Linear Extensions of N-free Orders

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Abstract. We consider the number of linear extensions of an N-free order P. We give upper and lower bounds on this number in terms of parameters of the corresponding arc diagram. We propose a dynamic programming algorithm to calculate the number. The algorithm is polynomial if a new parameter called activity is bounded by a constant. The activity can be bounded in terms of parameters of the arc diagram.

1 Introduction

The number of linear extensions is one of the most fundamental combinatorial parameters of an order (poset). Explicit formulas and efficient algorithms have been found for several classes of orders, see e.g. [1] [2], [3] or [7]. Brightwell and Winkler [4] have shown that for general orders the problem is #P-complete. Hence, unless P=NP, there is no polynomial algorithm for this problem.

Problems that are hard for general orders may become tractable when restricted to more structured classes of orders, see [11] for a survey on the topic.

One of the most prominent classes of orders is the class of N-free ordered sets. This class was introduced by Grillet [5]. Leclerc and Montjardet [10] characterized N-free orders as the chain-antichains-complete orders. A series of papers investigated algorithmic problems on N-free orders : Habib and Jegou [6] and independently Sysło [14] showed that the jump number of an N-free order can be computed in linear time. For general orders the jump-number problem is NP-hard. Habib and Möhring [7] proved that the isomorphism problem remains isomorphism-complete when restricted to N-free orders. More recent contributions to N-free orders are the enumeration of N-free orders of a given height [8] and the computation of their page number [9]. Zaguia [16] verified the $1/3 - 2/3$ conjecture for N-free orders. We are not aware of previous results concerning the number of linear extensions of N-free orders.

The next section provides definitions and background information about N-free orders. In Section 2 we present upper and lower bounds for the number of linear extensions of an N-free order.

Section 3 is devoted to the dynamic programming algorithm for calculating the number of linear extensions of N-free orders. Dynamic programming was used to show that the number of linear extensions of orders of bounded width can be computed in polynomial time, see [1]. In our approach the size of an antichain is replaced by the size of an *active set*. In particular our algorithm has a polynomial running time for some N-free orders of unbounded width.

We conclude with a discussion of some open problems.

1.1   Definitions and background on N-free orders

We are concerned here with combinatorial problems for partially ordered sets. We assume some familiarity with concepts and results in this area, including linear extensions, comparability graphs and diagrams. For readers who are new to the subject, we suggest consulting one of the books on the topic (Trotter [15], Schröder [13], Neggers and Kim [12]).

An order is *N-free* if its diagram does not contain an N (see Figure 1). There are many characterizations for N-free orders, an overview is given by Möhring [11]. The relevant characterization in our context is the following :
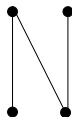

Figure 1: The order N.

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Theorem 1.1 *An order is N-free if and only if its diagram is the line digraph of a directed acyclic graph (dag).*

Theorem 1.1 enables us to represent N-free orders, using an *arc diagram.* The arc diagram $A(P)$ of an N-free order P is the digraph of which the order is the line digraph. Figure 2 shows an example. Such a digraph is not unique, but if we require that $A(P)$ has a unique source $\hat{0}$ and a unique sink $\hat{1}$, then it becomes unique. Sysło [14] used $A(P)$ to deal with jump number of N-free orders. Since $A(P)$ can easily be computed we will assume that P and $A(P)$ are both available. Depending on the context we then think of elements of P as vertices of P or as edges of $A(P)$, whatever is more convenient.
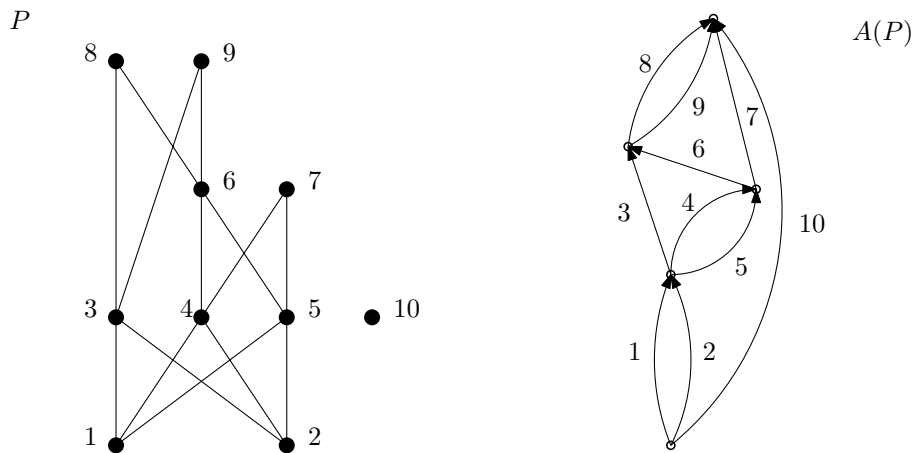

Figure 2: An N-free order and its arc diagram.

Being acyclic the arc diagram $A(P)$ or more precisely its transitive closure is a partial order relation on the vertex set of $A(P)$. In the following, we will switch hence and forth between the arc diagram as a directed graph and the arc diagram as an order. The interpretation will be clear from the context. It is important to notice that in general the digraph $A(P)$ is not the diagram of the order $A(P)$, simply because $A(P)$ may have transitive edges.

2 Bounds for the linear extensions of an N-free order

Using the arc diagram representation of an N-free order, we prove the following

Proposition 2.1 *Let P be an N-free order and $e(P)$ be its number of linear*

extensions, then

$$e(A(P)) \prod_{v \in V_{A(P)}} o_v! \quad \leq \quad e(P) \quad \leq \quad |P|! \prod_{v \in V_{A(P)}} \binom{i_v + o_v}{o_v}^{-1}$$

where $V_{A(P)}$ is the set of the vertices of $A(P)$, o_v is the out-degree of $v \in V_{A(P)}$, and i_v is the in-degree of $v \in V_{A(P)}$.

Proof. For the lower bound we construct an appropriate set of linear extensions of P. Let $L = (v_1, v_2, .., v_k)$ be a linear extension (topological ordering) of the arc diagram $A(P)$. For each vertex v of $A(P)$, choose a permutation $\sigma_v = (x_1^v, x_2^v, .., x_{o_v}^v)$ of the out-edges of v. Concatenating these permutations in the order given by L yields $L' = (x_1^{v_1}, x_2^{v_1}, .., x_{o_v}^{v_1}, \ldots, x_1^{v_i}, .., x_{o_{v_i}}^{v_i}, \ldots, x_{o_{v_k}}^{v_k})$, a permutation of the elements of P. We claim that L' is a linear extension of P. If $x < y$ in P, then there is a directed path in $A(P)$ whose first and last edge are x and y respectively. Let v_x and v_y be the source vertices of the edges x and y. The directed path implies the relation $v_x < v_y$ in $A(P)$, whence v_x precedes v_y in L. This in turn implies that x precedes y in L'. We have thus shown that L' respects the order relations of P, i.e., it is a linear extension.

Hence, there is a linear extension L' for every tuple $(L, \sigma_1, \ldots, \sigma_k)$. Obviously, if L' and L'' are constructed using different tuples, then they are different. The lower bound is just the number of tuples.

The upper bound is based on a probabilistic argument. Let σ be a randomly chosen permutation of the elements of P. The upper bound will be obtained by considering the probability that σ is a linear extension of P.

A vertex $v \in A(P)$ is *good* with respect to σ, in symbols $v \blacktriangleright \sigma$, if in σ all in-edges of v precede all out-edges of v. The probability that v is good with respect to a random permutation σ is $\binom{i_v + o_v}{o_v}^{-1}$: just consider the induced permutation σ_v on the edges incident to v, out of the $(i_v + o_v)!$ possible permutations σ_v exactly $i_v! o_v!$ make v good.

Now we try to see what happens if we have to deal with more than one vertex. Let $\Pr(A)$ denote the probability of A and $\Pr(A|B) = \Pr(A \cap B)\Pr(B)^{-1}$ be the conditional probability of A given B. Since a permutation is a linear extension of P if and only if all the vertices of $A(P)$ are good for this permutation, we get:

$$\Pr(\sigma \in \mathcal{L}(P)) = \prod_{j=1}^{k} \Pr(v_j \blacktriangleright \sigma | v_1 \blacktriangleright \sigma, \ldots v_{j-1} \blacktriangleright \sigma) \tag{1}$$

Where $(v_1, \ldots, v_k)$ is an arbitrary enumeration of the vertices of $A(P)$ and $\mathcal{L}(P)$ is the set of the linear extensions of P. To shorten the proof a little, we will assume that $(v_1, \ldots, v_k)$ is a linear extension of $A(P)$.

We claim that for all $j \in [k]$

$$\mathsf{Pr}(v_j \blacktriangleright \sigma | v_1 \blacktriangleright \sigma, \ldots, v_{j-1} \blacktriangleright \sigma) \le \mathsf{Pr}(v_j \blacktriangleright \sigma) = \binom{i_{v_j} + o_{v_j}}{o_{v_j}}^{-1}. \qquad (2)$$

Plugging these inequalities into (1) yields the upper bound.

Permutations σ and π of P are called *equivalent for j* if:

1. all $x \in P$ that are not incident to v_j are at the same position in σ and π,

2. both permutations induce the same orderings (permutations) on the in-edges and the out-edges of v_j, i.e., $\sigma_j^{\mathsf{in}} = \pi_j^{\mathsf{in}}$ and $\sigma_j^{\mathsf{out}} = \pi_j^{\mathsf{out}}$.

Claim. If an equivalence class $\mathcal{C}$ contains a permutation that makes v_j good, then

$$\mathsf{Pr}(v_j \blacktriangleright \sigma | \sigma \in \mathcal{C}) = \binom{i_{v_j} + o_{v_j}}{o_{v_j}}^{-1}, \text{ otherwise } \quad \mathsf{Pr}(v_j \blacktriangleright \sigma | \sigma \in \mathcal{C}) = 0.$$

To see that the claim implies the correlation inequality (2) we propose an experiment in two phases. First we pick a class with a probability proportional to its size, then we pick a random element from the class. When the probability space for the experiment consists of all permutations, the probability for picking a σ with $v_j \blacktriangleright \sigma$ is $\binom{i_{v_j} + o_{v_j}}{o_{v_j}}^{-1}$. For a proof of (2) we are interested in a probability space for the experiment that only consists of permutations satisfying $v_1 \blacktriangleright \sigma, \ldots, v_{j-1} \blacktriangleright \sigma$. In this space we only get a σ with $v_j \blacktriangleright \sigma$ if we are lucky in the first phase of the experiment. From the claim we know that, conditioned on success in the first phase, the probability of success in the second phase is again $\binom{i_{v_j} + o_{v_j}}{o_{v_j}}^{-1}$. This yields (2).

Proof of the claim. Let $\mathcal{C}$ be a class. With a permutation $\sigma \in \mathcal{C}$ consider the induced permutation σ_j on the edges incident to v_j. From the two properties in the definition of equivalence it follows that σ is uniquely determined if we know which positions of σ_j are used by in-edges and which by out-edges. This can be encoded by a 0-1 vector $s_\sigma = (s_1, s_2, \ldots, s_{i_{v_j} + o_{v_j}})$ with $s_i = 1$ if the ith entry of σ_j is an in-edge and $s_i = 0$ if it is an out-edge.

Now consider an adjacent inversion $(s_i, s_{i+1}) = (1, 0)$ in s_σ and let σ' be the permutation obtained from σ by swapping the out-edge x and the in-edge y of v_j that are at positions i and $i + 1$ in σ_j. To show that σ' belongs to the same class it suffices to show that $v_\ell \blacktriangleright \sigma'$ for all $1 \le \ell < j$. Let x be the edge $v_j \to v_{j_x}$ and y be the edge $v_{j_y} \to v_j$. Since $(v_1, \ldots, v_k)$ is a topological order of $A(P)$ we have $j_y < j < j_x$. We don't care whether v_{j_x} is good with respect to σ'. Vertex v_{j_y} is good for σ and in σ' an out-edge of v_{j_y} has been moved further to the right compared to σ, hence, v_{j_y} is good for σ'. For all the other $\ell < j$ the property $v_\ell \blacktriangleright \sigma'$ is directly inherited from

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σ. Note that the encoding vector $s_{\sigma'}$ of σ' has exactly one inversion pair less than s_σ.

Let s and s' be two 0-1 vectors of length t with r ones such that s' is majorized by s, i.e., $\sum_{i=1}^{j} s'_i \leq \sum_{i=1}^{j} s_i$ for all $j = 0, .., t$. Sequence s' can be reached from s by a sequence of steps that remove a single inversion each. Now, suppose class $\mathcal{C}$ contains a permutation σ that makes v_j good. Being good is equivalent to $s_\sigma = (1, .., 1, 0, .., 0)$ which is the maximum in the majorization order. Hence, starting from σ we can find a sequence of inversion reducing swaps to reach every permutation σ' such that $s_{\sigma'}$ is majorized by s_σ. Since $\sigma \in \mathcal{C}$ and the steps do not leave $\mathcal{C}$ we find that $|\mathcal{C}| = \binom{i_{v_j} + o_{v_j}}{o_{v_j}}$. This completes the proof of the claim and of the proposition. $\qquad\square$

There are few examples of N-free orders where one of the bounds is sharp. In special cases, however, there may be many ways for improving the bounds. We exemplify this with the lower bound. First note that the dual P^* of an N-free order P is N-free again and $A(P^*)$ is the dual of $A(P)$. Indeed $A(P^*) = A(P)^*$ holds in both interpretations, as a digraph and as an order. Hence the dual lower bound can be written as $e(A(P)) \prod_{v \in V_{A(P)}} i_v!$ and also provides a lower bound on $e(P)$. Moreover, in many cases the set of linear extensions of P that are obtained from the lower bound proof and the dual are disjoint. This is always the case if there is a vertex v in $A(P)$ such that the out-edges of v are always separated if we keep the in-edges of all vertices together. In particular, if $A(P)$ has a subdigraph of one of the types shown in Figure 3 then $e(A(P))(\prod_{v \in V_{A(P)}} o_v! + \prod_{v \in V_{A(P)}} i_v!)$ is a lower bound on $e(P)$.
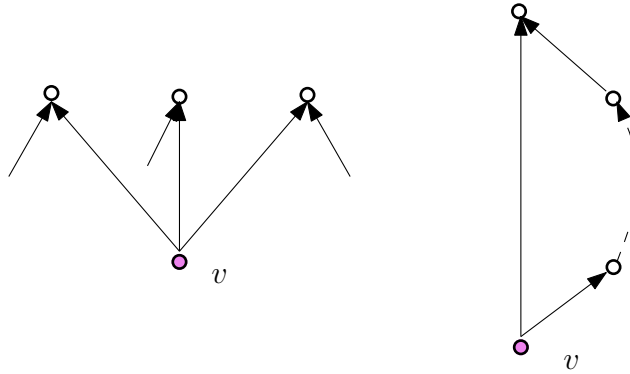

Figure 3: Subdigraphs of $A(P)$ that allow the addition of the lower bound and its dual.

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3 A dynamic programming algorithm for enumeration

In this section we describe a dynamic programming algorithm for the enumeration of linear extensions of an N-free order.

Let P be an N-free order with arc diagram $A(P)$. Given a linear extension $(x_1, \ldots, x_n)$ of P the algorithm computes the number of linear extensions of the induced suborders $P_i = P[x_1, .., x_i]$ for i increasing from 1 to n. Denote the edge of $A(P)$ that represents x_j by (v_j^-, v_j^+). The *active set* A_i corresponding to P_i consists of all vertices of $A(P)$ that have incident edges from P_i and from the complement of P_i, i.e., $A_i = \{v \; : \; \exists \, j' \leq i < j''$ with $v_{j'}^+ = v = v_{j''}^-\}$. Let L be a linear extension of P_{i-1}, extending L to a linear extension of P_i means that x_i is placed at a position behind all of its immediate predecessors in L. The immediate predecessors of x_i are exactly the elements that point into v_i^-, whence v_i^- belongs to the active set A_{i-1}.

We are now ready to describe the algorithm. For $i = 1, \ldots, n$ and a function $f : A_i \to [i]$ we define

$$z_i(f) \;\; = \;\; \#\big(L : \text{linear extension of } P_i \text{ such that in } L \text{ the last element } x_j \text{ with } v_j^+ = v$$
$$\text{is at position } f(v), \text{ for all } v \in A_i\big).$$

Let $z_{i-1}(f)$ be known for all $f : A_{i-1} \to [i-1]$, then we can compute $z_i(f)$ for all $f : A_i \to [i]$. The first thing that has to be done is to find A_i. Only two events have to be checked for the update from A_{i-1} to A_i :

- If all element incident to v_i^- belong to P_i, then v_i^- is removed from the active set.

- If x_i is the only element of P_i incident to v_i^+, then v_i^+ is added to the active set.

For fixed f let consider a linear extension L counted by $z_{i-1}(f)$. Inserting x_i in L after the tth element yields a linear extension of P_i if and only if $f(v_i^-) \leq t < i$. If t respects these bounds we obtain a linear extension of P_i counted by $z_i(f')$ where for $v \in A_i$ the value $f'(v)$ is defined as follows:

$$f'(v) = \begin{cases} \max(f(v), t) + 1 & \text{if } v = v_i^+ \\ f(v) & \text{if } f(v) \leq t \\ f(v) + 1 & \text{if } f(v) > t. \end{cases}$$

We write $f' = \Gamma_i(f, t)$. The initialization for the dynamic program is $z_i(f) = 0$ for all i and f except $z_1(f_0) = 1$, where $f_0(v_1^+) = 1$ (note that $A_1 = \{v_1^+\}$). The outer loop of the algorithm is increasing i from 2 to n. For each i we go through all f and all t with $f(v_i^-) \leq t \leq i$ and update $z_i(\Gamma_i(f, t)) \leftarrow z_i(\Gamma_i(f, t)) + z_{i-1}(f)$.

Let α be an upper bound on the size of the active sets, i.e, if $|A_i| \leq \alpha$ for all i. For each of the n levels i of the computation we have to consider at most n^α functions f. For each function f we have to consider $\leq n$ derived functions f', if in the loop for f we decrease t in steps of -1, then updates can be done in $O(1)$. Altogether this yields a running time of $O(n^{\alpha+2})$.

We can improve the running time by stripping off a factor of n. The key observation is that in each relevant f at level i, i.e., each f with $z_i(f) \neq 0$, we have some $v \in A_i$ with $f(v) = i$. Hence, if we mark this element v we can describe the rest of f as a function in $[n]^{\alpha-1}$. This shows that we only have to consider $\alpha n^{\alpha-1}$ functions on each level.

In the following subsection we discuss conditions that allow to bound the size of the active set.

3.1 Bounding the size of active sets

Since each active set A_i is a subset of the vertex set of the arc diagram we easily get:

Proposition 3.1 *The number of linear extensions of N-free orders with n elements and an arc diagram with $\leq k$ vertices can be computed in $O(n^{k+1})$.*

The initial linear extension $L_0 = (x_1, \ldots, x_n)$ of P determines the sequence $A_1, \ldots, A_n$ of active sets. Different choices for L_0 may lead to active sets of very different size. Let us define the *activity $\alpha(L)$ of a linear extension* as the maximum size of an active set in its sequence. The *activity an N-free order* is

$$\alpha(P) = \min\big(\alpha(L) \ : \ L \text{ a linear extension of } P\big)$$

In some sense the strongest result that can be stated on the basis of our algorithm is the following

Proposition 3.2 *The number of linear extensions of an N-free order P of bounded activity can be computed efficiently if a linear extension L with $\alpha(P) = \alpha(L)$ is given.*

However, we do not know how to compute the activity of an N-free order. We propose the activity as a new parameter of N-free ordered sets for further studies.

It would be nice to improve on Proposition 3.1 by only bounding the width of the arc diagram. This, however, is not enough. In Figure 4 we show the arc diagram $A(R_\ell)$ of an N-free order R_ℓ with 3ℓ elements and $\mathsf{width}(A(R_\ell)) = 1$, the activity of R_ℓ is at least ℓ. Indeed if $\hat{x}$ is at position i in L, then A_i contains at least one of $\{u_i, w_i\}$ for $i = 1, .., \ell$.

For two vertices u and v of a directed acyclic graph D we define $\mathsf{spread}(u, v)$ by looking at pairs of interiorly disjoint u to v paths. If there is no such pair, then $\mathsf{spread}(u, v) = 0$, otherwise we let $\mathsf{spread}(u, v)$ be the maximal
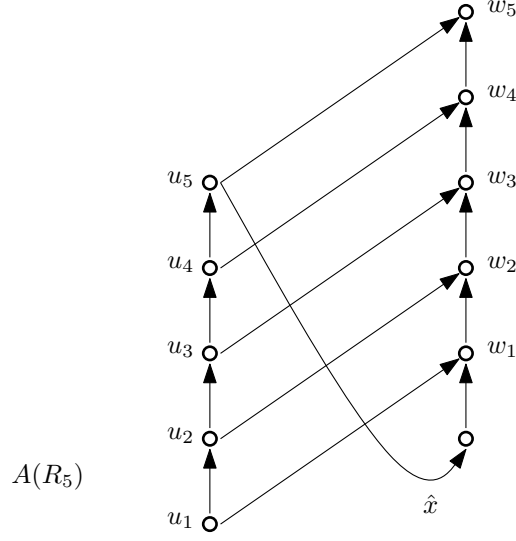
Figure 4: The arc diagram of an N-free order R_5 with $\alpha(R_5) = 6$.

difference in length of two such paths. The *spread* of D is $\mathsf{spread}(D) = \max(\mathsf{spread}(u, v) : u, v \in D)$. Since shortest and longest paths in directed acyclic graphs are efficiently computable the spread is also tractable.

Note that $\mathsf{spread}(A(R_\ell)) = \ell + 1$. In fact, the activity of an N-free order P is small if width and spread of $A(P)$ are small.

Lemma 3.3 *For N-free orders P we have $\alpha(P) \leq \mathsf{width}(A(P)) \cdot (\mathsf{spread}(A(P)) + 2)$.*

Proof. The rank $\mathsf{rk}(v)$ of a vertex of $A(P)$ is the length of a longest path ending in v. Let $T = (v_1, \ldots, v_s)$ be a topological order of $A(P)$ such that $i < j$ implies $\mathsf{rk}(v_i) \leq \mathsf{rk}(v_j)$. Define a linear extension L_T such that for $i < j$ the out-edges of v_i precede the out-edges of v_j.

Claim. $\alpha(L_T) \leq \mathsf{width}(A(P)) \cdot (\mathsf{spread}(A(P)) + 2)$.

Partition the vertices of $A(P)$ into ranks $S_j = \{v \in V_{A(P)} : \mathsf{rk}(v) = j\}$. By definition each S_j is an antichain of $A(P)$, hence $|S_j| \leq \mathsf{width}(A(P))$. If x_i, the ith element of L_T, has its source $v_{x_i}^-$ in S_j, then $A_i \subset S_j \cup S_{j+1} \cup \ldots \cup S_{j+\mathsf{spread}(A(P))+1}$. To see this consider $v \in A_i$ with $\mathsf{rk}(v) = k$. Let p be a path of length k from $\hat{0}$ to v in $A(P)$. From $v \in A_i$ and the construction of L_T we conclude that there is $y \in P$ with $\mathsf{rk}(v_y^-) \leq j$ and $v_y^+ = v$. Let q be a path from $\hat{0}$ through v_y^- to v. The length of q is at most $j + 1$. Note that after clipping p and q from $\hat{0}$ to the last common vertex u we obtain a pair of disjoint u to v paths whose difference in length is at least $k - (j+1)$, hence $\mathsf{spread}(A(P)) \geq k - j - 1$. $\qquad\square$

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Theorem 3.4 *The number of linear extensions of N-free orders with n elements and an arc diagram with width $\leq k$ and spread $\leq s$ can be computed in $O(n^{k(s+2)+1})$.*

Conclusion

We have shown that the number of linear extensions of N-free orders of bounded activity can be computed in polynomial time. It would be interesting to understand the class of N-free orders of bounded activity better. Is membership in this class testable in polynomial time? Are there simple additional sufficient conditions or necessary conditions for membership in this class?

We do not expect that there is a polynomial algorithm that solves the problem for general N-free orders. Indeed we conjecture that counting linear extensions of N-free orders is #P-complete.

Brightwell and Winkler [4] have shown #P-completeness for general orders of height 3. They expect that the problem remains hard for orders of height 2. Since N-free orders of height 2 are disjoint unions of weak orders of height 2 the counting problem for this class is easy. We think that the counting problem should already be hard for N-free orders of height 3.

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