Lower bounds for partial matchings in regular bipartite graphs and applications to the monomer-dimer entropy

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Abstract

We derive here the Friedland-Tverberg inequality for positive hyperbolic polynomials. This inequality is applied to give lower bounds for the number of matchings in r-regular bipartite graphs. It is shown that some of these bounds are asymptotically sharp. We improve the known lower bound for the three dimensional monomer-dimer entropy.

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1 Introduction

The aim of this paper is to explore the connections between the problem of counting the number of partial matchings in bipartite graphs and positive hyperbolic polynomials. Let $G := (V_1 \cup V_2, E)$, where $E \subset V_1 \times V_2$ and $n = \#V_1 = \#V_2$. (We allow graphs with multiple edges.) We want to compute or estimate the number of m-matchings in G, i.e. the number of subsets M of edges E, where $\#M = m$, and no two edges in M have a common vertex.

Let $\mathbb{Z}_+$ be the set of nonnegative integers. Assume that $A(G) = [a_{ij}]_{i,j=1}^n \in \mathbb{Z}_+^{n\times n}$ is the incidence matrix of the bipartite graph G, i.e. a_{ij} is the number of edges connecting $i \in V_1, j \in V_2$. Then the number of m-matchings in G is equal to $\mathrm{perm}_m A(G)$, where $\mathrm{perm}_m A$ is the sum of permanents of all $m \times m$ submatrices of $A \in \mathbb{R}^{n\times n}$. For $m = n$, $\mathrm{perm}\, A(G)$, the permanent of $A(G)$ is the number of perfect matchings in G.

We now describe our main results for $\mathrm{perm}_m A$, where A is doubly stochastic, and their applications to lower bounds on partial matchings in bipartite graphs. Recall that the minimum of the permanent of $n \times n$ doubly stochastic matrices, denoted by Ω_n, is achieved only for the flat matrix J_n, whose all entries equal to $\frac{1}{n}$. Thus $\mathrm{perm}\, B \geq \mathrm{perm}\, J_n = \frac{n!}{n^n}$ for any $B \in \Omega_n$ and this inequality was conjectured by van der Waerden [22]. This conjecture was independently proved by Egorichev [3] and Falikman [5]. We call the above inequality Egorichev-Falikman-van der Waerden (EFW) inequality. The asymptotic behavior of EFW inequality is captured by the inequality $\mathrm{perm}\, B \geq e^{-n}$ for any $B \in \Omega_n$. This inequality was shown by the first name author [6] two years before [3, 5]. Let $\Gamma(n, r)$ be the set of all r-regular bipartite graphs G on $2n$ vertices, (multiple edges are allowed). For $G \in \Gamma(n, r)$ the matrix $B := \frac{1}{r} A(G)$ is doubly stochastic. Hence the number of perfect matchings in

G is at least $(\frac{r}{e})^n$. Thus for $r \geq 3$, the number of perfect matchings in r-regular bipartite graphs grows exponentially, which proves a conjecture by Erdös-Rényi [4]. Schrijver [18] improved the EFW inequality for r-regular bipartite graphs by showing that $\mathrm{perm}_m A(G) \geq (\frac{(r-1)^{r-1}}{r^{r-2}})^n$ for any $G \in \Gamma(n, r)$. Schrijver's inequality is asymptotically sharp. Recently, the second name author [14] improved Schrijver's inequality. Moreover, the proof in [14] is significantly simpler and transparent. One of the main tools in the proof in [14] is the use of the classical theory of hyperbolic polynomials.

It was shown by the first named author that $\mathrm{perm}_m A \geq \mathrm{perm}_m J_n$ for any $A \in \Omega_n$, and for $m \in [2, n]$ equality holds only if and only if $A = J_n$ [7]. ($\mathrm{perm}_1 A = n$ for each $A \in \Omega_n$.) This inequality was conjectured by Tverberg [21], and it is called here the Friedland-Tverberg (FT) inequality. FT inequality gives a lower bound on the number of partial matchings in any $G \in \Gamma(n, r)$.

We derive here the Schrijver type inequalities for m matchings in r-regular bipartite graphs on $2n$ vertices. This is done using the results and techniques of [14]. In particular we give a generalized versions of FT inequality to positive homogeneous hyperbolic polynomials, which are of independent interest.

These inequalities yield new lower bounds for the d-dimensional monomer-dimer entropy of dimer density $h_d(p), p \in [0, 1]$ in the lattice $\mathbb{Z}^d$. In particular we obtain the best known lower bound for the three dimensional monomer dimer entropy h_3, which combined with the known upper bound in [11] gives the tight result $h_3 \in [.7845, .7863]$.

We now list briefly the contents of this paper. In §2 we discuss briefly the notion of positive hyperbolic polynomials and examples used in this paper. In §3 we bring the generalized version of FT inequality for positive hyperbolic polynomials. In §4 we give an analog of the Schrijver-Gurvits inequality to $\mathrm{perm}_m B$, where B is a doubly stochastic matrix with at most r nonzero entries in each column. In §5 we discuss the asymptotic lower matching conjecture (ALMC) and the asymptotic lower r-permanent matching conjecture (ALPMC), which is a generalization of ALMC. We show that the main result in §4 proves the ALMC and ALPMC for a countable values of densities for each $r \geq 2$. In the last section we state the asymptotic upper matching conjectures (AUMC). We illustrate the relations of ALMC and AUMC to the monomer-dimer entropy in statistical mechanics by plotting the corresponding graphs for the dimensions $d = 2, 3$. We thank Uri Peled for supplying us with the Figures 1 and 2.

2 Positive hyperbolic polynomials

Definitions and Notations

1. A vector $\mathbf{x} := (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$ is called *positive* or *nonnegative*, and denoted by $\mathbf{x} > \mathbf{0} := (0, \ldots, 0)^\top$ or $\mathbf{x} \geq \mathbf{0}$ if $x_i > 0$ or $x_i \geq 0$ for $i = 1, \ldots, n$ respectively. A nonnegative vector $\mathbf{x} \neq \mathbf{0}$ is denoted by $\mathbf{x} \gneq \mathbf{0}$. $\mathbf{y} \geq \mathbf{x} \iff \mathbf{y} - \mathbf{x} \geq \mathbf{0}$. The cone of all nonnegative vectors in $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

2. A polynomial $p = p(\mathbf{x}) = p(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ is called *positive hyperbolic* if the following conditions hold:

   - p is a homogeneous polynomial of degree $m \geq 0$.
   - $p(\mathbf{x}) > 0$ for all $\mathbf{x} > 0$.
   - $\phi(t) := p(\mathbf{x} + t\mathbf{u})$, for $t \in \mathbb{R}$, has m-real t-roots for each $\mathbf{u} > \mathbf{0}$ and each $\mathbf{x}$.

3. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree m. For each integer $i \in [0, n]$ the *i-th degree* of p is the integer $r_i \in [0, m]$ such that

$$\frac{\partial^{r_i} p}{\partial x_i^{r_i}}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \not\equiv 0, \text{ and } \frac{\partial^{r_i+1} p}{\partial x_i^{r_i+1}}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \equiv 0.$$

Let $\deg_i p := r_i$ for $i = 1, \ldots, n$.

4. Let $\mathbf{e}_i := (\delta_{i1}, \ldots, \delta_{in})^\top \in \mathbb{R}^n,\ i = 1, \ldots, n$ be the standard basis in $\mathbb{R}^n$.

5. Let $\mathbf{1} := (1, \ldots, 1)^\top \in \mathbb{R}^n$ and denote by $J_n \in \mathbb{R}^{n\times n}$ the $n \times n$ matrix whose all entries are equal to $\frac{1}{n}$.

We refer to [8, 13, 14] for properties of positive hyperbolic polynomials used here.

Examples of positive hyperbolic polynomials

1. Let $A = (a_{ij})_{i=j=1}^{m,n} \in \mathbb{R}^{m\times n}$ be a nonnegative matrix, denoted by $A \geq \mathbf{0}$, where each row of A is nonzero. Fix a positive integer $k \in [1, m]$. Then

$$p_{k,A}(\mathbf{x}) := \sum_{1\leq i_1<\ldots i_k\leq m} \prod_{j=1}^{k} (A\mathbf{x})_{i_j}, \mathbf{x} \in \mathbb{R}^n, \tag{2.1}$$

is positive hyperbolic of degree k in n variables.

2. Let $A_1, \ldots, A_n \in \mathbb{C}^{m\times m}$ hermitian, nonnegative definite matrices such that $A_1 + \ldots + A_n$ is a positive definite matrix. Let $p(\mathbf{x}) = \det \sum_{i=1}^{n} x_i A_i$. Then $p(\mathbf{x})$ is positive hyperbolic.

Proof.

1. First note that $p_{k,A}(\mathbf{x}) > 0$ for $\mathbf{x} > \mathbf{0}$. The hyperbolicity of $p_{m,A}$ and $p_{1,A}$ is obvious. Assume that $k \in (1, m)$. Let $\mathbf{z} = (z_1, \ldots, z_{n+m-k})^\top \in \mathbb{R}^{n+m-k}$ and define $P(\mathbf{z}) := \prod_{i=1}^{m}(\sum_{j=1}^{n} a_{ij} z_j + \sum_{j=n+1}^{n+m-k} z_j)$. Then

$$p_{k,A}(\mathbf{x}) = \binom{m}{k}^{-1} \frac{\partial^{m-k} P}{\partial z_{n+1} \ldots \partial z_{n+m-k}}((x_1, \ldots, x_n, 0, \ldots, 0)).$$

Hence by [8, Lemma 2.1] $p_{k,A}$ positive hyperbolic.

2. This is a standard example and the proof is straightforward.

□

Let $p(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \geq 1$. As in [14] define

$$\mathrm{Cap}\, p := \inf_{\mathbf{x}>\mathbf{0}, x_1\ldots x_n=1} p(\mathbf{x}) = \inf_{\mathbf{x}>\mathbf{0}} \frac{p(\mathbf{x})}{(x_1 \ldots x_n)^{\frac{m}{n}}}. \tag{2.2}$$

It is possible that $\mathrm{Cap}\, p = 0$. For example let $p = x_1^{m_1} \ldots x_n^{m_n}$ where $m_1, \ldots, m_n$ are nonnegative integer whose sum is m and $(m_1, \ldots, m_n) \neq k\mathbf{1}$.

Proposition 2.1 *Let $A \in \mathbb{R}^{n\times n}$ be a doubly stochastic matrix. Let $p_{k,A}, k \in [1, n]$ be positive hyperbolic defined as part 1 of the above example. Then $\mathrm{Cap}\, p_{k,A} = \binom{n}{k}$. Let $B \in \mathbb{R}^n$ be a matrix with positive entries. Then there exists two positive definite diagonal matrices D_1, D_2, unique up to $tD_1, t^{-1}D_2, t > 0$, such that $A := D_1BD_2$ is a doubly stochastic matrix [20]. Let $p_{n,B}$ be defined as above. Then $\mathrm{Cap}\, p_{n,B} = \frac{1}{\det D_1D_2}$.*

Proof. Consider first $p_{n,A}$. Since A is row stochastic $p_{n,A}(\mathbf{1}) = 1$. Hence $\mathrm{Cap}\, p_{n,A} \leq 1$. Let $\mathbf{u} = (u_1, \ldots, u_n)^\top \gneq \mathbf{0}$ be a probability vector. Then for any $\mathbf{x} = (x_1, \ldots, x_n) > \mathbf{0}$ the generalized arithmetic-geometric inequality states $\mathbf{u}^\top\mathbf{x} \geq \prod_{i=1}^{n} x_i^{u_i}$. Use this inequality for each $(A\mathbf{x})_i$. The assumption that A is doubly stochastic yields that $p_{n,A} \geq x_1 \ldots x_n \Rightarrow \mathrm{Cap}\, p_{n,A} \geq 1$. Hence $\mathrm{Cap}\, p_{n,A} = 1$.

Let $k \in [1, n)$. Then $p_{k,A}(\mathbf{1}) = \binom{n}{k}$. Hence Hence Cap $p_{k,A} \le \binom{n}{k}$. Apply the arithmetic-geometric inequality to $\frac{p_{k,A}}{\binom{n}{k}}$ to deduce that $p_{k,A} \ge \binom{n}{k} p_{n,A}^{\frac{m}{n}}$. Hence Cap $p_{k,A} \ge \binom{n}{k}$.

It is straightforward to show that $\frac{p_{n,B}(\mathbf{x})}{x_1 \ldots x_n} = \frac{p_{n,A}(\mathbf{y})}{\det(D_1 D_2) y_1 \ldots y_n}$, where $\mathbf{y} = D_2^{-1}\mathbf{x}$. Hence Cap $p_{n,B} = \frac{1}{\det D_1 D_2}$. □

The following result is taken from [14].

Lemma 2.2 *Let $k \ge 1$ be an integer, $\mathbf{u} := (u_1, \ldots, u_k)^\top > \mathbf{0}, \mathbf{v} := (v_1, \ldots, v_k)^\top > \mathbf{0}$ and define $f(t) := \prod_{i=1}^k (u_i t + v_i)$. Let $K(f) := \inf_{t>0} \frac{f(t)}{t}$. Then $f'(0) = K$ for $k = 1$ and $f'(0) \ge (\frac{k-1}{k})^{k-1} K$ for $k \ge 2$. For $k \ge 2$ equality holds if and only if $\frac{v_1}{u_1} = \ldots = \frac{v_k}{u_k}$.*

The following proposition follows straightforward from [8, Lemma 2.1, part *3*].

Proposition 2.3 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree m. Let $i \in [1, n]$ be an integer. Then*

1. $\deg_i p = 0 \iff p(\mathbf{x}) = (p(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n))$.
2. *For each integer $j \in [0, \deg_i p]$ $\frac{\partial^j p}{\partial x_i^j}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ is a positive hyperbolic polynomial of degree $m - j$.*
3. *For each integer $j \in [1, n], j \ne i$,*

$$\deg_j \frac{\partial p}{\partial x_i}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \le \min(\deg_j p, n - 1).$$

The following result is crucial for the proof of a generalized Friedland-Tverberg inequality and is due essentially to the second author in [14].

Lemma 2.4 *Let $p : \mathbb{R}^n \to \mathbb{R}$ be a positive hyperbolic polynomial of degree $m \ge 1$. Assume that* Cap $p > 0$. *Then $\deg_i p \ge 1$ for $i = 1, \ldots, n$. For $m = n \ge 2$*

$$\text{Cap } \frac{\partial p}{\partial x_i}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \ge (\frac{\deg_i p - 1}{\deg_i p})^{\deg_i p - 1} \text{Cap } p \textit{ for } i = 1, \ldots, n,$$

where $0^0 = 1$.

Proof. It is enough to prove the result for $i = n$. Suppose to the contrary that p does not depend on x_n. Then let $\mathbf{x}(t) = (1, \ldots, 1, t)^\top$ and $t \to \infty$ in (2.2) to deduce that Cap $p = 0$ contrary to our assumption.

Assume that $m = n > 1$. Let $k = \deg_n p \ge 1$. Let $\mathbf{x}_0 := (x_1, \ldots, x_{n-1}, 0)^\top, \mathbf{x}_1 := (x_1, \ldots, x_{n-1})^\top$. Proposition 2.3 yields that $g(\mathbf{x}_1) := \frac{\partial^k p}{\partial x_i^k}(\mathbf{x}_0)$ is a positive hyperbolic function in $n - 1$ variables of degree $m - l$. Hence $g(\mathbf{x}_1) > 0$ for $\mathbf{x}_1 > \mathbf{0}$. Thus for $\mathbf{x}_1 > \mathbf{0}$

$$p(\mathbf{x}_0 + t\mathbf{e}_n) = k! g(\mathbf{x}_1) t^l + \ldots = k! g(\mathbf{x}_1) \prod_{i=1}^k (t + \lambda_i(\mathbf{x}_1)), \quad \lambda_i(x) > 0, \text{ for } i = 1, \ldots, k. \quad (2.3)$$

The second equality follows from [8, Lemma 2.1, part *2*]. Assume in addition that $x_1 \ldots x_{n-1} = 1$. Then $\inf_{t>0} \frac{p(\mathbf{x}_0 + t\mathbf{e}_n)}{t} \ge$ Cap p. Apply Lemma 2.2 to the right-hand side of (2.3) to deduce that $\frac{\partial p}{\partial x_n}(\mathbf{x}_0) \ge (\frac{k-1}{k})^{k-1}$Cap p. Since we assumed that $x_1 \ldots x_{n-1} = 1$ it follows that Cap $\frac{\partial p}{\partial x_n}(\mathbf{x}_0) \ge (\frac{k-1}{k})^{k-1}$Cap p. □

3 Friedland-Tverberg inequality

Theorem 3.1 *Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1, n]$. Assume that $\deg_i p \le r_i \in [1, m]$ for $i = 1, \ldots, n$. Rearrange the sequence $r_1, \ldots, r_n$ in an increasing order $1 \le r_1^* \le r_2^* \le \ldots \le r_n^*$. Let $k \in [1, n]$ be the smallest integer such that $r_k^* > m - k$. Then*

$$\sum_{1 \le i_1 < \ldots < i_m \le n} \frac{\partial^m p}{\partial x_{i_1} \ldots \partial x_{i_m}}(\mathbf{0}) \ge$$

$$\frac{n^{n-m}}{(n-m)!} \frac{(n-k+1)!}{(n-k+1)^{n-k+1}} \prod_{j=1}^{k-1} \left(\frac{r_j^* + n - m - 1}{r_j^* + n - m}\right)^{r_j^* + n - m - 1} \mathrm{Cap}\, p. \tag{3.1}$$

(Here $0^0 = 1$, and the empty product for $k = 1$ is assumed to be 1.) If $\mathrm{Cap} > 0$ and $r_i = m$ for $i = 1, \ldots, m$ equality holds if and only if $p = C(\frac{x_1 + \ldots + x_n}{n})^m$ for each $C > 0$.

Proof. Suppose that $\mathrm{Cap}\, p = 0$. [8, Lemma 2.1, part *3*] yields that the left-hand side of (3.1) is nonnegative and the theorem holds in this case.

Clearly, it is enough to assume the case $\mathrm{Cap}\, p = 1$. The case $m = n$ is essentially proven in [14] and we repeat its proof for the convenience of the reader. Permute the coordinates of $x_1, \ldots, x_n$ such that $\deg_n p = \min_{i \in [1,n]} \deg_i p \le r_1^*$. Assume that $\deg_n p = l$. Then Lemma 2.4 yields that $r((x_1, \ldots, x_{n-1})) := \frac{\partial p}{\partial x_n}((x_1, \ldots, x_{n-1}, 0))$ is positive hyperbolic of degree $n - 1$ and $\mathrm{Cap}\, r \ge (\frac{l-1}{l})^{l-1} \mathrm{Cap}\, p$. Since the sequence $(\frac{i-1}{i})^{i-1}, i = 1, \ldots,$ is decreasing to have the lowest possible lower bound we have to assume $l = r_1^*$. Suppose first that $r_1^* = n$. Repeating this process n times we get that

$$\frac{\partial^n p}{\partial x_1 \ldots \partial x_n}(\mathbf{0}) \ge \mathrm{Cap}\, p \prod_{j=2}^{n} \left(\frac{j-1}{j}\right)^{j-1} = \frac{n!}{n^n} \mathrm{Cap}\, p.$$

This inequality to corresponds to the case $r_i^* = n$ for $i = 1, \ldots, n$. The equality case is discussed in [14].

Let $m \in [1, n-1]$. Put $P(\mathbf{x}) = p(\mathbf{x})(\frac{1}{n}\sum_{i=1}^n x_i)^{n-m}$. Clearly, P is positive hyperbolic of degree n. Since $\frac{1}{n}\sum_{i=1}^n x_i \ge (x_1 \ldots x_n)^{\frac{1}{n}}$ for each $\mathbf{x} \ge 0$, it follows that $\mathrm{Cap}\, P \ge \mathrm{Cap}\, p$. Apply (3.1) to P for $m = n$ to deduce (3.1) in the general case. Since the equality case for P holds if and only $P = (\frac{1}{n}\sum_{i=1}^n x_i)^n$ it follows that the equality in (3.1) holds if and only if $p = (\frac{1}{n}\sum_{i=1}^n x_i)^m$. □

Let $A \in \mathbb{R}^{n \times n}$ be a doubly stochastic matrix. Apply this theorem to $p_{m,A}$ defined Proposition 2.1 to deduce the Friedland-Tverberg inequality for the sum of all $m \times m$ permanents of A:

Corollary 3.2 *Let $A \in \mathbb{R}_+^{n \times n}$ be a doubly stochastic matrix. Then $\mathrm{perm}_m A \ge \binom{n}{m}^2 \frac{m!}{n^m}$ for any $m \in [2, n]$. equality holds if and only $A = J_n$.*

Theorem 3.3 *(Gurvits) Let $A \in \mathbb{R}_+^{n \times n}$ be a doubly stochastic matrix, such that each column contains at most $r \in [1, n]$ nonzero entries. Then*

$$\mathrm{perm}\, A \ge \frac{r!}{r^r} \left(\frac{r-1}{r}\right)^{(r-1)(n-r)} = \frac{r!}{r^r} \left(\frac{r}{r-1}\right)^{r(r-1)} \left(\frac{r-1}{r}\right)^{(r-1)n}. \tag{3.2}$$

Proof. Note that for $p(\mathbf{x}) = \prod_{i=1}^n (A\mathbf{x})_i$ we have that $\deg_i p = r$ for $i = 1, \ldots, n$. Apply (3.1) to this case, i.e. $m = n$, $r_j^* = r, j = 1, \ldots, n$ and $k = n - r + 1$ to deduce the theorem. □

4 A lower bound for partial matchings in $\Gamma(n,r)$

The FT inequality yields the following lower bound on m matchings in $\Gamma(n,r)$

$$\operatorname{perm}_m(A(G)) \geq r^m \operatorname{perm}_m J_n = \binom{n}{m}^2 \frac{m! r^m}{n^m}, \quad \text{for all } G \in \Gamma(n,r). \tag{4.1}$$

In order to improve this inequality, as done by Schrijver, one has to use the fact that $A(G)$ has at most r nonzero entries in each column. Unfortunately, Theorem 3.1 does not improve the inequality (4.1) for large values of n, where $\frac{m}{n} \approx t \in (0,1)$. This is due to the fact that the proof of Theorem 3.1 for a polynomial p is obtained by considering the polynomial $q = p(x_1 + \ldots + x_n)^{n-m}$. (Note that $\deg_i q = \deg_i p + n - m$ for $i = 1, \ldots, n$.)

In order to use efficiently the fact $\deg_i p_{m,A(G)} \leq r$ for $i = 1, \ldots, n$, where $A(G), G \in \Gamma(n,r)$, one needs to consider the product $p_{m,A(G)} q(x)$, where $\deg q = n - m$ and $\deg_i q \leq s$ for $i = 1, \ldots, n$. This q should be a highly symmetric polynomial, similar to $(x_1 + \ldots + x_n)^{n-m}$. We find such q by averaging $p_{n-m,A(H)}, H \in \Gamma(n,s)$ with respect to a certain "natural" measure on $\Gamma(n,s)$, [17]. (Note that q may not be hyperbolic, but since it is a convex combination of hyperbolic polynomials, we can use Theorem 3.1.)

Let S_N be the permutation group on $\{1, \ldots, N\}$. We now give a map $\tau : S_{nr} \to \Gamma(n,r)$. Fix $\mu \in S_{nr}$. Let $e_1, ..., e_{nr}$ be nr edges going from vertices $\{1, ..., n\}$ in the group V_1 to vertices $\{1, ..., n\}$ to the group V_2 as follows. Each e_i connects the vertex $\lceil \frac{i}{r} \rceil$ in group V_1 to $\lceil \frac{\mu(i)}{r} \rceil$ in group V_2 for $i = 1, ..., rn$. Note that the vertex i in group V_1 has r edges labeled $r(i-1)+1, ..., ri$. It is straightforward to see that each vertex j in the group V_2 has r different edges connected to it, i.e. the equation $j = \lceil \frac{\mu(i)}{r} \rceil$ has exactly r integers $\mu^{-1}(\{j(r-1)+1, ..., jr\})$. It is not difficult to show that τ is onto. Let $\#\tau^{-1}(G)$ be the number of preimages of $G \in \Gamma(n,r)$ in S_{nr}. Denote by $\nu(n,r)$ the probability measure on $\Gamma(n,r)$ given by $\frac{\#\tau^{-1}(G)}{(rn)!}$. The following lemma is taken from [10] and we bring its proof for completeness.

Lemma 4.1 *Let $\nu(n,r)$ be the probability measure on $\Gamma(n,r)$ defined above. Then*

$$E_{\nu(n,r)}(\operatorname{perm}_m A(G)) = \frac{\binom{n}{m}^2 r^{2m} m!(rn-m)!}{(rn)!}. \tag{4.2}$$

Proof. We adopt the arguments of [18] to our case. First choose subset $\alpha \subset \{1, ..., n\}$ of m vertices in the group V_1. There are $\binom{n}{m}$ choices for α. α induces the set $I = \cup_{i \in \alpha} \{r(i-1)+1, ..., ir\}$ of edges of cardinality rm. From I choose a set $J = \{1 \leq j_1 < \ldots < j_m \leq nr\}$ of m edges, so that $e_j, j \in J$ corresponds to the choice of one element in the group $\{r(i-1)+1, ..., ir\}$, for each $i \in \alpha$. There are r^m of the choices of J. Now we want to choose μ so that $\lceil \frac{\mu(j)}{r} \rceil, j \in J$ will be a subset of m distinct elements $\beta = \{\lceil \frac{\mu(j_1)}{r} \rceil, \ldots, \lceil \frac{\mu(j_m)}{r} \rceil\} \subset \{1, ..., n\}$. There are $\binom{n}{m}$ such choices of β. Given $\beta \subset \{1, \ldots, n\}$ we can permute the order of elements in β in $m!$ ways. Altogether we have $m!\binom{n}{m}$ choices of $\lceil \frac{\mu(j_1)}{r} \rceil, \ldots, \lceil \frac{\mu(j_m)}{r} \rceil$. Then $\mu(j) \in \{\lceil \frac{\mu(j)}{r} \rceil (r-1)+1, ..., \lceil \frac{\mu(j)}{r} \rceil r\}$ for each $j \in J$. Again there are r^m such choices. Thus we chose μ by determining the image of the elements in J in $\{1, ..., nr\}$, which is denoted by $\beta := \mu(J)$. The rest of the of elements $\{1, ..., rn\} \backslash J$ is mapped to $\{1, ..., rn\} \backslash \beta$. The number of choices here is $(nr-m)!$. Multiply all these choices to get the numerator of the right-hand side of (4.2). Divide these number of choices by the number of permutations of $\{1, ..., rn\}$ to deduce the lemma. □

The case $m = n$ in (4.2) is given in [17]. The proof of the above Lemma yields:

Corollary 4.2 *For $\beta \subset \langle n \rangle, \#\beta = m$ and $G \in \Gamma(n,r)$ let $\psi(G,\beta)$ be all m-matching in G that cover the set $\beta \subset V_2$. Then*

$$E_{\nu(n,r)}(\psi(G,\beta)) = \frac{\binom{n}{m} r^{2m} m!(rn-m)!}{(rn)!}, \quad m = 0, \ldots, n. \tag{4.3}$$

Remark 4.3 *The probability measure $\nu(n,r)$ on $\Gamma(n,r)$ was used in [19] to get an upper bound: $\min_{G\in\Gamma(n,r)} \mathrm{perm}(A(G)) \le E_{\nu(n,r)}(\mathrm{perm}(A(G))$. The proof of the lower bound in [18], substantially harder result, had no connection to $\nu(n,r)$.*
Quite surprisingly, we use in this paper the measure $\nu(n,s)$ to obtain a lower bound. This combination of the "hyperbolic polynomials approach" and the probabilistic method is the main contribution of our paper.

Theorem 4.4 *Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1,n)$. Assume that $\deg_i p \le r_i \in [1,m]$ for $i = 1,\dots,n$. Rearrange the sequence $r_1,\dots,r_n$ in an increasing order $1 \le r_1^* \le r_2^* \le \dots \le r_n^*$. Let $s \in \mathbb{N}$. Let $k \in [1,n]$ be the smallest integer such that $r_k^* + s > n-k$. Then*

$$\sum_{1\le i_1<\dots<i_m\le n} \frac{\partial^m p}{\partial x_{i_1}\dots\partial x_{i_m}}(\mathbf{0}) \ge$$
$$\frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!}\frac{(n-k+1)!}{(n-k+1)^{n-k+1}}\prod_{j=1}^{k-1}\Big(\frac{r_j^*+s-1}{r_j^*+s}\Big)^{r_j^*+s-1}\mathrm{Cap}\, p. \tag{4.4}$$

Proof. Let $q : \mathbb{R}^n \to \mathbb{R}$ be positive hyperbolic of degree $n-m$ with $\deg_i q \le s$ for $i = 1,\dots,n$ and $\mathrm{Cap}\, q = 1$. Then $f = pq : \mathbb{R}^n \to \mathbb{R}$ is positive hyperbolic of degree n with $\mathrm{Cap}\, f \ge \mathrm{Cap}\, p$ and $\deg_i f \le r_i + s$ for $i = 1,\dots,n$. Apply Theorem 3.1 to f to deduce

$$\sum_{1\le i_1<\dots<i_m\le n} \frac{\partial^m p}{\partial x_{i_1}\dots\partial x_{i_m}}(\mathbf{0})\ \frac{\partial^{n-m} q}{\partial x_{i'_1}\dots\partial x_{i'_{n-m}}}(\mathbf{0}) \ge$$
$$\frac{(n-k+1)!}{(n-k+1)^{n-k+1}}\prod_{j=1}^{k-1}\Big(\frac{r_j^*+s-1}{r_j^*+s}\Big)^{r_j^*+s-1}\mathrm{Cap}\, p, \tag{4.5}$$

where $I' := \{1 \le i'_1 < \dots < i'_{n-m} \le n\}$ and $\{i_1,\dots,i_m,i'_1,\dots,i'_{n-m}\} = \langle n\rangle$.

Let $A := A(G), G \in \Gamma(n,s)$ and choose $q = \binom{n}{n-m}^{-1} p_{n-m,\frac{1}{s}A}(\mathbf{x})$ as in (2.1). Note

$$\frac{\partial^{n-m} q}{\partial x_{i'_1}\dots\partial x_{i'_{n-m}}}(\mathbf{0}) = \frac{1}{\binom{n}{n-m}s^{n-m}}\psi(G,I').$$

Now take the expected value of the left-hand side of the inequalities (4.5) corresponding to all $G \in \Gamma(n,s)$. Use Corollary 4.2 to deduce that the coefficient of each $\frac{\partial^m p}{\partial x_{i_1}\dots\partial x_{i_m}}(\mathbf{0})$ is $\frac{s^{n-m}(n-m)!((s-1)n+m)!}{(sn)!}$. □

Corollary 4.5 *Let $p : R^n \to \mathbb{R}$ be positive hyperbolic of degree $m \in [1,n)$. Assume that $\deg_i p \le r \in [1,m]$ for $i = 1,\dots,n$. Let $s \in \mathbb{N}$ and $k = n-r-s+1 \ge 1$. Then*

$$\sum_{1\le i_1<\dots<i_m\le n} \frac{\partial^m p}{\partial x_{i_1}\dots\partial x_{i_m}}(\mathbf{0}) \ge$$
$$\frac{(sn)!}{s^{n-m}(n-m)!((s-1)n+m)!}\frac{(r+s)!}{(r+s)^{r+s}}\Big(\frac{r+s-1}{r+s}\Big)^{(r+s-1)(n-r-s)}\mathrm{Cap}\, p. \tag{4.6}$$

Theorem 4.6 *Let $B \in \mathbb{R}_+^{n\times n}$ be a doubly stochastic matrix with at most r nonzero entries in each column. Let $s \in \mathbb{N}$ and $k = n-r-s+1 \ge 1$. Then for each $m \in [1,n)$*

$$\mathrm{perm}_m B \ge \frac{(sn)!\binom{n}{m}}{s^{n-m}(n-m)!((s-1)n+m)!}\frac{(r+s)!}{(r+s)^{r+s}}\Big(\frac{r+s-1}{r+s}\Big)^{(r+s-1)(n-r-s)}. \tag{4.7}$$

Proof. Let $p = p_{m,B}(\mathbf{x})$ as defined by (2.1). Then (4.7) follow from Corollary 4.5. □

5 ALMC and ALPMC

Let $G = (V, E)$ be a general, (not necessary bipartite), graph with the set of vertices V and edges E. A *matching* in G is a subset $M \subseteq E$ such that no two edges in M share a common endpoint. The endpoints of the edges in M are said to be *covered* by M. We can think of each edge $e = (u, v) \in M$ as occupied by a *dimer*, consisting of two neighboring atoms at u and v forming a bond, and of each vertex not covered by M as a *monomer*, which is an atom not forming any bond. For this reason a matching in G is also called a *monomer-dimer cover* of G. If there are no monomers, M is said to be a *perfect matching.* Note that if a perfect matching exists then $\#V$ is even. A matching M with $\#M = k$ is called an *k-matching.* We denote by $\phi_G(k)$ be the number of k-matchings in G (in particular $\phi_G(0) = 1$), and by $\Phi_G(x) := \sum_k \phi_G(k)x^k$ the matching generating polynomial of G. It is known that all the roots of matching polynomial are real negative numbers [17]. Assume that G is a bipartite graph $G = (V, E)$, where $V = V_1 \cup V_2$ and $\#V_1 = \#V_2 = n$. Then $\phi_G(k) = \mathrm{perm}_k\, A(G)$.

The following asymptotic result is shown in [10]. (It follows straightforward from Lemma 4.1.)

Theorem 5.1 *Let $\nu(n, r)$ be the probability measure on $\Gamma(n, r)$ defined in §4. Let $j_n \in [1, n]$, $n = 1, 2, \ldots$ be a sequence of integers with $\lim_{n\to\infty} \frac{j_n}{n} = t \in [0, 1]$. Then*

$$\lim_{n\to\infty} \frac{\log E_{\nu(n,r)}(\mathrm{perm}_{j_n} A(G))}{2n} = gh_r(t), \tag{5.1}$$

where

$$gh_r(t) := \tfrac{1}{2}\left(t \log r - t \log t - 2(1-t)\log(1-t) + (r-t)\log\left(1 - \tfrac{t}{r}\right)\right), \tag{5.2}$$

An equivalent form of the following conjecture is stated in [9].

Conjecture 5.2 (The Asymptotic Lower Matching Conjecture) *Let $r \geq 2$ be an integer, $\{k_l\}_{l=1}^{\infty}, \{n_l\}_{l=1}^{\infty}$ be two increasing sequences of positive integers such that $k_l \leq n_l, l = 1, \ldots,$ and $\lim_{l\to\infty} \frac{k_l}{n_l} = t \in [0, 1]$. Then*

$$\lim_{l\to\infty} \frac{\log \min_{G\in\Gamma(r,n_l)} \mathrm{perm}_{k_l} A(G)}{2n_l} = gh_r(t). \tag{5.3}$$

For $r = 1$ this conjecture holds trivially. For $r = 2$ this conjecture is proved in [10]. The inequality (4.1) implies that under the conditions of Conjecture 5.2 the following inequality holds, see [11]

$$\liminf_{l\to\infty} \frac{\log \min_{G\in\Gamma(r,n_l)} \mathrm{perm}_{k_l} A(G)}{2n_l} \geq fh_r(t), \tag{5.4}$$

where

$$fh_r(t) := \tfrac{1}{2}(-t\log t - 2(1-t)\log(1-t) + t\log r - t). \tag{5.5}$$

Definition 5.3 *For $r \in \mathbb{N}$ let $\Omega_{n,r}$ be the set of all $n \times n$ doubly stochastic matrices with at most r nonzero entries in each column.*

Note that for $G \in \Gamma(n, r)$ $B := \frac{1}{r}A(G) \in \Omega_{n,r}$. Hence, the ALMC conjecture follows from the following stronger conjecture:

Conjecture 5.4 (The Asymptotic Lower r-Permanent Conjecture)

Let $r \geq 2$ be an integer, $\{k_l\}_{l=1}^{\infty}, \{n_l\}_{l=1}^{\infty}$ be two increasing sequences of positive integers such that $k_l \leq n_l, l = 1, \ldots,$ and $\lim_{l\to\infty} \frac{k_l}{n_l} = t \in [0, 1]$. Then

$$\lim_{l\to\infty} \frac{\log \min_{B\in\Omega_{n_l,r}} \mathrm{perm}_{k_l} B}{2n_l} = gh_r(t) - \frac{t}{2}\log r. \tag{5.6}$$

Theorem 5.5 *Let $r \geq 2, s \geq 1$ be integers. Let $\{k_l\}_{l=1}^{\infty}, \{n_l\}_{l=1}^{\infty}$ be two increasing sequences of positive integers such that $k_l \leq n_l, l = 1, \ldots,$ and $\lim_{l\to\infty} \frac{k_l}{n_l} = t \in [0,1]$. Assume that $B_{n_l} \in \Omega_{n_l,r}, l = 1, 2, \ldots$. Then*

$$\liminf_{l\to\infty} \frac{\log \mathrm{perm}_{k_l} B_{n_l}}{2n_l} \geq \frac{1}{2}\left(-t\log t - 2(1-t)\log(1-t)\right) + \tag{5.7}$$
$$\frac{1}{2}\left((r+s-1)\log(1-\frac{1}{r+s}) - (s-1+t)\log(1-\frac{1-t}{s})\right).$$

Hence the Asymptotic Lower r-Permanent Conjecture 5.4 and ALMC holds for $t_s = \frac{r}{r+s}, s = 0, 1, 2, \ldots,$ and $t = 0$.

Proof. Apply the inequality (4.7) to B_{n_l} for $m = k_l$. Take the logarithm of the both sides of this inequality and let $l \to \infty$. A straightforward calculation for the right-hand side, using the Stirling's formula, yields the inequality (5.7). Assume that $t = t_s = \frac{r}{r+s}$. Then $\frac{1-t_s}{s} = \frac{1}{r+s} = \frac{t_s}{r}$. Then the right-and side of (5.7) is equal to $gh_r(t_s) - \frac{t_s}{2}\log r$.

We now show that the Asymptotic Lower r-Permanent Conjecture 5.4 holds for $t_s = \frac{r}{r+s}$, where s is any nonnegative integer, and for $t_\infty := 0$. Assume that conditions of Conjecture 5.4. Lemma 4.1 implies that there exists $G_l \in \Gamma(r, n_l)$ so that

$$\mathrm{perm}_{k_l} \frac{1}{r} A(G_l) \leq \frac{\binom{n_l}{k_l}^2 r^{k_l}(k_l)!(rn_l - k_l)!}{(rn_l)!}.$$

Clearly, $B_{n_l} := \frac{1}{r}A(G_l) \in \Omega_{n_l,r}$ for each l. Use Theorem 5.1 to deduce that for this sequence $B_{n_l}, l \in \mathbb{N}$, lim sup of the left-hand side of (5.6) is at most $gh_r(t) - \frac{t}{2}\log r$.

Let $s \in \mathbb{N}$ and $t_s = \frac{r}{r+s}$. (5.7) implies that lim inf of the sequence given by the left-hand side of (5.6) is not less than $gh_r(t_s) - \frac{t_s}{2}\log r$. Hence (5.6) holds for $t = t_s$.

We now discuss the case $s = 0$, i.e. $t = t_0 = 1$. Let $B = (b_{ij})_{i,j=1}^n$ be any $n \times n$ nonnegative matrix. Denote by $G(B) = (V, E)$ the bipartite graph induced by B, i.e. the edge (i,j) is in E, if and only if $b_{ij} > 0$. Then B induces the weighted graph on G, where the weight of the edge (i,j) is b_{ij}. Let $p_B(x) = x^n + \sum_{m=1}^n (-1)^m \mathrm{perm}_m B$. $p_B(x)$ is called the matching polynomial of the weighted graph G. Heilmann and Lieb showed in [16] that $p_B(x)$ has nonnegative roots. (See also [17].) Hence the arithmetic-geometric inequality for the elementary symmetric polynomials of the nonnegative roots of $p_B(x)$ yields the inequality $\mathrm{perm}_m B \geq \binom{n}{m}(\mathrm{perm}\, B)^{\frac{m}{n}}$. (See [23] for the case of m-matchings in bipartite graphs.)

Use Theorem 3.3 to deduce that $\mathrm{perm}\, B_n \geq \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{r-1}{r}\right)^{(r-1)n}$ for any $B \in \Omega_{n,r}$. Apply the above two inequalities for the sequence B_{n_l} and $m = k_l$ for $l = 1, 2, \ldots$ to obtain that lim inf of the sequence given by the left-hand side of (5.6) is not less than $gh_r(1) - \frac{1}{2}\log r$ for $t = 1$. Hence (5.6) for $t = 1$.

The case $t = 0$ follows from (5.4) for $t = 0$. As we noted above the Asymptotic Lower r-Permanent Conjecture 5.4 for $t = \tau$ implies the ALMC for $t = \tau$. $\square$

Let C_n a cycle on n vertices, and let $T_{n,d} = (V_n, E_n) := \underbrace{C_n \times \ldots \times C_n}_{d}, n = 3, \ldots$ be a sequence of d dimensional torii. Note that each $T_{n,d}$ is $2d$ regular graph. It is a classical result that the following limit exists for any $t \in [0,1]$:

$$\lim_{n\to\infty} \frac{\log \phi_{T_{n,d}}(j_n)}{\#V_n} = h_d(t), \text{ if } \lim_{n\to\infty} \frac{2j_n}{\#V_n} = t \in [0,1]. \tag{5.8}$$

$h_d(t)$ is the d-dimensional monomer-dimer entropy of dimer density $t \in [0,1]$ in the lattice $\mathbb{Z}^d$ [15] and [11]. Let $h_d := \max_{t\in[0,1]} h_d(t)$. (The quantities h_d and $\tilde{h}_d := h_d(1)$ are called the d-monomer-dimer entropy and the 2-dimer entropy, respectively, in [11]).

Let $d = 6$ and $t_3 := \frac{6}{9} = \frac{2}{3}$. The validity of ALMC for t_3 yields that $h_3(\frac{2}{3}) \geq .7845241927$, which implies that $h_3 = \max_{t\in[0,1]} h_3(t) \geq .7845241927$. This improves the lower bound implied by (5.4) $h_3 \geq .7652789557$ [11]. The computations in [11] yield that $h_3 \leq .7862023450$. Thus $h_3 \in [.7845, .7863]$.

6 The AUMC and related graphs

Denote by $\mathbb{R}[x]$ the algebra of polynomials in x with real coefficients, by $0 \in \mathbb{R}[x]$ the zero polynomial, and by $\mathbb{R}_+[x] \subset \mathbb{R}[x]$ the subalgebra of polynomials with non-negative coefficients. We partially order $\mathbb{R}[x]$ by writing, for $f, g \in \mathbb{R}[x]$, $g \succeq f$ when $g - f \in \mathbb{R}_+[x]$, and $g \succ f$ when $g - f \in \mathbb{R}_+[x] \setminus \{0\}$. Clearly, if $g_1 \succeq f_1 \succ 0$ and $g_2 \succeq f_2 \succ 0$, then $g_1 g_2 \succ f_1 f_2$ unless $g_1 = f_1$ and $g_2 = f_2$.

Let $qK_{r,r}$ denote the union of q complete bipartite graphs $K_{r,r}$ having r vertices of each color class. It is straightforward to show that any finite graphs G, G' satisfy

$$\Phi_{G \cup G'}(x) = \Phi_G(x)\Phi_{G'}(x), \tag{6.1}$$

and that

$$\Phi_{K_{r,r}}(x) = \sum_{k=0}^{r} \binom{r}{k}^2 k!\, x^k. \tag{6.2}$$

The following conjecture is stated in [9]

Conjecture 6.1 (The Upper Matching Conjecture) *Let G be a bipartite r-regular graph on $2qr$ vertices where $q, r \geq 2$. Then $\Phi_G \preceq \Phi_{qK_{r,r}}$, equality holding only if $G = qK_{r,r}$.*

For $k = 2$ this conjecture is proved in [10]. The above conjecture implies the following Asymptotic Upper Matching Conjecture [9]. Denote by $K(r)$ be the countably infinite union of $K_{r,r}$. Let $P_{K(r)}(u)$ and $h_{K(r)}(t), t \in (0,1)$ be the pressure and the t-matching entropy associated and the with $K(r)$ [12]:

$$P_{K(r)}(u) = \frac{\log \sum_{k=0}^{r} \binom{r}{k}^2 k!\, e^{2ku}}{2r}, \qquad u \in \mathbb{R}. \tag{6.3}$$

$$h_{K(r)}(t(u)) = P_{K(r)}(u) - ut(u), \qquad u \in \mathbb{R} \tag{6.4}$$

where

$$t(u) = P'_{K(r)}(u) = \frac{\sum_{k=0}^{r} \binom{r}{k}^2 k!\,(2k) e^{2ku}}{2r \sum_{k=0}^{r} \binom{r}{k}^2 k!\, e^{2ku}}, \qquad u \in \mathbb{R}. \tag{6.5}$$

Conjecture 6.2 (The Asymptotic Upper Matching Conjecture) *Let $r \geq 2$ be an integer, $\{k_l\}_{l=1}^{\infty}, \{n_l\}_{l=1}^{\infty}$ be two increasing sequences of positive integers such that $k_l \leq n_l, l = 1, \ldots,$ and $lim_{l\to\infty} \frac{k_l}{n_l} = t \in [0,1]$. Then Then*

$$\lim_{l\to\infty} \frac{\log \max_{G \in \Gamma(n_l, r)} \mathrm{perm}_{k_l} A(G)}{2n_l} = h_{K(r)}(t). \tag{6.6}$$

For $r = 2$ the AUMC is proven in [10]. For $t = 1$ and any $r \in \mathbb{N}$ the AUMC follows from the proof of Minc conjecture by Bregman [2]. Some computations performed in [9] support the ALMC and AUMC.

The following plots illustrating the Asymptotic Matching Conjectures for $r = 4, 6$. Figure 1 shows various bounds and values for the monomer-dimer entropy $h_2(p)$ of dimer density $p \in [0,1]$ in the 4-regular 2-dimensional grid. FT is the Friedland-Tverberg lower bound $fh_4(p)$ of (5.5), h2 is the true monomer-dimer entropy equal to $\max_{p \in [0,1]} h_2(p)$ (it is known to a precision much greater than the picture resolution). The crosses marked B are Baxter's computed values [1]. (Baxter's computations, based on heuristic arguments, where confirmed by theoretical rigorous computations in [12].) ALMC is the function $gh_4(p)$ of (5.2), conjectured to be a lower bound in the Asymptotic Lower Matching Conjecture. AUMC is the monomer-dimer entropy $h_K(p)$ of dimer density t in a countable union of $K_{4,4}$, given by (6.3)–(6.5) and conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture. Notice that AUMC goes a little over h2: a countable union of $K_{4,4}$ has a higher monomer-dimer entropy than an infinite planar grid.

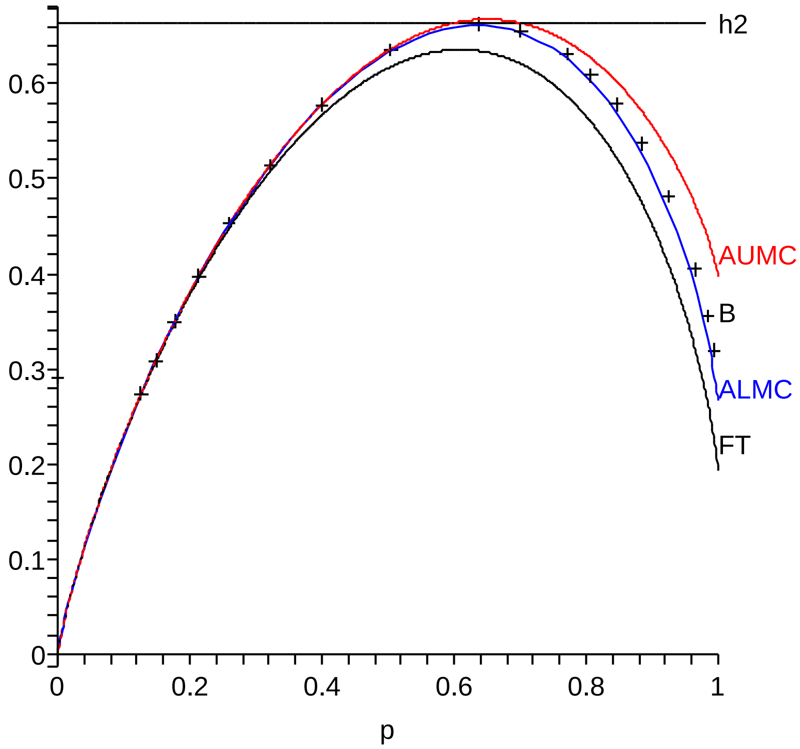

Figure 1: Monomer-dimer tiling of the 2-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h2 is the true monomer-dimer entropy. B are Baxter's computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

Figure 2 shows similarly various bounds and values for the monomer-dimer entropy $h_3(p)$ of dimer density $p \in [0,1]$ in the 6-regular 3-dimensional grid. FT is the Friedland-Tverberg lower bound $fh_6(p)$ of (5.5). h3High is the best known upper bounds for the true monomer-dimer entropy equal to $\max_{p\in[0,1]} h_3(p)$, given in [11]. h3Low is a lower bound implied by the maximal value of FT lower bound. ALMC is the function $gh_6(p)$ of (5.2), conjectured to be a lower bound in the Asymptotic Lower Matching Conjecture. AUMC is the monomer-dimer entropy $h_K(p)$ of dimer density p in a countable union of $K_{6,6}$, given by (6.3)–(6.5) and conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture. Notice that AUMC goes a little over h3High: a countable union of $K_{6,6}$ has a higher monomer-dimer entropy than an infinite cubic grid.

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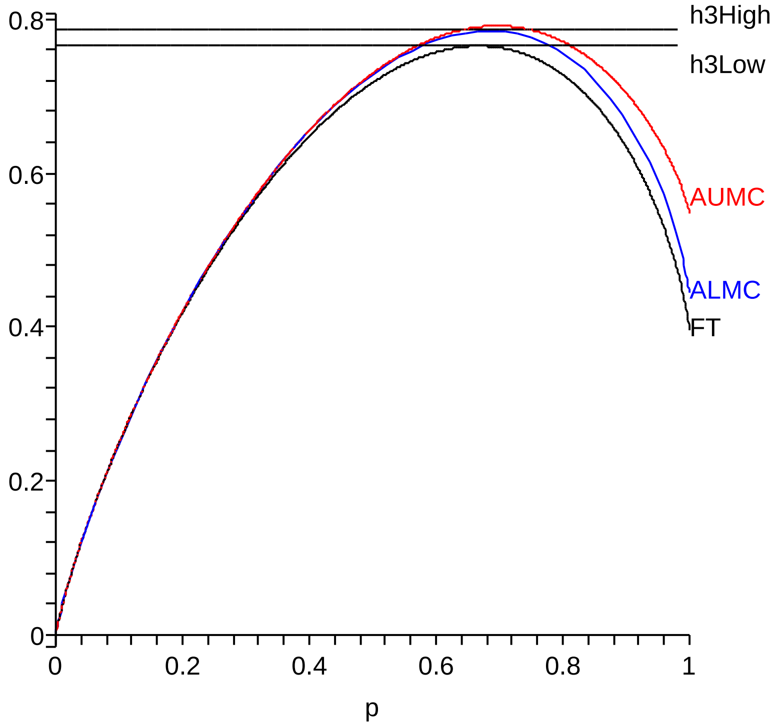

Figure 2: Monomer-dimer tiling of the 3-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h3Low and h3High are the known bounds for the monomer-dimer entropy. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{6,6}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

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