Near Optimal Colourability on Hereditary Graph Families

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Abstract

In this paper, we initiate a systematic study on a new notion called near optimal colourability which is closely related to perfect graphs and the Lovász theta function. A graph family $\mathcal{G}$ is *near optimal colourable* if there is a constant number c such that every graph $G \in \mathcal{G}$ satisfies $\chi(G) \leq \max\{c, \omega(G)\}$, where $\chi(G)$ and $\omega(G)$ are the chromatic number and clique number of G, respectively. The near optimal colourable graph families together with the Lovász theta function are useful for the study of the chromatic number problems for hereditary graph families. We investigate the near optimal colourability for (H_1, H_2)-free graphs. Our main result is an almost complete characterization for the near optimal colourability for (H_1, H_2)-free graphs with two exceptional cases, one of which is the celebrated Gyárfás conjecture. As an application of our results, we show that the chromatic number problem for $(2K_2, P_4 \vee K_\mathsf{n})$-free graphs is polynomial time solvable, which solves an open problem in [K. K. Dabrowski and D. Paulusma. On colouring ($2P_2$, H)-free and (P_5, H)-free graphs. Information Processing Letters, 134:35-41, 2018].

1 Introduction

All graphs in this paper are finite and simple. For general graph theory notation we follow [1]. Let P_n, C_n and K_n denote the path, cycle and complete graph on n vertices, respectively. We denote the complement graph of G by $\overline{G}$. For two vertex disjoint graphs G and H, we write $G + H$ to denote the *disjoint union* of G and H, and $G \vee H$ to denote the graph obtained from $G + H$ by adding an edge between every vertex in G and every vertex in H. For a positive integer r, we use rG to denote the disjoint union of r copies of G. A *hole* is an induced cycle on four or more vertices. An *antihole* is the complement of a hole. A hole or antihole is *odd* if it has an odd number of vertices. The graph $K_n - e$ is obtained from K_n by removing an edge. The graph $K_1 \vee 3K_1$, $(K_2 + K_1) \vee K_1$, $K_4 - e$, $P_4 \vee K_1$, $(K_2 + K_1) \vee K_2$ are usually called *claw*, *paw*, *diamond*, *gem* and *HVN*, respectively. A *linear forest* is a disjoint union of paths.

We say that a graph G *contains* a graph H if G has an induced subgraph that is isomorphic to H. A graph G is *H-free* if G does not contain H. For a family $\mathcal{H}$ of graphs, G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. We write $(H_1, \ldots, H_n)$-free instead of $\{H_1, \ldots, H_n\}$-free. Each graph in $\mathcal{H}$ is called a *forbidden induced subgraph* of the family of $\mathcal{H}$-free graphs. A graph family $\mathcal{G}$ is *hereditary* if $G \in \mathcal{G}$ implies that every induced subgraph of G belongs to $\mathcal{G}$. Clearly, a graph family is hereditary if and only if it is the family of $\mathcal{H}$-free graphs for some graph set $\mathcal{H}$.

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A *q-colouring* of a graph G is an assignment of colours from $\{1, 2, \ldots, q\}$ to each vertex of G such that adjacent vertices receive different colours. We say that a graph G is *q-colourable* if G admits a q-colouring. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number q for which G is q-colourable. The *clique number* of G, denoted by $\omega(G)$, is the size of a largest clique in G. Clearly, every graph G satisfies $\chi(G) \geq \omega(G)$. A graph G is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. As a generalisation of perfect graphs, Gyárfás [11] introduced the χ-bounded graph families. A graph family $\mathcal{G}$ is *χ-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$. The function f is called a *χ-binding function* for $\mathcal{G}$. It follows from a classical result of Erdős [7] that if a graph H is not a forest, then the family of H-free graphs is not χ-bounded. Gyárfás [10] conjectured that if a graph H is a forest, then the family of H-free graphs is χ-bounded (known as the Gyárfás conjecture).

To determine whether a general graph is q-colourable is NP-complete when $q \geq 3$ [13]. This implies that the chromatic number problem for general graphs is NP-complete. However, for some graph families, there exist polynomial time algorithms for the chromatic number problem. For example, Grötschel, Lovász and Schrijver [9] proved that the chromatic number of a perfect graph can be computed in polynomial time via the Lovász theta function, which is defined as follows:

$$\vartheta(G) := \max\{\sum_{i,j=1}^{n} b_{ij} : B = (b_{ij}) \text{ is positive semidefinite with trace at most } 1,$$
$$\text{and } b_{ij} = 0 \text{ if } ij \in E\}.$$

The Lovász theta function can be computed in polynomial time, and satisfies that $\omega(G) \leq \vartheta(\overline{G}) \leq \chi(G)$ for any graph G [9]. It follows that $\chi(G) = \vartheta(\overline{G})$ if G is perfect.

In this paper, we define a new notion called near optimal colourable that relates χ-boundedness with computational complexity of graph colouring via the Lovász theta function.

Definition 1 (Main Notion). *A graph family $\mathcal{G}$ is said to be near optimal colourable if there is a constant number c such that every graph $G \in \mathcal{G}$ has $\chi(G) \leq \max\{c, \omega(G)\}$.*

The following theorem shows that near optimal colourability allows one to reduce the chromatic number problem to k-colouring for fixed k.

Theorem 1. *Suppose that $\mathcal{G}$ is a near optimal colourable family with c being such that every graph $G \in \mathcal{G}$ satisfies that $\chi(G) \leq max\{\omega(G), c\}$. If the k-colouring problem for $\mathcal{G}$ is polynomial time solvable for every fixed positive integer $k \leq c - 1$, then the chromatic number problem for $\mathcal{G}$ is polynomial time solvable.*

Proof. Let $G \in \mathcal{G}$. We may calculate $\vartheta(\overline{G})$ in polynomial time [9]. If $\vartheta(\overline{G}) \geq c$, then $\vartheta(\overline{G}) \geq \max\{c, \omega(G)\} \geq \chi(G) \geq \vartheta(\overline{G})$ and so $\chi(G) = \vartheta(\overline{G})$. Now we assume that $\vartheta(\overline{G}) < c$. Then $\omega(G) < c$ and so $\chi(G) \leq c$. So the chromatic number problem is reduced to the k-colouring problems for $k \leq c - 1$. ☐

Therefore, it is useful to study which hereditary graph classes are near optimal colourable. It is known that [15] the family of H-free graph is linearly χ-bounded if and only if H is an induced of P_4. This implies that the family of H-free graphs is near optimal colourable if and only if H is an induced subgraph of P_4. So it is natural to study hereditary graph families defined by two forbidden induced subgraphs. For example, we (together with Goedgebeur and Merkel) proved that every (P_6, diamond)-free graph G satisfies that $\chi(G) \leq \max\{6, \omega(G)\}$, and that the 5-colouring problem for (P_6, diamond)-free graphs is polynomial time solvable [8].

It then follows from Theorem 1 and the known result on the polynomial-time solvibility of 4-colouring P_6-free graphs [5] that the chromatic number problem for (P_6, diamond)-free graphs is polynomial time solvable [8].

Our Contributions

Our main result is an almost complete characterization for the near optimal colourability for (H_1, H_2)-free graphs. Let $\mathcal{X}$ be the set of all graphs listed in (1), (2), (3), and (4). Let $\mathcal{X}'$ be the set of all graphs listed in (1) and (2) except the paw.

(1) $G = P_4 \vee K_n$ for $n \geq 1$.

(2) $G = (K_2 + K_1) \vee K_n$ for $n \geq 1$.

(3) $G = K_n - e$ for $n \geq 4$.

(4) $G = K_n$ for $n \geq 3$.

The following is our main result.

Theorem 2. *Let H_1, H_2 be graphs that are not induced subgraphs of P_4. Then the family of (H_1, H_2)-free graphs is near optimal colourable only if H_1 is a forest and $H_2 \in \mathcal{X}$. Moreover if $H_2 \in \mathcal{X}'$, then the family of (H_1, H_2)-free graphs is near optimal colourable if and only if $H_1 = 2K_2$. (H_1 and H_2 are interchangeable.)*

Our theorem leaves only two open cases for the near optimal colourability for (H_1, H_2)-free graphs: the Gyárfás conjecture and Problem 1 in Section 5 (where we explain why this is the case). Moreover, Problem 1 is also related to Gyárfás conjecture and so the open cases in some sense are as hard as Gyárfás conjecture. As an application of our result, we prove the following Theorem 3. This solves an open problem by Konrad and Paulusma [6].

Theorem 3. *For every positive integer n, the chromatic number problem of $(2K_2, P_4 \vee K_n)$-free graphs is polynomial time solvable.*

Proof. By Theorem 2, the family of $(2K_2, P_4 \vee K_n)$-free graphs is near optimal colourable. Since the k-colouring problem of P_5-free graphs is polynomial time solvable [12], it follows from Theorem 1 that the chromatic number problem of $(2K_2, P_4 \vee K_n)$-free graphs is polynomial time solvable. □

The remainder of the paper is organized as follows. In Section 2 we present some preliminaries. In Section 3 we reduce Theorem 2 to the case that $H_1 = 2K_2$, $H_2 = P_4 \vee K_n$. In Section 4 we prove the family of $(2K_2, P_4 \vee K_n)$-free graphs is near optimal colourable (Theorem 7). In Section 5 we make some concluding remarks and list some open problems.

2 Preliminaries

Let $G = (V, E)$. We say a vertex u is a *neighbour* (*nonneighbour*) of another vertex v in G, if u and v are adjacent (nonadjacent). The set of neighbours of a vertex v in G is denoted by $N_G(v)$, and we often write $N(v)$ if the context is clear. We write $\overline{N}(v)$ to denote the set $V \setminus (N(v) \cup \{v\})$. For a set $S \subseteq V$, we write $N_S(v)$ to denote $N(v) \cap S$, and $\overline{N_S}(v)$ to denote $\overline{N}(v) \cap S$. For $S \subseteq V$, let $G[S]$ denote the subgraph of G induced by S, and we often write S for $G[S]$ if the context is clear. We say that S induces a graph H if $G[S]$ is isomorphic to

H. For two sets $S, T \subseteq V$, we say S and T are *complete*(*anti-complete*), if every vertex in S is adjacent (nonadjacent) to every vertex in T. (If S or T has only one vertex v, we may write v for $\{v\}$.) A vertex set $S \subseteq V$ is a *stable set* if the vertices in S are pairwise nonadjacent.

We will use the following three known results:

Theorem 4 ([4])**.** *A graph is perfect if and only if it does not contain an odd hole or an odd antihole.*

Theorem 5 ([18])**.** *If G is a $2K_2$-free graph, then $\chi(G) \le \binom{\omega(G)+1}{2}$.*

Theorem 6 ([3])**.** *If G is a ($2K_2$,gem)-free graph, then $\chi(G) \le \max\{3, \omega(G)\}$.*

3 The Main Result

In this section we prove Theorem 2. We start with two lemmas.

Lemma 1. *Let H be a forest which is not an induced subgraph of P_4. Then either H contains a $3K_1$ or H is a $2K_2$, and $\overline{H}$ is not a forest.*

Proof. Assume that H is $3K_1$-free, then H is a linear forest and has at most two components. Since P_5 contains a $3K_1$ and H is not an induced subgraph of P_4, H has two components. Then each component of H has at most two vertices. This implies that H is a $2K_2$ since H is not an induced subgraph of P_4. So H contains a $3K_1$ or is a $2K_2$, which implies that $\overline{H}$ contains a C_3 or is a C_4. □

Lemma 2. *Let $\mathcal{G}$ be a χ-bounded graph family. Then $\mathcal{G}$ is near optimal colourable if and only if there is a constant number g such that every graph $G \in \mathcal{G}$ with $\omega(G) \ge g$ has $\chi(G) = \omega(G)$.*

Proof. The necessity is obvious. Suppose that every graph $G \in \mathcal{G}$ with $\omega(G) \ge g$ has $\chi(G) = \omega(G)$. Since $\mathcal{G}$ is χ-bounded, we may assume that every graph $G \in \mathcal{G}$ with $\omega(G) < g$ has $\chi(G) \le c$. So every graph $G \in \mathcal{G}$ has $\chi(G) \le \max\{c, \omega(G)\}$. □

Now we define two families of graphs (see Figure 1). For every positive integer n, let $X_n = C_5 \vee K_n$, then $\chi(X_n) = n + 3$ and $\omega(X_n) = n + 2$. The graph Y_n is obtained from a C_5 by blowing up two nonadjacent vertices to K_n, then $\chi(Y_n) = n + 2$ and $\omega(Y_n) = n + 1$. (To *blow up* a vertex v in G to a graph H, is to remove v and add a graph H and then make H complete to $N_G(v)$ and anti-complete to $\overline{N_G}(v)$.) Note that every X_n is ($2K_2, 3K_1, C_4$)-free, and every Y_n is (gem, HVN, $3K_1, C_4$)-free. (The constructions of X_n and Y_n can also be found in [2] and [19].)

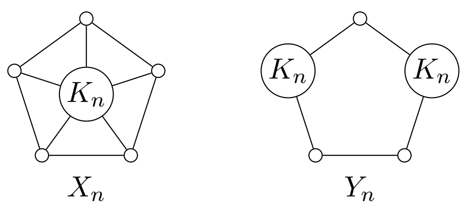

Figure 1

To prove Theorem 2, we need to handle a particular graph families, namely ($2K_2, P_4 \vee K_n$)-free graphs.

Theorem 7. *For every positive integer n, the family of ($2K_2, P_4 \vee K_n$)-free graphs is near optimal colourable.*

We shall prove Theorem 7 in Section 4. Now we use Theorem 7 to prove Theorem 2.

Proof of Theorem 2. Let $\mathcal{G}$ denote the family of (H_1, H_2)-free graphs. Assume that $\mathcal{G}$ is near optimal colourable. By the result of Erdős [7], if neither H_1 nor H_2 is a forest, then $\mathcal{G}$ is not χ-bounded and so not near optimal colourable. By symmetry let H_1 be a forest.

Assume that H_2 is not a complement of a linear forest. By Lemma 1, H_1 is not a complement of a linear forest, either. By Lemma 2, there exists a constant number g such that every graph $G \in \mathcal{G}$ with $\omega(G) \geq g$ has $\chi(G) = \omega(G)$. Since neither H_1 nor H_2 is a complement of a linear forest, each of $\overline{H_1}$ and $\overline{H_2}$ contains a cycle or a claw. Assume that the longest induced cycle in $\overline{H_1}$ and $\overline{H_2}$ has m vertices, if it exists. If neither $\overline{H_1}$ nor $\overline{H_2}$ contains a cycle, then let m be 0. We consider an odd antihole $L = \overline{C_{2n+1}}$ with $n \geq \max\{g, \frac{m}{2}\}$. Then $\chi(L) = n + 1$ and $\omega(L) = n$. Since $\overline{L} = C_{2n+1}$ does not contain a cycle of size m or less (if $m > 0$) or a claw, $\overline{L}$ is $(\overline{H_1}, \overline{H_2})$-free. So $L \in \mathcal{G}$, which contradicts our assumption that every graph $G \in \mathcal{G}$ with $\omega(G) \geq g$ has $\chi(G) = \omega(G)$. So H_2 is a complement of a linear forest.

By Lemma 1, H_1 contains either a $3K_1$ or a $2K_2$. If H_2 contains a C_4, then $\mathcal{G}$ contains X_n for arbitrarily large n, which contradicts that $\mathcal{G}$ is near optimal colourable. So H_2 must be C_4-free, and then $\overline{H_2}$ is a $2K_2$-free linear forest. So $\overline{H_2}$ is an induced subgraph of $P_4 + nK_1$ for some n. It is easy to check that $\mathcal{X}$ are all graphs satisfying the conditions for H_2. Now we assume that H_2 is C_4-free and contains a gem or an HVN, that is, $H_2 \in \mathcal{X}'$. If H_1 contains a $3K_1$, then $\mathcal{G}$ contains Y_n for arbitrarily large n, which contradicts that $\mathcal{G}$ is near optimal colourable. By Lemma 1, H_1 can only be $2K_2$. This proves the necessity. The sufficiency follows from Theorem 7 (note that every $(2K_2, (K_2 + K_1) \vee K_n)$-free graph is $(2K_2, P_4 \vee K_n)$-free). □

4 $(2K_2, P_4 \vee K_n)$-free Graphs

In this section we prove Theorem 7. Brause, Randerath, Schiermeyer, and Vumar [3] proved that the family of $(2K_2, P_4 \vee K_1)$-free graphs is near optimal colourable (Theorem 6). Our Theorem 7 is a generalization of their result.

We start with a simple proposition.

Lemma 3. *For every positive integer n, there exists a constant number c such that every $(2K_2, P_4 \vee K_n)$-free graph G which contains an antihole on 6 or more vertices has $\chi(G) \leq c$.*

Proof. Let $G = (V, E)$ be $(2K_2, P_4 \vee K_n)$-free. Let $Q = \{v_1, v_2, \ldots, v_r\} (r \geq 6)$ be an antihole in G with $v_i v_{i+1} \notin E$ for $i = 1, 2, \ldots, r$, with all indices modulo r. Since $\overline{C_{2n+5}}$ contains a $P_4 \vee K_n$, we have $r \leq 2n + 4$. Then $\omega(Q) = \lfloor \frac{r}{2} \rfloor \leq n + 2$.

For each subset S of Q, we denote $N_S = \{u \in V \setminus Q : N_Q(u) = S\}$. Then every vertex in $V \setminus Q$ belongs to exactly one of these 2^r sets. For any subset S of Q we consider the following two cases:

Case 1. $Q \setminus S$ contains a K_2. Then N_S is a stable set since G is $2K_2$-free.

Case 2. $Q \setminus S$ is a stable set. Then there exist an i such that $Q \setminus S \subseteq \{v_i, v_{i+1}\}$, and $\{v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\} \subseteq S$ induces a P_4. So $\omega(N_S) \leq n - 1$ since G is $P_4 \vee K_n$-free.

So for every S, $\omega(N_S) \leq \max\{1, n - 1\} \leq n$, then $\omega(G) \leq 2^r \cdot n + \omega(Q) \leq 2^{2n+4} \cdot n + n + 2$ and $\chi(G) \leq \binom{2^{2n+4} \cdot n+n+3}{2}$ by Theorem 5. □

To prove Theorem 7, it remains to prove the following theorem on $(2K_2, P_4 \vee K_n)$-free graphs with no antihole of length 6 or more.

Theorem 8. *For every positive integer n, the family of $(2K_2, P_4 \vee K_n)$-free graph G which contains no antiholes on 6 or more vertices is near optimal colourable.*

Proof. By Theorem 5 and Lemma 2, it suffices to prove that for every positive integer n, there exists a constant number g such that every $(2K_2, P_4 \vee K_n)$-free graph G which contains no antiholes on 6 or more vertices with $\omega(G) \geq g$ has $\chi(G) = \omega(G)$. We prove by induction on n. By Theorem 6, this theorem is true for $n = 1$. Now we assume that $n \geq 2$, and every $(2K_2, P_4 \vee K_{n-1})$-free graph G which contains no antiholes on 6 or more vertices with $\omega(G) \geq g_{n-1}$ has $\chi(G) = \omega(G)$.

Let $G = (V, E)$ be a $(2K_2, P_4 \vee K_n)$-free graph which contains no antiholes on 6 or more vertices. We assume that $\omega(G) \geq \max\{g_{n-1}, 4n + 2\} + 10n$, and we prove that $\chi(G) = \omega(G)$ in the following.

We say two nonadjacent vertices u, v are *comparable* if $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$. If $u_1, u_2 \in V$ are comparable with $N(u_1) \subseteq N(u_2)$, then $\chi(G-u_1) = \chi(G)$ and $\omega(G-u_1) = \omega(G)$. So we may assume that G has no pairs of comparable vertices.

If G is C_5-free, then G is perfect by Theorem 4. So in the following we assume that $Q = \{v_1, v_2, v_3, v_4, v_5\}$ induces a C_5 in G with $v_i v_{i+1} \in E$ for $i = 1, 2, \ldots, 5$, with all indices modulo 5. We define some vertex sets:

$A_i = \{u \in V \setminus Q : N_Q(u) = \{v_{i-1}, v_{i+1}\}\}$.

$B_i = \{u \in V \setminus Q : N_Q(u) = \{v_i, v_{i-2}, v_{i+2}\}\}$.

$D_i = \{u \in V \setminus Q : N_Q(u) = Q \setminus \{v_i\}\}$.

$F = \{u \in V \setminus Q : N_Q(u) = Q\}$.

$Z = \{u \in V \setminus Q : N_Q(u) = \emptyset\}$.

$A = \bigcup_{i=1}^{5} A_i$, $B = \bigcup_{i=1}^{5} B_i$, $D = \bigcup_{i=1}^{5} D_i$.

$S_i = A_i \cup \{v_i\}$, $S = \bigcup_{i=1}^{5} S_i$.

We prove some properties of these sets.

Claim 1. *$V = Q \cup A \cup B \cup D \cup F \cup Z$.*

Proof. Let $u \in V \setminus Q$. If u has no neighbours in Q, then $u \in Z$. By symmetry assume that $uv_1 \in E$. Since $\{u, v_1, v_3, v_4\}$ cannot induce a $2K_2$, u must be adjacent to at least one of v_3 and v_4. By symmetry let $uv_3 \in E$. If $uv_2 \notin E$, then depending on uv_4 and uv_5, there are four cases that u belongs to A_2, B_1, B_3 or D_2. If $uv_2 \in E$, then sinse $\{u, v_2, v_4, v_5\}$ cannot induce a $2K_2$, u must be adjacent to at least one of v_4 and v_5. Then there are three cases that u belongs to D_4, D_5 or F. □

Claim 2. *Each $S_i \cup Z$ is a stable set. As a corollary, Z and S are anti-complete.*

Proof. If $u_1, u_2 \in S_i \cup Z$ are adjacent, then $\{u_1, u_2, v_{i-2}, v_{i+2}\}$ induces a $2K_2$. □

Claim 3. *Each $D_i \cup F$ has clique number at most $n - 1$.*

Proof. If $K \subseteq D_i \cup F$ is a clique on n vertices, then $K \cup \{v_{i+1}, v_{i+2}, v_{i-2}, v_{i-1}\}$ induces a $P_4 \vee K_n$. □

Claim 4. *B_i and $B_{i-1} \cup B_{i+1} \cup A_i$ are complete.*

Proof. If $u_1 \in B_i$ and $u_2 \in B_{i+1} \cup A_i$ are nonadjacent, then $\{u_1, v_{i+2}, u_2, v_{i-1}\}$ induces a $2K_2$. □

Claim 5. *Let $u \in A_{i+2} \cup A_{i-2} \cup B_{i+2} \cup B_{i-2} \cup D \cup F$, then $\overline{N_{B_i}}(u)$ is a stable set.*

Proof. It is clear that u is adjacent to v_{i+1} or v_{i-1}. By symmetry let $uv_{i+1} \in E$. Assume that $u_1, u_2 \in \overline{N_{B_i}}(u)$ are adjacent, then $\{u, v_{i+1}, u_1, u_2\}$ induces a $2K_2$. □

Claim 6. *If $\omega(B_i) \geq n+1$, then $B_{i+2} \cup B_{i-2} \cup D_{i+2} \cup D_{i-2} = \emptyset$.*

Proof. By symmetry suppose that $u \in B_{i+2} \cup D_{i-2}$. By Claim 5, there is a clique K on n vertices in $N_{B_i}(u)$, then $\{v_i, u, v_{i+2}, v_{i-2}\} \cup K$ induces a $P_4 \vee K_n$. □

Claim 7. *Let $u \in A_{i+1} \cup A_{i-1}$, then $\omega(N_{B_i}(u)) \leq n-1$.*

Proof. If $K \subseteq N_{B_i}(u)$ is a clique on n vertices, then $\{v_i, u, v_{i+2}, v_{i-2}\} \cup K$ induces a $P_4 \vee K_n$. □

If every B_i has $\omega(B_i) \leq n$, then $\omega(G) \leq \omega(S) + \omega(B) + \omega(D) + \omega(F) + \omega(Z) \leq 5 + 5n + 5(n-1) + (n-1) + 1 = 11n$, which contradicts our assumption that $\omega(G) \geq 14n+2$. By symmetry we may assume that $\omega(B_1) \geq n+1$. By Claim 6, $B_3 \cup B_4 \cup D_3 \cup D_4$ is empty. Moreover, at most one of B_2 and B_5 has clique number at least $n+1$. By symmetry we assume that $\omega(B_5) \leq n$. Then we discuss two cases based on whether $\omega(B_2) \geq n+1$.

Case 1. $\omega(B_2) \geq n+1$.

By Claim 6, $B = B_1 \cup B_2$ and $D = D_1 \cup D_2$. So $B \cup D \cup F \subseteq N(v_4)$. By Claim 2, for every $u \in Z$, $N(u) \subseteq N(v_4)$. So $Z = \emptyset$ since G has no pairs of comparable vertices.

Claim 8. *$D_2 \cup F$ and B_2 are complete. (resp. $D_1 \cup F$ and B_1 are complete.)*

Proof. Suppose that $u_1 \in D_2 \cup F$ and $u_2 \in B_2$ are nonadjacent. By Claim 5, there is a clique $K \subseteq N_{B_1}(u_1)$ on n vertices. By Claim 4, u_2 and K are complete, then $\{v_1, u_1, v_4, u_2\} \cup K$ induces a $P_4 \vee K_n$. □

Claim 9. *D_1, D_2, F are pairwise complete.*

Proof. By symmetry suppose that $u_1 \in D_1$ and $u_2 \in D_2 \cup F$ are nonadjacent. By Claim 5, there is a vertex $u_3 \subseteq B_2$ such that $u_1u_3, u_2u_3 \in E$, and there is a clique $K \subseteq N_{B_1}(u_2)$ on n vertices. By Claim 8 and Claim 4, $\{u_1, u_3\}$ and K are complete, then $\{v_1, u_2, u_3, u_1\} \cup K$ induces a $P_4 \vee K_n$. □

Claim 10. *$\omega(B_1 \cup D_2 \cup S_2 \cup S_5) = \omega(B_1 \cup D_2)$. (resp. $\omega(B_2 \cup D_1 \cup S_1 \cup S_3) = \omega(B_2 \cup D_1)$.)*

Proof. Suppose that $K \subseteq B_1 \cup D_2 \cup S_2 \cup S_5$ is a clique on $\omega(B_1 \cup D_2) + 1$ vertices. If $|K \cap (B_1 \cup D_2)| \leq \omega(B_1 \cup D_2) - 2$, then $|K| \leq \omega(S_2 \cup S_5) + |K \cap (B_1 \cup D_2)| \leq \omega(B_1 \cup D_2)$, a contradiction. So $|K \cap (B_1 \cup D_2)| \geq \omega(B_1 \cup D_2) - 1 \geq n$. Let $L \subseteq K \cap (B_1 \cup D_2)$ be a clique on n vertices. Since $|K| > \omega(B_1 \cup D_2)$, there is a vertex $u \in K \cap (S_2 \cup S_5)$, then $\{v_1, u, v_3, v_4\} \cup L$ induces a $P_4 \vee K_n$. □

Claim 11. $\omega(G - S_4) = \omega(B_1 \cup D_2) + \omega(B_2 \cup D_1) + \omega(F) = \omega(G) - 1$.

Proof. Since $B_1 \cup D_2$, $B_2 \cup D_1$ and F are pairwise complete by Claim 8 and Claim 9, we have that $\omega(G - S_4) \geq \omega(B_1 \cup D_2) + \omega(B_2 \cup D_1) + \omega(F)$. By Claim 10,

$$\begin{aligned}\omega(G - S_4) &\leq \omega(B_1 \cup D_2 \cup S_2 \cup S_5) + \omega(B_2 \cup D_1 \cup S_1 \cup S_3) + \omega(F)\\ &= \omega(B_1 \cup D_2) + \omega(B_2 \cup D_1) + \omega(F)\end{aligned}$$

So $\omega(G - S_4) = \omega(B_1 \cup D_2) + \omega(B_2 \cup D_1) + \omega(F)$. Therefore, there is a clique $L \subseteq B_1 \cup B_2 \cup D_1 \cup D_2 \cup F$ on $\omega(G - S_4)$ vertices.

Since S_4 is a stable set, we have that $\omega(G - S_4) \geq \omega(G) - 1$. If $\omega(G - S_4) = \omega(G)$, then $L \cup \{v_4\}$ is a clique on $\omega(G) + 1$ vertices since v_4 is complete to $B_1 \cup B_2 \cup D_1 \cup D_2 \cup F$. This proves that $\omega(G - S_4) = \omega(G) - 1$. □

Claim 12. *Each of $B_1 \cup D_2 \cup S_2 \cup S_5$, $B_2 \cup D_1 \cup S_1 \cup S_3$ and F is C_5-free.*

Proof. Since F and B_1 are complete, and $\omega(B_1) \geq n$, we have that F is P_4-free.

Suppose that $R \subseteq B_1 \cup D_2 \cup S_2 \cup S_5$ induces a C_5. Since S_5 is a stable set, R has at most two vertices in S_5. Choose an induced $P = P_4 \subseteq R$ such that $|P \cap S_5| \leq 1$. Note that $B_1 \cup D_2 \cup S_2$ and B_2 are complete. If $P \cap S_5 = \emptyset$, then $P \cup B_2$ contains a $P_4 \vee K_n$. If $P \cap S_5 = \{u\}$, then by Claim 5, there is a clique $K \subseteq N_{B_2}(u)$ on n vertices. Then $P \cup K$ induces a $P_4 \vee K_n$. □

Then,

$$\begin{aligned}\chi(G) &\leq \chi(B_1 \cup D_2 \cup S_2 \cup S_5) + \chi(B_2 \cup D_1 \cup S_1 \cup S_3) + \chi(F) + \chi(S_4)\\ &= \omega(B_1 \cup D_2 \cup S_2 \cup S_5) + \omega(B_2 \cup D_1 \cup S_1 \cup S_3) + \omega(F) + 1\\ &= \omega(B_1 \cup D_2) + \omega(B_2 \cup D_1) + \omega(F) + 1\\ &= \omega(G).\end{aligned}$$

Case 2. $\omega(B_2) \leq n$.

By Claim 6, $B = B_1 \cup B_2 \cup B_5$ and $D = D_1 \cup D_2 \cup D_5$. Since $\omega(G - B_1) \leq \omega(S) + \omega(B_2 \cup B_5) + \omega(D) + \omega(F) + \omega(Z) \leq 5 + 2n + 3(n-1) + (n-1) + 1 = 6n + 2$, we have that $\omega(B_1) \geq \omega(G) - (6n+2) \geq \max\{g_{n-1}, 4n+2\} + (4n-2)$.

Claim 13. *Let $u \in Z$, then $\omega(N_{B_1}(u)) \leq n - 1$.*

Proof. If u and B_2 are anti-complete, then by Claim 2, $N(u) \subseteq B_1 \cup B_5 \cup D \cup F \subseteq N(v_3)$, which contradicts our assumption that G has no pairs of comparable vertices. So there is a vertex b in $N_{B_2}(u)$. Suppose that $K \subseteq N_{B_1}(u)$ is a clique on n vertices, then $\{u, b, v_4, v_3\} \cup K$ induces a $P_4 \vee K_n$. □

Let $H = B \cup D \cup F \cup S_1 \cup S_3 \cup S_4 = V \setminus (S_2 \cup S_5 \cup Z)$. By Claim 4 and Claim 5, $S_1 \cup B_2 \cup B_5$ and B_1 are complete, and for every $u \in S_3 \cup S_4 \cup D \cup F$, $\overline{N_{B_1}}(u)$ is a stable set. Let K be a maximum clique of G. Since $\omega(G - B_1) \leq 6n + 2$, we have $|K \cap B_1| \geq n$. By Claim 7 and Claim 13, $K \cap (S_2 \cup S_5 \cup Z) = \emptyset$. So $\omega(H) = \omega(G)$.

Claim 14. *Let $K \subseteq B_1$ be a clique on $n+2$ vertices. If there are $a_1, a_2 \in S_3 \cup S_4 \cup D \cup F$ and $b_1, b_2 \in K$ such that $a_1b_1, a_2b_2 \notin E$, then $a_1a_2 \in E$.*

Proof. By Claim 5, $a_1b_2, a_2b_1 \in E$, and $\{a_1, a_2\}$ and $K \setminus \{b_1, b_2\}$ are complete. Suppose that $a_1a_2 \notin E$, then $\{a_1, b_2, b_1, a_2\}$ induces a P_4, and then $\{a_1, b_2, b_1, a_2\} \cup (K \setminus \{b_1, b_2\})$ induces a $P_4 \vee K_n$. □

Claim 15. *Let $K \subseteq B_1$ be a clique on m vertices with $m \geq 4n-1$, then there are at least $m-(4n-2)$ vertices in K which are complete to $H \setminus B_1$.*

Proof. Suppose that there are $b_1, b_2, \dots, b_{4n-1} \in K$, such that for every $i \in \{1, 2, \dots, 4n-1\}$, b_i and $H \setminus B_1$ are not complete. By Claim 5, there are $a_1, a_2, \dots, a_{4n-1} \in S_3 \cup S_4 \cup D \cup F$, such that $a_ib_i \notin E$ for every $i \in \{1, 2, \dots, 4n-1\}$. By Claim 14, $\{a_1, a_2, \dots, a_{4n-1}\}$ induces a K_{4n-1}. But $\omega(S_3 \cup S_4 \cup D \cup F) \leq 2 + 3(n-1) + (n-1) = 4n-2$, a contradiction. □

Claim 16. *If $\chi(H) = \omega(H)$, then $\chi(G) = \omega(G)$.*

Proof. Since $\omega(B_1) \geq \max\{g_{n-1}, 4n+2\} + (4n-2)$, there is a clique $K \subseteq B_1$ on $\max\{g_{n-1}, 4n+2\}$ vertices which is complete to $H \setminus B_1$ by Claim 15. Suppose that H is already coloured by $\omega(H)$ colours. Let $L = \{u \in H : u \text{ has the same colour with some vertex in } K.\}$ and $M = H \setminus L$. Then $\chi(L) \leq \max\{g_{n-1}, 4n+2\}$, and $\chi(M) \leq \omega(H) - \max\{g_{n-1}, 4n+2\}$. Since $K \subseteq L$, we have $\omega(L) \geq |K| = \max\{g_{n-1}, 4n+2\}$, and so $\chi(L) = \omega(L) = \max\{g_{n-1}, 4n+2\}$. Since K and $H \setminus B_1$ are complete, we have $L \subseteq B_1$.

By Claim 7 and Claim 13, $\omega(L \cup S_2 \cup S_5 \cup Z) = \omega(L)$. Suppose that $P \subseteq L \cup S_2 \cup S_5 \cup Z$ induces a $P_4 \vee K_{n-1}$. If $P \subseteq L \cup S_2 \cup S_5$, then $P \subseteq N(v_1)$, and so $P \cup \{v_1\}$ induces a $P_4 \vee K_n$. So there is a vertex $u \in P \cap Z$. Note that $\omega(N_P(u)) = n$. By Claim 2, $\omega(N_{P\cap B_1}(u)) = n$. But $\omega(N_{B_1}(u)) \leq n-1$, a contradiction. So $L \cup S_2 \cup S_5 \cup Z$ is $P_4 \vee K_{n-1}$-free. Since $\omega(L) \geq g_{n-1}$, we have $\chi(L \cup S_2 \cup S_5 \cup Z) = \omega(L)$. Then,

$$\begin{aligned}
\chi(G) &\leq \chi(L \cup S_2 \cup S_5 \cup Z) + \chi(M) \\
&\leq \max\{g_{n-1}, 4n+2\} + (\omega(H) - \max\{g_{n-1}, 4n+2\}) \\
&= \omega(G).
\end{aligned}$$

□

Claim 17. *Suppose that L_1, L_2 are disjoint cliques in G, such that $|L_1| = |L_2| = \omega(L_1 \cup L_2) \geq 3n+3$. Then every vertex in L_1 (L_2) has exactly one nonneighbour in L_2 (L_1).*

Proof. Suppose that $u \in L_1$ and L_2 are complete, then $\omega(L_1 \cup L_2) \geq |L_2| + 1$, a contradiction.

Now we prove that:

There is a vertex $s_1 \in L_1$, such that for any $t_1, t_2 \in L_1 \setminus \{s_1\}$, $|N_{L_2}(t_1) \cap N_{L_2}(t_2)| \geq n$. Moreover if there is a vertex $t_1 \in L_1 \setminus \{s_1\}$ such that $|N_{L_2}(t_1) \cap N_{L_2}(s_1)| \leq n-1$, then $|N_{L_2}(s_1)| \leq \lfloor \frac{|L_1|+n+1}{2} \rfloor$. (By symmetry, there is a vertex $s_2 \in L_2$, such that for any $t_1, t_2 \in L_2 \setminus \{s_2\}$, $|N_{L_1}(t_1) \cap N_{L_1}(t_2)| \geq n$.) (1)

If for any $r_1, r_2 \in L_1$, $|N_{L_2}(r_1) \cap N_{L_2}(r_2)| \geq n$, then we are done. Suppose that $a_1, a_2 \in L_1$ such that $|N_{L_2}(r_1) \cap N_{L_2}(r_2)| \leq n-1$, then,

$$\begin{aligned}
|N_{L_2}(r_1)| + |N_{L_2}(r_2)| &= |N_{L_2}(r_1) \cup N_{L_2}(r_2)| + |N_{L_2}(r_1) \cap N_{L_2}(r_2)| \\
&\leq |L_2| + |N_{L_2}(r_1) \cap N_{L_2}(r_2)| \\
&\leq |L_1| + n - 1.
\end{aligned}$$

By symmetry we may assume that $|N_{L_2}(r_1)| \leq \lfloor \frac{|L_1|+n+1}{2} \rfloor$. Then $|\overline{N_{L_2}}(r_1)| \geq \lceil \frac{|L_1|+n+1}{2} \rceil \geq n+2$. Since G is $2K_2$-free, every vertex in $L_1 \setminus r_1$ has at most one nonneighbour in $\overline{N_{L_2}}(r_1)$. So for any $r_3, r_4 \in L_1 \setminus \{r_1\}$, we have $|N_{\overline{N_{L_2}(r_1)}}(r_3) \cap N_{\overline{N_{L_2}(r_1)}}(r_4)| \geq |\overline{N_{L_2}}(r_1)| - 2 \geq n$. Let r_1 be the s_1 we are finding, then we complete the proof of (1).

Now suppose that $u_1 \in L_1$ has nonneighbours $u_3, u_4 \in L_2$. Since G is $2K_2$-free, every vertex in $L_1 \setminus \{u_1\}$ is adjacent to u_3 or u_4. If $L_1 \setminus \{u_1\}$ and $\{u_3, u_4\}$ are complete, then

$\omega(L_1 \cup L_2) \geq |L_1| + 1$. So there is a vertex $u_2 \in L_1$ which is adjacent to exactly one of u_3 and u_4, say u_3, then $\{u_1, u_2, u_3, u_4\}$ induces a P_4. So $|N_{L_2}(u_1) \cap N_{L_2}(u_2)| \leq n - 1$, or else a clique on n vertices in $N_{L_2}(u_1) \cap N_{L_2}(u_2)$ together with $\{u_1, u_2, u_3, u_4\}$ induces a $P_4 \vee K_n$. Since u_1 is an arbitrary vertex in L_1 which has two or more nonneighbours in L_2, we may assume that $u_1 = s_1$. Since u_3, u_4 are two arbitrary vertices in $\overline{N_{L_2}}(u_1)$, and $|\overline{N_{L_2}}(u_1)| \geq 3$, we may assume that $u_3, u_4 \neq s_2$, then $|N_{L_1}(u_3) \cap N_{L_1}(u_4)| \geq n$, and so a clique on n vertices in $N_{L_1}(u_3) \cap N_{L_1}(u_4)$ together with $\{u_1, u_2, u_3, u_4\}$ induces a $P_4 \vee K_n$. □

Claim 18. *Suppose that L_1, L_2 are cliques in G, such that $|L_1| = |L_2| = \omega(L_1 \cup L_2) \geq 4n + 2$, Then every vertex in $L_1 \setminus L_2$ ($L_2 \setminus L_1$) has exactly one nonneighbour in $L_2 \setminus L_1$ ($L_1 \setminus L_2$).*

Proof. If $|L_1 \cap L_2| \leq n-1$, then it follows from Claim 17. So we may assume that $|L_1 \cap L_2| \geq n$. Since $\omega(L_1 \cup L_2) = |L_2|$, each vertex in $L_1 \setminus L_2$ has at least one nonneighbour in $L_2 \setminus L_1$. Suppose that $u_1 \in L_1 \setminus L_2$ has two nonneighbours $u_3, u_4 \in L_2 \setminus L_1$. Since G is $2K_2$-free and $\omega(L_1 \cup L_2) = |L_2|$, there is a vertex $u_2 \in L_1 \setminus L_2$ which is adjacent to exactly one of u_3 and u_4, say u_3. Therefore, $\{u_1, u_2, u_3, u_4\}$ induces a P_4, then $\{u_1, u_2, u_3, u_4\} \cup (L_1 \cap L_2)$ contains a $P_4 \vee K_n$. □

Let $X_0 = \{u \in B_1 : u$ and $H \setminus B_1$ are complete.$\}$, then $\omega(X_0) \geq \max\{g_{n-1}, 4n + 2\}$ by Claim 15. Let X be a maximal set such that $X_0 \subseteq X \subseteq B_1$ and $\omega(X) = \omega(X_0)$, and $Y = H \setminus X$. By the maximality, for every $u \in Y \cap B_1$ we have $\omega(N_X(u)) = \omega(X)$.

Claim 19. *Let $L_1 \subseteq X$ be a clique on $\omega(X)$ vertices, and $u \in Y \cap B_1$. There is a clique $L_2 \subseteq N_X(u)$ on $\omega(X)$ vertices such that $|L_1 \cap L_2| \geq \omega(X) - 1$.*

Proof. If u and L_1 are complete, then we are done. Now we assume that u and L_1 are not complete. Let $L_3 \in N_X(u)$ be a clique on $\omega(X)$ vertices. Suppose that u has two nonneighbours $a_1, a_2 \in L_1$. By Claim 18, there are $a_3, a_4 \in L_3$ such that $a_1a_3, a_2a_4 \notin E$. Let $K \subseteq L_3 \setminus \{a_3, a_4\}$ be a clique on n vertices, then $\{u, a_3, a_2, a_1\} \cup K$ induces a $P_4 \vee K_n$.

So u has exactly one nonneighbour in L_1, say a_1. Let $a_3 \in L_3$ such that $a_1a_3 \notin E$, then $(L_1 \setminus \{a_1\}) \cup \{a_3\}$ is a clique on $\omega(X)$ vertices in $N_X(u)$. □

Claim 20. *Let $L \subseteq Y$ be a clique, then there is a clique $K \subseteq X$ on $\omega(X)$ vertices such that K and L are complete.*

Proof. If $L \cap B_1 = \emptyset$, then L and X_0 are complete, and then we are done since $X_0 \subseteq X$ and $\omega(X) = \omega(X_0)$. Now we assume that $L \cap B_1 \neq \emptyset$. We prove by induction on $|L|$. The claim is true for $|L| = 1$ by the maximality of X. Now we consider the case that $|L| = m$ ($m \geq 2$), and assume that the claim is true for $|L| \leq m - 1$. Let $u_1 \in L \cap B_1$. Let $L_1 \subseteq X$ be a clique on $\omega(X)$ vertices which is complete to $L \setminus \{u_1\}$. If u_1 and L_1 are complete then we are done. Now we assume that u_1 and L_1 are not complete. By Claim 19, there is a clique $L_2 \in N_X(u_1)$ on $\omega(X)$ vertices such that $L_1 \setminus L_2 = \{r_1\}$ and $L_2 \setminus L_1 = \{r_2\}$, and $r_1r_2, r_1u_1 \notin E$. If r_2 and $L \setminus \{u_1\}$ are complete then we are done. So assume that $u_2 \in L \setminus \{u_1\}$ is nonadjacent to r_2, then $\{r_2, u_1, u_2, r_1\} \cup (L_1 \cap L_2)$ contains a $P_4 \vee K_n$. □

As a corollary of Claim 20, $\omega(H) = \omega(X) + \omega(Y)$. Suppose that $P \subseteq Y$ induces a P_4, by Claim 19, there is a clique in X on n vertices which is complete to P. So Y is P_4-free and therefore is perfect. Since $X \subseteq B_1 \subseteq N(v_1)$, we have that X is $P_4 \vee K_{n-1}$-free, and $\chi(X) = \omega(X)$ since $\omega(X) \geq g_{n-1}$. So $\chi(H) \leq \chi(X) + \chi(Y) = \omega(X) + \omega(Y) = \omega(H)$, and then $\chi(G) = \omega(G)$ by Claim 16. □

5 Conclusions

Theorem 2 is an almost complete characterization for the near optimal colourability for (H_1, H_2)-free graphs. The open cases left are that H_1 is a forest while $H_2 \in \mathcal{X}\backslash\mathcal{X}'$, that is, $H_2 = K_n, K_n - e$ or a paw. Since a graph G is paw-free if and only if each component of G is K_3-free or complete multipartite [14], the case that H_2 is a paw can be reduced to the case that $H_2 = K_3$. Clearly, the family of (H_1, K_n)-free graphs is near optimal colourable for every n if and only if the family of H_1-free graphs is χ-bounded. So the Gyárfás conjecture is equivalent to that each graph family of (H_1, H_2)-free graphs with H_1 being a forest and $H_2 = K_n$ is near optimal colourable. The other open case is that $H_2 = K_n - e$ with $n \geq 4$. Since $K_n - e$ is an induced subgraph of $P_4 \vee K_{n-2}$, we conclude that the family of $(2K_2, K_n - e)$-free graphs is near optimal colourable by Theorem 7. By Lemma 1, it suffices to consider the case that H_1 is a forest with independent number at least 3 and $H_2 = K_n - e$.

Problem 1. *Decide whether the family of (H_1, H_2)-free graphs is near optimal colourable when H_1 is a forest with independent number at least 3 and $H_2 = K_n - e$.*

Gyárfás conjecture is a major open problem in graph colouring, and only few partial results are known. See [16, 17] for more results on the Gyárfás conjecture. Since K_n is an induced subgraph of $K_{n+1} - e$, we believe that Problem 1 is as difficult as the Gyárfás conjecture. Our results on (P_6,diamond)-free graphs [8] solves a subproblem of Problem 1.

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