Complexity and Algorithms for Semipaired Domination in Graphs

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Abstract

For a graph $G = (V, E)$ with no isolated vertices, a set $D \subseteq V$ is called a semipaired dominating set of G if (i) D is a dominating set of G, and (ii) D can be partitioned into two element subsets such that the vertices in each two element set are at distance at most two. The minimum cardinality of a semipaired dominating set of G is called the semipaired domination number of G, and is denoted by $\gamma_{pr2}(G)$. The MINIMUM SEMIPAIRED DOMINATION problem is to find a semipaired dominating set of G of cardinality $\gamma_{pr2}(G)$. In this paper, we initiate the algorithmic study of the MINIMUM SEMIPAIRED DOMINATION problem. We show that the decision version of the MINIMUM SEMIPAIRED DOMINATION problem is NP-complete for bipartite graphs and split graphs. On the positive side, we present a linear-time algorithm to compute a minimum cardinality semipaired dominating set of interval graphs and trees. We also propose a $1 + \ln(2\Delta + 2)$-approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem, where Δ denote the maximum degree of the graph and show that the MINIMUM SEMIPAIRED DOMINATION problem cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME$(|V|^{O(\log \log |V|)})$.

Keywords: Domination, Semipaired Domination, Bipartite Graphs, Chordal Graphs, Graph algorithm, NP-complete, Approximation algorithm.

1 Introduction

A *dominating set* in a graph G is a set D of vertices of G such that every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. The MINIMUM DOMINATION problem is to find a dominating set of cardinality $\gamma(G)$. More thorough treatment of domination, can be found in the books [6, 7]. A dominating set D is called a *paired dominating set* if $G[D]$ contains a perfect matching. The *paired domination number* of G, denoted by $\gamma_{pr}(G)$ is the minimum cardinality of paired dominating set of G. The concept of paired domination was introduced by Haynes and Slater in [11].

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A relaxed form of paired domination called semipaired domination was introduced by Haynes and Henning [8] and studied further in [12, 9, 10]. A set S of vertices in a graph G with no isolated vertices is a *semipaired dominating set*, abbreviated a semi-PD-set, of G if S is a dominating set of G and S can be partitioned into 2-element subsets such that the vertices in each 2-element set are at distance at most 2. In other words, the vertices in the dominating set S can be partitioned into 2-element subsets such that if $\{u, v\}$ is a 2-set, then the distance between u and v is either 1 or 2. We say that u and v are *semipaired*. The *semipaired domination number* of G, denoted by $\gamma_{pr2}(G)$, is the minimum cardinality of a semi-PD-set of G. Since every paired dominating set is a semi-PD-set, and every semi-PD-set is a dominating set, we have the following observation.

Observation 1.1. ([8]) *For every isolate-free graph G, $\gamma(G) \leq \gamma_{pr2}(G) \leq \gamma_{pr}(G)$.*

By Observation 1.1, the semipaired domination number is squeezed between two fundamental domination parameters, namely the domination number and the paired domination number.

More formally, the minimum semipaired domination problem and its decision version are defined as follows:

MINIMUM SEMIPAIRED DOMINATION problem (MSPDP)

Instance: A graph $G = (V, E)$.

Solution: A semi-PD-set D of G.

Measure: Cardinality of the set D.

SEMIPAIRED DOMINATION DECISION problem (SPDDP)

Instance: A graph $G = (V, E)$ and a positive integer $k \leq |V|$.

Question: Does there exist a semi-PD-set D in G such that $|D| \leq k$?

In this paper, we initiate the algorithmic study of the semipaired domination problem. The main contributions of the paper are summarized below. In Section 2, we discuss some definitions and notations. In Section 3, we discuss the difference between the complexity of paired domiantion and semipaired domination in graphs. In Section 4, we show that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for bipartite and split graphs. In Section 5 and Section 6, we propose a linear-time algorithms to solve the MINIMUM SEMIPAIRED DOMINATION problem in interval graphs and trees respectively. In Section 7, we propose an approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem in general graphs. In Section 8, we discuss an approximation hardness result. Finally, Section 9, concludes the paper.

2 Terminology and Notation

For notation and graph theory terminology, we in general follow [13]. Specifically, let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$, and let v be a vertex in V. The *open neighborhood* of v is the set $N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. Thus, a set D of vertices in G is a dominating set of G if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V \setminus D$, while D is a total dominating set of G if $N_G(v) \cap D \neq \emptyset$ for every vertex $v \in V$. The *distance* between two vertices u and v in a connected graph G, denoted by $d_G(u, v)$, is the length of a shortest (u, v)-path in G. If the graph G is clear from the context, we omit it in the above expressions. We write $N(v)$, $N[v]$ and $d(u, v)$ rather than $N_G(v)$, $N_G[v]$ and $d_G(u, v)$, respectively.

For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. If $G[C]$, where $C \subseteq V$, is a complete subgraph of G, then C is a *clique* of G. A set $S \subseteq V$ is an *independent set* if $G[S]$ has no edge. A graph G is *chordal* if every cycle in G of length at least four has a *chord*, that is, an edge joining two non-consecutive vertices of the cycle. A chordal graph $G = (V, E)$ is a *split graph* if V can be partitioned into two sets I and C such that C is a clique and I is an independent set. A vertex $v \in V(G)$ is a *simplicial* vertex of G if $N_G[v]$ is a clique of G. An ordering $\alpha = (v_1, v_2, ..., v_n)$ is a *perfect elimination ordering* (PEO) of vertices of G if v_i is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, ..., v_n\}]$ for all i, $1 \leq i \leq n$. Fulkerson and Gross [4] characterized chordal graphs, and showed that a graph G is chordal if and only if it has a PEO. A graph $G = (V, E)$ is *bipartite* if V can be partitioned into two disjoint sets X and Y such that every edge of G joins a vertex in X to a vertex in Y, and such a partition (X, Y) of $V(G)$ is called a *bipartition* of G. Further, we denote such a bipartite graph G by $G = (X, Y, E)$. A graph G is an *interval graph* if there exists a one-to-one correspondence between its vertex set and a family of closed intervals in the real line, such that two vertices are adjacent if and only if their corresponding intervals intersect. Such a family of intervals is called an *interval model* of a graph.

In the rest of the paper, all graphs considered are simple connected graphs with at least two vertices, unless otherwise mentioned specifically. We use the standard notation $[k] = \{1, \ldots, k\}$. For most of the approximation related terminologies, we refer to [1, 14].

3 Complexity difference between paired domination and semipaired domination

In this section, we make an observation on complexity difference between paired domination and semipaired domination. We show that the decision version of the Minimum paired domination problem is NP-complete for GP4 graphs, but the Minimum Semipaired Domination problem is easily solvable for GP4 graphs. The class of GP4 graphs was introduced by Henning and Pandey in [15]. Below we recall the definition of GP4 graphs.

Definition 3.1 (GP4-graph)**.** *A graph $G = (V, E)$ is called a* GP4-graph *if it can be obtained from a general connected graph $H = (V_H, E_H)$ where $V_H = \{v_1, v_2, \ldots, v_{n_H}\}$, by adding a path of length 3 to every vertex of H. Formally, $V = V_H \cup \{w_i, x_i, y_i, z_i \mid 1 \leq i \leq n_H\}$ and $E = E_H \cup \{v_iw_i, w_ix_i, x_iy_i, y_iz_i \mid 1 \leq i \leq n_H\}$.*

Theorem 3.1. *If G is a* GP4-graph*, then $\gamma_{pr2}(G) = \frac{2}{5}|V(G)|$.*

Lemma 3.1. *If G is a GP4-graph constructed from a graph H as in Definition 3.1, then H has a paired dominating set of cardinality k, $k \leq n_H$ if and only if G has a semi-PD-set of cardinality $2n_H + k$.*

Since the decision version of the Minimum Paired Domination problem is known to be NP-complete for general graphs [11], the following theorem follows directly from Lemma 3.1.

Theorem 3.2. *The decision version of the* Minimum Paired Domination *problem is NP-complete for GP4-graphs.*

4 NP-completeness Results

In this section, we study the NP-completeness of the SEMIPAIRED DOMINATION DECISION problem. We show that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for bipartite graphs and split graphs.

4.1 NP-completeness proof for bipartite graphs

Theorem 4.1. *The* SEMIPAIRED DOMINATION DECISION *problem is NP-complete for bipartite graphs.*

Proof. Clearly, the SEMIPAIRED DOMINATION DECISION problem is in NP for bipartite graphs. To show the hardness, we give a polynomial reduction from the MINIMUM VERTEX COVER problem. Given a non-trivial graph $G = (V, E)$, where $V = \{v_1, v_2, \dots, v_n\}$ and $E = \{e_1, e_2, \dots, e_m\}$, we construct a graph $H = (V_H, E_H)$ in the following way:

Let $V_k = \{v_i^k \mid i \in [n]\}$ and $E_k = \{e_j^k \mid j \in [m]\}$ for $k \in [2]$. Also assume that $A = \{a_i \mid i \in [n]\}$, $B = \{b_i \mid i \in [n]\}$, $C = \{c_i \mid i \in [n]\}$, and $F = \{f_i \mid i \in [n]\}$.

Now define $V_H = V_1 \cup V_2 \cup E_1 \cup E_2 \cup A \cup B \cup C \cup F$,
and $E_H = \{v_i^1 f_i, v_i^2 f_i, a_i b_i, b_i c_i, a_i f_i \mid i \in [n]\} \cup \{v_p^k e_i^k, v_q^k e_i^k \mid k \in [2], i \in [m], v_p, v_q$ are endpoints of edge e_i in $G\}$. Fig. 2 illustrates the construction of H from G.

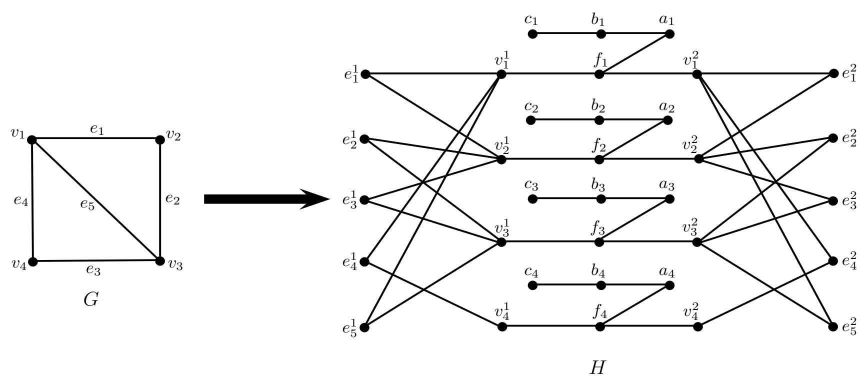


Figure 1: An illustration of the construction of H from G in the proof of Theorem 4.1.

Note that the set $I_1 = V_1 \cup V_2 \cup A \cup C$ is an independent set in H. Also, the set $I_2 = E_1 \cup E_2 \cup F \cup B$ is an independent set in H. Since $V_H = I_1 \cup I_2$, the graph H is a bipartite graph. Now to complete the proof, it suffices for us to prove the following claim:

Claim 4.1. *The graph G has a vertex cover of cardinality at most k if and only if the graph H has a semi-PD-set of cardinality at most $2n + 2k$.*

Proof. Let $V_c = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$ be a vertex cover of G of cardinality k. Then $D_p = \{v_{i_1}^1, v_{i_2}^1, \dots, v_{i_k}^1\} \cup \{v_{i_1}^2, v_{i_2}^2, \dots, v_{i_k}^2\} \cup B \cup F$ is a semi-PD-set of H of cardinality $2n + 2k$.

Conversely, suppose that H has a semi-PD-set D of cardinality at most $2n + 2k$. Note that $D \cap \{a_i.b_i, c_i, f_i\}| \geq 2$ for each $i \in [n]$. Hence, without loss of generality, we may assume that $\{b_i, f_i \mid i \in [n]\} \subseteq D$, where b_i and f_i are semipaired. Hence $|D \cap (E_1 \cup E_2 \cup V_1 \cup V_2)| \leq 2k$. Let $S = (V_1 \cup E_1) \cap D$. Without loss of generality, we may also assume that $|S| \leq k$. Now, if $e_i^1 \in S$ for some $i \in [m]$, and none of its neighbors belongs to D, then e_i^1 must be semipaired with some vertex e_j^1 where $j \in [m] \setminus \{i\}$, and also there must exists a vertex v_k^1 which is a common neighbor of e_i^1 and e_j^1. In this case, we replace the vertex e_i^1 in the set S with the vertex v_k^1 and so $S \leftarrow (S \setminus \{e_i^1\}) \cup \{v_k^1\}$ where v_k^1 and e_j^1 are semipaired. We do this for each vertex $e_i^1 \in S$ where $i \in [m]$ with none of its neighbors in the set D. For the resulting set S, $|S \cap V_1| \leq k$ and every vertex e_i^1 has a neighbor in $V_1 \cap S$. The set $V_c = \{v_i \mid v_i^1 \in S\}$ is a vertex cover of G of cardinality at most k. This completes the proof of the claim. □

Hence, the theorem is proved. □

4.2 NP-completeness result for split graphs

Theorem 4.2. *The* SEMIPAIRED DOMINATION DECISION *problem is NP-complete for split graphs.*

Proof. Clearly, the SEMIPAIRED DOMINATION DECISION problem is in NP. To show the hardness, we give a polynomial time reduction from the DOMINATION DECISION problem, which is well known NP-complete problem. Given a non-trivial graph $G = (V, E)$, where $V = \{v_i \mid i \in [n]\}$ and $E = \{e_j \mid j \in [m]\}$, we construct a split graph $G' = (V_{G'}, E_{G'})$ as follows:

Let $V_k = \{v_i^k \mid i \in [n]\}$ and $U_k = \{u_i^k \mid i \in [n]\}$ for $k \in [2]$. Now define $V_{G'} = V_1 \cup V_2 \cup U_1 \cup U_2$, and $E_{G'} = \{uv \mid u, v \in V_1 \cup U_1, u \neq v\} \cup \{v_i^2 v_j^1, u_i^2 u_j^1 \mid i \in [n] \text{ and } v_j \in N_G[v_i]\}$. Note that the set $A = V_1 \cup U_1$ is a clique in G' and the set $B = V_2 \cup U_2$ is an independent set in G'. Since $V_{G'} = A \cup B$, the constructed graph G' is a split graph. Fig. 2 illustrates the construction of G' from G.

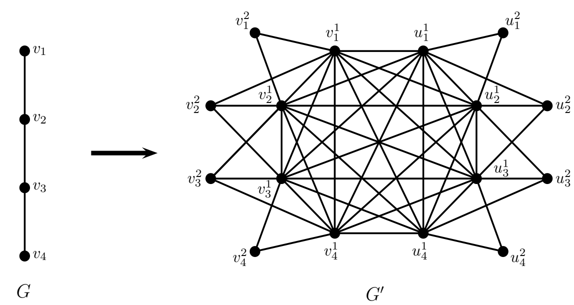


Figure 2: An illustration to the construction of G' from G in the proof of Theorem 4.2.

Now, to complete the proof of the theorem, we only need to prove the following claim.

Claim 4.2. *G has a dominating set of cardinality k if and only if G' has a semi-PD-set of size cardinality $2k$.*

Proof. Let $D = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$ be a dominating set of size atmost k of G. Then $D_{sp} = \{v_{i_1}^1, v_{i_2}^1, \ldots, v_{i_k}^1\} \cup \{u_{i_1}^1, u_{i_2}^1, \ldots, u_{i_k}^1\}$ is a semi-PD-set of G' of size atmost $2k$.

Conversely, suppose that G has a semi-PD-set D_{sp} of cardinality at most $2k$. Let $S_1 = (V_1 \cup V_2) \cap D_{sp}$ and $S_2 = U_1 \cup U_2 \cap D_{sp}$. Then either $|S_1| \leq k$ or $|S_2| \leq k$. Without loss of generality, let us assume that $|S_1| \leq k$. Note that if $v_i^2 \in S_1$ and none of neighbors belong to S_1 then we replace v_i^2 by some of its neighbor v_j^1 in the set S_1. So, we may assume that $S_1 \cap V_2 = \phi$. Now the set $D = \{v_i \mid v_i^1 \in S_1\}$ is a dominating set of G of size atmost k. Hence, the result follows. □

Hence, the theorem is proved. □

5 Algorithm for Interval Graphs

In this section, we present a linear-time algorithm to compute a minimum cardinality semi-PD-set of an interval graph.

A linear time recognition algorithm exists for interval graphs, and for an interval graph an interval family can also be constructed in linear time [2, 5]. Let $G = (V, E)$ be an interval graph and I be its interval model. For a vertex $v_i \in V$, let I_i be the corresponding interval. Let a_i and b_i denote the left and right end points of the interval I_i. Without loss of generality, we may assume that no two intervals share a common end point. Let $\alpha = (v_1, v_2, \ldots, v_n)$ be the *left end ordering* of vertices of G, that is, $a_i < a_j$ whenever $i < j$. Now we first prove the following lemmas.

Lemma 5.1. *Let $\alpha = (v_1, v_2, \ldots, v_n)$ be the left end ordering of vertices of G. If $v_iv_j \in E$ for $i < j$, then $v_iv_k \in E$ for every $i < k < j$.*

Proof. The proof directly follows from the left end ordering of vertices of G. $\square$

Define the set $V_i = \{v_1, v_2, \ldots, v_i\}$, for each $i \in [n]$.

Lemma 5.2. *If G is a connected interval graph, then $G[V_i]$ is also connected.*

Proof. The proof can easily be done using induction on i. $\square$

Let $F(v_i)$ be the least index vertex adjacent to v_i, that is, if $F(v_i) = v_p$, then $p = \min\{k \mid v_kv_i \in E\}$. In particular, we define $F(v_1) = v_1$. Let $L(v_i) = v_q$, where $q = \max\{k \mid v_kv_i \notin E \text{ and } k < i\}$. In particular, if $L(v_i)$ does not exist, we assume that $L(v_i) = v_0$ ($v_0 \notin V$). Let $G_i = G[V_i]$ and D_i denote a semi-PD-set of G_i of minimum cardinality. Recall that we only consider connected graphs with at least two vertices.

Lemma 5.3. *For $i \geq 2$, if $F(v_i) = v_1$, then $D_i = \{v_1, v_i\}$.*

Proof. Note that every vertex in G_i is dominated by v_1, and $d_{G_i}(v_1, v_i) = 1$. Hence, $D_i = \{v_1, v_i\}$. $\square$

Lemma 5.4. *For $i > 1$, if $F(v_i) = v_j$, $j > 1$ and $F(v_j) = v_1$, then $D_i = \{v_1, v_j\}$.*

Proof. Note that every vertex in G_i is dominated by some vertex in the set $\{v_1, v_j\}$, and $d_{G_i}(v_1, v_i) = 1$. Hence, $D_i = \{v_1, v_i\}$. $\square$

Lemma 5.5. *For $r < k < j < i$, let $F(v_i) = v_j$, $F(v_j) = v_k$ $F(v_k) = v_r$. If every vertex v_l where $k < l < j$, is adjacent to at least one vertex in the set $\{v_j, v_r\}$, then the following holds:*
(a) $\{v_j, v_r\} \subseteq D_i$.
(b) v_j is semipaired with v_r in D_i.
(c) $D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\} = \{v_j, v_r\}$.

Proof. (a) To dominate v_i, either $v_i \in D_i$ or $v_{i1} \in D_i$, where $j \leq i1 < i$ and $v_{i1} \in N_{G_i}(v_i)$. If $i1 \neq j$ and v_{i1} is semipaired with some vertex v_{j1}, then $N_{G_i}(v_{i1}) \subseteq N_{G_i}(v_j)$, and $d_{G_i}(v_j, v_{j1}) \leq 2$. Hence, we can update the set D_i as $D_i = (D_i \setminus \{v_{i1}\}) \cup \{v_j\}$ and semipair v_j with v_{j1}. This proves that $v_j \in D_i$.

If v_r also belongs to D_i, then we are done. Otherwise, if v_j is semipaired with v_{j1} (where $j1 \neq r$), then $j1 > r$. Also, $N_G[v_{j1}] \subseteq N_G[v_j] \cup N_G[v_r]$. In that case, we can update the set D_i as $D_i = (D_i \setminus \{v_{j1}\}) \cup \{v_r\}$. Hence, $\{v_j, v_r\} \subseteq D_i$.

(b) Suppose $\{v_j, v_r\} \subseteq D_i$. If v_j is semipaired with v_r in D_i, then we are done. Otherwise, if v_j is not semipaired with v_r, assume that v_j is semipaired with v_{j1} and v_r is semipaired with v_{r1}. Note that $j1$ must be greater than r, and $N_{G_i}[v_{j1}] \subseteq N_{G_i}[v_j] \cup N_{G_i}[v_r]$. Therefore, the set $D_i \setminus \{v_{j1}\}$ also dominates all the vertices of G_i.

Suppose that $N_{G_i}(v_{r1}) \subseteq D_i$. In this case, $D' = D_i \setminus \{v_{j1}, v_{r1}\}$ is a semi-PD-set of G_i where v_j and v_r are semipaired. This contradicts the fact that D_i is a semi-PD-set of G_i of minimum cardinality. Hence, $N_{G_i}(v_{r1}) \nsubseteq D_i$.

Let $v_{r2} \in D_i \setminus N_{G_i}(v_{r1})$. Now update the set D_i as follows: remove v_{j1} from D_i, add v_{r2} in the set D_i, semipair v_j with v_r and v_{r1} with v_{r2}. Clearly, the updated set is also a semi-PD-set of G_i of minimum cardinality. This proves that there always exists a semi-PD-set D_i of G_i such that $\{v_j, v_r\} \subseteq D_i$, and v_j is semipaired with v_r in D_i.

(c) We know that $\{v_j, v_r\} \subseteq D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}$. We need to show that $D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\} = \{v_j, v_r\}$, that is, there is no other vertex from the set $\{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}$ belongs to D_i. Suppose, to the contrary, that there does not exist any D_i for which $D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\} = \{v_j, v_r\}$. So, for each D_i, $|D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}| \geq 3$. Consider a set D_i for which $|D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}|$ is minimum.

Let $|D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}| = l$. Also, assume that $v_p \in D_i$, where $p \neq j, r$ and $s+1 \leq p \leq i$. Also, assume that v_p is semipaired with v_{p1} in D_i. Now consider the following two cases.
Case 1. $p1 > s$. If $v_s \in D_i$, then if, some vertex of the set $\{v_1, v_2, \ldots, v_s\}$ is dominated by v_p or v_{p1}, then that vertex is also dominated by v_s. In that case, $D_i \setminus \{v_p, v_{p1}\}$ is also a semi-PD-set of G_i, which is a contradiction. If $v_s \notin D_i$ and $N_{G_i}(v_s) \subseteq D_i$, then also $D_i \setminus \{v_p, v_{p1}\}$ is a semi-PD-set of G_i, which is again a contradiction. Hence, $v_s \notin D_i$ and $N_{G_s}(v_s) \nsubseteq D_i$. Suppose $v_q \in N_{G_s}(v_s) \cap D_i$. Then, update the set D_i as $D_i = (D_i \setminus \{v_p, v_{p1}\}) \cup \{v_s, v_q\}$. Note that D_i is still a semi-PD-set of G_i of minimum cardinality, and $|D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}| < l$, a contradiction.
Case 2. $p1 \leq s$. If $v_s \notin D_i$, then the updated set $D_i = (D_i \setminus \{v_p\}) \cup \{v_s\}$ is also a semi-PD-set of G_i of minimum cardinality. If $v_s \in D_i$ and $N_{G_s}(v_{p1}) \subseteq D_i$, then the updated set $D_i = D_i \setminus \{v_p, v_{p1}\}$ is also a semi-PD-set of G_i, a contradiction. If $v_s \in D_i$ and $N_{G_s}(v_{p1}) \nsubseteq D_i$, let $v_q \in N_{G_s}(v_{p1}) \setminus D_i$. Then, update D_i as $D_i = (D_i \setminus \{v_p\}) \cup \{v_q\}$. Note that D_i is still a semi-PD-set of G_i of minimum cardinality, and $|D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}| < l$, a contradiction.

Since both Case 1 and Case 2 produce a contradiction, there exists a semi-PD-set D_i of G_i of minimum cardinality, for which the set $D_i \cap \{v_{s+1}, \ldots, v_r, v_{r+1}, \ldots, v_i\}$ contains only v_j and v_r. ☐

Lemma 5.6. *For $r < k < j < i$, let $F(v_i) = v_j$, $F(v_j) = v_k$ $F(v_k) = v_r$. If every vertex v_l where $k < l < j$, is adjacent to at least one vertex in the set $\{v_j, v_r\}$, then the following holds.*
(a) $D_i = \{v_j, v_r\}$ if $L(v_r) = v_0$.
(b) $D_i = \{v_1, v_2, v_j, v_r\}$ if $L(v_r) = v_1$.
(c) $D_i = D_s \cup \{v_j, v_r\}$ if $L(v_r) = v_s$ with $s \geq 2$.

Proof. (a) Clearly $D_i = \{v_j, v_r\}$.

(b) From Lemma 5.5, we know that $\{v_j, v_r\} \subseteq D_i$. Also, other than v_1, all vertices are dominated by the set $\{v_j, v_r\}$. Hence, $D_i = \{v_1, v_2, v_j, v_r\}$.

(c) Clearly $D_s \cup \{v_j, v_r\}$ is a semi-PD-set of G_i. Hence $|D_i| \leq |D_s| + 2$. We also know that there exists a semi-PD-set D_i of G_i of minimum cardinality such that $D_i \cap \{v_{s+1}, v_{s+2}, \ldots, v_i\} = \{v_j, v_r\}$ (where v_j and v_r are semipaired in D_i). Hence $D_i \setminus \{v_j, v_r\} \subseteq V(G_s)$. Also, $\{v_j, v_r\}$ dominates the set

$\{v_{s+1}, v_{s+2}, \ldots, v_n\}$, implying that the set $\{v_1, v_2, \ldots, v_s\}$ is dominated by the vertices in $D_i \setminus \{v_j, v_r\}$. Hence, the set $D_i \setminus \{v_j, v_r\}$ is semi-PD-set of G_s. Therefore, $|D_s| \leq |D_i| - 2$. This proves that $|D_i| = |D_s| + 2$. Hence, $D_i = D_s \cup \{v_j, v_r\}$. □

Lemma 5.7. *For $r < k < j < i$, let $F(v_i) = v_j$, $F(v_j) = v_k$ $F(v_k) = v_r$, and $\{v_l \mid k < l < j\} \nsubseteq N_{G_i}[v_r] \cup N_{G_i}[v_j]$. Let $t = \max\{l \mid k < l < j$ and $v_l v_j \notin E\}$ (assume that such a t exists). Let $F(v_t) = v_b$. Then, the following holds.*
(a) $\{v_j, v_b\} \subseteq D_i$.
(b) v_j is semipaired with v_b in D_i.
(c) $D_i \cap \{v_{s+1}, \ldots, v_b, v_{b+1}, \ldots, v_i\} = \{v_j, v_b\}$.

Proof. (a) First we show that $v_j \in D_i$. Suppose $v_j \notin D_i$. Let v_p be the vertex dominating v_i in D_i. Note that $j < p \leq i$ and $N_{G_i}[v_p] \subseteq N_{G_i}[v_j]$. Let v_q be the vertex semipaired with v_p in D_i. Since $N[v_p] \subseteq N[v_j]$, any vertex which is within distance 2 from v_p is also within distance 2 from v_j. We can update D_i as $D_i \setminus \{v_p\} \cup \{v_j\}$ with v_j semipaired with v_q. Hence, D_i contains v_j. Similarly, we can show that D_i also contains v_b. So, $\{v_j, v_b\} \subseteq D_i$.

(b) If v_j is semipaired with v_b in D_i, then we are done. Suppose, to the contrary, that v_j is not semipaired with v_b in D_i. So, assume that v_j is semipaired with v_p and v_b is semipaired with v_q in D_i. We consider the four cases based on the values of the indices p and q.
Case 1. $p > b$ and $q > b$. Here, $N_{G_i}[v_p] \cup N_{G_i}[v_q] \subseteq N_{G_i}[v_j] \cup N_{G_i}[v_b]$. Hence, the set $D_i \setminus \{v_p, v_q\}$ is also a semi-PD-set of G_i, a contradiction.
Case 2. $p < b$ and $q < b$. Since the distance between v_p and v_j is at most 2, $p \geq r$. If $q < b$ and $d_{G_i}(v_q, v_b) \leq 2$, then $d_{G_i}(v_q, v_p) \leq 2$. So, in the set D_i, v_j can be semipaired with v_b, and v_p can be semipaired with v_q.
Case 3. $p > b$ and $q < b$. Here, $N_{G_i}[v_p] \subseteq N_{G_i}[v_j] \cup N_{G_i}[v_b]$. If $N_{G_i}(v_q) \subseteq D_i$, then the set $D_i \setminus \{v_p, v_q\}$ is also a semi-PD-set of G_i, a contradiction. If $N_{G_i}(v_q) \nsubseteq D_i$, let $v_x \in N_{G_i}(v_q) \setminus D_i$. Then update D_i as $D_i = (D_i \setminus \{v_p\}) \cup \{v_x\}$, and semipair v_q with v_x and v_j with v_b.
Case 4. $p < b$ and $q > b$. Since the distance between v_p and v_j is at most 2, $p \geq r$. Also $N_{G_i}[v_q] \subseteq N_{G_i}[v_j] \cup N_{G_i}[v_b]$. If $N_{G_i}(v_p) \subseteq D_i$, then the set $D_i \setminus \{v_p, v_q\}$ is also a semi-PD-set of G_i, a contradiction. If $N_{G_i}(v_p) \nsubseteq D_i$, let $v_y \in N_{G_i}(v_p) \setminus D_i$. Then update D_i as $D_i = (D_i \setminus \{v_q\}) \cup \{v_y\}$, and semipair v_p with v_y and v_j with v_b.

By the above four cases, there always exists a semi-PD-set D_i of G_i of minimum cardinality such that v_j is semipaired with v_b in D_i. This completes the proof of part (b).

(c) The proof is similar to the proof of Lemma 5.5(c), and hence is omitted. □

Lemma 5.8. *For $r < k < j < i$, let $F(v_i) = v_j$, $F(v_j) = v_k$ $F(v_k) = v_r$, and $\{v_l \mid k < l < j\} \nsubseteq N_{G_i}[v_r] \cup N_{G_i}[v_j]$. Let $t = \max\{l \mid k < l < j$ and $v_l v_j \notin E\}$ (assume that such a t exists). Let $F(v_t) = v_b$. Then, the following holds.*
(a) $D_i = \{v_j, v_b\}$ if $L(v_b) = v_0$.
(b) $D_i = \{v_1, v_2, v_j, v_b\}$ if $L(v_b) = v_1$.
(c) $D_i = D_s \cup \{v_j, v_b\}$ if $L(v_b) = v_s$ with $s \geq 2$.

Proof. The proof is similar to the proof of Lemma 5.6, and hence is omitted. □

Based on above lemmas, we present an algorithm to compute a minimum semi-PD-set of an interval graph.

Algorithm 1 SEMI-PAIRED-DOM-IG(G)

Input: An interval graph $G = (V, E)$ with a left end ordering $\alpha = (v_1, v_2, \ldots, v_n)$ of vertices of G.
Output: A semi-PD-set D of G of minimum cardinality.

```
V' = V;
while (V' ≠ φ) do
    Let i = max{k | v_k ∈ V'}. if (F(v_i) = v_1) then
        D = D ∪ {v_1, v_i};
        V' = V' \ {v_1, v_2, ..., v_i};
    else if (F(v_i) = v_j and F(v_j) = v_1 where j > 1) then
        D = D ∪ {v_1, v_j}; V' = V' \ {v_1, v_2, ..., v_i};
    else if (F(v_i) = v_j and F(v_j) = v_k where k ≥ 2) then
        Let F(v_k) = v_r. if {v_{k+1}, v_{k+2}, ..., v_{j-1}} ⊆ N_G[v_j] ∪ N_G[v_r] then
            if (L(v_r) = v_0) then
                D = D ∪ {v_j, v_r};
                V' = V' \ {v_1, v_2, ..., v_i};
            else if (L(v_r) = v_1) then
                D = D ∪ {v_1, v_2, v_j, v_r};
                V' = V' \ {v_1, v_2, ..., v_i};
            else
                Let (L(v_r) = v_s) where s ≥ 2.
                D = D ∪ {v_j, v_r};
                V' = V' \ {v_{s+1}, v_{s+2}, ..., v_i};
        else
            Let t = max{l | k < l < j and v_l ∉ N_G(v_j)} and F(v_t) = v_b. if (L(v_b) = v_0) then
                D = D ∪ {v_j, v_b};
                V' = V' \ {v_1, v_2, ..., v_i};
            else if (L(v_b) = v_1) then
                D = D ∪ {v_1, v_2, v_j, v_b};
                V' = V' \ {v_1, v_2, ..., v_i};
            else
                Let (L(v_b) = v_s) where s ≥ 2.
                D = D ∪ {v_j, v_b};
                V' = V' \ {v_{s+1}, v_{s+2}, ..., v_i};
```

Here, we illustrate the algorithm SEMI-PAIRED-DOM-IG, with the help of an example. An interval graph G and its interval model I is shown in Fig 3.

For the interval graph G given in Fig. 3, the algorithm SEMI-PAIRED-DOM-IG computes a semi-PD-set of minimum cardinality in 3 iterations. Below, we illustrate all the 3 iterations of the algorithm.

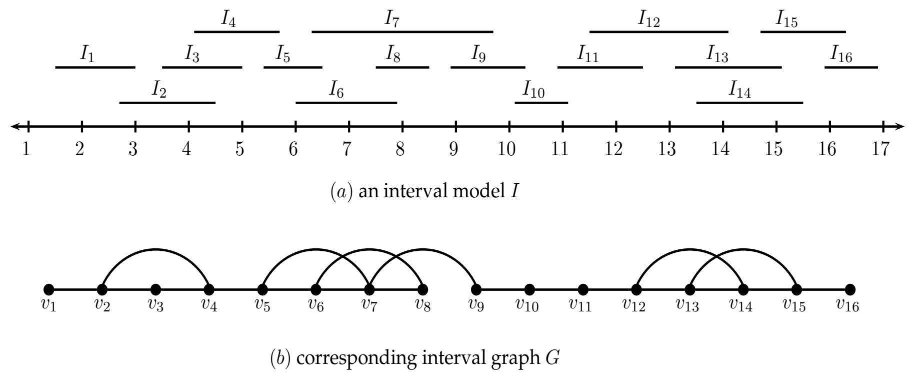


Figure 3: An interval model I and corresponding interval graph G.

INITIALLY

$V' = \{v_1, v_2, \ldots, v_{16}\}$ and $D = \phi$.

ITERATION 1

$i = 16$ and $F(v_i) = F(v_{16}) = v_{15} \neq v_1$

$j = 15$ and $F(v_j) = F(v_{15}) = v_{13} \neq v_1$

$k = 13$ and $F(v_k) = F(v_{13}) = v_{12}$

$r = 12$ and $\{v_{k+1} \ldots, v_{j-1}\} = \{v_{14}\} \subseteq N_G[v_j] \cup N_G[v_r]$

Since $L(v_r) = L(v_{13}) = v_{10}$ and $s = 10 > 2$,

$D = D \cup \{v_{13}, v_{15}\}$ and $V' = V' \setminus \{v_{11} \ldots v_{16}\}$.

AFTER ITERATION 1

$D = \{v_{13}, v_{15}\}$ and $V' = \{v_1, v_2 \ldots v_{10}\}$

ITERATION 2

$i = 10$ and $F(v_i) = F(v_{10}) = v_9 \neq v_1$

$j = 9$ and $F(v_j) = F(v_9) = v_7 \neq v_1$

$k = 7$ and $F(v_k) = F(v_7) = v_5$

$r = 5$ and $\{v_{k+1} \ldots, v_{j-1}\} = \{v_9\} \nsubseteq N_G[v_j] \cup N_G[v_r]$

In this case $t = \max\{l \mid k < l < j \text{ and } v_l \notin N_G(v_j)\} = 8$ and

$F(v_t) = F(v_8) = v_6$ (clearly $b = 6$)

Since $L(v_b) = L(v_6) = v_4$ and $s = 4 > 2$,

$D = D \cup \{v_6, v_9\}$ and $V' = V' \setminus \{v_5 \ldots v_{10}\}$.

AFTER ITERATION 2

$D = \{v_6, v_9, v_{13}, v_{15}\}$ and $V' = \{v_1, v_2, v_3, v_4\}$

ITERATION 3

$i = 4$ and $F(v_i) = F(v_4) = v_2 \neq v_1$

$j = 2$ and $F(v_j) = F(v_2) = v_1$, hence

$D = D \cup \{v_1, v_2\}$ and $V' = V' \setminus \{v_1, v_2, v_3, v_4\}$.

AFTER ITERATION 3

$D = \{v_1, v_2, v_6, v_9, v_{13}, v_{15}\}$ and $V' = \phi$

As $V' = \phi$ hence, loop terminates.

Our algorithm returns the set $D = \{v_1, v_2, v_6, v_9, v_{13}, v_{15}\}$, which is a minimum cardinality semi-PD-set of the interval graph G.

Theorem 5.1. *Given a left end ordering of vertices of G, the algorithm SEMI-PAIRED-DOM-IG computes a semi-PD-set of G of minimum cardinality in linear-time.*

Proof. By Lemmas 5.3, 5.4, 5.6 and 5.8, we can ensure that the algorithm SEMI-PAIRED-DOM-IG computes a semi-PD-set of G of minimum cardinality. Also, it can be easily seen that the algorithm can be implemented in $O(m+n)$ time, where $n = |V(G)|$ and $m = |E(G)|$. □

6 Algorithm for Trees

In this section, we present a linear-time algorithm to compute a minimum cardinality semipaired dominating set in trees.

Let $T = (V, E)$ be a tree, and $\beta = (v_n, v_{n-1}, \ldots, v_1)$ be the BFS ordering of vertices of T starting at a pendant vertex v_n. Let $\alpha = (v_1, v_2, \ldots, v_n)$ be the reverse ordering of β. In our algorithm, we process the vertices in the order they appear in α. Let $p(v_i)$ denote the parent of vertex v_i. If v_i is the root vertex, we assume $p(v_i) = v_i$.

The idea behind our algorithm is the following. We start with an empty set D, an array L and an array M. Initially $L[v_i] = 0$ and $M[v_i] = 0$ for all $v_i \in V$. We process the vertices one by one in the order $\alpha = (v_1, v_2, \ldots, v_n)$. During each of the iterations, we update D, L and M suitably. During the iterations, $L[v_i] = 0$ if v_i is not selected in D, $L[v_i] = 1$ if v_i is selected in D but not semipaired, and $L[v_i] = 2$ if v_i is selected in D and semipaired. Also, $M[v_i] = k$ if v_k need to be semipaired with some vertex in $N_T[v_i] \setminus D$. At the end of the algorithm D becomes a minimum cardinality semi-PD-set of the given tree T. At the i^{th} iteration, we process the vertex v_i. While processing v_i, we update D, L and M as follows.

Case 1: $i \neq n, n-1$ and v_i is not dominated by D.

Subcase 1.1: For every $v_r \in N_T[p(v_i)]$, $M[v_r] = 0$.

Update $D = D \cup \{p(v_i)\}$, $L[p(v_i)] = 1$ and $M[p(v_j)] = j$, where $v_j = p(v_i)$.

Subcase 1.2: For some $v_r \in N_T[p(v_i)]$, $M[v_r] \neq 0$.

Let $C = \{v_r \in N_T[p(v_i)] \mid M[w] \neq 0\}$. Let v_k be the least index vertex in C and $m[v_k] = v_s$. Update $L[p(v_i)] = L[v_s] = 2$, and $D = D \cup \{p(v_i)\}$.

Case 2: $i \in \{n, n-1\}$ and v_i is not dominated by D.

Update $L[v_{n-1}] = L[v_n] = 2$, and $D = D \cup \{v_{n-1}, v_n\}$.

Case 3: v_i is dominated by D and $M[v_i] = 0$.

No Update in D, L and M are made.

Case 4: v_i is dominated by D and $M[v_i] = k \neq 0$ (that is, v_k need to be semipaired with some vertex in $N_T[v_i] \setminus D$).

Subcase 4.1: $L[p(v_i)] = 0$.

Update $L[p(v_i)] = L[v_k] = 2$, $M[v_i] = 0$ and $D = D \cup \{p(v_i)\}$.

Subcase 4.1: $L[p(v_i)] = 1$.

This case will not arrive.

Subcase 4.3: $L[p(v_i)] = 2$.

Update $L[v_i] = L[v_k] = 2$, $M[v_i] = 0$ and $D = D \cup \{v_i\}$.

Theorem 6.1. *The* MINIMUM SEMIPAIRED DOMINATION *problem is linear-time solvable in trees.*

7 Approximation Algorithm

In this section, we present a greedy approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem in graphs. We also provide an upper bound on the approximation ratio of this algorithm. The greedy algorithm is described as follows.

Algorithm 2 : APPROX-SEMI-PAIRED-DOM-SET(G)

Input: A graph $G = (V, E)$ with no isolated vertex.
Output: A semi-PD-set D of G.
begin
  $D = \emptyset$;
  $i = 0$; $D_0 = \emptyset$;
  while $(V \setminus (D_0 \cup D_1 \cup \ldots \cup D_i) \neq \emptyset)$ **do**
    $i = i + 1$;
    choose two distinct vertices $u, v \in V$ such that $d_G(u, v) \leq 2$ and $|(N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup \ldots \cup D_{i-1})|$ is maximized;
    $D_i = (N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup \ldots \cup D_{i-1})$;
    $D = D \cup \{u, v\}$;
  return D;

Lemma 7.1. *The algorithm APPROX-SEMI-PAIRED-DOM-SET produces a semi-PD-set of G in polynomial time.*

Proof. Clearly, the output set D produced by the algorithm APPROX-SEMI-PAIRED-DOM-SET is a semi-PD-set of G. Also, each step of the algorithm can be computed in polynomial time. Hence, the lemma follows. □

Lemma 7.2. *For each vertex $v \in V$, there exists exactly one set D_i which contains v.*

Proof. We note that $V = D_0 \cup D_1 \cup \ldots D_{|D|/2}$. Also, if $v \in D_i$, then $v \notin D_j$ for $i < j$. Hence, the lemma follows. □

By Lemma 7.2, there exists only one index $i \in [|D|/2]$ such that $v \in D_i$ for each $v \in V$. We now define $d_v = \frac{1}{|D_i|}$. Now we are ready to prove the main theorem of this section.

Theorem 7.1. *The* MINIMUM SEMIPAIRED DOMINATION *problem for a graph G with maximum degree Δ can be approximated with an approximation ratio of* $1 + \ln(2\Delta + 2)$.

Proof. For any finite set $X \neq \emptyset$, $\displaystyle\sum_{x \in X} \frac{1}{|X|} = 1$. Hence, we have

$$|D| = 2 \sum_{i=1}^{\frac{|D|}{2}} \sum_{w \in D_i} \frac{1}{|D_i|} = 2 \sum_{w \in V} d_w.$$

Let $D^* = \{u_1, v_1, u_2, v_2, \ldots, u_{\frac{|D^*|}{2}}, v_{\frac{|D^*|}{2}}\}$ be a semi-PD-set of G of minimum cardinality, where u_i is semipaired with v_i, for each $i \in [\frac{|D^*|}{2}]$. Define $M = \{\{u_1, v_1\}, \{u_2, v_2\}, \ldots, \{u_{\frac{|D^*|}{2}}, v_{\frac{|D^*|}{2}}\}\}$. Note that for each vertex w, there exists a pair $\{u_i, v_i\} \in M$ such that $w \in N_G[u_i] \cup N_G[v_i]$. Hence, the following inequality follows.

$$\sum_{w \in V} d_w \leq \sum_{\{u_i,v_i\} \in M} \sum_{w \in N_G[u_i] \cup N_G[v_i]} d_w.$$

Consider a pair $\{u, v\} \in M$ and define $z_k = |(N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup D_2 \cup \ldots D_k)|$ for $k \in \{0\} \cup [\frac{|D|}{2}]$. Clearly, $z_{k-1} \geq z_k$ for $k \in [\frac{|D|}{2}]$. Suppose l is the smallest index such that $z_l = 0$. At the k^{th} step of the algorithm, D_k contains $z_{k-1} - z_k$ vertices from the set $N_G[u] \cup N_G[v]$. Hence

$$\sum_{w \in N_G[u] \cup N_G[v]} d_w = \sum_{k=1}^{l} (z_{k-1} - z_k) \cdot \frac{1}{|D_k|}.$$

At the k^{th} step of the algorithm, we choose the pair u_k, v_k such that $|D_k| = |(N_G[u_k] \cup N_G[v_k]) \setminus (D_0 \cup D_1 \cup \cdots \cup D_{k-1})|$ is maximum. Hence $|D_k| \geq |(N_G[u] \cup N_G[v]) \setminus (D_0 \cup D_1 \cup \cdots D_{k-1})| = z_{k-1}$. Therefore the following inequality follows.

$$\sum_{w \in N_G[u] \cup N_G[v]} d_w \leq \sum_{k=1}^{l} \frac{z_{k-1} - z_k}{z_{k-1}}.$$

For all integers $a < b$, we know that $H(b) - H(a) \geq \frac{b-a}{b}$, where $H(b) = \sum_{i=1}^{b} \frac{1}{i}$ and $H(0) = 0$. Therefore

$$\sum_{w \in N_G[u] \cup N_G[v]} d_w \leq \sum_{k=1}^{l} H(z_{k-1}) - H(z_k) = H(z_0) = H(|N_G[u] \cup N_G[v]|) \leq H(2\Delta + 2).$$

It follows that

$$|D| = 2 \sum_{w \in V} d_w \leq \sum_{\{u,v\} \in M} H(2\Delta + 2) = |D^*| H(2\Delta + 2) \leq (\ln(2\Delta + 2) + 1) \cdot |D^*|.$$

This shows that the MINIMUM SEMIPAIRED DOMINATION problem can be approximated with an approximation ratio of $1 + \ln(2\Delta + 2)$. □

8 Lower bound on approximation ratio

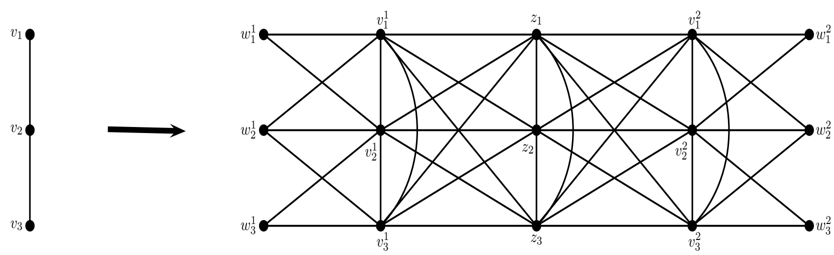
To obtain the lower bound on the approximation ratio of the MINIMUM SEMIPAIRED DOMINATION problem, we give an approximation preserving reduction from the MINIMUM DOMINATION problem. The following approximation hardness result is already known for the MINIMUM DOMINATION problem.

Theorem 8.1. *[3] For a graph $G = (V, E)$, the* MINIMUM DOMINATION *problem cannot be approximated within $(1 - \epsilon) \ln |V|$ for any $\epsilon > 0$ unless NP $\subseteq$ DTIME $(|V|^{O(\log \log |V|)})$.*

Now, we are ready to prove the following theorem.

Theorem 8.2. *For a graph $G = (V, E)$, the* MINIMUM SEMIPAIRED DOMINATION *problem cannot be approximated within* $(1 - \epsilon) \ln |V|$ *for any* $\epsilon > 0$ *unless NP* $\subseteq$ *DTIME* $(|V|^{O(\log \log |V|)})$.

Proof. Let $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ be an arbitrary instance of the MINIMUM DOMINATION problem. Now, we construct a graph $H = (V_H, E_H)$, an instance of the MINIMUM SEMIPAIRED DOMINATION problem in the following way: $V_H = \{v_i^1, v_i^2, w_i^1, w_i^2, z_i \mid i \in [n]\}$ and $E_H = \{w_i^1 v_j^1, w_i^2 v_j^2 \mid v_j \in N_G[v_i]\} \cup \{v_i^1 v_j^1, v_i^2 v_j^2, z_i z_j \mid 1 \le i < j \le n\} \cup \{v_i^1 z_j, v_i^2 z_j \mid i \in [n], j \in [n]\}$. Fig. 4 illustrates the construction of H from G.



G $\qquad$ H

Figure 4: An illustration of the construction of H from G in the proof of Theorem 8.2.

Let $V^k = \{v_i^k \mid i \in [n]\}$ and $W^k = \{w_i^k \mid i \in [n]\}$ for $k = 1, 2$. Also, assume that $Z = \{z_i \mid i \in [n]\}$. Note that $V^1 \cup Z$ is a clique in H. Also $V^2 \cup Z$ is a clique in H.

Let D^* denote a minimum dominating set of G. Then the set $D' = \{v_i^1, v_i^2 \mid v_i \in D^*\}$ is a semi-PD-set of H. Hence, if D_{sp}^* denotes a semi-PD-set of H of minimum cardinality, then $|D_{sp}^*| \le 2|D^*|$.

Suppose that the MINIMUM SEMIPAIRED DOMINATION problem can be approximated within a ratio of α, where $\alpha = (1 - \epsilon) \ln(|V_H|)$ for some fixed $\epsilon > 0$, by some polynomial time approximation algorithm, say **Algorithm A**. Next, we propose an algorithm, which we call **APPROX-DOMINATING-SET**, to compute a dominating set of a given graph G in polynomial time.

Algorithm 3 : APPROX-DOMINATING-SET(G)

Input: A graph $G = (V, E)$.
Output: A dominating set D of G.
begin
    Initialize $k = 0$;
    Construct the graph H;
    Compute a semi-PD-set D_{sp} of H using Algorithm A;
    Define $D'_{sp} = D_{sp}$;
    if $(|D'_{sp} \cap (V^1 \cup W^1)| \le |D_{sp}|/2)$ **then**
        k=1;
    else
        k=2;
    for *i=1 to n* **do**
        if $(N_H(w_i^k) \cap D'_{sp} == \emptyset)$ **then**
            $D'_{sp} = (D'_{sp} \setminus w_i^k) \cup \{v_i^k\}$;
    $D = \{v_i \mid v_i^k \in D'_{sp} \cap V^k\}$;
    return D;

Next, we show that the set D returned by Algorithm 3 is a dominating set of G. If D_{sp} is any semi-PD-set of H, then clearly either $|D_{sp} \cap (V^1 \cup W^1)| \leq |D_{sp}|/2$ or $|D_{sp} \cap (V^2 \cup W^2)| \leq |D_{sp}|/2$. Assume that $|D_{sp} \cap (V^k \cup W^k)| \leq |D_{sp}|/2$ for some $k \in [2]$. Now, to dominate a vertex $w_i^k \in W^k$, either $w_i^k \in D_{sp}$ or $v_j^k \in D_{sp}$ where $v_j^k \in N_H(w_i)$. If $N_H(w_i^k) \cap D_{sp}$ is an empty set, then we update D_{sp} by removing w_i^k and adding v_j^k for some $v_j^k \in N_H(w_i)$, and call the updated set D'_{sp}. We do this for each i from 1 to n. Note that even for the updated set D'_{sp}, we have $|D'_{sp} \cap (V^k \cup W^k)| \leq |D_{sp}|/2$. Also, in the updated set D'_{sp}, for each w_i^k, $N_H(w_i^k) \cap (D_{sp} \cap V^k)$ is non-empty. Hence $|D'_{sp} \cap V^k| \leq |D_{sp}|/2$ and $D'_{sp} \cap V^k$ dominates W^k. Therefore the set $D = \{v_i \mid v_i^k \in D'_{sp} \cap V^k\}$ is a dominating set of G. Also $|D| \leq |D_{sp}|/2$.

By above arguments, we may conclude that the Algorithm 3 produces a dominating set D of the given graph G in polynomial time, and $|D| \leq |D_{sp}|/2$. Hence, $|D| \leq \frac{|D_{sp}|}{2} \leq \alpha \frac{|D^*_{sp}|}{2} \leq \alpha |D^*|$.

Also $\alpha = (1-\epsilon)\ln(|V_H|) \approx (1-\epsilon)\ln(|V|)$ where $|V_H| = 5|V|$. Therefore the Algorithm **APPROX-DOMINATING-SET** approximates the minimum dominating set within ratio $(1-\epsilon)\ln(|V|)$ for some $\epsilon > 0$. By Theorem 8.1, if the minimum dominating set can be approximated within ratio $(1-\epsilon)\ln(|V|)$ for some $\epsilon > 0$, then NP $\subseteq$ DTIME $(|V|^{O(\log\log|V|)})$. Hence, if the MINIMUM SEMIPAIRED DOMINATION problem can be approximated within ratio $(1-\epsilon)\ln(|V_H|)$ for some $\epsilon > 0$, then NP $\subseteq$ DTIME $(|V_H|^{O(\log\log|V_H|)})$. This proves that the MINIMUM SEMIPAIRED DOMINATION problem cannot be approximated within $(1-\epsilon)\ln(|V_H|)$ unless NP $\subseteq$ DTIME $(|V_H|^{O(\log\log|V_H|)})$. $\square$

9 Conclusion

In this paper, we initiate the algorithmic study of the MINIMUM SEMIPAIRED DOMINATION problem. We have resolved the complexity status of the problem for bipartite graphs, chordal graphs and interval graphs. We have proved that the SEMIPAIRED DOMINATION DECISION problem is NP-complete for bipartite graphs and split graphs. We also present a linear-time algorithm to compute a semi-PD-set of minimum cardinality for interval graphs and trees. A $1+\ln(2\Delta+2)$ approximation algorithm for the MINIMUM SEMIPAIRED DOMINATION problem in general graphs is given, and we prove that it can not be approximated within any sub-logarithmic factor. It will be interesting to study better approximation algorithms for this problem for bipartite graphs, chordal graphs and other important graph classes.

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