Discrete-Time Fractional Variational Problems

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Abstract

We introduce a discrete-time fractional calculus of variations on the time scale $h\mathbb{Z}$, $h > 0$. First and second order necessary optimality conditions are established. Examples illustrating the use of the new Euler-Lagrange and Legendre type conditions are given. They show that solutions to the considered fractional problems become the classical discrete-time solutions when the fractional order of the discrete-derivatives are integer values, and that they converge to the fractional continuous-time solutions when h tends to zero. Our Legendre type condition is useful to eliminate false candidates identified via the Euler-Lagrange fractional equation.

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1. Introduction

The *Fractional Calculus* (calculus with derivatives of arbitrary order) is an important research field in several different areas such as physics (including classical and quantum mechanics as well as thermodynamics), chemistry, biology, economics, and control theory [3, 10, 40, 42, 48]. It has its origin more than 300 years ago when L'Hopital asked Leibniz what should be the meaning of a derivative of non-integer order. After that episode several more famous mathematicians contributed to the development of Fractional Calculus: Abel, Fourier, Liouville, Riemann, Riesz, just to mention a few names [30, 47]. In the last decades, considerable research has been done in fractional calculus. This is particularly true in the area of the calculus of variations, which is being subject

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to intense investigations during the last few years [10, 11, 44, 45]. Applications include fractional variational principles in mechanics and physics, quantization, control theory, and description of conservative, nonconservative, and constrained systems [10, 15, 16, 45]. Roughly speaking, the classical calculus of variations and optimal control is extended by substituting the usual derivatives of integer order by different kinds of fractional (non-integer) derivatives. It is important to note that the passage from the integer/classical differential calculus to the fractional one is not unique because we have at our disposal different notions of fractional derivatives. This is, as argued in [10, 44], an interesting and advantage feature of the area. Most part of investigations in the fractional variational calculus are based on the replacement of the classical derivatives by fractional derivatives in the sense of Riemann–Liouville, Caputo, Riesz, and Jumarie [1, 4, 10, 27]. Independently of the chosen fractional derivatives, one obtains, when the fractional order of differentiation tends to an integer order, the usual problems and results of the calculus of variations. Although the fractional Euler–Lagrange equations are obtained in a similar manner as in the standard variational calculus [44], some classical results are extremely difficult to be proved in a fractional context. This explains, for example, why a fractional Legendre type condition is absent from the literature of fractional variational calculus. In this work we give a first result in this direction (cf. Theorem 3.6).

Despite its importance in applications, less is known for discrete-time fractional systems [44]. In [39] Miller and Ross define a fractional sum of order $\nu > 0$ *via* the solution of a linear difference equation. They introduce it as (see §2 for the notations used here)

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t - \sigma(s))^{(\nu-1)} f(s). \tag{1}$$

Definition (1) is analogous to the Riemann-Liouville fractional integral

$${}_a\mathbf{D}_x^{-\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_a^x (x-s)^{\nu-1} f(s) ds$$

of order $\nu > 0$, which can be obtained *via* the solution of a linear differential equation [39, 40]. Basic properties of the operator $\Delta^{-\nu}$ in (1) were obtained in [39]. More recently, Atici and Eloe introduced the fractional difference of order $\alpha > 0$ by $\Delta^\alpha f(t) = \Delta^m(\Delta^{\alpha-m} f(t))$, where m is the integer part of α, and developed some of its properties that allow to obtain solutions of certain fractional difference equations [8, 9].

The fractional differential calculus has been widely developed in the past few decades due mainly to its demonstrated applications in various fields of science and engineering [30, 40, 43]. The study of necessary optimality conditions for fractional problems of the calculus of variations and optimal control is a fairly recent issue attracting an increasing attention – see [1, 2, 7, 22, 23, 25, 26, 41] and references therein – but available results address only the continuous-time case. It is well known that discrete analogues of differential equations can be very useful in applications [13, 31, 29] and that fractional Euler-Lagrange differential equations are extremely difficult to solve, being necessary to discretize them [2, 11]. Therefore, it is pertinent to develop a fractional discrete-time theory of the calculus of variations for the time scale $(h\mathbb{Z})_a$, $h > 0$ (cf. definitions in Section 2). Computer simulations show that this time scale is particularly interesting because when h tends to zero one recovers previous fractional continuous-time results.

Our objective is two-fold. On one hand we proceed to develop the theory of *fractional difference calculus*, namely, we introduce the concept of left and right fractional sum/difference (cf. Definition 2.8). On the other hand, we believe that the present work will potentiate research not only in the fractional calculus of variations but also in solving fractional difference equations, specifically, fractional equations in which left and right fractional differences appear. Because the theory of fractional difference calculus is still in its infancy [8, 9, 39], the paper is self contained. In §2 we introduce notations, we give necessary definitions, and prove some preliminary results needed in the sequel. Main results of the paper appear in §3: we prove a fractional formula of h-summation by parts (Theorem 3.2), and necessary optimality conditions of first and second order (Theorems 3.5 and 3.6, respectively) for the proposed h-fractional problem of the calculus of variations (17). Section 4 gives some illustrative examples, and we end the paper with §5 of conclusions and future perspectives.

The results of the paper are formulated using standard notations of the theory of time scales [20, 32, 33]. It remains an interesting open question how to generalize the present results to an arbitrary time scale $\mathbb{T}$. This is a difficult and challenging problem since our proofs deeply rely on the fact that in $\mathbb{T} = (h\mathbb{Z})_a$ the graininess function is a constant.

2. Preliminaries

We begin by recalling the main definitions and properties of time scales (cf. [18, 20] and references therein). A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and is denoted by $\mathbb{T}$. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for all $t \in \mathbb{T}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$ for all $t \in \mathbb{T}$, with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). A point $t \in \mathbb{T}$ is called *right-dense*, *right-scattered*, *left-dense*, or *left-scattered*, if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, or $\rho(t) < t$, respectively. Throughout the text we let $\mathbb{T} = [a, b] \cap \tilde{\mathbb{T}}$ with $a < b$ and $\tilde{\mathbb{T}}$ a time scale. We define $\mathbb{T}^\kappa = \mathbb{T}\backslash(\rho(b), b]$, $\mathbb{T}^{\kappa^2} = (\mathbb{T}^\kappa)^\kappa$ and more generally $\mathbb{T}^{\kappa^n} = \left(\mathbb{T}^{\kappa^{n-1}}\right)^\kappa$, for $n \in \mathbb{N}$. The following standard notation is used for σ (and ρ): $\sigma^0(t) = t$, $\sigma^n(t) = (\sigma \circ \sigma^{n-1})(t)$, $n \in \mathbb{N}$. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *delta differentiable* at $t \in \mathbb{T}^\kappa$ if there is a number $f^\Delta(t)$ such that for all $\varepsilon > 0$ there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta derivative* of f at t. The $r^{th}-$*delta derivative* ($r \in \mathbb{N}$) of f is defined to be the function $f^{\Delta^r} : \mathbb{T}^{\kappa^r} \to \mathbb{R}$, provided $f^{\Delta^{r-1}}$ is delta differentiable on $\mathbb{T}^{\kappa^{r-1}}$. For delta differentiable f and g and for an arbitrary time scale $\mathbb{T}$ the next formulas hold: $f^\sigma(t) = f(t) + \mu(t)f^\Delta(t)$ and

$$(fg)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t), \tag{2}$$

where we abbreviate $f \circ \sigma$ by f^σ. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at right-dense points and if its left-sided limit exists at left-dense points. The set of all rd-continuous functions is denoted by $C_{\rm rd}$ and the set of all delta differentiable functions with rd-continuous derivative by $C^1_{\rm rd}$. It is known that rd-continuous functions possess an *antiderivative*, i.e., there exists a function $F \in C^1_{\rm rd}$ with $F^\Delta = f$. The delta *integral* is then defined by $\int_a^b f(t)\Delta t = F(b) - F(a)$. It satisfies the equality $\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t)$. We make use of the following properties of the delta integral:

Lemma 2.1. *(cf. [20, Theorem 1.77]) If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}$, then*

1. $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(t)\Delta t$;
2. $\int_a^b f(t)g^\Delta(t)\Delta t = (fg)(t)|_{t=a}^{t=b} - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$.

One way to approach the Riemann-Liouville fractional calculus is through the theory of linear differential equations [43]. Miller and Ross [39] use an analogous methodology to introduce fractional discrete operators for the case $\mathbb{T} = \mathbb{Z}_a = \{a, a+1, a+2, \ldots\}, a \in \mathbb{R}$. Here we go a step further: we use the theory of time scales in order to introduce fractional discrete operators to the more general case $\mathbb{T} = (h\mathbb{Z})_a = \{a, a+h, a+2h, \ldots\}$, $a \in \mathbb{R}$, $h > 0$.

For $n \in \mathbb{N}_0$ and rd-continuous functions $p_i : \mathbb{T} \to \mathbb{R}$, $1 \leq i \leq n$, let us consider the nth order linear dynamic equation

$$Ly = 0\,, \qquad \text{where } Ly = y^{\Delta^n} + \sum_{i=1}^{n} p_i y^{\Delta^{n-i}}\,. \tag{3}$$

A function $y : \mathbb{T} \to \mathbb{R}$ is said to be a solution of equation (3) on $\mathbb{T}$ provided y is n times delta differentiable on $\mathbb{T}^{\kappa^n}$ and satisfies $Ly(t) = 0$ for all $t \in \mathbb{T}^{\kappa^n}$.

Lemma 2.2. *[20, p. 239] If $z = (z_1, \ldots, z_n) : \mathbb{T} \to \mathbb{R}^n$ satisfies for all $t \in \mathbb{T}^\kappa$*

$$z^\Delta = A(t)z(t), \qquad \textit{where} \qquad A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -p_n & \ldots & \ldots & -p_2 & -p_1 \end{pmatrix} \tag{4}$$

then $y = z_1$ is a solution of equation (3)*. Conversely, if y solves* (3) *on $\mathbb{T}$, then $z = \left(y, y^\Delta, \ldots, y^{\Delta^{n-1}}\right) : \mathbb{T} \to \mathbb{R}$ satisfies* (4) *for all $t \in \mathbb{T}^{\kappa^n}$*

Definition 2.3. *[20, p. 239] We say that equation (3) is* regressive *provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, where A is the matrix in* (4).

Definition 2.4. *[20, p. 250] We define the Cauchy function $y : \mathbb{T} \times \mathbb{T}^{\kappa^n} \to \mathbb{R}$ for the linear dynamic equation (3) to be, for each fixed $s \in \mathbb{T}^{\kappa^n}$, the solution of the initial value problem*

$$Ly = 0, \quad y^{\Delta^i}((\sigma(s), s) = 0, \quad 0 \leq i \leq n-2, \quad y^{\Delta^{n-1}}((\sigma(s), s) = 1\,. \tag{5}$$

Theorem 2.5. *[20, p. 251] Suppose* $\{y_1, \ldots, y_n\}$ *is a fundamental system of the regressive equation (3). Let* $f \in C_{rd}$. *Then the solution of the initial value problem*

$$Ly = f(t), \quad y^{\Delta^i}(t_0) = 0, \quad 0 \le i \le n-1,$$

is given by $y(t) = \int_{t_0}^t y(t,s) f(s)\Delta s$, *where* $y(t,s)$ *is the Cauchy function for (3).*

It is known that $y(t,s) := H_{n-1}(t, \sigma(s))$ is the Cauchy function for $y^{\Delta^n} = 0$, where H_{n-1} is a time scale generalized polynomial [20, Example 5.115]. The generalized polynomials H_k are the functions $H_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows:

$$H_0(t,s) \equiv 1, \quad H_{k+1}(t,s) = \int_s^t H_k(\tau, s)\Delta\tau, \quad k = 1, 2, \ldots$$

for all $s, t \in \mathbb{T}$. If we let $H_k^{\Delta}(t,s)$ denote, for each fixed s, the derivative of $H_k(t,s)$ with respect to t, then (cf. [20, p. 38])

$$H_k^{\Delta}(t,s) = H_{k-1}(t,s) \quad \text{for } k \in \mathbb{N},\ t \in \mathbb{T}^{\kappa}.$$

From now on we restrict ourselves to the time scale $\mathbb{T} = (h\mathbb{Z})_a$, $h > 0$, for which the graininess function is the constant h. Our main goal is to propose and develop a discrete-time fractional variational theory in $\mathbb{T} = (h\mathbb{Z})_a$. We borrow the notations from the recent calculus of variations on time scales [18, 24, 32]. How to generalize our results to an arbitrary time scale $\mathbb{T}$, with the graininess function μ depending on time, is not clear and remains a challenging question.

Let $a \in \mathbb{R}$ and $h > 0$, $(h\mathbb{Z})_a = \{a, a+h, a+2h, \ldots\}$, and $b = a + kh$ for some $k \in \mathbb{N}$. We have $\sigma(t) = t + h$, $\rho(t) = t - h$, $\mu(t) \equiv h$, and we will frequently write $f^\sigma(t) = f(\sigma(t))$. We put $\mathbb{T} = [a,b] \cap (h\mathbb{Z})_a$, so that $\mathbb{T}^\kappa = [a, \rho(b)] \cap (h\mathbb{Z})_a$ and $\mathbb{T}^{\kappa^2} = [a, \rho^2(b)] \cap (h\mathbb{Z})_a$. The delta derivative coincides in this case with the forward h-difference: $f^{\Delta}(t) = \dfrac{f^\sigma(t) - f(t)}{\mu(t)}$. If $h = 1$, then we have the usual discrete forward difference $\Delta f(t)$. The delta integral gives the h-sum (or h-integral) of f: $\displaystyle\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h$. If we have a function f of two variables, $f(t,s)$, its partial forward h-differences will be denoted by $\Delta_{t,h}$ and $\Delta_{s,h}$, respectively. We will make use of the standard conventions $\sum_{t=c}^{c-1} f(t) = 0$, $c \in \mathbb{Z}$, and $\prod_{i=0}^{-1} f(i) = 1$. Often, *left fractional delta integration* (resp., *right fractional delta integration*) of order $\nu > 0$ is denoted by ${}_a\Delta_t^{-\nu} f(t)$ (resp. ${}_t\Delta_b^{-\nu} f(t)$). Here, similarly as in Ross *et. al.* [46], where the authors omit the subscript t on the operator (the operator itself cannot depend on t), we write ${}_a\Delta_h^{-\nu} f(t)$ (resp. ${}_h\Delta_b^{-\nu} f(t)$).

Before giving an explicit formula for the generalized polynomials H_k on $h\mathbb{Z}$ we introduce the following definition:

Definition 2.6. *For arbitrary* $x, y \in \mathbb{R}$ *the* h*-factorial function is defined by*

$$x_h^{(y)} := h^y \frac{\Gamma(\frac{x}{h} + 1)}{\Gamma(\frac{x}{h} + 1 - y)},$$

where Γ *is the well-known Euler gamma function, and we use the convention that division at a pole yields zero.*

Remark 2.1. *For $h = 1$, and in accordance with the previous literature* (1)*, we write $x^{(y)}$ to denote $x_h^{(y)}$.*

Proposition 2.1. *For the time-scale $\mathbb{T} = (h\mathbb{Z})_a$ one has*

$$H_k(t,s) := \frac{(t-s)_h^{(k)}}{k!} \quad \textit{for all} \quad s,t \in \mathbb{T} \textit{ and } k \in \mathbb{N}_0 \,. \tag{6}$$

To prove (6) we use the following technical lemma. Throughout the text the basic property $\Gamma(x+1) = x\Gamma(x)$ of the gamma function will be frequently used.

Lemma 2.7. *Let $s \in \mathbb{T}$. Then, for all $t \in \mathbb{T}^\kappa$ one has*

$$\Delta_{t,h}\left\{\frac{(t-s)_h^{(k+1)}}{(k+1)!}\right\} = \frac{(t-s)_h^{(k)}}{k!} \,.$$

Proof. The equality follows by direct computations:

$$\begin{aligned}
\Delta_{t,h}\left\{\frac{(t-s)_h^{(k+1)}}{(k+1)!}\right\} &= \frac{1}{h}\left\{\frac{(\sigma(t)-s)_h^{(k+1)}}{(k+1)!} - \frac{(t-s)_h^{(k+1)}}{(k+1)!}\right\} \\
&= \frac{h^{k+1}}{h(k+1)!}\left\{\frac{\Gamma((t+h-s)/h+1)}{\Gamma((t+h-s)/h+1-(k+1))} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h+1-(k+1))}\right\} \\
&= \frac{h^k}{(k+1)!}\left\{\frac{((t-s)/h+1)\Gamma((t-s)/h+1)}{((t-s)/h-k)\Gamma((t-s)/h-k)} - \frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h-k)}\right\} \\
&= \frac{h^k}{k!}\left\{\frac{\Gamma((t-s)/h+1)}{\Gamma((t-s)/h+1-k)}\right\} = \frac{(t-s)_h^{(k)}}{k!} \,.
\end{aligned}$$

□

Proof. (of Proposition 2.1) We proceed by mathematical induction. For $k = 0$

$$H_0(t,s) = \frac{1}{0!}h^0 \frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}+1-0)} = \frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}+1)} = 1 \,.$$

Assume that (6) holds for k replaced by m. Then by Lemma 2.7

$$H_{m+1}(t,s) \quad = \quad \int_s^t H_m(\tau,s)\Delta\tau = \int_s^t \frac{(\tau-s)_h^{(m)}}{m!}\Delta\tau = \frac{(t-s)_h^{(m+1)}}{(m+1)!},$$

which is (6) with k replaced by $m+1$. □

Let $y_1(t), \ldots, y_n(t)$ be n linearly independent solutions of the linear homogeneous dynamic equation $y^{\Delta^n} = 0$. From Theorem 2.5 we know that the solution of (5) (with $L = \Delta^n$ and $t_0 = a$) is

$$y(t) = \Delta^{-n}f(t) = \int_a^t \frac{(t-\sigma(s))_h^{(n-1)}}{\Gamma(n)} f(s)\Delta s = \frac{1}{\Gamma(n)} \sum_{k=a/h}^{t/h-1} (t-\sigma(kh))_h^{(n-1)} f(kh)h \,.$$

Since $y^{\Delta_i}(a) = 0$, $i = 0, \ldots, n-1$, then we can write that

$$\begin{aligned} \Delta^{-n} f(t) &= \frac{1}{\Gamma(n)} \sum_{k=a/h}^{t/h-n} (t - \sigma(kh))_h^{(n-1)} f(kh) h \\ &= \frac{1}{\Gamma(n)} \int_a^{\sigma(t-nh)} (t - \sigma(s))_h^{(n-1)} f(s) \Delta s \,. \end{aligned} \tag{7}$$

Note that function $t \to (\Delta^{-n} f)(t)$ is defined for $t = a + nh \bmod(h)$ while function $t \to f(t)$ is defined for $t = a \bmod(h)$. Extending (7) to any positive real value ν, and having as an analogy the continuous left and right fractional derivatives [40], we define the left fractional h-sum and the right fractional h-sum as follows. We denote by $\mathcal{F}_{\mathbb{T}}$ the set of all real valued functions defined on a given time scale $\mathbb{T}$.

Definition 2.8. *Let $a \in \mathbb{R}$, $h > 0$, $b = a + kh$ with $k \in \mathbb{N}$, and put $\mathbb{T} = [a, b] \cap (h\mathbb{Z})_a$. Consider $f \in \mathcal{F}_{\mathbb{T}}$. The left and right fractional h-sum of order $\nu > 0$ are, respectively, the operators ${}_a\Delta_h^{-\nu} : \mathcal{F}_{\mathbb{T}} \to \mathcal{F}_{\tilde{\mathbb{T}}_\nu^+}$ and ${}_h\Delta_b^{-\nu} : \mathcal{F}_{\mathbb{T}} \to \mathcal{F}_{\tilde{\mathbb{T}}_\nu^-}$, $\tilde{\mathbb{T}}_\nu^\pm = \{t \pm \nu h : t \in \mathbb{T}\}$, defined by*

$${}_a\Delta_h^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t-\nu h)} (t - \sigma(s))_h^{(\nu-1)} f(s) \Delta s = \frac{1}{\Gamma(\nu)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}-\nu} (t - \sigma(kh))_h^{(\nu-1)} f(kh) h$$

$${}_h\Delta_b^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{t+\nu h}^{\sigma(b)} (s - \sigma(t))_h^{(\nu-1)} f(s) \Delta s = \frac{1}{\Gamma(\nu)} \sum_{k=\frac{t}{h}+\nu}^{\frac{b}{h}} (kh - \sigma(t))_h^{(\nu-1)} f(kh) h.$$

Remark 2.2. *In Definition 2.8 we are using summations with limits that are reals. For example, the summation that appears in the definition of operator ${}_a\Delta_h^{-\nu}$ has the following meaning:*

$$\sum_{k=\frac{a}{h}}^{\frac{t}{h}-\nu} G(k) = G(a/h) + G(a/h + 1) + G(a/h + 2) + \cdots + G(t/h - \nu),$$

where $t \in \{a + \nu h, a + h + \nu h, a + 2h + \nu h, \ldots, \underbrace{a + kh}_{b} + \nu h\}$ with $k \in \mathbb{N}$.

Lemma 2.9. *Let $\nu > 0$ be an arbitrary positive real number. For any $t \in \mathbb{T}$ we have: (i) $\lim_{\nu \to 0} {}_a\Delta_h^{-\nu} f(t + \nu h) = f(t)$; (ii) $\lim_{\nu \to 0} {}_h\Delta_b^{-\nu} f(t - \nu h) = f(t)$.*

Proof. Since

$$\begin{aligned} {}_a\Delta_h^{-\nu} f(t + \nu h) &= \frac{1}{\Gamma(\nu)} \int_a^{\sigma(t)} (t + \nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s \\ &= \frac{1}{\Gamma(\nu)} \sum_{k=\frac{a}{h}}^{\frac{t}{h}} (t + \nu h - \sigma(kh))_h^{(\nu-1)} f(kh) h \\ &= h^\nu f(t) + \frac{\nu}{\Gamma(\nu + 1)} \sum_{k=\frac{a}{h}}^{\frac{\rho(t)}{h}} (t + \nu h - \sigma(kh))_h^{(\nu-1)} f(kh) h \,, \end{aligned}$$

it follows that $\lim_{\nu \to 0} {}_a\Delta_h^{-\nu} f(t + \nu h) = f(t)$. The proof of (ii) is similar. □

For any $t \in \mathbb{T}$ and for any $\nu \geq 0$ we define ${}_a\Delta_h^0 f(t) := {}_h\Delta_b^0 f(t) := f(t)$ and write

$$\begin{aligned} {}_a\Delta_h^{-\nu} f(t+\nu h) &= h^\nu f(t) + \frac{\nu}{\Gamma(\nu+1)} \int_a^t (t+\nu h - \sigma(s))_h^{(\nu-1)} f(s) \Delta s\,, \\ {}_h\Delta_b^{-\nu} f(t) &= h^\nu f(t-\nu h) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^{\sigma(b)} (s+\nu h - \sigma(t))_h^{(\nu-1)} f(s) \Delta s\,. \end{aligned} \tag{8}$$

Theorem 2.10. *Let $f \in \mathcal{F}_\mathbb{T}$ and $\nu \geq 0$. For all $t \in \mathbb{T}^\kappa$ we have*

$$ {}_a\Delta_h^{-\nu} f^\Delta(t+\nu h) = ({}_a\Delta_h^{-\nu} f(t+\nu h))^\Delta - \frac{\nu}{\Gamma(\nu+1)} (t+\nu h - a)_h^{(\nu-1)} f(a)\,. \tag{9}$$

To prove Theorem 2.10 we make use of a technical lemma:

Lemma 2.11. *Let $t \in \mathbb{T}^\kappa$. The following equality holds for all $s \in \mathbb{T}^\kappa$:*

$$\begin{aligned} &\Delta_{s,h}\left((t+\nu h - s)_h^{(\nu-1)} f(s))\right) \\ &\qquad = (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) - (v-1)(t+\nu h - \sigma(s))_h^{(\nu-2)} f(s)\,. \end{aligned} \tag{10}$$

Proof. Direct calculations give the intended result:

$$\begin{aligned} &\Delta_{s,h}\left((t+\nu h - s)_h^{(\nu-1)} f(s)\right) \\ &= \Delta_{s,h}\left((t+\nu h - s)_h^{(\nu-1)}\right) f(s) + (t+\nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= \frac{f(s)}{h}\left[h^{\nu-1}\frac{\Gamma\left(\frac{t+\nu h-\sigma(s)}{h}+1\right)}{\Gamma\left(\frac{t+\nu h-\sigma(s)}{h}+1-(\nu-1)\right)} - h^{\nu-1}\frac{\Gamma\left(\frac{t+\nu h - s}{h}+1\right)}{\Gamma\left(\frac{t+\nu h-s}{h}+1-(\nu-1)\right)}\right] \\ &\qquad + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= f(s)\left[h^{\nu-2}\left[\frac{\Gamma(\frac{t+\nu h-s}{h})}{\Gamma(\frac{t-s}{h}+1)} - \frac{\Gamma(\frac{t+\nu h-s}{h}+1)}{\Gamma(\frac{t-s}{h}+2)}\right]\right] + (t+\nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= f(s)h^{\nu-2}\frac{\Gamma(\frac{t+\nu h-s-h}{h}+1)}{\Gamma(\frac{t-s+\nu h-h}{h}+1-(\nu-2))}(-(\nu-1)) + (t+\nu h-\sigma(s))_h^{(\nu-1)} f^\Delta(s) \\ &= -(\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s) + (t+\nu h - \sigma(s))_h^{(\nu-1)} f^\Delta(s)\,, \end{aligned}$$

where the first equality follows directly from (2). □

Remark 2.3. *Given an arbitrary $t \in \mathbb{T}^\kappa$ it is easy to prove, in a similar way as in the proof of Lemma 2.11, the following equality analogous to* (10)*: for all $s \in \mathbb{T}^\kappa$*

$$\begin{aligned} &\Delta_{s,h}\left((s+\nu h-\sigma(t))_h^{(\nu-1)} f(s))\right) \\ &\qquad = (\nu-1)(s+\nu h - \sigma(t))_h^{(\nu-2)} f^\sigma(s) + (s+\nu h-\sigma(t))_h^{(\nu-1)} f^\Delta(s)\,. \end{aligned} \tag{11}$$

Proof. (of Theorem 2.10) From Lemma 2.11 we obtain that

$$\begin{aligned}
{}_a\Delta_h^{-\nu} f^{\Delta}(t+\nu h) &= h^{\nu} f^{\Delta}(t) + \frac{\nu}{\Gamma(\nu+1)} \int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f^{\Delta}(s)\Delta s \\
&= h^{\nu} f^{\Delta}(t) + \frac{\nu}{\Gamma(\nu+1)} \left[(t+\nu h-s)_h^{(\nu-1)} f(s)\right]_{s=a}^{s=t} \\
&\quad + \frac{\nu}{\Gamma(\nu+1)} \int_a^{\sigma(t)} (\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\Delta s \qquad (12)\\
&= -\frac{\nu(t+\nu h-a)_h^{(\nu-1)}}{\Gamma(\nu+1)} f(a) + h^{\nu} f^{\Delta}(t) + \nu h^{\nu-1} f(t) \\
&\quad + \frac{\nu}{\Gamma(\nu+1)} \int_a^t (\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\Delta s.
\end{aligned}$$

We now show that $({}_a\Delta_h^{-\nu} f(t+\nu h))^{\Delta}$ equals (12):

$$\begin{aligned}
({}_a\Delta_h^{-\nu} f(t+\nu h))^{\Delta} &= \frac{1}{h}\left[h^{\nu} f(\sigma(t)) + \frac{\nu}{\Gamma(\nu+1)} \int_a^{\sigma(t)} (\sigma(t)+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right. \\
&\quad \left. -h^{\nu} f(t) - \frac{\nu}{\Gamma(\nu+1)} \int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right] \\
&= h^{\nu} f^{\Delta}(t) + \frac{\nu}{h\Gamma(\nu+1)} \left[\int_a^t (\sigma(t)+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right. \\
&\quad \left. - \int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)} f(s)\Delta s\right] + h^{\nu-1}\nu f(t) \\
&= h^{\nu} f^{\Delta}(t) + \frac{\nu}{\Gamma(\nu+1)} \int_a^t \Delta_{t,h}\left((t+\nu h-\sigma(s))_h^{(\nu-1)}\right) f(s)\Delta s + h^{\nu-1}\nu f(t) \\
&= h^{\nu} f^{\Delta}(t) + \frac{\nu}{\Gamma(\nu+1)} \int_a^t (\nu-1)(t+\nu h-\sigma(s))_h^{(\nu-2)} f(s)\Delta s + \nu h^{\nu-1} f(t)\,.
\end{aligned}$$

□

Follows the counterpart of Theorem 2.10 for the right fractional h-sum:

Theorem 2.12. *Let $f \in \mathcal{F}_{\mathbb{T}}$ and $\nu \geq 0$. For all $t \in \mathbb{T}^{\kappa}$ we have*

$$ {}_h\Delta_{\rho(b)}^{-\nu} f^{\Delta}(t-\nu h) = \frac{\nu}{\Gamma(\nu+1)} (b+\nu h-\sigma(t))_h^{(\nu-1)} f(b) + ({}_h\Delta_b^{-\nu} f(t-\nu h))^{\Delta}\,. \quad (13)$$

Proof. From (11) we obtain from integration by parts (item 2 of Lemma 2.1) that

$$\begin{aligned}
{}_h\Delta_{\rho(b)}^{-\nu} f^{\Delta}(t-\nu h) &= \frac{\nu(b+\nu h-\sigma(t))_h^{(\nu-1)}}{\Gamma(\nu+1)} f(b) + h^{\nu} f^{\Delta}(t) - \nu h^{\nu-1} f(\sigma(t)) \\
&\quad - \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (\nu-1)(s+\nu h-\sigma(t))_h^{(\nu-2)} f^{\sigma}(s)\Delta s.
\end{aligned} \quad (14)$$

We show that $({}_h\Delta_b^{-\nu} f(t-\nu h))^\Delta$ equals (14):

$$
\begin{aligned}
&({}_h\Delta_b^{-\nu} f(t-\nu h))^\Delta\\
&\quad = h^\nu f^\Delta(t) + \frac{\nu}{h\Gamma(\nu+1)}\left[\int_{\sigma^2(t)}^{\sigma(b)} (s+\nu h-\sigma^2(t)))_h^{(\nu-1)} f(s)\Delta s\right.\\
&\qquad \left. - \int_{\sigma^2(t)}^{\sigma(b)} (s+\nu h-\sigma(t))_h^{(\nu-1)} f(s)\Delta s\right] - \nu h^{\nu-1} f(\sigma(t))\\
&\quad = h^\nu f^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)}\int_{\sigma^2(t)}^{\sigma(b)} \Delta_{t,h}\left((s+\nu h-\sigma(t))_h^{(\nu-1)}\right) f(s)\Delta s - \nu h^{\nu-1} f(\sigma(t))\\
&\quad = h^\nu f^\Delta(t) - \frac{\nu}{\Gamma(\nu+1)}\int_{\sigma^2(t)}^{\sigma(b)} (\nu-1)(s+\nu h-\sigma^2(t))_h^{(\nu-2)} f(s)\Delta s - \nu h^{\nu-1} f(\sigma(t))\\
&\quad = h^\nu f^\Delta(t) - \frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^{b} (\nu-1)(s+\nu h-\sigma(t))_h^{(\nu-2)} f(s)\Delta s - \nu h^{\nu-1} f(\sigma(t)).
\end{aligned}
$$

□

Definition 2.13. *Let* $0 < \alpha \leq 1$ *and set* $\gamma := 1-\alpha$. *The* left fractional difference ${}_a\Delta_h^\alpha f(t)$ *and the* right fractional difference ${}_h\Delta_b^\alpha f(t)$ *of order* α *of a function* $f \in \mathcal{F}_\mathbb{T}$ *are defined as*

$$
{}_a\Delta_h^\alpha f(t) := ({}_a\Delta_h^{-\gamma} f(t+\gamma h))^\Delta \quad \text{and} \quad {}_h\Delta_b^\alpha f(t) := -({}_h\Delta_b^{-\gamma} f(t-\gamma h))^\Delta
$$

for all $t \in \mathbb{T}^\kappa$.

3. Main Results

Our aim is to introduce the h-fractional problem of the calculus of variations and to prove corresponding necessary optimality conditions. In order to obtain an Euler-Lagrange type equation (cf. Theorem 3.5) we first prove a fractional formula of h-summation by parts.

3.1. Fractional h-summation by parts

A big challenge was to discover a fractional h-summation by parts formula within the time scale setting. Indeed, there is no clue of what such a formula should be. We found it eventually, making use of the following lemma.

Lemma 3.1. *Let* f *and* k *be two functions defined on* $\mathbb{T}^\kappa$ *and* $\mathbb{T}^{\kappa^2}$, *respectively, and* g *a function defined on* $\mathbb{T}^\kappa \times \mathbb{T}^{\kappa^2}$. *The following equality holds:*

$$
\int_a^b f(t)\left[\int_a^t g(t,s)k(s)\Delta s\right]\Delta t = \int_a^{\rho(b)} k(t)\left[\int_{\sigma(t)}^b g(s,t)f(s)\Delta s\right]\Delta t\,.
$$

Proof. Consider the matrices $R = [f(a+h), f(a+2h), \cdots, f(b-h)]$,

$$
C_1 = \begin{bmatrix} g(a+h,a)k(a) \\ g(a+2h,a)k(a) + g(a+2h,a+h)k(a+h) \\ \vdots \\ g(b-h,a)k(a) + g(b-h,a+h)k(a+h) + \cdots + g(b-h,b-2h)k(b-2h) \end{bmatrix}
$$

$$C_2 = \begin{bmatrix} g(a+h,a) \\ g(a+2h,a) \\ \vdots \\ g(b-h,a) \end{bmatrix}, \quad C_3 = \begin{bmatrix} 0 \\ g(a+2h,a+h) \\ \vdots \\ g(b-h,a+h) \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ g(b-h,b-2h) \end{bmatrix}.$$

Direct calculations show that

$$\begin{aligned}
&\int_a^b f(t)\left[\int_a^t g(t,s)k(s)\Delta s\right]\Delta t = h^2 \sum_{i=a/h}^{b/h-1} f(ih) \sum_{j=a/h}^{i-1} g(ih,jh)k(jh) = h^2 R\cdot C_1 \\
&\quad = h^2 R\cdot [k(a)C_2 + k(a+h)C_3 + \cdots + k(b-2h)C_4] \\
&\quad = h^2 \left[k(a)\sum_{j=a/h+1}^{b/h-1} g(jh,a)f(jh) + k(a+h)\sum_{j=a/h+2}^{b/h-1} g(jh,a+h)f(jh)\right. \\
&\qquad\qquad \left. + \cdots + k(b-2h)\sum_{j=b/h-1}^{b/h-1} g(jh,b-2h)f(jh)\right] \\
&\quad = \sum_{i=a/h}^{b/h-2} k(ih)h \sum_{j=\sigma(ih)/h}^{b/h-1} g(jh,ih)f(jh)h = \int_a^{\rho(b)} k(t)\left[\int_{\sigma(t)}^b g(s,t)f(s)\Delta s\right]\Delta t.
\end{aligned}$$

□

Theorem 3.2 (fractional h-summation by parts)**.** *Let f and g be real valued functions defined on $\mathbb{T}^\kappa$ and $\mathbb{T}$, respectively. Fix $0 < \alpha \leq 1$ and put $\gamma := 1-\alpha$. Then,*

$$\begin{aligned}
&\int_a^b f(t)_a\Delta_h^\alpha g(t)\Delta t = h^\gamma f(\rho(b))g(b) - h^\gamma f(a)g(a) + \int_a^{\rho(b)} {}_h\Delta_{\rho(b)}^\alpha f(t) g^\sigma(t)\Delta t \\
&+\frac{\gamma}{\Gamma(\gamma+1)}g(a)\left(\int_a^b (t+\gamma h - a)_h^{(\gamma-1)} f(t)\Delta t - \int_{\sigma(a)}^b (t+\gamma h-\sigma(a))_h^{(\gamma-1)} f(t)\Delta t\right).
\end{aligned} \tag{15}$$

Proof. By (9) we can write

$$\begin{aligned}
&\int_a^b f(t)_a\Delta_h^\alpha g(t)\Delta t = \int_a^b f(t)({}_a\Delta_h^{-\gamma} g(t+\gamma h))^\Delta \Delta t \\
&\quad = \int_a^b f(t)\left[{}_a\Delta_h^{-\gamma} g^\Delta(t+\gamma h) + \frac{\gamma}{\Gamma(\gamma+1)}(t+\gamma h-a)_h^{(\gamma-1)} g(a)\right]\Delta t \\
&\quad = \int_a^b f(t)_a\Delta_h^{-\gamma} g^\Delta(t+\gamma h)\Delta t + \int_a^b \frac{\gamma}{\Gamma(\gamma+1)}(t+\gamma h-a)_h^{(\gamma-1)} f(t)g(a)\Delta t.
\end{aligned} \tag{16}$$

Using (8) we get

$$\int_a^b f(t)_a\Delta_h^{-\gamma} g^\Delta(t+\gamma h)\Delta t$$
$$= \int_a^b f(t)\left[h^\gamma g^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)} g^\Delta(s)\Delta s\right]\Delta t$$
$$= h^\gamma \int_a^b f(t)g^\Delta(t)\Delta t + \frac{\gamma}{\Gamma(\gamma+1)}\int_a^{\rho(b)} g^\Delta(t)\int_{\sigma(t)}^b (s+\gamma h-\sigma(t))_h^{(\gamma-1)} f(s)\Delta s\Delta t$$
$$= h^\gamma f(\rho(b))[g(b)-g(\rho(b))] + \int_a^{\rho(b)} g^\Delta(t)_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h)\Delta t,$$

where the third equality follows by Lemma 3.1. We proceed to develop the right hand side of the last equality as follows:

$$h^\gamma f(\rho(b))[g(b)-g(\rho(b))] + \int_a^{\rho(b)} g^\Delta(t)_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h)\Delta t$$
$$= h^\gamma f(\rho(b))[g(b)-g(\rho(b))] + \left[g(t)_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h)\right]_{t=a}^{t=\rho(b)}$$
$$- \int_a^{\rho(b)} g^\sigma(t)({}_h\Delta_{\rho(b)}^{-\gamma} f(t-\gamma h))^\Delta \Delta t$$
$$= h^\gamma f(\rho(b))g(b) - h^\gamma f(a)g(a)$$
$$- \frac{\gamma}{\Gamma(\gamma+1)} g(a)\int_{\sigma(a)}^b (s+\gamma h-\sigma(a))_h^{(\gamma-1)} f(s)\Delta s + \int_a^{\rho(b)} \left({}_h\Delta_{\rho(b)}^\alpha f(t)\right) g^\sigma(t)\Delta t,$$

where the first equality follows from Lemma 2.1. Putting this into (16) we get (15). □

3.2. Necessary optimality conditions

We begin to fix two arbitrary real numbers α and β such that $\alpha,\beta\in(0,1]$. Further, we put $\gamma := 1-\alpha$ and $\nu := 1-\beta$.

Let a function $L(t,u,v,w) : \mathbb{T}^\kappa \times \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be given. We consider the problem of minimizing (or maximizing) a functional $\mathcal{L} : \mathcal{F}_\mathbb{T} \to \mathbb{R}$ subject to given boundary conditions:

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y^\sigma(t), {}_a\Delta_h^\alpha y(t), {}_h\Delta_b^\beta y(t))\Delta t \longrightarrow \min,\ y(a)=A,\ y(b)=B\,. \tag{17}$$

Our main aim is to derive necessary optimality conditions for problem (17).

Definition 3.3. *For $f\in\mathcal{F}_\mathbb{T}$ we define the norm*

$$\|f\| = \max_{t\in\mathbb{T}^\kappa} |f^\sigma(t)| + \max_{t\in\mathbb{T}^\kappa} |{}_a\Delta_h^\alpha f(t)| + \max_{t\in\mathbb{T}^\kappa} |{}_h\Delta_b^\beta f(t)|.$$

A function $\hat{y}\in\mathcal{F}_\mathbb{T}$ with $\hat{y}(a)=A$ and $\hat{y}(b)=B$ is called a local minimum for problem (17) *provided there exists $\delta>0$ such that $\mathcal{L}(\hat{y}) \le \mathcal{L}(y)$ for all $y\in\mathcal{F}_\mathbb{T}$ with $y(a)=A$ and $y(b)=B$ and $\|y-\hat{y}\|<\delta$.*

Definition 3.4. *A function $\eta\in\mathcal{F}_\mathbb{T}$ is called an admissible variation provided $\eta\neq 0$ and $\eta(a)=\eta(b)=0$.*

From now on we assume that the second-order partial derivatives L_{uu}, L_{uv}, L_{uw}, L_{vw}, L_{vv}, and L_{ww} exist and are continuous.

3.2.1. First order optimality condition

Next theorem gives a first order necessary condition for problem (17), i.e., an Euler-Lagrange type equation for the fractional h-difference setting.

Theorem 3.5 (The h-fractional Euler-Lagrange equation for problem (17))**.** *If $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ is a local minimum for problem (17), then the equality*

$$L_u[\hat{y}](t) + {}_h\Delta^{\alpha}_{\rho(b)} L_v[\hat{y}](t) + {}_a\Delta^{\beta}_{h} L_w[\hat{y}](t) = 0 \tag{18}$$

holds for all $t \in \mathbb{T}^{\kappa^2}$ with operator $[\cdot]$ defined by $[y](s) = (s, y^\sigma(s), {}_a\Delta^\alpha_s y(s), {}_s\Delta^\beta_b y(s))$.

Proof. Suppose that $\hat{y}(\cdot)$ is a local minimum of $\mathcal{L}[\cdot]$. Let $\eta(\cdot)$ be an arbitrarily fixed admissible variation and define a function $\Phi : \left(-\frac{\delta}{\|\eta(\cdot)\|}, \frac{\delta}{\|\eta(\cdot)\|}\right) \rightarrow \mathbb{R}$ by

$$\Phi(\varepsilon) = \mathcal{L}[\hat{y}(\cdot) + \varepsilon\eta(\cdot)]. \tag{19}$$

This function has a minimum at $\varepsilon = 0$, so we must have $\Phi'(0) = 0$, i.e.,

$$\int_a^b \left[L_u[\hat{y}](t)\eta^\sigma(t) + L_v[\hat{y}](t)_a\Delta^\alpha_h\eta(t) + L_w[\hat{y}](t)_h\Delta^\beta_b\eta(t)\right]\Delta t = 0,$$

which we may write equivalently as

$$hL_u[\hat{y}](t)\eta^\sigma(t)|_{t=\rho(b)} + \int_a^{\rho(b)} L_u[\hat{y}](t)\eta^\sigma(t)\Delta t + \int_a^b L_v[\hat{y}](t)_a\Delta^\alpha_h\eta(t)\Delta t + \int_a^b L_w[\hat{y}](t)_h\Delta^\beta_b\eta(t)\Delta t = 0. \tag{20}$$

Using Theorem 3.2 and the fact that $\eta(a) = \eta(b) = 0$, we get

$$\int_a^b L_v[\hat{y}](t)_a\Delta^\alpha_h\eta(t)\Delta t = \int_a^{\rho(b)} \left({}_h\Delta^\alpha_{\rho(b)}\left(L_v[\hat{y}]\right)(t)\right)\eta^\sigma(t)\Delta t \tag{21}$$

for the third term in (20). Using (13) it follows that

$$\begin{aligned}
&\int_a^b L_w[\hat{y}](t)_h\Delta^\beta_b\eta(t)\Delta t \\
&= -\int_a^b L_w[\hat{y}](t)({}_h\Delta^{-\nu}_b\eta(t-\nu h))^\Delta\Delta t \\
&= -\int_a^b L_w[\hat{y}](t)\left[{}_h\Delta^{-\nu}_{\rho(b)}\eta^\Delta(t-\nu h) - \frac{\nu}{\Gamma(\nu+1)}(b+\nu h-\sigma(t))_h^{(\nu-1)}\eta(b)\right]\Delta t \\
&= -\int_a^b L_w[\hat{y}](t)_h\Delta^{-\nu}_{\rho(b)}\eta^\Delta(t-\nu h)\Delta t + \frac{\nu\eta(b)}{\Gamma(\nu+1)}\int_a^b (b+\nu h-\sigma(t))_h^{(\nu-1)}L_w[\hat{y}](t)\Delta t.
\end{aligned} \tag{22}$$

We now use Lemma 3.1 to get

$$\begin{aligned}
&\int_a^b L_w[\hat{y}](t)_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t-\nu h)\Delta t\\
&\quad=\int_a^b L_w[\hat{y}](t)\left[h^\nu\eta^\Delta(t)+\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^b (s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]\Delta t\\
&\quad=\int_a^b h^\nu L_w[\hat{y}](t)\eta^\Delta(t)\Delta t\\
&\qquad+\frac{\nu}{\Gamma(\nu+1)}\int_a^{\rho(b)}\left[L_w[\hat{y}](t)\int_{\sigma(t)}^b (s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^\Delta(s)\Delta s\right]\Delta t\\
&\quad=\int_a^b h^\nu L_w[\hat{y}](t)\eta^\Delta(t)\Delta t\\
&\qquad+\frac{\nu}{\Gamma(\nu+1)}\int_a^b\left[\eta^\Delta(t)\int_a^t (t+\nu h-\sigma(s))_h^{(\nu-1)}L_w[\hat{y}](s)\Delta s\right]\Delta t\\
&\quad=\int_a^b \eta^\Delta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\Delta t.
\end{aligned} \tag{23}$$

We apply again the time scale integration by parts formula (Lemma 2.1), this time to (23), to obtain,

$$\begin{aligned}
&\int_a^b \eta^\Delta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\Delta t\\
&\quad=\int_a^{\rho(b)} \eta^\Delta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\Delta t\\
&\qquad+(\eta(b)-\eta(\rho(b)))_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}\\
&\quad=\left[\eta(t)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)\right]_{t=a}^{t=\rho(b)}-\int_a^{\rho(b)}\eta^\sigma(t)({}_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h))^\Delta\Delta t\\
&\qquad+\eta(b)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}-\eta(\rho(b))_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}\\
&\quad=\eta(b)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=\rho(b)}-\eta(a)_a\Delta_h^{-\nu}\left(L_w[\hat{y}]\right)(t+\nu h)|_{t=a}\\
&\qquad-\int_a^{\rho(b)}\eta^\sigma(t)_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\Delta t.
\end{aligned} \tag{24}$$

Since $\eta(a)=\eta(b)=0$ we obtain, from (23) and (24), that

$$\int_a^b L_w[\hat{y}](t)_h\Delta_{\rho(b)}^{-\nu}\eta^\Delta(t)\Delta t=-\int_a^{\rho(b)}\eta^\sigma(t)_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\Delta t\,,$$

and after inserting in (22), that

$$\int_a^b L_w[\hat{y}](t)_h\Delta_b^\beta\eta(t)\Delta t=\int_a^{\rho(b)}\eta^\sigma(t)_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\Delta t. \tag{25}$$

By (21) and (25) we may write (20) as

$$\int_a^{\rho(b)}\left[L_u[\hat{y}](t)+{}_h\Delta_{\rho(b)}^\alpha\left(L_v[\hat{y}]\right)(t)+{}_a\Delta_h^\beta\left(L_w[\hat{y}]\right)(t)\right]\eta^\sigma(t)\Delta t=0\,.$$

Since the values of $\eta^\sigma(t)$ are arbitrary for $t \in \mathbb{T}^{\kappa^2}$, the Euler-Lagrange equation (18) holds along $\hat{y}$. □

The next result is a direct corollary of Theorem 3.5.

Corollary 3.1 (The h-Euler-Lagrange equation – cf., e.g., [18, 24])**.** *Let $\mathbb{T}$ be the time scale $h\mathbb{Z}$, $h > 0$, with the forward jump operator σ and the delta derivative Δ. Assume $a, b \in \mathbb{T}$, $a < b$. If $\hat{y}$ is a solution to the problem*

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \longrightarrow \min,\ y(a) = A,\ y(b) = B\,,$$

then the equality $L_u(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t)) - \left(L_v(t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))\right)^\Delta = 0$ holds for all $t \in \mathbb{T}^{\kappa^2}$.

Proof. Choose $\alpha = 1$ and a L that does not depend on w in Theorem 3.5. □

Remark 3.1. *If we take $h = 1$ in Corollary 3.1 we have that*

$$L_u(t, \hat{y}^\sigma(t), \Delta\hat{y}(t)) - \Delta L_v(t, \hat{y}^\sigma(t), \Delta\hat{y}(t)) = 0$$

holds for all $t \in \mathbb{T}^{\kappa^2}$. This equation is usually called the discrete Euler-Lagrange equation*, and can be found, e.g., in [29, Chap. 8].*

3.2.2. Natural boundary conditions

If the initial condition $y(a) = A$ is not present in problem (17) (i.e., $y(a)$ is free), besides the h-fractional Euler-Lagrange equation (18) the following supplementary condition must be fulfilled:

$$\begin{split} - h^\gamma L_v[\hat{y}](a) + \frac{\gamma}{\Gamma(\gamma+1)} \Bigg(\int_a^b (t + \gamma h - a)_h^{(\gamma-1)} L_v[\hat{y}](t)\Delta t \\ - \int_{\sigma(a)}^b (t + \gamma h - \sigma(a))_h^{(\gamma-1)} L_v[\hat{y}](t)\Delta t \Bigg) + L_w[\hat{y}](a) = 0. \end{split} \tag{26}$$

Similarly, if $y(b) = B$ is not present in (17) ($y(b)$ is free), the extra condition

$$\begin{split} hL_u[\hat{y}](\rho(b)) + h^\gamma L_v[\hat{y}](\rho(b)) - h^\nu L_w[\hat{y}](\rho(b)) \\ + \frac{\nu}{\Gamma(\nu+1)} \Bigg(\int_a^b (b + \nu h - \sigma(t))_h^{(\nu-1)} L_w[\hat{y}](t)\Delta t \\ - \int_a^{\rho(b)} (\rho(b) + \nu h - \sigma(t))_h^{(\nu-1)} L_w[\hat{y}](t)\Delta t \Bigg) = 0 \end{split} \tag{27}$$

is added to Theorem 3.5. We leave the proof of the *natural boundary conditions* (26) and (27) to the reader. We just note here that the first term in (27) arises from the first term of the left hand side of (20).

3.2.3. Second order optimality condition

We now obtain a second order necessary condition for problem (17), i.e., we prove a Legendre optimality type condition for the fractional h-difference setting.

Theorem 3.6 (The h-fractional Legendre necessary condition)**.** *If $\hat{y} \in \mathcal{F}_{\mathbb{T}}$ is a local minimum for problem* (17)*, then the inequality*

$$\begin{aligned} &h^2 L_{uu}[\hat{y}](t) + 2h^{\gamma+1} L_{uv}[\hat{y}](t) + 2h^{\nu+1}(\nu-1) L_{uw}[\hat{y}](t) + h^{2\gamma}(\gamma-1)^2 L_{vv}[\hat{y}](\sigma(t)) \\ &\quad + 2h^{\nu+\gamma}(\gamma-1) L_{vw}[\hat{y}](\sigma(t)) + 2h^{\nu+\gamma}(\nu-1) L_{vw}[\hat{y}](t) + h^{2\nu}(\nu-1)^2 L_{ww}[\hat{y}](t) \\ &\quad + h^{2\nu} L_{ww}[\hat{y}](\sigma(t)) + \int_a^t h^3 L_{ww}[\hat{y}](s) \left(\frac{\nu(1-\nu)}{\Gamma(\nu+1)} (t+\nu h - \sigma(s))_h^{(\nu-2)} \right)^2 \Delta s \\ &\quad + h^{\gamma} L_{vv}[\hat{y}](t) + \int_{\sigma(\sigma(t))}^b h^3 L_{vv}[\hat{y}](s) \left(\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)} (s+\gamma h - \sigma(\sigma(t)))_h^{(\gamma-2)} \right)^2 \Delta s \geq 0 \end{aligned} \tag{28}$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\hat{y}](t) = (t, \hat{y}^\sigma(t), {}_a\Delta_t^\alpha \hat{y}(t), {}_t\Delta_b^\beta \hat{y}(t))$.

Proof. By the hypothesis of the theorem, and letting Φ be as in (19), we have as necessary optimality condition that $\Phi''(0) \geq 0$ for an arbitrary admissible variation $\eta(\cdot)$. Inequality $\Phi''(0) \geq 0$ is equivalent to

$$\begin{aligned} &\int_a^b \Big[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\hat{y}](t)\eta^\sigma(t)_a\Delta_h^\alpha \eta(t) + 2L_{uw}[\hat{y}](t)\eta^\sigma(t)_h\Delta_b^\beta \eta(t) \\ &+ L_{vv}[\hat{y}](t)({}_a\Delta_h^\alpha \eta(t))^2 + 2L_{vw}[\hat{y}](t)_a\Delta_h^\alpha \eta(t)_h\Delta_b^\beta \eta(t) + L_{ww}(t)({}_h\Delta_b^\beta \eta(t))^2 \Big] \Delta t \geq 0. \end{aligned} \tag{29}$$

Let $\tau \in \mathbb{T}^{\kappa^2}$ be arbitrary, and choose $\eta : \mathbb{T} \to \mathbb{R}$ given by $\eta(t) = \begin{cases} h & \text{if } t = \sigma(\tau); \\ 0 & \text{otherwise.} \end{cases}$ It follows that $\eta(a) = \eta(b) = 0$, i.e., η is an admissible variation. Using (9) we get

$$\begin{aligned} &\int_a^b \left[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 + 2L_{uv}[\hat{y}](t)\eta^\sigma(t)_a\Delta_h^\alpha \eta(t) + L_{vv}[\hat{y}](t)({}_a\Delta_h^\alpha \eta(t))^2 \right] \Delta t \\ &\quad = \int_a^b \Bigg[L_{uu}[\hat{y}](t)(\eta^\sigma(t))^2 \\ &\qquad + 2L_{uv}[\hat{y}](t)\eta^\sigma(t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right) \\ &\qquad + L_{vv}[\hat{y}](t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \Bigg] \Delta t \\ &\quad = h^3 L_{uu}[\hat{y}](\tau) + 2h^{\gamma+2} L_{uv}[\hat{y}](\tau) + h^{\gamma+1} L_{vv}[\hat{y}](\tau) \\ &\qquad + \int_{\sigma(\tau)}^b L_{vv}[\hat{y}](t) \left(h^\gamma \eta^\Delta(t) + \frac{\gamma}{\Gamma(\gamma+1)} \int_a^t (t+\gamma h - \sigma(s))_h^{(\gamma-1)} \eta^\Delta(s) \Delta s \right)^2 \Delta t. \end{aligned}$$

Observe that

$$\begin{aligned}
&h^{2\gamma+1}(\gamma-1)^2 L_{vv}[\hat{y}](\sigma(\tau)) \\
&\qquad + \int_{\sigma^2(\tau)}^{b} L_{vv}[\hat{y}](t)\left(\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)^2 \Delta t \\
&= \int_{\sigma(\tau)}^{b} L_{vv}[\hat{y}](t)\left(h^{\gamma}\eta^{\Delta}(t)+\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)^2 \Delta t.
\end{aligned}$$

Let $t \in [\sigma^2(\tau), \rho(b)] \cap h\mathbb{Z}$. Since

$$\begin{aligned}
&\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s \\
&\qquad = \frac{\gamma}{\Gamma(\gamma+1)}\left[\int_a^{\sigma(\tau)} (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right. \\
&\qquad\qquad\qquad \left. + \int_{\sigma(\tau)}^{t} (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right] \\
&\qquad = h\frac{\gamma}{\Gamma(\gamma+1)}\left[(t+\gamma h-\sigma(\tau))_h^{(\gamma-1)} - (t+\gamma h-\sigma(\sigma(\tau)))_h^{(\gamma-1)}\right] \\
&\qquad = \frac{\gamma h^{\gamma}}{\Gamma(\gamma+1)}\left[\frac{\left(\frac{t-\tau}{h}+\gamma-1\right)\Gamma\left(\frac{t-\tau}{h}+\gamma-1\right)-\left(\frac{t-\tau}{h}\right)\Gamma\left(\frac{t-\tau}{h}+\gamma-1\right)}{\left(\frac{t-\tau}{h}\right)\Gamma\left(\frac{t-\tau}{h}\right)}\right] \\
&\qquad = h^2\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)}(t+\gamma h-\sigma(\sigma(\tau)))_h^{(\gamma-2)},
\end{aligned} \tag{30}$$

we conclude that

$$\begin{aligned}
&\int_{\sigma^2(\tau)}^{b} L_{vv}[\hat{y}](t)\left(\frac{\gamma}{\Gamma(\gamma+1)}\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)^2 \Delta t \\
&\qquad\qquad = \int_{\sigma^2(\tau)}^{b} L_{vv}[\hat{y}](t)\left(h^2\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)}(t+\gamma h-\sigma^2(\tau))_h^{(\gamma-2)}\right)^2 \Delta t.
\end{aligned}$$

Note that we can write ${}_t\Delta_b^{\beta}\eta(t) = -{}_h\Delta_{\rho(b)}^{-\nu}\eta^{\Delta}(t-\nu h)$ because $\eta(b)=0$. It is not difficult to see that the following equality holds:

$$\begin{aligned}
\int_a^b 2L_{uw}[\hat{y}](t)\eta^{\sigma}(t)_h\Delta_b^{\beta}\eta(t)\Delta t &= -\int_a^b 2L_{uw}[\hat{y}](t)\eta^{\sigma}(t)_h\Delta_{\rho(b)}^{-\nu}\eta^{\Delta}(t-\nu h)\Delta t \\
&= 2h^{2+\nu}L_{uw}[\hat{y}](\tau)(\nu-1)\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\int_a^b 2L_{vw}[\hat{y}](t)_a\Delta_h^{\alpha}\eta(t)_h\Delta_b^{\beta}\eta(t)\Delta t \\
&\quad = -2\int_a^b L_{vw}[\hat{y}](t)\left\{\left(h^{\gamma}\eta^{\Delta}(t)+\frac{\gamma}{\Gamma(\gamma+1)}\cdot\int_a^t (t+\gamma h-\sigma(s))_h^{(\gamma-1)}\eta^{\Delta}(s)\Delta s\right)\right. \\
&\qquad\qquad \left.\cdot\left[h^{\nu}\eta^{\Delta}(t)+\frac{\nu}{\Gamma(\nu+1)}\int_{\sigma(t)}^{b}(s+\nu h-\sigma(t))_h^{(\nu-1)}\eta^{\Delta}(s)\Delta s\right]\right\}\Delta t \\
&\quad = 2h^{\gamma+\nu+1}(\nu-1)L_{vw}[\hat{y}](\tau)+2h^{\gamma+\nu+1}(\gamma-1)L_{vw}[\hat{y}](\sigma(\tau)).
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
&\int_a^b L_{ww}[\hat{y}](t)({}_h\Delta_b^\beta \eta(t))^2 \Delta t \\
&= \int_a^{\sigma(\sigma(\tau))} L_{ww}[\hat{y}](t) \left[h^\nu \eta^\Delta(t) + \frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (s+\nu h - \sigma(t))_h^{(\nu-1)} \eta^\Delta(s)\Delta s \right]^2 \Delta t \\
&= \int_a^\tau L_{ww}[\hat{y}](t) \left[\frac{\nu}{\Gamma(\nu+1)} \int_{\sigma(t)}^b (s+\nu h - \sigma(t))_h^{(\nu-1)} \eta^\Delta(s)\Delta s \right]^2 \Delta t \\
&\quad + hL_{ww}[\hat{y}](\tau)(h^\nu - \nu h^\nu)^2 + h^{2\nu+1} L_{ww}[\hat{y}](\sigma(\tau)) \\
&= \int_a^\tau L_{ww}[\hat{y}](t) \left[h \frac{\nu}{\Gamma(\nu+1)} \left\{ (\tau + \nu h - \sigma(t))_h^{(\nu-1)} - (\sigma(\tau)+\nu h - \sigma(t))_h^{(\nu-1)} \right\} \right]^2 \\
&\quad + hL_{ww}[\hat{y}](\tau)(h^\nu - \nu h^\nu)^2 + h^{2\nu+1} L_{ww}[\hat{y}](\sigma(\tau)).
\end{aligned}$$

Similarly as we did in (30), we can prove that

$$\begin{aligned}
&h \frac{\nu}{\Gamma(\nu+1)} \left\{ (\tau+\nu h - \sigma(t))_h^{(\nu-1)} - (\sigma(\tau)+\nu h - \sigma(t))_h^{(\nu-1)} \right\} \\
&\qquad\qquad = h^2 \frac{\nu(1-\nu)}{\Gamma(\nu+1)} (\tau + \nu h - \sigma(t))_h^{(\nu-2)}.
\end{aligned}$$

Thus, we have that inequality (29) is equivalent to

$$\begin{aligned}
h\Bigg\{ & h^2 L_{uu}[\hat{y}](t) + 2h^{\gamma+1} L_{uv}[\hat{y}](t) + h^\gamma L_{vv}[\hat{y}](t) + L_{vv}(\sigma(t))(\gamma h^\gamma - h^\gamma)^2 \\
&+ \int_{\sigma(\sigma(t))}^b h^3 L_{vv}(s) \left(\frac{\gamma(\gamma-1)}{\Gamma(\gamma+1)} (s+\gamma h - \sigma(\sigma(t)))_h^{(\gamma-2)} \right)^2 \Delta s \\
&+ 2h^{\nu+1} L_{uw}[\hat{y}](t)(\nu-1) + 2h^{\gamma+\nu}(\nu-1) L_{vw}[\hat{y}](t) \\
&+ 2h^{\gamma+\nu}(\gamma-1) L_{vw}(\sigma(t)) + h^{2\nu} L_{ww}[\hat{y}](t)(1-\nu)^2 + h^{2\nu} L_{ww}[\hat{y}](\sigma(t)) \\
&+ \int_a^t h^3 L_{ww}[\hat{y}](s) \left(\frac{\nu(1-\nu)}{\Gamma(\nu+1)} (t+\nu h - \sigma(s))^{\nu-2} \right)^2 \Delta s \Bigg\} \geq 0. \qquad (31)
\end{aligned}$$

Because $h > 0$, (31) is equivalent to (28). The theorem is proved. □

The next result is a simple corollary of Theorem 3.6.

Corollary 3.2 (The h-Legendre necessary condition – cf. Result 1.3 of [18])**.** *Let $\mathbb{T}$ be the time scale $h\mathbb{Z}$, $h > 0$, with the forward jump operator σ and the delta derivative Δ. Assume $a, b \in \mathbb{T}$, $a < b$. If $\hat{y}$ is a solution to the problem*

$$\mathcal{L}(y(\cdot)) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \longrightarrow \min, \;\; y(a) = A, \;\; y(b) = B\,,$$

then the inequality

$$h^2 L_{uu}[\hat{y}](t) + 2hL_{uv}[\hat{y}](t) + L_{vv}[\hat{y}](t) + L_{vv}[\hat{y}](\sigma(t)) \geq 0 \qquad (32)$$

holds for all $t \in \mathbb{T}^{\kappa^2}$, where $[\hat{y}](t) = (t, \hat{y}^\sigma(t), \hat{y}^\Delta(t))$.

Proof. Choose $\alpha = 1$ and a Lagrangian L that does not depend on w. Then, $\gamma = 0$ and the result follows immediately from Theorem 3.6. □

Remark 3.2. *When h goes to zero we have $\sigma(t) = t$ and inequality* (32) *coincides with Legendre's classical necessary optimality condition $L_{vv}[\hat{y}](t) \geq 0$ (cf., e.g., [50]).*

4. Examples

In this section we present some illustrative examples.

Example 4.1. *Let us consider the following problem:*

$$\mathcal{L}(y) = \frac{1}{2} \int_0^1 \left({}_0\Delta_h^{\frac{3}{4}} y(t) \right)^2 \Delta t \longrightarrow \min , \quad y(0) = 0 , \quad y(1) = 1 . \qquad (33)$$

We consider (33) with different values of h. Numerical results show that when h tends to zero the h-fractional Euler-Lagrange extremal tends to the fractional continuous extremal: when $h \to 0$ (33) tends to the fractional continuous variational problem in the Riemann-Liouville sense studied in [1, Example 1], with solution given by

$$y(t) = \frac{1}{2} \int_0^t \frac{dx}{[(1-x)(t-x)]^{\frac{1}{4}}} . \qquad (34)$$

This is illustrated in Figure 1. In this example for each value of h there is

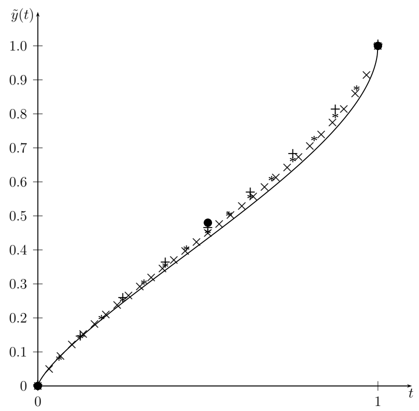

Figure 1: Extremal $\tilde{y}(t)$ for problem of Example 4.1 with different values of h: $h = 0.50$ (•); $h = 0.125$ (+); $h = 0.0625$ (∗); $h = 1/30$ (×). The continuous line represent function (34).

a unique h-fractional Euler-Lagrange extremal, solution of (18)*, which always verifies the h-fractional Legendre necessary condition* (28)*.*

Example 4.2. *Let us consider the following problem:*

$$\mathcal{L}(y) = \int_0^1 \left[\frac{1}{2} \left({}_0\Delta_h^{\alpha} y(t) \right)^2 - y^{\sigma}(t) \right] \Delta t \longrightarrow \min , \quad y(0) = 0 , \quad y(1) = 0 . \quad (35)$$

We begin by considering problem (35) with a fixed value for α and different values of h. The extremals $\tilde{y}$ are obtained using our Euler-Lagrange equation

(18). As in Example 4.1 the numerical results show that when h tends to zero the extremal of the problem tends to the extremal of the corresponding continuous fractional problem of the calculus of variations in the Riemann-Liouville sense. More precisely, when h approximates zero problem (35) tends to the fractional continuous problem studied in [2, Example 2]. For $\alpha = 1$ and $h \to 0$ the extremal of (35) is given by $y(t) = \frac{1}{2}t(1-t)$, which coincides with the extremal of the classical problem of the calculus of variations

$$\mathcal{L}(y) = \int_0^1 \left(\frac{1}{2}y'(t)^2 - y(t)\right) dt \longrightarrow \min\,, \quad y(0) = 0\,, \quad y(1) = 0\,.$$

This is illustrated in Figure 2 for $h = \frac{1}{2^i}$, $i = 1, 2, 3, 4$. In this example, for

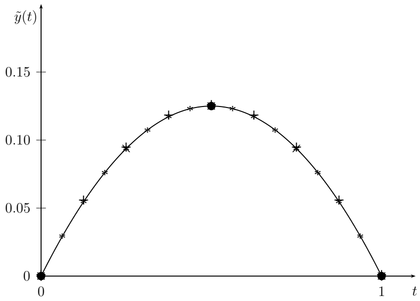

Figure 2: Extremal $\tilde{y}(t)$ for problem (35) with $\alpha = 1$ and different values of h: $h = 0.5$ (•); $h = 0.25$ (×); $h = 0.125$ (+); $h = 0.0625$ (∗).

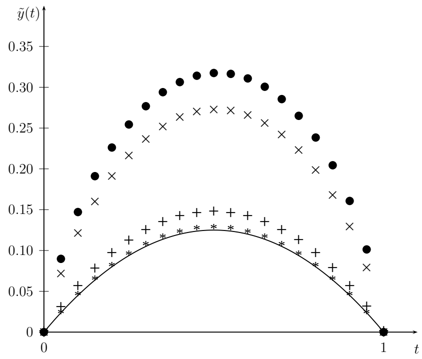

Figure 3: Extremal $\tilde{y}(t)$ for (35) with $h = 0.05$ and different values of α: $\alpha = 0.70$ (•); $\alpha = 0.75$ (×); $\alpha = 0.95$ (+); $\alpha = 0.99$ (∗). The continuous line is $y(t) = \frac{1}{2}t(1-t)$.

each value of α and h, we only have one extremal (we only have one solution to (18) for each α and h). Our Legendre condition (28) *is always verified along the extremals. Figure 3 shows the extremals of problem* (35) *for a fixed value of h ($h = 1/20$) and different values of α. The numerical results show that when α tends to one the extremal tends to the solution of the classical (integer order) discrete-time problem.*

Our last example shows that the h-fractional Legendre necessary optimality condition can be a very useful tool. In Example 4.3 we consider a problem for which the h-fractional Euler-Lagrange equation gives several candidates but just a few of them verify the Legendre condition (28).

Example 4.3. *Let us consider the following problem:*

$$\mathcal{L}(y) = \int_a^b \left({}_a\Delta_h^\alpha y(t)\right)^3 + \theta\left({}_h\Delta_b^\alpha y(t)\right)^2 \Delta t \longrightarrow \min\,, \quad y(a) = 0\,, \quad y(b) = 1\,. \tag{36}$$

For $\alpha = 0.8$, $\beta = 0.5$, $h = 0.25$, $a = 0$, $b = 1$, and $\theta = 1$, problem (36) has eight different Euler-Lagrange extremals. As we can see on Table 1 only two of the candidates verify the Legendre condition. To determine the best candidate we compare the values of the functional $\mathcal{L}$ along the two good candidates. The extremal we are looking for is given by the candidate number five on Table 1.

#	$\tilde{y}\left(\frac{1}{4}\right)$	$\tilde{y}\left(\frac{1}{2}\right)$	$\tilde{y}\left(\frac{3}{4}\right)$	$\mathcal{L}(\tilde{y})$	Legendre condition (28)
1	-0.5511786	0.0515282	0.5133134	9.3035911	Not verified
2	0.2669091	0.4878808	0.7151924	2.0084203	Verified
3	-2.6745703	0.5599360	-2.6730125	698.4443232	Not verified
4	0.5789976	1.0701515	0.1840377	12.5174960	Not verified
5	1.0306820	1.8920322	2.7429222	-32.7189756	Verified
6	0.5087946	-0.1861431	0.4489196	10.6730959	Not verified
7	4.0583690	-1.0299054	-5.0030989	2451.7637948	Not verified
8	-1.7436106	-3.1898449	-0.8850511	238.6120299	Not verified

Table 1: There exist 8 Euler-Lagrange extremals for problem (36) with $\alpha = 0.8$, $\beta = 0.5$, $h = 0.25$, $a = 0$, $b = 1$, and $\theta = 1$, but only 2 of them satisfy the fractional Legendre condition (28).

#	$\tilde{y}(0.1)$	$\tilde{y}(0.2)$	$\tilde{y}(0.3)$	$\tilde{y}(0.4)$	$\mathcal{L}(\tilde{y})$	(28)
1	-0.305570704	-0.428093486	0.223708338	0.480549114	12.25396166	No
2	-0.427934654	-0.599520948	0.313290997	-0.661831134	156.2317667	No
3	0.284152257	-0.227595659	0.318847274	0.531827387	8.669645848	No
4	-0.277642565	0.222381632	0.386666793	0.555841555	6.993518478	No
5	0.387074742	-0.310032839	0.434336603	-0.482903047	110.7912605	No
6	0.259846344	0.364035314	0.463222456	0.597907505	5.104389191	Yes
7	-0.375094681	0.300437245	0.522386246	-0.419053781	93.95316858	No
8	0.343327771	0.480989769	0.61204299	-0.280908953	69.23497954	No
9	0.297792192	0.417196073	-0.218013689	0.460556635	14.12227593	No
10	0.41283304	0.578364133	-0.302235104	-0.649232892	157.8272685	No
11	-0.321401682	0.257431098	-0.360644857	0.400971272	19.87468886	No
12	0.330157414	-0.264444122	-0.459803086	0.368850105	24.84475504	No
13	-0.459640837	0.368155651	-0.515763025	-0.860276767	224.9964788	No
14	-0.359429958	-0.50354835	-0.640748011	0.294083676	34.43515839	No
15	0.477760586	-0.382668914	-0.66536683	-0.956478654	263.3075289	No
16	-0.541587541	-0.758744525	-0.965476394	-1.246195157	392.9592508	No

Table 2: There exist 16 Euler-Lagrange extremals for problem (36) with $\alpha = 0.3$, $h = 0.1$, $a = 0$, $b = 0.5$, and $\theta = 0$, but only 1 (candidate #6) satisfy the fractional Legendre condition (28).

For problem (36) with $\alpha = 0.3$, $h = 0.1$, $a = 0$, $b = 0.5$, and $\theta = 0$, we obtain the results of Table 2: there exist sixteen Euler-Lagrange extremals but only one satisfy the fractional Legendre condition. The extremal we are looking for is given by the candidate number six on Table 2.

The numerical results show that the solutions to our discrete-time fractional variational problems converge to the classical discrete-time solutions when the fractional order of the discrete-derivatives tend to integer values, and to the fractional Riemann-Liouville continuous-time solutions when h tends to zero.

5. Conclusion

The discrete fractional calculus is a recent subject under strong current development due to its importance as a modeling tool of real phenomena. In this work we introduce a new fractional difference variational calculus in the time-scale $(h\mathbb{Z})_a$, $h > 0$ and a a real number, for Lagrangians depending on left and right discrete-time fractional derivatives. Our objective was to introduce the concept of left and right fractional sum/difference (cf. Definition 2.8) and to develop the theory of fractional difference calculus. An Euler–Lagrange type equation (18), fractional natural boundary conditions (26) and (27), and a second order Legendre type necessary optimality condition (28), were obtained. The results are based on a new discrete fractional summation by parts formula (15) for $(h\mathbb{Z})_a$. Obtained first and second order necessary optimality conditions were implemented computationally in the computer algebra systems Maple and Maxima. Our numerical results show that:

1. the solutions of our fractional problems converge to the classical discrete-time solutions in $(h\mathbb{Z})_a$ when the fractional order of the discrete-derivatives tend to integer values;
2. the solutions of the considered fractional problems converge to the fractional Riemann–Liouville continuous solutions when $h \to 0$;
3. there are cases for which the fractional Euler–Lagrange equation give only one candidate that does not verify the obtained Legendre condition (so the problem at hands does not have a minimum);
4. there are cases for which the Euler–Lagrange equation give only one candidate that verify the Legendre condition (so the extremal is a candidate for minimizer, not for maximizer);
5. there are cases for which the Euler–Lagrange equation give us several candidates and just a few of them verify the Legendre condition.

We can say that the obtained Legendre condition can be a very practical tool to conclude when a candidate identified via the Euler–Lagrange equation is really a solution of the fractional variational problem. It is worth to mention that a fractional Legendre condition for the continuous fractional variational calculus is still an open question.

Undoubtedly, much remains to be done in the development of the theory of discrete fractional calculus of variations in $(h\mathbb{Z})_a$ here initiated. Moreover, we trust that the present work will initiate research not only in the area of the discrete-time fractional calculus of variations but also in solving fractional difference equations containing left and right fractional differences. One of the subjects that deserves special attention is the question of existence of solutions to the discrete fractional Euler–Lagrange equations. Note that the obtained fractional equation (18) involves both the left and the right discrete fractional derivatives. Other interesting directions of research consist to study optimality conditions for more general variable endpoint variational problems [28, 34, 35]; isoperimetric problems [5, 6]; higher-order problems of the calculus of variations [12, 24, 38]; to obtain fractional sufficient optimality conditions of Jacobi type and a version of Noether's theorem [17, 21, 25, 27] for discrete-time fractional variational problems; direct methods of optimization for absolute extrema [19, 36, 49]; to generalize our fractional first and second order optimality conditions for a fractional Lagrangian possessing delay terms [14, 37]; and to generalize the results from $(h\mathbb{Z})_a$ to an arbitrary time scale $\mathbb{T}$.

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[1] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. **272** (2002), no. 1, 368–379.

[2] O. P. Agrawal, A general finite element formulation for fractional variational problems, J. Math. Anal. Appl. **337** (2008), no. 1, 1–12.

[3] O. P. Agrawal and D. Baleanu, A Hamiltonian formulation and a direct numerical scheme for fractional optimal control problems, J. Vib. Control **13** (2007), no. 9-10, 1269–1281.

[4] R. Almeida, A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, J. Math. Phys. **51** (2010), no. 3, 033503, 12 pp. `arXiv:1001.2722`

[5] R. Almeida and D. F. M. Torres, Hölderian variational problems subject to integral constraints, J. Math. Anal. Appl. **359** (2009), no. 2, 674–681. `arXiv:0807.3076`

[6] R. Almeida and D. F. M. Torres, Isoperimetric problems on time scales with nabla derivatives, J. Vib. Control **15** (2009), no. 6, 951–958. `arXiv:0811.3650`

[7] R. Almeida and D. F. M. Torres, Calculus of variations with fractional derivatives and fractional integrals, Appl. Math. Lett. **22** (2009), no. 12, 1816–1820. `arXiv:0907.1024`

[8] F. M. Atici and P. W. Eloe, A transform method in discrete fractional calculus, Int. J. Difference Equ. **2** (2007), no. 2, 165–176.

[9] F. M. Atici and P. W. Eloe, Initial value problems in discrete fractional calculus, Proc. Amer. Math. Soc. **137** (2009), no. 3, 981–989.

[10] D. Baleanu, New applications of fractional variational principles, Rep. Math. Phys. **61** (2008), no. 2, 199–206.

[11] D. Baleanu, O. Defterli and O. P. Agrawal, A central difference numerical scheme for fractional optimal control problems, J. Vib. Control **15** (2009), no. 4, 583–597.

[12] D. Baleanu and F. Jarad, Discrete variational principles for higher-order Lagrangians, Nuovo Cimento Soc. Ital. Fis. B **120** (2005), no. 9, 931–938.

[13] D. Baleanu and F. Jarad, Difference discrete variational principles, in *Mathematical analysis and applications*, 20–29, Amer. Inst. Phys., Melville, NY, 2006.

[14] D. Baleanu, T. Maaraba and F. Jarad, Fractional variational principles with delay, J. Phys. A **41** (2008), no. 31, 315403, 8 pp.

[15] D. Baleanu and S. I. Muslih, Nonconservative systems within fractional generalized derivatives, J. Vib. Control **14** (2008), no. 9-10, 1301–1311.

[16] D. Baleanu, S. I. Muslih and E. M. Rabei, On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative, Nonlinear Dynam. **53** (2008), no. 1-2, 67–74.

[17] Z. Bartosiewicz and D. F. M. Torres, Noether's theorem on time scales, J. Math. Anal. Appl. **342** (2008), no. 2, 1220–1226. arXiv:0709.0400

[18] M. Bohner, Calculus of variations on time scales, Dynam. Systems Appl. **13** (2004), 339–349.

[19] M. Bohner, R. A. C. Ferreira and D. F. M. Torres, *Integral inequalities and their applications to the calculus of variations on time scales*, Math. Inequal. Appl. 13 (2010), no. 3, 511–522. arXiv:1001.3762

[20] M. Bohner and A. Peterson, *Dynamic equations on time scales*, Birkhäuser, Boston, MA, 2001.

[21] J. Cresson, G. S. F. Frederico and D. F. M. Torres, Constants of motion for non-differentiable quantum variational problems, Topol. Methods Nonlinear Anal. **33** (2009), no. 2, 217–231. arXiv:0805.0720

[22] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β), Math. Methods Appl. Sci. **30** (2007), no. 15, 1931–1939. arXiv:math-ph/0702099

[23] R. A. El-Nabulsi and D. F. M. Torres, Fractional actionlike variational problems, J. Math. Phys. **49** (2008), no. 5, 053521, 7 pp. arXiv:0804.4500

[24] R. A. C. Ferreira and D. F. M. Torres, Higher-order calculus of variations on time scales, in *Mathematical control theory and finance*, 149–159, Springer, Berlin, 2008. arXiv:0706.3141

[25] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. **334** (2007), no. 2, 834–846. arXiv:math.OC/0701187

[26] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. **53** (2008), no. 3, 215–222. arXiv:0711.0609

[27] G. S. F. Frederico and D. F. M. Torres, Fractional Noether's theorem in the Riesz-Caputo sense, Appl. Math. Comput. (2010), in press. DOI:10.1016/j.amc.2010.01.100 arXiv:1001.4507

[28] R. Hilscher and V. Zeidan, Calculus of variations on time scales: weak local piecewise $C^1_{\rm rd}$ solutions with variable endpoints, J. Math. Anal. Appl. **289** (2004), no. 1, 143–166.

[29] W. G. Kelley and A. C. Peterson, *Difference equations*, Academic Press, Boston, MA, 1991.

[30] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[31] F. Jarad and D. Baleanu, Discrete variational principles for Lagrangians linear in velocities, Rep. Math. Phys. **59** (2007), no. 1, 33–43.

[32] F. Jarad, D. Baleanu and T. Maraaba, Hamiltonian formulation of singular Lagrangians on time scales, Chinese Phys. Lett. **25** (2008), no. 5, 1720–1723.

[33] A. B. Malinowska and D. F. M. Torres, Strong minimizers of the calculus of variations on time scales and the Weierstrass condition, Proc. Est. Acad. Sci. **58** (2009), no. 4, 205–212. arXiv:0905.1870

[34] A. B. Malinowska and D. F. M. Torres, Natural boundary conditions in the calculus of variations, Math. Methods Appl. Sci. (2010), in press. DOI:10.1002/mma.1289 arXiv:0812.0705

[35] A. B. Malinowska and D. F. M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, Comput. Math. Appl. **59** (2010), no. 9, 3110–3116. arXiv:1002.3790

[36] A. B. Malinowska and D. F. M. Torres, Leitmann's direct method of optimization for absolute extrema of certain problems of the calculus of variations on time scales, Appl. Math. Comput. (2010), in press. DOI: 10.1016/j.amc.2010.01.015 arXiv:1001.1455

[37] T. Maraaba, F. Jarad and D. Baleanu, Variational optimal-control problems with delayed arguments on time scales, Adv. Difference Equ. **2009**, Art. ID 840386, 15 pp.

[38] N. Martins and D. F. M. Torres, Calculus of variations on time scales with nabla derivatives, Nonlinear Anal. **71** (2009), no. 12, e763–e773. arXiv:0807.2596

[39] K. S. Miller and B. Ross, Fractional difference calculus, in *Univalent functions, fractional calculus, and their applications (Kōriyama, 1988)*, 139–152, Horwood, Chichester, 1989.

[40] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

[41] S. I. Muslih and D. Baleanu, Fractional Euler-Lagrange equations of motion in fractional space, J. Vib. Control **13** (2007), no. 9-10, 1209–1216.

[42] M. D. Ortigueira, Fractional central differences and derivatives, J. Vib. Control **14** (2008), no. 9-10, 1255–1266.

[43] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999.

[44] E. M. Rabei, K. I. Nawafleh, R. S. Hijjawi, S. I. Muslih and D. Baleanu, The Hamilton formalism with fractional derivatives, J. Math. Anal. Appl. **327** (2007), no. 2, 891–897.

[45] E. M. Rabei, D. M. Tarawneh, S. I. Muslih and D. Baleanu, Heisenberg's equations of motion with fractional derivatives, J. Vib. Control **13** (2007), no. 9-10, 1239–1247.

[46] B. Ross, S. G. Samko and E. R. Love, Functions that have no first order derivative might have fractional derivatives of all orders less than one, Real Anal. Exchange **20** (1994), 140–157.

[47] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[48] M. F. Silva, J. A. Tenreiro Machado and R. S. Barbosa, Using fractional derivatives in joint control of hexapod robots, J. Vib. Control **14** (2008), no. 9-10, 1473–1485.

[49] D. F. M. Torres and G. Leitmann, Contrasting two transformation-based methods for obtaining absolute extrema, J. Optim. Theory Appl. **137** (2008), no. 1, 53–59. `arXiv:0704.0473`

[50] B. van Brunt, *The calculus of variations*, Springer, New York, 2004.