# Avoiding long Berge cycles, the missing cases $k = r + 1$ and $k = r + 2$

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**Abstract**

The maximum size of an $r$-uniform hypergraph without a Berge cycle of length at least $k$ has been determined for all $k \geq r + 3$ by Füredi, Kostochka and Luo and for $k < r$ (and $k = r$, asymptotically) by Kostochka and Luo. In this paper, we settle the remaining cases: $k = r + 1$ and $k = r + 2$, proving a conjecture of Füredi, Kostochka and Luo.

Given a hypergraph $\mathcal{H}$, let $V(\mathcal{H})$ and $E(\mathcal{H})$ denote the set of vertices and hyperedges of $\mathcal{H}$, respectively, and let $e(\mathcal{H}) = |E(\mathcal{H})|$. A hypergraph is called $r$-uniform if all of its hyperedges have size $r$. For convenience, we refer to an $r$-uniform hypergraph as an $r$-graph. Berge introduced the following definitions of a cycle and a path in a hypergraph.

**Definition 1.** *A* Berge cycle *of length $l$ in a hypergraph is a set of $l$ distinct vertices $\{v_1, \dots, v_l\}$ and $l$ distinct hyperedges $\{e_1, \dots, e_l\}$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ with indices taken modulo $l$.*

*A* Berge path *of length $l$ in a hypergraph is a set of $l+1$ distinct vertices $v_1, \dots, v_{l+1}$ and $l$ distinct hyperedges $e_1, \dots, e_l$ such that $\{v_i, v_{i+1}\} \subseteq e_i$ for all $1 \leq i \leq l$. We say that such a Berge path is between $v_1$ and $v_{l+1}$.*

**Notation 1.** *Let $\mathcal{H}$ be a hypergraph. Then its* 2-shadow, *$\partial_2\mathcal{H}$, is the collection of pairs that lie in some hyperedge of $\mathcal{H}$. Given a set $S \subseteq V(\mathcal{H})$, the subhypergraph of $\mathcal{H}$ induced by $S$ is denoted by $\mathcal{H}[S]$.*

*We say $\mathcal{H}$ is* connected *if $\partial_2(\mathcal{H})$ is a connected graph. A hyperedge $h \in E(\mathcal{H})$ is called a* cut-hyperedge *of $\mathcal{H}$ if $\mathcal{H} \setminus \{h\} := (V(\mathcal{H}), E(\mathcal{H}) \setminus \{h\})$ is not connected.*

*When we say $D$ is a block of $\partial_2(\mathcal{H})$, we may either mean $D$ is the vertex-set of the block, or $D$ is the edge-set of the block depending on the context.*

# 1 Background and our results

Győri, Katona and Lemons extended the well-known Erdős-Gallai theorem to hypergraphs by showing the following.

**Theorem 1** (Győri, Katona, Lemons [8])**.** *Let $\mathcal{H}$ be an $r$-uniform hypergraph with no Berge path of length $k$. If $k > r + 1 > 3$, we have*

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r}.$$

*If $r \geq k > 2$, we have*

$$e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}.$$

For the case $k = r + 1$, Győri, Katona and Lemons conjectured that the upper bound should have the same form as the $k > r + 1$ case. This was settled by Davoodi, Győri, Methuku and Tompkins [1] who showed the following.

**Theorem 2** (Davoodi, Győri, Methuku, Tompkins [1])**.** *Fix $k = r + 1 > 2$ and let $\mathcal{H}$ be an $r$-uniform hypergraph containing no Berge path of length $k$. Then,*

$$e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n.$$

The bounds in the above two theorems are sharp for each $k$ and $r$ for infinitely many $n$. Győri, Methuku, Salia, Tompkins and Vizer [9] proved a significantly smaller upper bound on the maximum number of hyperedges in an $n$-vertex $r$-graph with no Berge path of length $k$ under the assumption that it is connected. Their bound is asymptotically exact when $r$ is fixed and $k$ and $n$ are sufficiently large. The notion of Berge cycles and Berge paths was generalized to arbitrary Berge graphs $F$ by Gerbner and Palmer in [5], and the (3-uniform) Turán number of Berge-$K_{2,t}$ was determined asymptotically in [6]. The general behaviour of the Turán number of Berge-$F$, as the uniformity increases, was studied in [7].

Recently, Füredi, Kostochka and Luo [3] proved exact bounds similar to Theorem 1 for hypergraphs avoiding long Berge cycles.

**Theorem 3** (Füredi, Kostochka, Luo [3])**.** *Let $r \geq 3$ and $k \geq r+3$, and suppose $\mathcal{H}$ is an $n$-vertex $r$-graph with no Berge cycle of length $k$ or longer. Then $e(\mathcal{H}) \leq \frac{n-1}{k-2}\binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block $D$ of $\partial_2(\mathcal{H})$, $D = K_{k-1}$ and $\mathcal{H}[D] = K^r_{k-1}$.*

Moreover, Kostochka and Luo [10] found exact bounds for $k \leq r-1$ and asymptotic bounds for $k = r$. Let us remark that their asymptotic bound in the case $k = r$ also follows from Theorem 5 stated below. (More recently, extending [3], Füredi, Kostochka, Luo [4] proved exact bounds and determined the extremal examples for all $n$ when $k \geq r+4$.)

The two cases $k = r+2$ and $k = r+1$ remained open. For the case $k = r+2$, Füredi, Kostochka and Luo conjectured [3] that a similar statement as that of Theorem 3 holds and mentioned the answer is unknown for the case $k = r+1$. In this paper, we prove this conjecture.

**Theorem 4.** *Let $r \geq 3$ and $n \geq 1$, and suppose $\mathcal{H}$ is an $n$-vertex $r$-graph with no Berge cycle of length $r+2$ or longer. Then $e(\mathcal{H}) \leq \frac{r+1}{r}(n-1)$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block $D$ of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D] = K^r_{r+1}$.*

In the case $k = r+1$, we prove the following exact result, and characterize the extremal examples.

**Theorem 5.** *Let $r \geq 3$ and $n \geq 1$, and suppose $\mathcal{H}$ is an $n$-vertex $r$-graph with no Berge cycle of length $r+1$ or longer. Then $e(\mathcal{H}) \leq n-1$. Moreover, equality is achieved if and only if $\partial_2(\mathcal{H})$ is connected and for every block $D$ of $\partial_2(\mathcal{H})$, $D = K_{r+1}$ and $\mathcal{H}[D]$ consists of $r$ hyperedges.*

Note that Theorem 5 easily implies Theorem 2. In fact, it gives the following stronger form. Here we quickly prove this implication.

**Theorem 6.** *Fix $k = r+1 > 2$ and let $\mathcal{H}$ be an $r$-uniform hypergraph containing no Berge path of length $k$. Then, $e(\mathcal{H}) \leq \frac{n}{k}\binom{k}{r} = n$. Moreover, equality holds if and only if each connected component $D$ of $\partial_2(\mathcal{H})$ is $K_{r+1}$, and $\mathcal{H}[D] = K^r_{r+1}$.*

*Proof.* We proceed by induction on $n$. The base cases $n \leq r+1$ are easy to check. Let $\mathcal{H}$ be an $r$-uniform hypergraph containing no Berge path of length $k = r+1$ such that $e(\mathcal{H}) \geq n$. Then by Theorem 5, $\mathcal{H}$ contains a Berge cycle $\mathcal{C}$ of length $r+1$ or longer. $\mathcal{C}$ must be of length exactly $r+1$, otherwise it would contain a Berge path of length $r+1$. Let $v_1, \dots, v_{r+1}$ and $e_1, \dots, e_{r+1}$ be the vertices and edges of $\mathcal{C}$ where $\{v_i, v_{i+1}\} \subseteq e_i$ (indices are taken modulo $r+1$). For any $i$ with $1 \leq i \leq r+1$, if $e_i$ contains a vertex $v \notin \{v_1, \dots, v_{r+1}\}$, then $v_{i+1}e_{i+1}v_{i+2}e_{i+2} \dots e_{i-1}v_ie_iv$ forms a Berge path of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, all of the edges $e_i$ (for $1 \leq i \leq r+1$) are contained in the set $S := \{v_1, \dots, v_{r+1}\}$. That is, $\mathcal{H}[S] = K^r_{r+1}$. It is easy to see that $S$ forms a connected component in $\partial_2(\mathcal{H})$ because if any hyperedge $h$ of $\mathcal{H}$ (with $h \notin \mathcal{C}$) contains a vertex of $\mathcal{C}$, then $\mathcal{C}$ can be extended to form a Berge path of length $r+1$.

Let $S_1, S_2, \dots, S_t$ be the vertex sets of connected components of $\partial_2(\mathcal{H})$. As noted before, one of them, say $S_1$, is equal to $S$. We delete the vertices of $S_1$ from $\mathcal{H}$ to form a new hypergraph $\mathcal{H}'$; note that $|V(\mathcal{H}')| = |V(\mathcal{H})| - (r+1)$ and $|E(\mathcal{H}')| = |E(\mathcal{H})| - (r+1)$ and the connected components of $\partial_2(\mathcal{H}')$ are $S_2, \dots, S_t$. By induction $|E(\mathcal{H}')| \leq |V(\mathcal{H}')|$. Thus $|E(\mathcal{H})| = |E(\mathcal{H}')| + (r+1) \leq |V(\mathcal{H}')| + (r+1) = |V(\mathcal{H})|$. Moreover, if $|E(\mathcal{H})| = |V(\mathcal{H})|$, then $|E(\mathcal{H}')| = |V(\mathcal{H}')|$, so by the induction hypothesis each connected component $S_i$ ($i \geq 2$) of $\partial_2(\mathcal{H}')$ is $K_{r+1}$, and $\mathcal{H}'[S_i] = K^r_{r+1}$, proving the theorem. □

**Structure of the paper:** In Section 2, we prove some basic lemmas which are used in our proofs. In Section 3, we prove Theorem 4, and in Section 4, we prove Theorem 5.

# 2 Basic Lemmas

We will use the following two lemmas.

**Lemma 1.** *For any $r \geq 3$, if a set $S$ of size $r+1$ contains $r$ hyperedges of size $r$, then between any two vertices $u, v \in S$, there is a Berge path of length $r$ consisting of these hyperedges.*

*Proof.* Let $\mathcal{H}$ be the hypergraph consisting of $r$ hyperedges on $r+1$ vertices. First notice that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. (Indeed, there is at most one hyperedge that does not contain $x$ and at most one hyperedge that does not contain $y$.) This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least 3 hyperedges contained in $S$. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2, \dots, x_{r+1}$ of length $r$ in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_{r+1}$. We want to show that there are distinct hyperedges containing the pairs $x_ix_{i+1}$ for each $1 \leq i \leq r$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \dots, x_rx_{r+1}\}$ in one class and the $r$ hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We will show that Hall's condition holds: As noted before, every pair is contained in a hyperedge. Given any two distinct pairs $x_ix_{i+1}$ and $x_jx_{j+1}$, there is at most one hyperedge that does not contain either of them; i.e., at least $r-1$ hyperedges contain one of them. Thus we need $2 \leq r-1$ for Hall's condition to hold, but this is true as we assumed $r \geq 3$. Moreover, if we take any $3 \leq j \leq r$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \leq r$, but this is true by assumption. This finishes the proof of the lemma. □

**Lemma 2.** *For any $r \geq 4$, if a set $S$ of size $r+1$ contains $r-1$ hyperedges of size $r$, then between any two vertices $u, v \in S$, there is a Berge path of length $r-1$ consisting of these hyperedges.*

*Proof.* The proof is similar to that of Lemma 1. Let $\mathcal{H}$ be the hypergraph consisting of $r-1$ hyperedges on $r+1$ vertices. First notice that for any pair of vertices $x, y \in S$, the number of hyperedges $h \subset S$ such that $\{x, y\} \not\subset h$ is at most 2. This means that every pair $x, y \in S$ is contained in some hyperedge, as there are at least $r-1 \geq 3$ hyperedges contained in $S$. In other words, $\partial_2(\mathcal{H}) = K_{r+1}$.

Consider an arbitrary path $x_1x_2\dots x_r$ of length $r-1$ in the $\partial_2(\mathcal{H})$ connecting $u = x_1$ and $v = x_r$. We want to show that there are distinct hyperedges containing the pairs $x_ix_{i+1}$ for each $1 \le i \le r-1$. To this end, we consider an auxiliary bipartite graph with pairs $\{x_1x_2, x_2x_3, \dots, x_{r-1}x_r\}$ in one class and the $r-1$ hyperedges $h \subset S$ in the other class, and a pair is connected to a hyperedge if it is contained in the hyperedge. We show that Hall's condition holds: As noted before, every pair is contained in a hyperedge. Given any two distinct pairs $x_ix_{i+1}$ and $x_jx_{j+1}$, there is at most one hyperedge that does not contain either of them; i.e., at least $r-2$ hyperedges contain one of them. Thus we need $2 \le r-2$ for Hall's condition to hold, but this is true as we assumed $r \ge 4$. Moreover, if we take any $3 \le j \le r-1$ distinct pairs, then every hyperedge contains one of them. Therefore, we need $j \le r-1$ for Hall's condition to hold, and this is true by assumption. This finishes the proof of the lemma. □

# 3 Proof of Theorem 4 ($k = r+2$)

We will prove the theorem by induction on $n$. For the base cases, note that if $1 \le n \le r$ then the statement of the theorem is trivially true. If $n = r+1$, the statement is true since there are at most $r+1$ hyperedges of size $r$ on $r+1$ vertices. Moreover, equality holds if and only if $\mathcal{H} = K^r_{r+1}$.

We will show the statement is true for $n \ge r+2$ assuming it is true for all smaller values. Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices having no Berge cycle of length $r+2$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S \subseteq V(\mathcal{H})$ of vertices, the number of hyperedges of $\mathcal{H}$ incident to the vertices of $S$ is at least $|S|$.

Indeed, suppose there is a set $S \subseteq V(\mathcal{H})$ with fewer than $|S|$ hyperedges incident to the vertices of $S$. If $|S| = n$ we immediately have the required bound on $e(\mathcal{H})$, so assume $n > |S|$. We can delete the vertices of $S$ from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n-|S|$ vertices. By induction, $\mathcal{H}'$ contains at most $\frac{r+1}{r}(n-|S|-1)$ hyperedges, so $\mathcal{H}$ contains less than $\frac{r+1}{r}(n-1-|S|)+|S| < \frac{r+1}{r}(n-1)$ hyperedges, as desired.

(2) There is no cut-hyperedge in $\mathcal{H}$.

Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are non-empty disjoint sets $V_1$ and $V_2$ such that $V(\mathcal{H}) = V_1 \cup V_2$, and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between $V_1$ and $V_2$. So both hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge cycle of length $r+2$ or longer. By induction, $e(\mathcal{H}[V_1]) \le \frac{r+1}{r}(|V_1|-1)$ and $e(\mathcal{H}[V_2]) \le \frac{r+1}{r}(|V_2|-1)$. In total, $e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \le \frac{r+1}{r}(|V_1|+|V_2|-2)+1 < \frac{r+1}{r}(|V(\mathcal{H})|-1)$, as desired.

Consider an auxiliary bipartite graph $B$ consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ on the other class. Then property (1) shows that Hall's condition holds. Therefore, there is a perfect matching in $B$. In other words, there exists an injection $f : V(\mathcal{H}) \to E(\mathcal{H})$ such that $v \in f(v)$.

Given an injection $f : V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge path of the form $v_1 f(v_1) v_2 f(v_2) \dots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \le i \le l-1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1\phi(v_1)v_2\phi(v_2)\dots v_{l-1}\phi(v_{l-1})v_l$ is a longest path.

**Claim 1.** $\phi(v_l) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$.

*Proof.* First notice that if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \dots, v_{l-r-1}\}$, then the Berge cycle $v_i\phi(v_i)v_{i+1}\phi(v_{i+1})\dots v_l\phi(v_l)v_i$ is of length $r+2$ or longer, a contradiction. Moreover, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then $\mathcal{P}_\phi$ can be extended to a longer path $v_1\phi(v_1)v_2\phi(v_2)\dots v_{l-1}\phi(v_{l-1})v_l\phi(v_l)v$, a contradiction. This completes the proof of the claim. $\square$

By Claim 1, we know that $\phi(v_l) = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\} \setminus \{v_j\}$ for some $l-r \le j \le l-1$.

**Claim 2.** *For any $i \in \{l-r, l-r+1, \dots, l\} \setminus \{j\}$, we have $\phi(v_i) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$.*

*Proof.* When $i = l$, we know the statement is true. Suppose $i \in \{l-r, l-r+1, \dots, l-1\} \setminus \{j\}$. Let us define a new injection $\psi : V(\mathcal{H}) \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{v_1, v_2, \dots, v_l\}$, and for every $v \in \{v_1, v_2, \dots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \ge k \ge i+1$.

Now consider the Berge path $v_1\phi(v_1)v_2\phi(v_2)\dots v_i\phi(v_l)v_l\phi(v_{l-1})\dots v_{i+2}\phi(v_{i+1})v_{i+1}$, equivalently $v_1\psi(v_1)v_2\psi(v_2)\dots v_i\psi(v_i)v_l\psi(v_l)\dots v_{i+2}\psi(v_{i+2})v_{i+1}$. This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, notice that the sets of last $r+1$ vertices of both paths are the same. Thus we can apply Claim 1 to conclude that $\phi(v_i) = \psi(v_{i+1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$, as desired. $\square$

Claim 2 shows that there are $r$ hyperedges (each of size $r$) contained in the set $S := \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ of size $r+1$. We will apply Lemma 1 to $S$.

**Claim 3.** *The set $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ induces a block of $\partial_2(\mathcal{H})$.*

*Proof.* Since the set $S = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ contains $r \ge 3$ hyperedges every pair $x, y \in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block $D$ of $\partial_2(\mathcal{H})$ containing $S$.

Suppose $D$ contains a vertex $t \notin S$. Then since $D$ is 2-connected, there are two paths $P_1, P_2$ in $\partial_2(\mathcal{H})$ between $t$ and $S$, which are vertex-disjoint besides $t$. Let $V(P_1) \cap S = \{u\}$ and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge $h_{xy}$ of $\mathcal{H}$ containing $xy$. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between $u$ and $v$.

On the other hand, by Lemma 1, there is a Berge path $\mathcal{P}'$ of length $r$ between $u$ and $v$ consisting of the $r$ hyperedges contained in $S$. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in $S$, while hyperedges of $\mathcal{P}'$

are contained in $S$). Therefore, $\mathcal{P} \cup \mathcal{P}'$ forms a Berge cycle of length $r+2$ or longer unless $\mathcal{P}$ consists of only one hyperedge, say $h$. Note that $h$ contains a vertex $x \notin S$ and $u, v \in h$; moreover by property (2), $h$ is not a cut-hyperedge of $\mathcal{H}$. So after deleting $h$ from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{h\}$ is still connected – so there is a (shortest) Berge path $\mathcal{Q}$ in $\mathcal{H} \setminus \{h\}$ between $x$ and a vertex $s \in S$ (note that the hyperedges of $\mathcal{Q}$ are not contained in $S$). The vertex $s$ is different from either $u$ or $v$, say $s \neq u$ without loss of generality. By Lemma 1, there is a Berge path $\mathcal{Q}'$ of length $r$ between $s$ and $u$ (consisting of hyperedges contained in $S$). Then, $\mathcal{Q}, \mathcal{Q}'$ and $h$ form a Berge cycle of length at least $r+2$, a contradiction. Therefore, $D$ contains no vertex outside $S$; thus $S$ induces a block of $\partial_2(\mathcal{H})$, as required. □

Let $D_1, D_2, \dots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 3 shows that one of these blocks, say $D_1$, is induced by $S$. Let us contract the vertices of $S$ to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \dots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length $r+2$ or longer, as well; moreover $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$. Thus, by induction, we have $e(\mathcal{H}') \leq \frac{r+1}{r}(|V(\mathcal{H}')|-1)$. Therefore,

$$e(\mathcal{H}) \leq \frac{r+1}{r}(|V(\mathcal{H}')|-1) + (r+1) = \frac{r+1}{r}(|V(\mathcal{H})|-r-1) + (r+1) = \frac{r+1}{r}(|V(\mathcal{H})|-1).$$

Now if $e(\mathcal{H}) = \frac{r+1}{r}(|V(\mathcal{H})|-1)$, then we must have $e(\mathcal{H}') = \frac{r+1}{r}(|V(\mathcal{H}')|-1)$ and $S$ must contain all $r+1$ subsets of size $r$ (i.e., $\mathcal{H}[S] = \mathcal{H}[D_1] = K_{r+1}^r$). Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block $D_i$ (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i] = K_{r+1}^r$. This means that for every block $D$ of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D] = K_{r+1}^r$, completing the proof.

# 4 Proof of Theorem 5 ($k = r+1$)

The proof is similar to that of Theorem 4 but there are many important differences.

We use induction on $n$. For the base cases, notice that the statement of the theorem is trivially true if $1 \leq n \leq r$. Moreover, if $n = r+1$, then $e(\mathcal{H}) \leq r$ because otherwise, $\mathcal{H} = K_{r+1}^r$ and then it is easy to see that there is a (Hamiltonian) Berge cycle of length $r+1$ in $\mathcal{H}$, a contradiction. Therefore, $e(\mathcal{H}) \leq r = n-1$. Moreover, equality holds if and only if $\partial_2(\mathcal{H}) = K_{r+1}$ and $\mathcal{H}$ consists of $r$ hyperedges.

We will show the statement is true for $n$ assuming it is true for all smaller values. Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices having no Berge cycle of length $r+1$ or longer. We show that we may assume the following two properties hold for $\mathcal{H}$.

(1) For any set $S \subseteq V(\mathcal{H})$ with $|S| \leq |V(\mathcal{H})|-1 = n-1$, the number of hyperedges of $\mathcal{H}$ incident to the vertices of $S$ is at least $|S|$.

Indeed, suppose there is a set $S \subset V(\mathcal{H})$ (i.e., $|S| \leq |V(\mathcal{H})|-1$) with fewer than $|S|$ hyperedges incident to the vertices of $S$. We delete the vertices of $S$ from $\mathcal{H}$ to obtain a new hypergraph $\mathcal{H}'$ on $n-|S|$ vertices. By induction, $\mathcal{H}'$ contains at most $(n-|S|-1)$ hyperedges, so $\mathcal{H}$ contains less than $(n-1-|S|)+|S| = (n-1)$ hyperedges, as required.

(2) There is no cut-hyperedge in $\mathcal{H}$.

Indeed, if $h \in E(\mathcal{H})$ is a cut-hyperedge, then $\partial_2(\mathcal{H} \setminus \{h\})$ is not a connected graph, so there are disjoint non-empty sets $V_1$ and $V_2$ such that $V(\mathcal{H}) = V_1 \cup V_2$ and there are no edges of $\partial_2(\mathcal{H} \setminus \{h\})$ between $V_1$ and $V_2$. So the hypergraphs $\mathcal{H}[V_1]$ and $\mathcal{H}[V_2]$ do not contain a Berge cycle of length $r + 1$ or longer. Therefore, by induction, $e(\mathcal{H}[V_1]) \leq |V_1|-1$ and $e(\mathcal{H}[V_2]) \leq |V_2|-1$. In total, $e(\mathcal{H}) = e(\mathcal{H}[V_1]) + e(\mathcal{H}[V_2]) + 1 \leq (|V_1|+|V_2|-2) + 1 = |V(\mathcal{H})|-1$, as desired.

Moreover, we claim that the equality $e(\mathcal{H}) = |V(\mathcal{H})|-1$ cannot hold in this case (i.e., if there is a cut-hyperedge). Indeed, if equality holds, then we must have $e(\mathcal{H}[V_1]) = |V_1|-1$ and $e(\mathcal{H}[V_2]) = |V_2|-1$. Notice that since $r \geq 3$, the hyperedge $h$ either contains at least two vertices $x, y \in V_1$ or two vertices $x, y \in V_2$. Without loss of generality, assume the former is true. By induction, $\partial_2(\mathcal{H}[V_1])$ is connected and for every block $D$ of $\partial_2(\mathcal{H}[V_1])$, we have $D = K_{r+1}$ and the subhypergraph induced by $D$ consists of $r$ hyperedges. So by Lemma 1, there is a Berge path of length $r$ (consisting of the $r$ hyperedges induced by $D$) between any two vertices of a block $D$. Then it is easy to see that since $\partial_2(\mathcal{H}[V_1])$ is connected, there is a Berge path $\mathcal{P}$ of length at least $r$ between any two vertices of $V_1$, so in particular between $x$ and $y$. Then $\mathcal{P}$ together with $h$ forms a Berge cycle of length $r + 1$ in $\mathcal{H}$, a contradiction.

Consider an auxiliary bipartite graph $B$ consisting of vertices of $\mathcal{H}$ in one class and hyperedges of $\mathcal{H}$ on the other class. Then property (1) shows that Hall's condition holds for all subsets of $V(\mathcal{H})$ of size up to $|V(\mathcal{H})|-1$. Therefore, there is a matching in $B$ that matches all the vertices in $V(\mathcal{H})$, except at most one vertex, say $x$. In other words, there exists an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ such that for every $v \in V(\mathcal{H}) \setminus \{x\}$, we have $v \in f(v)$. Given an injection $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, let $\mathcal{P}_f$ be a longest Berge path of the form $v_1 f(v_1) v_2 f(v_2) \ldots v_{l-1} f(v_{l-1}) v_l$ where for each $1 \leq i \leq l-1$, $v_{i+1} \in f(v_i)$. Moreover, among all injections $f : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ with $v \in f(v)$, suppose $\phi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ is an injection for which the path $\mathcal{P}_\phi = v_1 \phi(v_1) v_2 \phi(v_2) \ldots v_{l-1} \phi(v_{l-1}) v_l$ is a longest path.

Because of the way $\mathcal{P}_\phi$ was constructed, it is also clear that $x \notin \{v_1, v_2, \ldots, v_{l-1}\}$. We consider two cases depending on whether $v_l$ is equal to $x$ or not.

**Case 1:** $v_l \neq x$**.** Our aim is to get a contradiction, and show that this case is impossible.

**Claim 4.** *If $v_l \neq x$, then $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$.*

*Proof.* If $v_l \neq x$, then we claim $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$. Indeed, if $\phi(v_l)$ contains a vertex $v_i \in \{v_1, v_2, \ldots, v_{l-r}\}$, then the Berge cycle $v_i \phi(v_i) v_{i+1} \phi(v_{i+1}) \ldots v_l \phi(v_l) v_i$ is of length $r+1$ or longer, a contradiction. Moreover, if $\phi(v_l)$ contains a vertex $v \notin \{v_1, v_2, \ldots, v_l\}$, then $P_\phi$ can be extended to a longer path $v_1 \phi(v_1) v_2 \phi(v_2), \ldots, v_{l-1} \phi(v_{l-1}) v_l \phi(v_l) v$, a contradiction again, proving that $\phi(v_l) = \{v_{l-r+1}, v_{l-r+2}, \ldots, v_l\}$. □

Fix some $i \in \{l-r+1, l-r+2, \ldots, l-1\}$. Let us define a new injection $\psi : V(\mathcal{H}) \setminus \{x\} \to E(\mathcal{H})$ as follows: $\psi(v) = \phi(v)$ for every $v \notin \{x, v_1, v_2, \ldots, v_l\}$, and for every $v \in \{v_1, v_2, \ldots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_l)$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l \geq k \geq$

$i+1$. Now consider the Berge path $v_1\phi(v_1)v_2\phi(v_2)\ \dots\ v_i\phi(v_l)v_l\phi(v_{l-1})\dots v_{i+2}\phi(v_{i+1})v_{i+1}$ $= v_1\psi(v_1)v_2\psi(v_2)\ \dots\ v_i\psi(v_i)v_l\psi(v_l)\dots v_{i+2}\psi(v_{i+2})v_{i+1}$. This path has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, $v_{i+1}\neq x$, so we can apply Claim 4 to conclude that $\psi(v_{i+1}) = \{v_{l-r+1}, v_{l-r+2}, \dots, v_l\} = \phi(v_i)$. But then $\phi(v_i) = \phi(v_l)$, a contradiction to the fact that $\phi$ was an injection.

**Case 2:** $v_l = x$**.**

**Claim 5.** $\phi(v_{l-1}) \subset \{v_{l-r}, v_{l-r+1}, \dots, v_l\}$.

*Proof.* If $\phi(v_{l-1})$ contains a vertex $v \notin \{v_1, v_2, \dots, v_l\}$, then we consider the Berge path $v_1\phi(v_1)v_2\phi(v_2),\dots,v_{l-1}\phi(v_{l-1})v$. Since $v\neq x$, we get a contradiction by Case 1. Moreover, if $\phi(v_{l-1})$ contains a vertex $v_i$ with $i\in\{1,2,\dots,l-r-1\}$, then the Berge cycle $v_i\phi(v_i)v_{i+1}\phi(v_{i+1})\dots v_{l-1}\phi(v_{l-1})v_i$ is of length $r+1$ or longer, a contradiction. This finishes the proof of the claim. □

By Claim 5, we know that $\phi(v_{l-1}) = \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}\setminus\{v_j\}$ for some $j$ with $l-r\leq j\leq l-2$. (From now, in the rest of the proof we fix this $j$.)

**Claim 6.** *For any* $i\in\{l-r, l-r+1, \dots, l-1\}\setminus\{j\}$*, we have* $\phi(v_i)\subset\{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$.

*Proof.* When $i = l-1$, we know the statement is true by Claim 5.

Suppose $i\in\{l-r, l-r+1, \dots, l-2\}\setminus\{j\}$. Let us define a new injection $\psi : V(\mathcal{H})\setminus\{x\}\to E(\mathcal{H})$ as follows: $\psi(v)=\phi(v)$ for every $v\notin\{v_1,v_2,\dots,v_l\}$, and for every $v\in\{v_1, v_2, \dots, v_{i-1}\}$. Moreover, let $\psi(v_i) = \phi(v_{l-1})$ and $\psi(v_k) = \phi(v_{k-1})$ for each $l-1\geq k\geq i+1$. Now consider the Berge path $v_1\phi(v_1)v_2\phi(v_2)\ \dots\ v_i\phi(v_{l-1})v_{l-1}\phi(v_{l-2})\dots v_{i+1} = v_1\psi(v_1)v_2\psi(v_2)\ \dots\ v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\dots v_{i+1}$. (Note that when $i=l-2$, the Berge path is simply $v_1\phi(v_1)v_2\phi(v_2)\ \dots\ v_i\phi(v_{l-1})v_{l-1} = v_1\psi(v_1)v_2\psi(v_2)\dots v_i\psi(v_i)v_{l-1}$.)

If $\psi(v_{i+1})$ contains a vertex $v\notin\{v_1, v_2, \dots, v_l\}$, then the Berge path $v_1\psi(v_1)v_2\psi(v_2)\ \dots\ v_i\psi(v_i)v_{l-1}\psi(v_{l-1})\dots v_{i+2}\psi(v_{i+2})v_{i+1}\psi(v_{i+1})v$ has the same length as $\mathcal{P}_\phi$, so it is also a longest path. Moreover, since $v\neq x$, we get a contradiction by Case 1.

If $\psi(v_{i+1})$ contains a vertex $v_k\in\{v_1, v_2, \dots, v_{l-r-1}\}$ then one can see that the Berge cycle $v_k\psi(v_k)v_{k+1}\psi(v_{k+1})\ \dots\ v_{l-1}\psi(v_{l-1})v_k$ is of length $r+1$ or longer, a contradiction. Therefore, we have $\psi(v_{i+1})\subset\{v_{l-r}, v_{l-r+1}, \dots, v_l\}$. But we defined $\psi(v_{i+1}) = \phi(v_i)$, proving the claim. □

Note that Claim 6 shows that $r-1$ hyperedges of $\mathcal{H}$ are contained in a set $S := \{v_{l-r}, v_{l-r+1}, \dots, v_{l-1}, v_l\}$ of size $r+1$. The following claim shows that if we can find one more hyperedge of $\mathcal{H}$ contained in $S$, then $S$ must induce a block of $\partial_2(\mathcal{H})$.

**Claim 7.** *Suppose* $r\geq 3$*. If a set* $S$ *of size* $r+1$ *contains* $r$ *hyperedges of* $\mathcal{H}$ *then it induces a induces a block of* $\partial_2(\mathcal{H})$*.*

*Proof.* Since the set $S$ contains at least 3 hyperedges every pair $x, y\in S$ is contained in some hyperedge. Thus $\partial_2(\mathcal{H}[S]) = K_{r+1}$. Consider a (maximal) block $D$ of $\partial_2(\mathcal{H})$ containing $S$.

Suppose $D$ contains a vertex $t\notin S$. Then since $D$ is 2-connected, there are two paths $P_1, P_2$ in $\partial_2(\mathcal{H})$ between $t$ and $S$, which are vertex-disjoint besides $t$. Let $V(P_1)\cap S = \{u\}$

and $V(P_2) \cap S = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge $h_{xy}$ of $\mathcal{H}$ containing $xy$. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between $u$ and $v$.

On the other hand, by Lemma 1, there is a Berge path $\mathcal{P}'$ of length $r$ between $u$ and $v$ consisting of the $r$ hyperedges contained in $S$. Note that $\mathcal{P}$ and $\mathcal{P}'$ do not share any hyperedges (indeed, each hyperedge of $\mathcal{P}$ contains a vertex not in $S$, while hyperedges of $\mathcal{P}'$ are contained in $S$). Therefore, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length $r+1$ or longer, a contradiction. Therefore, $D$ contains no vertex outside $S$; thus $S$ induces a block of $\partial_2(\mathcal{H})$, as required. □

We will use the above claim several times later. At this point we need to distinguish the cases $r = 3$ and $r \geq 4$, since Lemma 2 only applies in the latter case.

## The case $r \geq 4$

Since $r \geq 4$, by Claim 6 and Lemma 2 there is a Berge path of length $r-1$ between any two vertices of $S = \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\}$. This will allow us to show the following.

**Claim 8.** $\phi(v_j) \subset \{v_{l-r}, v_{l-r+1}, \ldots, v_{l-1}, v_l\} = S$

*Proof.* Suppose for a contradiction that $\phi(v_j)$ contains a vertex $v \notin S$. The hyperedge $\phi(v_j)$ contains at least two vertices from $S$, namely $v_j$ and $v_{j+1}$. By property (2), $\phi(v_j)$ is not a cut-hyperedge of $\mathcal{H}$. So after deleting $\phi(v_j)$ from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{\phi(v_j)\}$ is still connected – so there is a (shortest) Berge path $\mathcal{Q}$ in $\mathcal{H} \setminus \{\phi(v_j)\}$ between $v$ and a vertex $s \in S$ (note that the hyperedges of $\mathcal{Q}$ are not contained in $S$). The vertex $s$ is different from either $v_j$ or $v_{j+1}$, say $s \neq v_j$, without loss of generality. By Lemma 2, there is a Berge path $\mathcal{Q}'$ of length $r-1$ between $s$ and $v_j$ (consisting of the hyperedges contained in $S$). Then $\mathcal{Q}, \mathcal{Q}'$ and $\phi(v_j)$ form a Berge cycle of length at least $r+1$ in $\mathcal{H}$, a contradiction. □

Claim 6 and Claim 8 together show that there are at least $r$ hyperedges of $\mathcal{H}$ contained in $S$. If all $r+1$ subsets of $S$ of size $r$ are hyperedges of $\mathcal{H}$, then $S$ induces $K_{r+1}^r$ and it is easy to show that it contains a Berge cycle of length $r+1$, a contradiction. This means $S$ contains exactly $r$ hyperedges of $\mathcal{H}$. Then by Claim 7, we know that $S$ induces a block of $\partial_2(\mathcal{H})$.

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. Claim 7 shows that one of these blocks, say $D_1$, is induced by $S$. Let us contract the vertices of $S$ to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length $r+1$ or longer, as well; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - r$ and $e(\mathcal{H}') = e(\mathcal{H}) - r$. By induction, we have $e(\mathcal{H}') \leq |V(\mathcal{H}')| - 1$. Therefore,

$$e(\mathcal{H}) = e(\mathcal{H}') + r \leq (|V(\mathcal{H}')| - 1) + r = (|V(\mathcal{H})| - r - 1) + r = |V(\mathcal{H})| - 1.$$

If $e(\mathcal{H}) = |V(\mathcal{H})| - 1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')| - 1$ and $S$ must contain exactly $r$ hyperedges. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block $D_i$ (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_{r+1}$ and $\mathcal{H}'[D_i]$ contains exactly $r$ hyperedges. This means that for every block $D$ of $\partial_2(\mathcal{H})$, we have $D = K_{r+1}$ and $\mathcal{H}[D]$ contains exactly $r$ hyperedges, completing the proof in the case $r \geq 4$.

## The case $r = 3$

Recall that using Claim 6 we can find a set $S$ of size 4 which contains 2 hyperedges of $\mathcal{H}$. Let $S = \{x, y, a, b\}$ and the two hyperedges be $xab$ and $yab$. By property (2), $xab$ is not a cut-hyperedge of $\mathcal{H}$. So after deleting $xab$ from $\mathcal{H}$, the hypergraph $\mathcal{H} \setminus \{xab\}$ is still connected – so there is a (shortest) Berge path $\mathcal{Q}$ between $x$ and $\{y, a, b\}$. If $\mathcal{Q}$ is of length at least 2, then it is easy to see that $\mathcal{Q}$ together with $yab$ and $xab$ form a Berge cycle of length at least 4, a contradiction. So $\mathcal{Q}$ consists of only one hyperedge, say $h$.

Our goal is to find a set of vertices which induces a block of $\partial_2(\mathcal{H})$, so that we can apply induction.

If $|h \cap \{y, a, b\}| = 2$ then $h, xab, yab$ are 3 hyperedges of $\mathcal{H}$ contained in $S$, so by Claim 7, we can conclude that $S$ induces a block of $\partial_2(\mathcal{H})$. (Notice that $S$ contains exactly $|S|-1 = 3$ hyperedges of $\mathcal{H}$, otherwise it is easy to find a Berge cycle of length 4; this will be useful later.) So we can suppose $|h \cap \{y, a, b\}| = 1$. We consider two cases depending on whether $h$ is either $xat$ or $xbt$, or whether $h$ is $xyt$ for some $t \notin S$.

**Case 1.** First suppose without loss of generality that $h = xat$ for some $t \notin S$. Consider the set $\mathcal{D}$ of all hyperedges of $\mathcal{H}$ containing the pairs $xa$, $ab$ or $xb$ and let $D$ be the set of vertices spanned by them. For each pair of vertices $i, j \in \{x, a, b\}$, let $V_{ij} = \{v \mid ijv \in \mathcal{H}\} \setminus \{x, a, b\}$. We claim that the sets $V_{xa}, V_{ab}, V_{xb}$ are pairwise disjoint. Suppose for the sake of a contradiction that $t' \in V_{xa} \cap V_{ab}$. Then the hyperedges $xat', abt', xab$ are contained in a set of 4 vertices $\{x, a, b, t'\}$. Thus by Claim 7, this set induces a block of $\partial_2(\mathcal{H})$ and we are done (we found the desired block!). Thus we can suppose $V_{xa} \cap V_{ab} = \emptyset$. Similarly $V_{ab} \cap V_{xb} = \emptyset$ and $V_{xa} \cap V_{xb} = \emptyset$. This shows that $|D| = 3 + |V_{xa}| + |V_{xb}| + |V_{ab}|$. On the other hand, $\mathcal{D}$ consists of $1 + |V_{xa}| + |V_{xb}| + |V_{ab}|$ hyperedges, so $|\mathcal{D}| = |D| - 2$.

We will now show that $D$ induces a block of $\partial_2(\mathcal{H})$. Let $D'$ be a (maximal) block of $\partial_2(\mathcal{H})$ containing $D$ and suppose for the sake of a contradiction that it contains a vertex $p \notin D$. Then since $D'$ is 2-connected, there are two paths $P_1, P_2$ in $\partial_2(\mathcal{H})$ between $p$ and $D$, which are vertex-disjoint besides $p$. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge $h_{xy}$ of $\mathcal{H}$ containing $xy$. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between $u$ and $v$. If $uv \notin \{xa, ab, xb\}$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 between $u$ and $v$ consisting of the hyperedges of $\mathcal{D}$. Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand if $uv \in \{xa, ab, xb\}$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$ but then $puv$ should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair $uv$); moreover, it is easy to check that between $u$ and $v$ there is a Berge path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, $D'$ contains no vertex outside $D$; so $D$

induces a block of $\partial_2(\mathcal{H})$ (which contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

**Case 2.** Finally suppose $h = xyt$ for some $t \notin S$. Let $\mathcal{D}$ be the set of all hyperedges of $\mathcal{H}$ containing the pair $xy$ plus the hyperedges $xab$ and $yab$, and let $D$ be the set of vertices spanned by the hyperedges of $\mathcal{D}$. Let $V_{xy} = \{v \mid xyv \in \mathcal{H}\}$. We claim that $a \notin V_{xy}$ and $b \notin V_{xy}$. Indeed suppose for the sake of a contradiction that $a \in V_{xy}$. Then the hyperedges $xab, yab, xya$ are contained in a set of 4 vertices $\{x, y, a, b\}$. So by Claim 7, this set induces a block of $\partial_2(\mathcal{H})$, and we are done. So $a \notin V_{xy}$. Similarly, we can conclude $b \notin V_{xy}$. Therefore, $|D| = |V_{xy}| + 4$. On the other hand, $|\mathcal{D}| = |V_{xy}| + 2$, so $|\mathcal{D}| = |D| - 2$.

We claim that $D$ induces a block of $\partial_2(\mathcal{H})$. The proof is very similar to that of **Case 1**, we still give it for completeness. Let $D'$ be a (maximal) block of $\partial_2(\mathcal{H})$ containing $D$ and suppose for the sake of a contradiction that it contains a vertex $p \notin D$. Then since $D'$ is 2-connected, there are two paths $P_1, P_2$ in $\partial_2(\mathcal{H})$ between $p$ and $D$, which are vertex-disjoint besides $p$. Let $V(P_1) \cap D = \{u\}$ and $V(P_2) \cap D = \{v\}$. For each edge $xy \in E(P_1) \cup E(P_2)$, fix an arbitrary hyperedge $h_{xy}$ of $\mathcal{H}$ containing $xy$. It is easy to see that a subset of the hyperedges $\{h_{xy} \mid xy \in E(P_1) \cup E(P_2)\}$ forms a Berge path $\mathcal{P}$ between $u$ and $v$.

If $uv \neq xy$, then it is easy to see that there is a path $\mathcal{P}'$ of length 3 or 4 between $u$ and $v$ consisting of the hyperedges of $\mathcal{D}$. (Indeed if $u, v \in V_{xy}$, then $\mathcal{P}'$ is of length 4, otherwise it is of length 3.) Then $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. On the other hand if $uv = xy$, then $\mathcal{P}$ must contain at least two hyperedges of $\mathcal{H}$ because otherwise $\mathcal{P} = \{puv\}$ but then $puv$ should have been in $\mathcal{D}$ (since by definition $\mathcal{D}$ must contain all the hyperedges of $\mathcal{H}$ containing the pair $uv$); moreover, it is easy to check that between $u$ and $v$ there is a Berge path $\mathcal{P}'$ of length 2 consisting of the hyperedges of $\mathcal{D}$. Then again, $\mathcal{P}$ together with $\mathcal{P}'$ forms a Berge cycle of length at least 4 in $\mathcal{H}$, a contradiction. Therefore, $D'$ contains no vertex outside $D$; so $D$ induces a block of $\partial_2(\mathcal{H})$ (and contains $|D|-2$ hyperedges of $\mathcal{H}$), as desired.

Let $D_1, D_2, \ldots, D_p$ be the unique decomposition of $\partial_2(\mathcal{H})$ into 2-connected blocks. In **Case 1** and **Case 2** we showed that one of these blocks, (say) $D_1 = D$ is such that $\mathcal{H}[D_1]$ contains $|D_1|-2$ hyperedges of $\mathcal{H}$, otherwise, $D_1$ is a set of 4 vertices such that $\mathcal{H}[D_1]$ contains exactly $|D_1|-1 = 3$ hyperedges of $\mathcal{H}$. In all these cases, note that $e(\mathcal{H}[D_1]) \leq |D_1|-1$.

Let us contract the vertices of $D_1$ to a single vertex, to produce a new hypergraph $\mathcal{H}'$. Then it is clear that the block decomposition of $\partial_2(\mathcal{H}')$ consists of the blocks $D_2, \ldots, D_p$. So $\mathcal{H}'$ does not contain any Berge cycle of length 4 or longer, as well; moreover, $|V(\mathcal{H}')| = |V(\mathcal{H})| - |D_1| + 1$ and $e(\mathcal{H}') = e(\mathcal{H}) - e(\mathcal{H}[D_1])$. By induction, we have $e(\mathcal{H}') \leq |V(\mathcal{H}')|-1$. Therefore,

$$e(\mathcal{H}) = e(\mathcal{H}') + e(\mathcal{H}[D_1]) \leq |V(\mathcal{H}')|-1+|D_1|-1 = (|V(\mathcal{H})|-|D_1|+1)-1+|D_1|-1 = |V(\mathcal{H})|-1.$$

If $e(\mathcal{H}) = |V(\mathcal{H})|-1$, then we must have $e(\mathcal{H}') = |V(\mathcal{H}')|-1$ and $\mathcal{H}[D_1]$ must contain exactly $|D_1|-1$ hyperedges. As noted before, this is only possible if $D_1$ has 4 vertices and induces exactly 3 hyperedges of $\mathcal{H}$. Moreover, since equality holds for $\mathcal{H}'$, by induction, $\partial_2(\mathcal{H}')$ is connected and for each block $D_i$ (with $2 \leq i \leq p$) of $\partial_2(\mathcal{H}')$, $D_i = K_4$ and $\mathcal{H}'[D_i]$ contains exactly 3 hyperedges. This means for every block $D$ of $\partial_2(\mathcal{H})$, we have $D = K_4$ and $\mathcal{H}[D]$ contains exactly 3 hyperedges of $\mathcal{H}$, completing the proof in the case $r = 3$.

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