $\overline{\mathsf{E}}$ is the $\mathcal{T}^2$-closure of E in $\mathcal{D}^2$. By the assumption of Theorem 15, $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$. This means that there exist $\overline{\mathsf{E}}$-classes of elements of $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ which include more than one E-class. We define the union of all those $\overline{\mathsf{E}}$-classes,

$$H = \{x \in \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L}) : \exists\, y \in \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})\ (x \,\overline{\mathsf{E}}\, y \ \&\ x \not\mathrel{\mathsf{E}} y)\}\,.$$

Obviously H is OD, nonempty, and E-invariant *inside* $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$, and moreover $H' = H^2 \cap \overline{\mathsf{E}} \neq \emptyset$, so that in particular $H' \in \mathbb{P}$ by Assertion 19.

Lemma 23 *Assume* λ-CUH. *If* $a, b \in H$ *and* $\langle a, b\rangle$ *is* $\mathbb{P}$*-generic over* L *then* $a \not\mathrel{\mathsf{E}} b$.

Proof Otherwise there exists a set $P \in \mathbb{P}$, $P \subseteq H \times H$ such that $a \mathrel{\mathsf{E}} b$ holds for *all* $\mathbb{P}$-generic $\langle a, b\rangle \in P$. We conclude that then $a \,\overline{\mathsf{E}}\, a' \longrightarrow a \mathrel{\mathsf{E}} a'$ for all OD-generic points $a,\ a' \in X = \mathrm{pr}_1\, P$; indeed, take b such that both $\langle a, b\rangle \in P$ and $\langle a', b\rangle \in P$ are $\mathbb{P}$-generic, by Lemma 22. In other words the relations E and $\overline{\mathsf{E}}$ coincide on the set $Y = \{x \in X : x$ is OD-generic over $\mathrm{L}\} \in \mathbb{X}$. (Y is nonempty by corollaries 17 and 18.)

Moreover, E and $\overline{\mathsf{E}}$ coincide on the set $Z = [Y]_{\mathsf{E}} \cap \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$. Indeed if $z,\ z' \in Z$, $z \,\overline{\mathsf{E}}\, z'$, then let $y,\ y' \in Y$ satisfy $z \mathrel{\mathsf{E}} y$ and $z' \mathrel{\mathsf{E}} y'$. Then $y \,\overline{\mathsf{E}}\, y'$, therefore $y \mathrel{\mathsf{E}} y'$, which implies $z \mathrel{\mathsf{E}} z'$.

We conclude that $Y \cap H = \emptyset$.

(Indeed, suppose that $x \in Y \cap H$. Then by definition there exists $y \in \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ such that $x \,\overline{\mathsf{E}}\, y$ but $x \not\mathrel{\mathsf{E}} y$. Then $y \notin Z$ because E and $\overline{\mathsf{E}}$ coincide on Z. Thus the pair $\langle x, y\rangle$ belongs to the OD set $P = Y \times [(\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})) \setminus Z]$. Notice that P does not intersect E by definition of Z. Therefore $\langle x, y\rangle$ cannot belong to the closure $\overline{\mathsf{E}}$ of E, contradiction.)

But $\emptyset \neq Y \subseteq X \subseteq H$, contradiction. □

Lemma 23 is a counterpart of the proposition in Harrington, Kechris, Louveau [2] that $\mathsf{E} | H$ is meager in $\overline{\mathsf{E}} | H$. But in fact the main content of this argument in [2] was implicitly taken by Lemma 22.

Lemma 24 *Assume* λ-CUH. *Let* $X, Y \subseteq H$ *be nonempty* OD *sets and* $X \,\overline{\mathsf{E}}\, Y$. *There exist nonempty* OD *sets* $X' \subseteq X$ *and* $Y' \subseteq Y$ *such that* $X' \cap Y' = \emptyset$ *but still* $X' \,\overline{\mathsf{E}}\, Y'$.

Proof There exist points $x_0 \in X$ and $y_0 \in Y$ such that $x_0 \neq y_0$ but $x_0 \,\overline{\mathsf{E}}\, y_0$. (Otherwise $X = Y$, and $\overline{\mathsf{E}}$ is the equality on X, which is impossible, see the previous proof.) Let U and V be disjoint Baire intervals in $\mathcal{D}$ containing resp. x_0 and y_0. The sets $X' = X \cap U \cap [Y \cap V]_{\overline{\mathsf{E}}}$ and $Y' = Y \cap V \cap [X \cap U]_{\overline{\mathsf{E}}}$ are as required. □

5 Embedding E_0 into E

In this section we end the proof of Theorem 15. Thus we prove, assuming λ-CUH and $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$ on $\mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$, that E embeds E_0 via a continuous special (see Definition 14) embedding.

5–A The embedding

By the assumption the set H of Subsection 4–E is nonempty; obviously H is OD. By lemmas 16, 20, and Proposition 6 there exists a nonempty $\mathcal{T}$-separable OD set $X_0 \subseteq H$ such that the set $P_0 = (X_0 \times X_0) \cap \overline{\mathsf{E}}$ belongs to $\mathbb{P}$ and is $\mathbb{P}$-separable. We observe that $\mathrm{pr}_1 P_0 = \mathrm{pr}_2 P_0 = X_0 \subseteq H \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$.

We define a family of sets X_u $(u \in 2^{<\omega})$ satisfying

(a) $X_u \subseteq X_0$, X_u is nonempty and OD, and $X_{u^\wedge i} \subseteq X_u$, for all u and i .

In addition to the sets X_u, we shall define relations $\mathsf{J}_{uv} \subseteq \mathcal{D}^2$ for *some* pairs $\langle u, v\rangle$, to provide important interconnections between branches in $2^{<\omega}$.

Let $u,\, v \in 2^n$. We say that $\langle u, v\rangle$ is a *neighbouring pair* iff $u = 0^{k\wedge}0^\wedge r$ and $v = 0^{k\wedge}1^\wedge r$ for some $k < n$ (0^k is the sequence of k terms each equal to 0) and some $r \in 2^{n-k-1}$ (possibly $k = n-1$, that is, $r = \Lambda$).

Thus we define sets $\mathsf{J}_{uv} \subseteq X_u \times X_v$ for all neighbouring pairs $\langle u, v\rangle$, so that the following requirements (b) and (c) will be satisfied.

(b) J_{uv} is OD, $\mathrm{pr}_1 \mathsf{J}_{uv} = X_u$, $\mathrm{pr}_2 \mathsf{J}_{uv} = X_v$, and $\mathsf{J}_{u^\wedge i\,,\,v^\wedge i} \subseteq \mathsf{J}_{uv}$ for every neighbouring pair $\langle u, v\rangle$ and each $i \in \{0,1\}$.

(c) For any k, the set $\mathsf{J}_k = \mathsf{J}_{0^{k\wedge}0\,,\,0^{k\wedge}1}$ is $\mathcal{T}$-separable, and $\mathsf{J}_k \subseteq \mathsf{E}_\alpha$ for some ordinal $\alpha = \alpha(k) < \omega_1$.

Notice that if $\langle u, v\rangle$ is neighbouring then $\langle u^\wedge i, v^\wedge i\rangle$ is neighbouring, but $\langle u^\wedge i, v^\wedge j\rangle$ is not neighbouring for $i \neq j$ (unless $u = v = 0^k$ for some k).

It follows that $X_u \,\mathsf{J}_{uv}\, X_v$, therefore $X_u \,\mathsf{E}\, X_v$, for all neighbouring pairs u, v. [10]

Remark 25 Every pair of $u,\, v \in 2^n$ can be tied in 2^n by a finite chain of neighbouring pairs. It follows that $X_u \,\mathsf{E}\, X_v$ and $X_u \,\overline{\mathsf{E}}\, X_v$ hold for *all* pairs $u,\, v \in 2^n$. □

Three more requirements will concern genericity.

Let $\{\mathcal{X}_n : n \in \omega\}$ be a fixed (not necessarily OD) enumeration of all dense in $\mathbb{X}$ below X_0 subsets of $\mathbb{X}$. Let $\{\mathcal{P}_n : n \in \omega\}$ be a fixed enumeration of all dense in $\mathbb{P}$ below P_0 subsets of $\mathbb{P}$. It is assumed that $\mathcal{X}_{n+1} \subseteq \mathcal{X}_n$ and $\mathcal{P}_{n+1} \subseteq \mathcal{P}_n$. Note that

[10] We recall that $X \,\mathsf{J}\, Y$ means that $\forall\, x \in X\; \exists\, y \in Y\; (x \,\mathsf{J}\, y)$ and $\forall\, y \in Y\; \exists\, x \in X\; (x \,\mathsf{J}\, y)$.

$\mathcal{X}' = \{P \in \mathbb{P} : P \subseteq P_0 \ \& \ \mathrm{pr}_1 P \cap \mathrm{pr}_2 P = \emptyset\}$ is dense in $\mathbb{P}$ below P_0 by Lemma 24, so we can suppose in addition that $\mathcal{P}_0 = \mathcal{X}'$.

In general, for any $\mathcal{T}$-separable set S let $\{\mathcal{X}_n(S) : n \in \omega\}$ be a fixed enumeration of all dense subsets in the algebra $\mathcal{P}^{\mathrm{OD}}(S) \setminus \{\emptyset\}$ such that $\mathcal{X}_{n+1}(S) \subseteq \mathcal{X}_n(S)$.

We now formulate the three additional requirements.

(g1) $X_u \in \mathcal{X}_n$ whenever $u \in 2^n$.

(g2) If $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ (that is, the last terms of u, v are different), then $P_{uv} = (X_u \times X_v) \cap \overline{\mathsf{E}} \in \mathcal{P}_n$.

(g3) If $\langle u, v\rangle = \langle 0^k{}^\wedge 0^\wedge r, 0^k{}^\wedge 1^\wedge r\rangle \in (2^n)^2$ then $\mathsf{J}_{uv} \in \mathcal{X}_n(\mathsf{J}_k)$.

In particular (g1) implies by Corollary 18 that for any $a \in 2^\omega$ the intersection $\bigcap_{n\in\omega} X_{a\,|\,n}$ contains a single point, denoted by $\phi(a)$, which is OD-generic over L, and the map ϕ is continuous in the Polish sense.

Assertion 26 *Assume* λ-CUH. ϕ *is a special continuous 1–1 embedding* E_0 *to* E *.*

Proof Let us prove that ϕ is 1–1. Suppose that $a \neq b \in 2^\omega$. Then $a(n-1) \neq b(n-1)$ for some n. Let $u = a\,|\,n$, $v = b\,|\,n$, so that we have $x = \phi(a) \in X_u$ and $y = \phi(b) \in X_v$. But then the set $P = (X_u \times X_v) \cap \overline{\mathsf{E}}$ belongs to $\mathcal{P}_n$ by (g2), therefore to $\mathcal{P}_0$. This implies $X_u \cap X_v = \emptyset$ by definition of $\mathcal{P}_0$, hence $\phi(a) \neq \phi(b)$ as required.

Furthermore if $a \not\mathsf{E}_0\, b$ (which means that $a(k) \neq b(k)$ for infinitely many numbers k) then $\langle\phi(a), \phi(b)\rangle$ is $\mathbb{P}$-generic by (g2), so $\phi(a) \not\mathsf{E}\, \phi(b)$ by Lemma 23.

Let us finally verify that $\langle\phi(0^k{}^\wedge 0^\wedge c), \phi(0^k{}^\wedge 1^\wedge c)\rangle \in \mathsf{E}_\alpha$ for all $c \in \mathcal{D}$ and $k \in \omega$, where $\alpha = \sup_k \alpha(k) < \omega_1$. The sequence of sets $W_m = \mathsf{J}_{0^k{}^\wedge 0^\wedge c\,|\,m,\, 0^k{}^\wedge 1^\wedge c\,|\,m}$ $(m \in \omega)$ is then generic over L by (g3) in the sense of the forcing $\mathcal{P}^{\mathrm{OD}}(\mathsf{J}_k) \setminus \{\emptyset\}$ (we recall that $\mathsf{J}_k = \mathsf{J}_{0^k{}^\wedge 0,\, 0^k{}^\wedge 1}$), which is simply a copy of $\mathbb{X}$, so that by Corollary 18 the intersection of all sets W_m is a singleton. Obviously the singleton can be only equal to $\langle\phi(0^k{}^\wedge 0^\wedge c)\,,\, \phi(0^k{}^\wedge 1^\wedge c)\rangle$. We conclude that $\phi(0^k{}^\wedge 0^\wedge c)\; \mathsf{E}_\alpha\; \phi(0^k{}^\wedge 1^\wedge c)$, as required. □

5–B Two preliminary lemmas

Thus the theorem is reduced to the construction of sets X_u and J_{uv}. Before the construction starts, we prove a couple of important lemmas.

Lemma 27 *Assume* λ-CUH. *Let* $X, Y \subseteq \mathcal{D} \cap \mathsf{Weak}_\lambda(\mathrm{L})$ *be* OD *sets such that* $(X \times Y) \cap \mathsf{E}$ *is nonempty. Then* $(X \times Y) \cap \mathsf{E}$ *contains a weak over* L *point* $\langle x, y\rangle$ *.*

Proof First of all, by Proposition 6 we can assume that $X = \mathsf{X}_t(\mathrm{L})$ and $Y = \mathsf{X}_{t'}(\mathrm{L})$, where t and t' belong to some $\mathrm{Term}_\alpha \cap \mathrm{L}$, $\alpha < \lambda$. Then, since λ is a limit L-cardinal, we have $X = \mathsf{X}_t(\mathrm{L}_\beta)$ and $Y = \mathsf{X}_{t'}(\mathrm{L}_\beta)$ for a suitable β, $\alpha \leq \beta < \lambda$. Take an arbitrary $\beta^{<\omega}$-generic over L function $f \in \beta^\omega$. Then the statement $(X \times Y) \cap \mathsf{E} \neq \emptyset$ turns out to be a $\varSigma^1_1$ formula with reals in $\mathrm{L}[f]$ (those coding f, t, t') as parameters. Notice that all sets in $\mathrm{L}[f]$ are weak over L, so it remains to apply Shoenfield. □

Lemma 28 *Assume* λ-CUH. *Let* $n \in \omega$, *and* X_u *be a nonempty* OD *set for each* $u \in 2^n$. *Assume that an* OD *set* $\mathsf{J}_{uv} \subseteq \mathcal{N}^2$ *is given for every neighbouring pair of* $u, v \in 2^n$ *so that* $X_u \, \mathsf{J}_{uv} \, X_v$.

1. *If* $u_0 \in 2^n$ *and* $X' \subseteq X_{u_0}$ *is* OD *and nonempty then there exists a system of* OD *nonempty sets* $Y_u \subseteq X_u$ $(u \in 2^n)$ *such that* $Y_u \, \mathsf{J}_{uv} \, Y_v$ *holds for all neighbouring pairs* u, v, *and in addition* $Y_{u_0} = X'$.
2. *Suppose that* $u_0, v_0 \in 2^n$ *is a neighbouring pair and nonempty* OD *sets* $X' \subseteq X_{u_0}$ *and* $X'' \subseteq X_{v_0}$ *satisfy* $X' \, \mathsf{J}_{u_0 v_0} X''$. *Then there exists a system of* OD *nonempty sets* $Y_u \subseteq X_u$ $(u \in 2^n)$ *such that* $Y_u \, \mathsf{J}_{uv} \, Y_v$ *holds for all neighbouring pairs* u, v, *and in addition* $Y_{u_0} = X'$, $Y_{v_0} = X''$.

Proof Notice that 1 follows from 2. Indeed take arbitrary v_0 such that either $\langle u_0, v_0 \rangle$ or $\langle v_0, u_0 \rangle$ is neighbouring, and put respectively $X'' = \{y \in X_{v_0} : \exists\, x \in X' \, (x \, \mathsf{J}_{u_0 v_0} \, y)\}$, or $X'' = \{y \in X_{v_0} : \exists\, x \in X' \, (y \, \mathsf{J}_{v_0 u_0} \, x)\}$.

To prove item 2, we use induction on n.

For $n = 1$ — then $u_0 = \langle 0 \rangle$ and $v_0 = \langle 1 \rangle$ — we take $Y_{u_0} = Y'$ and $Y_{v_0} = Y''$.

The step. We prove the lemma for $n+1$ provided it has been proved for n; $n \geq 1$. The principal idea is to divide 2^{n+1} on two copies of 2^n, minimally connected by neighbouring pairs, and handle them more or less separately using the induction hypothesis. The two "copies" are $U_0 = \{s^\wedge 0 : s \in 2^n\}$ and $U_1 = \{s^\wedge 1 : s \in 2^n\}$.

The only neighbouring pair that connects U_0 and U_1 is the pair of $\hat{u} = 0^{n\wedge}0$ and $\hat{v} = 0^{n\wedge}1$. If in fact $u_0 = \hat{u}$ and $v_0 = \hat{v}$ then we apply the induction hypothesis (item 1) independently for the families $\{X_u : u \in U_0\}$ and $\{X_u : u \in U_1\}$ and the given sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. Assembling the results, we get nonempty OD sets $Y_u \subseteq X_u$ $(u \in 2^{n+1})$ such that $Y_u \, \mathsf{J}_{uv} \, Y_v$ for all neighbouring pairs u, v, perhaps with the exception of the pair of $u = u_0 = \hat{u}$ and $v = v_0 = \hat{v}$, and in addition $Y_{u_0} = X'$ and $Y_{v_0} = X''$. Thus finally $Y_{\hat{u}} \, \mathsf{J}_{\hat{u}\hat{v}} \, Y_{\hat{v}}$ by the choice of X' and Y' .

It remains to consider the case when both u_0 and v_0 belong to one and the same domain, say to U_0. Then we first apply the induction hypothesis (item 2) to the family $\{X_u : u \in U_0\}$ and the sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. This results in a system of nonempty OD sets $Y_u \subseteq X_u$ $(u \in U_0)$; in particular we get an OD nonempty set $Y_{\hat{u}} \subseteq X_{\hat{u}}$. We put $Y_{\hat{v}} = \{y \in X_{\hat{v}} : \exists\, x \in Y_{\hat{u}} \, (x \, \mathsf{J}_{\hat{u}\hat{v}} \, y)\}$, so that $Y_{\hat{u}} \, \mathsf{J}_{\hat{u}\hat{v}} \, Y_{\hat{v}}$, and apply the induction hypothesis (item 1) to the family $\{X_u : u \in U_1\}$ and the set $Y_{\hat{v}} \subseteq X_{\hat{v}}$. $\square$

5–C The construction

We put $X_\Lambda = X_0$.

Now assume that the sets X_s $(s \in 2^n)$ and relations J_{st} for all neighbouring pairs of $s, t \in 2^{\leq n}$ have been defined, and expand the construction at level $n+1$.

We first put $A_{s^\wedge i} = X_s$ for all $s \in 2^n$ and $i \in \{0, 1\}$. We also define $\mathsf{Q}_{uv} = \mathsf{J}_{st}$ for any neighbouring pair of $u = s^\wedge i$, $v = t^\wedge i$ in 2^{n+1} other than the pair $\hat{u} = 0^{n\wedge}0$, $\hat{v} =$

$0^n{}^\wedge 1$. For the latter one (notice that $A_{\hat u} = A_{\hat v} = X_{0^n}$) we put $\mathsf{Q}_{\hat u \hat v} = \overline{\mathsf{E}}$, so that $A_u \,\mathsf{Q}_{uv}\, A_v$ holds for all neighbouring pairs of $u, v \in 2^{n+1}$ including the pair $\langle \hat u, \hat v\rangle$.

The sets A_u and relations Q_{uv} will be reduced in several steps to meet requirements (a), (b), (c) and (g1), (g2), (g3) of Subsection 5–A.

Part 1. After 2^{n+1} steps of the procedure of Lemma 28 (item 1) we obtain a system of nonempty OD sets $B_u \subseteq A_u$ ($u \in 2^{n+1}$) such that still $B_u \,\mathsf{Q}_{uv}\, B_v$ for all neighbouring pairs u, v in 2^{n+1}, but $B_u \in \mathcal{X}_{n+1}$ for all u. Thus (g1) is fixed.

Part 2. To fix (g2), consider an arbitrary pair of $u_0 = s_0{}^\wedge 0$, $v_0 = t_0{}^\wedge 1$, where $s_0, t_0 \in 2^n$. By Remark 25 and density of the set $\mathcal{P}_{n+1}$ there exist nonempty OD sets $B' \subseteq B_{u_0}$ and $B'' \subseteq B_{v_0}$ such that $P = (B' \times B'') \cap \overline{\mathsf{E}} \in \mathcal{P}_{n+1}$ and $\mathrm{pr}_1 P = B'$, $\mathrm{pr}_2 P = B''$, so in particular $B' \,\overline{\mathsf{E}}\, B''$. Now we apply Lemma 28 (item 1) separately for the two systems of sets, $\{B_{s^\wedge 0} : s \in 2^n\}$ and $\{B_{t^\wedge 1} : t \in 2^n\}$ (compare with the proof of Lemma 28 !), and the sets $B' \subseteq B_{s_0{}^\wedge 0}$, $B'' \subseteq B_{t_0{}^\wedge 1}$ respectively. This results in a system of nonempty OD sets $B'_u \subseteq B_u$ ($u \in 2^{n+1}$) satisfying $B'_{u_0} = B'$ and $B'_{v_0} = B''$, so that we have $(B'_{u_0} \times B'_{v_0}) \cap \overline{\mathsf{E}} \in \mathcal{P}_{n+1}$, and still $B'_u \,\mathsf{Q}_{uv}\, B'_v$ for all neighbouring pairs $u, v \in 2^{n+1}$, perhaps with the exception of the pair of $\hat u = 0^n{}^\wedge 0$, $\hat v = 0^n{}^\wedge 1$, which is the only one that connects the two domains. To handle this exceptional pair, note that $B'_{\hat u} \,\overline{\mathsf{E}}\, B'_{u_0}$ and $B'_{\hat v} \,\overline{\mathsf{E}}\, B'_{v_0}$ (Remark 25 is applied to each of the two domains), so that $B'_{\hat u} \,\overline{\mathsf{E}}\, B'_{\hat v}$ since $B' \,\overline{\mathsf{E}}\, B''$. Finally we observe that $\mathsf{Q}_{\hat u \hat v}$ is so far equal to $\overline{\mathsf{E}}$.

After 2^{n+1} steps (the number of pairs u_0, v_0 to be considered) we get a system of nonempty OD sets $C_u \subseteq B_u$ ($u \in 2^{n+1}$) such that $(C_u \times C_v) \cap \overline{\mathsf{E}} \in \mathcal{P}_{n+1}$ whenever $u(n) \neq v(n)$, and still $C_u \,\mathsf{Q}_{uv}\, C_v$ for all neighbouring pairs $u, v \in 2^{n+1}$. Thus (g2) is fixed.

Part 3. We fix (c) for the exceptional neighbouring pair of $\hat u = 0^n{}^\wedge 0$, $\hat v = 0^n{}^\wedge 1$. Since E is $\mathcal{T}^2$-dense in $\overline{\mathsf{E}}$, and $C_{\hat u} \,\overline{\mathsf{E}}\, C_{\hat v}$, the set $\mathsf{R} = (C_{\hat u} \times C_{\hat v}) \cap \mathsf{E}$ is nonempty. We observe that the OD set

$$\mathsf{R}' = \{\langle x, y\rangle \in \mathsf{R} : \langle x, y\rangle \text{ is weak over } \mathrm{L}\}$$

is nonempty, too, by Lemma 27. Then, since $\mathsf{R}' \subseteq \mathsf{R} \subseteq \mathsf{E}$, the intersection $\mathsf{R}'' = \mathsf{R}' \cap \mathsf{E}_\alpha$ is nonempty for some $\alpha < \omega_1$. (E_α is the α-th constituent of the Σ^1_1-set E.) Finally some nonempty OD set $\mathsf{Q} \subseteq \mathsf{R}''$ is $\mathcal{T}$-separable by Lemma 16. Consider the OD sets $C' = \mathrm{pr}_1\, \mathsf{Q}$ ($\subseteq C_{\hat u}$) and $C'' = \mathrm{pr}_2\, \mathsf{Q}$ ($\subseteq C_{\hat v}$); obviously $C' \,\mathsf{Q}\, C''$, so that $C' \,\mathsf{Q}_{\hat u \hat v}\, C''$. (We recall that at the moment $\mathsf{Q}_{\hat u \hat v} = \overline{\mathsf{E}}$.) Using Lemma 28 (item 2) again, we obtain a system of nonempty OD sets $Y_u \subseteq C_u$ ($u \in 2^{n+1}$) such that still $Y_u \,\mathsf{Q}_{uv}\, Y_v$ for all neighbouring pairs u, v in 2^{n+1}, and $Y_{\hat u} = C'$, $Y_{\hat v} = C''$. We re–define $\mathsf{Q}_{\hat u \hat v}$ by $\mathsf{Q}_{\hat u \hat v} = \mathsf{Q}$ (then $\mathsf{Q}_{\hat u \hat v} \subseteq \mathsf{E}_\alpha$), but this keeps $Y_{\hat u} \,\mathsf{Q}_{\hat u \hat v}\, Y_{\hat v}$.

Part 4. We fix (g3). Consider a neighbouring pair u_0, v_0 in 2^{n+1}. Then we have $u_0 = 0^k{}^\wedge 0^\wedge r$, $v_0 = 0^k{}^\wedge 1^\wedge r$ for some $k \le n$ and $r \in 2^{n-k}$. It follows that $\mathsf{Q}' = \mathsf{Q}_{u_0 v_0} \cap (Y_{u_0} \times Y_{v_0})$ is a nonempty (since $Y_{u_0} \,\mathsf{Q}_{u_0 v_0}\, Y_{v_0}$) OD subset of $\mathsf{J}_k =$

$\mathsf{J}_{0^k\wedge 0\,,\,0^k\wedge 1}$ by the construction. Let $\mathsf{Q} \subseteq \mathsf{Q}'$ be a nonempty OD set in $\mathcal{X}_{n+1}(\mathsf{J}_k)$. We now define $Y' = \mathrm{pr}_1\,\mathsf{Q}$ and $Y'' = \mathrm{pr}_2\,\mathsf{Q}$ (then $Y'\,\mathsf{Q}\,Y''$ and $Y'\,\mathsf{Q}_{u_0v_0}\,Y''$) and run Lemma 28 (item 2) for the system of sets Y_u $(u \in 2^{n+1})$ and the sets $Y' \subseteq Y_{u_0}$, $Y'' \subseteq Y_{v_0}$. After this define the "new" $\mathsf{Q}_{u_0v_0}$ by $\mathsf{Q}_{u_0v_0} = \mathsf{Q}$.

Do this consequtively for all neighbouring pairs; the finally obtained sets – let them be X_u $(u \in 2^{n+1})$ – are as required. The final relations J_{uv} $(u,\, v \in 2^{n+1})$ can be obtained as the restrictions of sets Q_{uv} to $X_u \times X_v$.

This ends the construction.

This also ends the proof of theorems 15 and 3, and Theorem 1 (the main theorem), see Subsection 4–B. □

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