Model-theoretic characterization of intuitionistic propositional formulas*

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Abstract. Notions of k-asimulation and asimulation are introduced as asymmetric counterparts to k-bisimulation and bisimulation, respectively. It is proved that a first-order formula is equivalent to a standard translation of an intuitionistic propositional formula iff it is invariant with respect to k-asimulations for some k, and then that a first-order formula is equivalent to a standard translation of an intuitionistic propositional formula iff it is invariant with respect to asimulations. Finally, it is proved that a first-order formula is intuitionistically equivalent to a standard translation of an intuitionistic propositional formula iff it is invariant with respect to asimulations between intuitionistic models.

Keywords. model theory, intuitionistic logic, propositional logic, bisimulation, Van Benthem's theorem.

Van Benthem's well-known modal characterization theorem (Theorem 3 below) states that a first-order formula is equivalent to a standard translation of a modal propositional formula iff it is invariant with respect to bisimulations. There is also a weaker 'parametrized' version of this result stating that a first-order formula is equivalent to a standard translation of a modal propositional formula iff this formula is invariant with respect to k-bisimulations for some k. Although both results yield a convenient model-theoretical technique distinguishing 'modal' first-order formulas from 'non-modal' ones, Van Benthem's characterization theorem, unlike its parametrized

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version, also isolates a single property defining expressive powers of modal propositional logic and thus gives us an important insight into its nature when this logic is viewed as a fragment of first-order logic.

It is somewhat surprising that results analogous to Van Benthem's modal characterization theorem and its parametrized version were not obtained thus far for the intuitionistic propositional logic, although the view of the latter as a fragment of modal propositional logic has a long and established tradition dating back to Tarski-Gödel translation of this logic into $S4$. The present paper fills this gap.

The layout of the paper is as follows. Starting from some notational conventions and preliminary remarks in section 1, we then move on to the proof of a 'parametrized' version of model-theoretic characterization of intuitionistic propositional logic in section 2 and finally prove the full unparametrized counterpart to Van Benthem's characterization theorem for intuitionistic propositional logic in section 3. From this latter result we derive a characterization of equivalence of a first-order formula to a standard translation of intuitionistic formula on the class of intuitionistic models. This latter result is of special interest, given that, unlike in the case of modal propositional logic, not every first-order model can be treated as a model of intuitionistic propositional logic. Finally, in section 4 we sum up and state some directions for further research.

1 Preliminaries

A formula is a formula of classical predicate logic with identity whose predicate letters are in vocabulary $\Sigma = \{ R^2, P_1^1, \dots P_n^1, \dots \}$. A model is a model of this logic. We refer to formulas with lower-case Greek letters distinct from α and β, and to sets of formulas with upper-case Greek letters distinct from Σ. If φ is a formula, then we associate with it the following finite vocabulary $\Sigma_\varphi \subseteq \Sigma$ such that $\Sigma_\varphi = \{ R^2 \} \cup \{ P_i \mid P_i \text{ occurs in } \varphi \}$. If ψ is a formula, $\Sigma' \subseteq \Sigma$ and every predicate letter occurring in ψ is in Σ', then we call ψ a Σ'-formula.

We refer to sequence $x_1, \dots, x_n$ of any objects as $\bar{x}_n$. If all free variables of a formula φ coincide with a variable x, we write $\varphi(x)$. If all free variables of formulas in Γ coincide with x, we write $\Gamma(x)$. We refer to the domain of a model M by $D(M)$. A pointed model is a pair (M, a), where M is a first-order model and $a \in D(M)$. If (M, a) is a pointed model, we write $M, a \models \varphi(x)$ and say that $\varphi(x)$ is true at (M, a) iff for any variable assignment f in M such that $f(x) = a$, we have $M, f \models \varphi(x)$. It follows from this convention that truth of a formula $\varphi(x)$ at a pointed model is to some extent independent from the choice of its only free variable.

An intuitionistic formula is a formula of intuitionistic propositional logic. We refer to intuitionistic formulas with letters i, j, k, possibly with primes or subscripts. We assume a standard Kripke semantics for intuitionistic propositional logic.

If x is an individual variable in a first-order language, then by standard x-translation of intuitionistic formulas into formulas we mean the following map ST defined by induction on the complexity of the corresponding intuitionistic formula. The induction goes as follows:

$$\begin{aligned}
&ST(p_n, x) = P_n(x);\\
&ST(\bot, x) = (x \neq x);\\
&ST(i \wedge j, x) = ST(i, x) \wedge ST(j, x);\\
&ST(i \vee j, x) = ST(i, x) \vee ST(j, x);\\
&ST(i \to j, x) = \forall y(R(x, y) \to (ST(i, y) \to ST(j, y))).
\end{aligned}$$

Standard conditions are imposed on the variables x, y.

By degree of a formula we mean the greatest number of nested quantifiers occurring in it. Degree of a formula φ is denoted by $r(\varphi)$. Its formal definition by induction on the complexity of φ goes as follows:

$$\begin{aligned}
&r(\varphi) = 0 && \text{for atomic } \varphi\\
&r(\neg\varphi) = r(\varphi) &&\\
&r(\varphi \circ \psi) = max(r(\varphi), r(\psi)) && \text{for } \circ \in \{\wedge, \vee, \to\}\\
&r(Qx\varphi) = r(\varphi) + 1 && \text{for } Q \in \{\forall, \exists\}
\end{aligned}$$

If $\Sigma' \subseteq \Sigma$, $k \in \mathbb{N}$ and $\varphi(x)$ is a Σ'-formula such that $r(\varphi) \leq k$, then φ is a (Σ', x, k)-formula.

2 A parametrized version of the main result

We start with the definition of an 'intuitionistic' counterpart of k-bisimulation.

Definition 1. *Let $\Sigma' \subseteq \Sigma$, $R^2 \in \Sigma'$, (M, a), (N, b) be two pointed Σ'-models. A binary relation*

$$A \subseteq \bigcup_{n>0} ((D(M)^n \times D(N)^n) \cup (D(N)^n \times D(M)^n)),$$

is called $\langle (M, a), (N, b)\rangle_k$-asimulation iff $(a)A(b)$ and for any $\alpha, \beta \in \{M, N\}$, any sequence $(\bar{a'}_m, a') \in D(\alpha)^{m+1}$ and any sequence $(\bar{b'}_m, b') \in D(\beta)^{m+1}$, whenever we have $(\bar{a'}_m, a')A(\bar{b'}_m, b')$, the following conditions hold:

$$\forall P \in \Sigma'(\alpha, a' \models P(x) \Rightarrow \beta, b' \models P(x)) \tag{1}$$

$$\begin{aligned}
&b'' \in D(\beta) \wedge b'R^{\beta}b'' \wedge m < k \Rightarrow\\
&\quad\Rightarrow \exists a'' \in D(\alpha)(a'R^{\alpha}a'' \wedge (\bar{b'}_m, b', b'')A(\bar{a'}_m, a', a'') \wedge (\bar{a'}_m, a', a'')A(\bar{b'}_m, b', b''))
\end{aligned} \tag{2}$$

Example 1. *Consider two $\{R^2, P^1\}$-models M and N such that $D(M) = \{a, b, c\}$, $R^M = \{(a, b), (a, c)\}$, $P^M = \{c\}$, and $D(N) = \{d, e\}$, $R^N = \{(d, e)\}$, $P^N = \{d\}$. Then binary relation A such that $(a)A(d)$, $(d, e)A(a, b)$ and $(a, b)A(d, e)$ is an $\langle (M, a), (N, d)\rangle_k$-asimulation for every $k \in \mathbb{N}$.*

Lemma 1. *Let $\varphi(x) = ST(i, x)$ for some intuitionistic formula i, and let $r(\varphi) = k$. Let $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, (M, a), (N, b) be two pointed Σ'-models, let A be an $\langle (M, a), (N, b)\rangle_l$-asimulation. Then*

$$\begin{aligned}
&\forall \alpha, \beta \in \{M, N\} \forall (\bar{a'}_m, a') \in D(\alpha)^{m+1} \forall (\bar{b'}_m, b') \in D(\beta)^{m+1}\\
&\qquad((\bar{a'}_m, a')A(\bar{b'}_m, b') \wedge m + k \leq l \wedge \alpha, a' \models \varphi(x) \Rightarrow \beta, b' \models \varphi(x)).
\end{aligned}$$

Proof. We proceed by induction on the complexity of i. In what follows we will abbreviate the induction hypothesis by IH.

Basis. Let $i = p_n$. Then $\varphi(x) = P_n(x)$ and we reason as follows:

$(\bar{a'}_m, a')A(\bar{b'}_m, b')$	(premise)	(3)
$\alpha, a' \models P_n(x)$	(premise)	(4)
$P_n \in \Sigma'$	(by $\Sigma_\varphi \subseteq \Sigma'$)	(5)
$\forall P \in \Sigma'(\alpha, a' \models P(x) \Rightarrow \beta, b' \models P(x))$	(from (3) by (1))	(6)
$\alpha, a' \models P_n(x) \Rightarrow \beta, b' \models P_n(x)$	(from (5) and (6))	(7)
$\beta, b' \models P_n(x)$	(from (4) and (7))	(8)

The case $i = \bot$ is obvious.

Induction step.

Case 1. Let $i = j \wedge k$. Then $\varphi(x) = ST(j,x) \wedge ST(k,x)$ and we reason as follows:

$(\bar{a'}_m, a')A(\bar{b'}_m, b')$	(premise)	(9)
$\alpha, a' \models ST(j,x) \wedge ST(k,x)$	(premise)	(10)
$m + r(ST(j,x) \wedge ST(k,x)) \leq l$	(premise)	(11)
$r(ST(j,x)) \leq r(ST(j,x) \wedge ST(k,x))$	(by df of r)	(12)
$r(ST(k,x)) \leq r(ST(j,x) \wedge ST(k,x))$	(by df of r)	(13)
$\alpha, a' \models ST(j,x)$	(from (10))	(14)
$\alpha, a' \models ST(k,x)$	(from (10))	(15)
$m + r(ST(j,x)) \leq l$	(from (11) and (12))	(16)
$m + r(ST(k,x)) \leq l$	(from (11) and (13))	(17)
$\beta, b' \models ST(j,x)$	(from (9), (14) and (16) by IH)	(18)
$\beta, b' \models ST(k,x)$	(from (9), (15) and (17) by IH)	(19)
$\beta, b' \models ST(j,x) \wedge ST(k,x)$	(from (18) and (19))	(20)

Case 2. Let $i = j \vee k$. Then $\varphi(x) = ST(j,x) \vee ST(k,x)$ and we have then $\alpha, a' \models ST(j,x) \vee ST(k,x)$. Assume, without a loss of generality, that $\alpha, a' \models ST(j,x)$. Then we reason as follows:

$\alpha, a' \models ST(j,x)$	(premise)	(21)
$(\bar{a'}_m, a')A(\bar{b'}_m, b')$	(premise)	(22)
$m + r(ST(j,x) \vee ST(k,x)) \leq l$	(premise)	(23)
$r(ST(j,x)) \leq r(ST(j,x) \vee ST(k,x))$	(by df of r)	(24)
$m + r(ST(j,x)) \leq l$	(from (23) and (24))	(25)
$\beta, b' \models ST(j,x)$	(from (21), (22) and (25) by IH)	(26)
$\beta, b' \models ST(j,x) \vee ST(k,x)$	(from (26))	(27)

Case 3. Let $i = j \to k$. Then

$$\varphi(x) = \forall y(R(x,y) \to (ST(j,y) \to ST(k,y))).$$

Let

$$\alpha, a' \models \forall y(R(x,y) \rightarrow (ST(j,y) \rightarrow (ST(k,y))),$$

and let

$$\beta, b' \models \exists y(R(x,y) \wedge (ST(j,y) \wedge \neg ST(k,y))).$$

This means that we can choose a $b'' \in D(\beta)$ such that $b'R^{\beta}b''$ and $\beta, b'' \models ST(j,y) \wedge \neg ST(k,y)$. We now reason as follows:

$$\beta, b'' \models ST(j,y) \wedge \neg ST(k,y) \quad \text{(by choice of } b''\text{)} \quad (28)$$

$$b'' \in D(\beta) \wedge b'R^{\beta}b'' \quad \text{(by choice of } b''\text{)} \quad (29)$$

$$(\bar{a'}_m, a')A(\bar{b'}_m, b') \quad \text{(premise)} \quad (30)$$

$$m + r(\varphi(x)) \leq l \quad \text{(premise)} \quad (31)$$

$$r(\varphi(x)) \geq 1 \quad \text{(by df of } r\text{)} \quad (32)$$

$$m < l \quad \text{(from (31) and (32))} \quad (33)$$

$$\exists a'' \in D(\alpha)(a'R^{\alpha}a'' \wedge (\bar{b'}_m, b', b'')A(\bar{a'}_m, a', a'') \wedge (\bar{a'}_m, a', a'')A(\bar{b'}_m, b', b'')) \quad (34)$$

(from (29), (30) and (33) by(2))

Now choose an a'' for which (34) is satisfied; we add the premises following from our choice of a'' and continue our reasoning as follows:

$$a'' \in D(\alpha) \wedge a'R^{\alpha}a'' \quad \text{(by choice of } a''\text{)} \quad (35)$$

$$(\bar{b'}_m, b', b'')A(\bar{a'}_m, a', a'') \quad \text{(by choice of } a''\text{)} \quad (36)$$

$$(\bar{a'}_m, a', a'')A(\bar{b'}_m, b', b'') \quad \text{(by choice of } a''\text{)} \quad (37)$$

$$r(ST(j,y)) \leq r(\varphi(x)) - 1 \quad \text{(by df of } r\text{)} \quad (38)$$

$$r(ST(k,y)) \leq r(\varphi(x)) - 1 \quad \text{(by df of } r\text{)} \quad (39)$$

$$m + 1 + r(ST(j,y)) \leq l \quad \text{(from (31) and (38))} \quad (40)$$

$$m + 1 + r(ST(k,y)) \leq l \quad \text{(from (31) and (39))} \quad (41)$$

$$\alpha, a'' \models ST(j,x) \quad \text{(from (28), (36), (40) by IH)} \quad (42)$$

$$\alpha, a'' \models \neg ST(k,x) \quad \text{(from (28), (37), (41) by IH)} \quad (43)$$

$$\alpha, a'' \models ST(j,y) \wedge \neg ST(k,y) \quad \text{(from (42), (43))} \quad (44)$$

$$\alpha, a' \models \exists y(R(x,y) \wedge (ST(j,y) \wedge \neg ST(k,y))) \quad \text{(from (35) and (44))} \quad (45)$$

The last line contradicts our initial assumption that

$$\alpha, a' \models \forall y(R(x,y) \rightarrow (ST(j,y) \rightarrow (ST(k,y))).$$

□

Definition 2. *A formula $\varphi(x)$ is invariant with respect to k-asimulations iff for any Σ' such that $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, any pointed Σ'-models (M, a) and (N, b), if there exists an $\langle(M, a), (N, b)\rangle_k$-asimulation A and $M, a \models \varphi(x)$, then $N, b \models \varphi(x)$.*

Example 2. *Consider again models M and N and binary relation A from Example 1. Formula $\exists y(R(x,y) \wedge P(y))$ is true at (M, a), but not at (N, d). So, since for every $k \in \mathbb{N}$ A is an $\langle(M, a), (N, d)\rangle_k$-asimulation, we get that there is no k such that this formula is invariant with respect to k-asimulations.*

Corollary 1. *If $\varphi(x)$ is a standard x-translation of an intuitionistic formula and $r(\varphi) = k$, then $\varphi(x)$ is invariant with respect to k-asimulations.*

Corollary 1 immediately follows from Lemma 1 setting $\alpha = M$, $\beta = N$, $m = 0$, $l = k$, $a' = a$ and $b' = b$.

Before we state and prove the parametrized version of our main result, we need to mention a fact from the classical model theory of first-order logic.

Lemma 2. *For any finite predicate vocabulary Σ', any variable x and any natural k there are, up to logical equivalence, only finitely many (Σ', x, k)-formulas.*

This fact is proved as Lemma 3.4 in [Ebbinghaus et al. 1984, pp. 189–190].

Definition 3. *Let $\varphi(x)$ be a formula. A conjunction of (Σ_φ, x, k)-formulas $\Psi(x)$ is called a complete (φ, x, k)-conjunction iff (1) every conjunct in $\Psi(x)$ is a standard x-translation of an intuitionistic formula; and (2) there is a pointed model (M, a) such that $M, a \models \Psi(x) \wedge \varphi(x)$ and for any (Σ_φ, x, k)-formula $\psi(x)$, if $\psi(x)$ is a standard x-translation of an intuitionistic formula and $M, a \models \psi(x)$, then $\Psi(x) \models \psi(x)$.*

Lemma 3. *For any formula $\varphi(x)$, any natural k, any Σ' such that $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$ and any pointed Σ'-model (M, a) such that $M, a \models \varphi(x)$ there is a complete (φ, x, k)-conjunction $\Psi(x)$ such that $M, a \models \Psi(x) \wedge \varphi(x)$.*

Proof. Let $\{\, \psi_1(x) \dots, \psi_n(x), \dots \}$ be the set of all (Σ_φ, x, k)-formulas that are standard x-translations of intuitionistic formulas true at (M, a). This set is non-empty since $ST(\bot \to \bot, x)$ will be true at (M, a). Due to Lemma 2, we can choose in this set a non-empty finite subset $\{\, \psi_{i_1}(x) \dots, \psi_{i_n}(x) \,\}$ such that any formula from the bigger set is logically equivalent to (and hence follows from) a formula in this subset. Therefore, every formula in the bigger set follows from $\psi_{i_1}(x) \wedge \dots \wedge \psi_{i_n}(x)$ and we also have $M, a \models \psi_{i_1}(x) \wedge \dots \wedge \psi_{i_n}(x)$, therefore, $\psi_{i_1}(x) \wedge \dots \wedge \psi_{i_n}(x)$ is a complete (φ, x, k)-conjunction. □

Lemma 4. *For any formula $\varphi(x)$ and any natural k there are, up to logical equivalence, only finitely many complete (φ, x, k)-conjunctions.*

Proof. It suffices to observe that for any formula $\varphi(x)$ and any natural k, a complete (φ, x, k)-conjunction is a (Σ_φ, x, k)-formula. Our lemma then follows from Lemma 2. □

In what follows we adopt the following notation for the fact that for any variable x all (Σ_φ, x, k)-formulas that are standard x-translations of intuitionistic formulas true at (M, a), are also true at (N, b):

$$(M, a) \leq_{\varphi,k} (N, b).$$

Theorem 1. *Let $r(\varphi(x)) = k$ and let $\varphi(x)$ be invariant with respect to k-asimulations. Then $\varphi(x)$ is equivalent to a standard x-translation of an intuitionistic formula.*

Proof. We may assume that both $\varphi(x)$ and $\neg\varphi(x)$ are satisfiable, since both $\bot$ and $\top$ are obviously invariant with respect to k-asimulations and we have, for example, the following valid formulas:

$$\bot \leftrightarrow ST(\bot, x), \top \leftrightarrow ST(\bot \to \bot, x).$$

We may also assume that there are two complete $(\varphi, x, k+2)$-conjunctions $\Psi(x), \Psi'(x)$ such that $\Psi'(x) \models \Psi(x)$, and both formulas $\Psi(x) \wedge \varphi(x)$ and $\Psi'(x) \wedge \neg\varphi(x)$ are satisfiable.

For suppose otherwise. Then take the set of all complete $(\varphi, x, k+2)$-conjunctions $\Psi(x)$ such that the formula $\Psi(x) \wedge \varphi(x)$ is satisfiable. This set is non-empty, because $\varphi(x)$ is satisfiable, and by Lemma 3, it can be satisfied only together with some complete $(\varphi, x, k+2)$-conjunction. Now, using Lemma 4, choose in it a finite non-empty subset $\{ \Psi_{i_1}(x) \ldots, \Psi_{i_n}(x) \}$ such that any complete $(\varphi, x, k+2)$-conjunction is equivalent to an element of this subset. We can show that $\varphi(x)$ is logically equivalent to $\Psi_{i_1}(x) \vee \ldots \vee \Psi_{i_n}(x)$. In fact, if $M, a \models \varphi(x)$ then, by Lemma 3, at least one complete $(\varphi, x, k+2)$-conjunction is true at (M, a) and therefore, its equivalent in $\{ \Psi_{i_1}(x) \ldots, \Psi_{i_n}(x) \}$ is also true at (M, a), and so, finally we have $M, a \models \Psi_{i_1}(x) \vee \ldots \vee \Psi_{i_n}(x)$. In the other direction, if $M, a \models \Psi_{i_1}(x) \vee \ldots \vee \Psi_{i_n}(x)$, then for some $1 \leq j \leq n$ we have $M, a \models \Psi_{i_j}(x)$. Then, since $\Psi_{i_j}(x) \models \Psi_{i_j}(x)$ and by the choice of $\Psi_{i_j}(x)$ the formula $\Psi_{i_j}(x) \wedge \varphi(x)$ is satisfiable, so, by our assumption, the formula $\Psi_{i_j}(x) \wedge \neg\varphi(x)$ must be unsatisfiable, and hence $\varphi(x)$ must follow from $\Psi_{i_j}(x)$. But in this case we will have $M, a \models \varphi(x)$ as well. So $\varphi(x)$ is logically equivalent to $\Psi_{i_1}(x) \vee \ldots, \vee \Psi_{i_n}(x)$ but the latter formula, being a disjunction of conjunctions of standard x-translations of intuitionistic formulas, is itself a standard x-translation of an intuitionistic formula, and so we are done.

If, on the other hand, one can take two complete $(\varphi, x, k+2)$-conjunctions $\Psi(x), \Psi'(x)$ such that $\Psi'(x) \models \Psi(x)$, and formulas $\Psi(x) \wedge \varphi(x)$ and $\Psi'(x) \wedge \neg\varphi(x)$ are satisfiable, we reason as follows. Take a pointed Σ_φ-model (M, a) such that $M, a \models \Psi(x) \wedge \varphi(x)$ and for any $(\Sigma_\varphi, x, k+2)$-formula $\psi(x)$, if $\psi(x)$ is a standard x-translation of an intuitionistic formula true at (M, a), then $\psi(x)$ follows from $\Psi(x)$, and take any pointed model (N, b) such that $N, b \models \Psi'(x) \wedge \neg\varphi(x)$.

We can construct an $\langle (M, a), (N, b) \rangle_k$-asimulation and thus obtain a contradiction in the following way.

Let $\alpha, \beta \in \{ M, N \}$ and let $(\bar{a'}_m, a')$ and $(\bar{b'}_m, b')$ be in $D(\alpha)^{m+1}$ and $D(\beta)^{m+1}$, respectively. Then

$$(\bar{a'}_m, a') A (\bar{b'}_m, b') \Leftrightarrow (m \leq k \wedge (\alpha, a') \leq_{\varphi, k-m+2} (\beta, b')).$$

By choice of $\Psi(x), \Psi'(x)$ and the independence of truth at a pointed model from the choice of a single free variable in a formula we obviously have $(a)A(b)$.

Further, since the degree of any atomic formula is 0, and the above condition implies that $k - m + 2 \geq 2$, it is evident that for any $(\bar{a'}_m, a') A (\bar{b'}_m, b')$ and any predicate letter $P \in \Sigma_\varphi$ we have $\alpha, a' \models P(x) \Rightarrow \beta, b' \models P(x)$.

To verify condition (2), take any $(\bar{a'}_m, a') A (\bar{b'}_m, b')$ such that $m < k$ and any $b'' \in D(\beta)$ such that $b' R^\beta b''$. In this case we will also have $m + 1 \leq k$.

Then consider the following two sets:

$$\Gamma = \{ ST(i, x) \mid ST(i, x) \text{ is a } (\Sigma_\varphi, x, k+1-m)\text{-formula, and } \beta, b'' \models ST(i, x) \};$$
$$\Delta = \{ ST(i, x) \mid ST(i, x) \text{ is a } (\Sigma_\varphi, x, k+1-m)\text{-formula, and } \beta, b'' \models \neg ST(i, x) \}.$$

These sets are non-empty, since by our assumption we have $k+1-m \geq 1$. Therefore, as we have $r(ST(\bot, x)) = 0$ and $r(ST(\bot \to \bot, x)) = 1$, we will also have $ST(\bot, x) \in \Delta$ and $ST(\bot \to \bot, x) \in \Gamma$. Then, according to our Lemma 2, there are finite non-empty sets of logical equivalents for both Γ and Δ. Choosing these finite sets, we in fact choose some finite $\{ ST(i_1, x) \ldots ST(i_t, x) \} \subseteq \Gamma$, $\{ ST(j_1, x) \ldots ST(j_u, x) \} \subseteq \Delta$ such that

$$\forall \psi(x) \in \Gamma(ST(i_1, x) \wedge \ldots \wedge ST(i_t, x) \models \psi(x));$$
$$\forall \chi(x) \in \Delta(\chi(x) \models ST(j_1, x) \vee \ldots \vee ST(j_u, x)).$$

But then we obtain that the formula

$$ST((i_1 \wedge \ldots \wedge i_t) \to (j_1 \vee \ldots \vee j_u), x)$$

is false at (β, b'). In fact, b'' disproves this implication for (β, b'). But every formula both in $\{ ST(i_1, x) \ldots ST(i_t, x) \}$ and $\{ ST(j_1, x) \ldots ST(j_u, x) \}$ is, by their choice, a $(\Sigma_\varphi, x, k+1-m)$-formula, and so the implication under consideration must be a $(\Sigma_\varphi, x, k+2-m)$-formula. Note, further, that by $(\bar{a'}_m, a')A(\bar{b'}_m, b')$ we have

$$(\alpha, a') \leq_{\varphi, k-m+2} (\beta, b')$$

and therefore this implication must be false at (α, a') as well. But then take any $a'' \in D(\alpha)$ such that $a' R^\alpha a''$ and a'' verifies the conjunction in the antecedent of the formula but falsifies its consequent. We must conclude then, by the choice of $\{ ST(i_1, x) \ldots ST(i_t, x) \}$, that $\alpha, a'' \models \Gamma$ and so, by the definition of A, and given that $m+1 \leq k$, that $(\bar{b'}_m, b', b'')A(\bar{a'}_m, a', a'')$. Since, in addition, a'' falsifies every formula from $\{ ST(j_1, x) \ldots ST(j_u, x) \}$, then, by the choice of this set, we must conclude that every $(\Sigma_\varphi, x, k+1-m)$-formula that is a standard x-translation of an intuitionistic formula false at (β, b'') is also false at (α, a''). But then, again by the definition of A, and given the fact that $m+1 \leq k$, we must also have $(\bar{a'}_m, a', a'')A(\bar{b'}_m, b', b'')$, and so condition (2) holds.

Therefore A is an $\langle (M, a), (N, b) \rangle_k$-asimulation and we have got our contradiction in place. □

Theorem 2. *A formula $\varphi(x)$ is equivalent to a standard x-translation of an intuitionistic formula iff there exists a $k \in \mathbb{N}$ such that $\varphi(x)$ is invariant with respect to k-asimulations.*

Proof. Let $\varphi(x)$ be equivalent to $ST(i, x)$. Then by Corollary 1, $ST(i, x)$ is invariant with respect to $r(ST(i, x))$-asimulations, and, therefore, so is $\varphi(x)$. In the other direction, let $\varphi(x)$ be invariant with respect to k-asimulations for some k. If $k \leq r(\varphi)$, then every $r(\varphi)$-asimulation is k-asimulation, so $\varphi(x)$ is invariant with respect to $r(\varphi)$-asimulations and hence, by Theorem 1, $\varphi(x)$ is equivalent to a standard x-translation of an intuitionistic formula. If, on the other hand, $r(\varphi) < k$, then set $l = k - r(\varphi)$ and consider variables $\bar{y}_l$ not occurring in $\varphi(x)$. Then $r(\forall \bar{y}_l \varphi(x)) = k$ and $\varphi(x)$ is logically equivalent to $\forall \bar{y}_l \varphi(x)$, so the latter formula is also invariant with respect to k-asimulations, and hence by Theorem 1 $\forall \bar{y}_l \varphi(x)$ is logically equivalent to a standard x-translation of an intuitionistic formula. But then $\varphi(x)$ is equivalent to this standard x-translation as well. □

3 The main result

We begin with a definition of an ‘intuitionistic’ counterpart to bisimulation:

Definition 4. *Let $\Sigma' \subseteq \Sigma$, $R^2 \in \Sigma'$, (M, a), (N, b) be two pointed Σ'-models. A binary relation*

$$A \subseteq (D(M) \times D(N)) \cup (D(N) \times D(M)),$$

is called $\langle (M, a), (N, b) \rangle$-asimulation iff aAb and for any $\alpha, \beta \in \{ M, N \}$, any $a' \in D(\alpha)$, $b' \in D(\beta)$ whenever we have $a'Ab'$, the following conditions hold:

$$\forall P \in \Sigma'(\alpha, a' \models P(x) \Rightarrow \beta, b' \models P(x)) \tag{46}$$

$$b'' \in D(\beta) \wedge b'R^{\beta}b'' \Rightarrow \exists a'' \in D(\alpha)(a'R^{\alpha}a'' \wedge b''Aa'' \wedge a''Ab'') \tag{47}$$

Example 3. *Consider again models M and N from Example 1. Binary relation $B = \{ (a, d), (b, e), (e, b) \}$ is an $\langle (M, a), (N, d) \rangle$-asimulation.*

Lemma 5. *Let A be an $\langle (M, a), (N, b) \rangle$-asimulation, and let*

$$A' = \{ \langle (\bar{c}_n, c'), (\bar{d}_n, d') \rangle \mid c'Ad' \}.$$

Then A' is an $\langle (M, a), (N, b) \rangle_k$-asimulation for any $k \in \mathbb{N}$.

Proof. We obviously have $(a)A'(b)$, and since for any $\alpha, \beta \in \{ M, N \}$, and any $(\bar{c}_n, c')$ in $D(\alpha)^{n+1}$, $(\bar{d}_n, d')$ in $D(\beta)^{n+1}$ such that $(\bar{c}_n, c')A'(\bar{d}_n, d')$ we have $c'Ad'$, condition (1) for A' follows from the fulfilment of condition (46) for A. Also, if $(\bar{c}_n, c')A'(\bar{d}_n, d')$ then $c'Ad'$, and if, further, $d'' \in D(\beta)$ and $d'R^{\beta}d''$ then by condition (47) we can choose $c'' \in D(\alpha)$ such that $c'R^{\alpha}c''$, $c''Ad''$ and $d''Ac''$. But then, by definition of A' we will also have $(\bar{c}_n, c', c'')A'(\bar{d}_n, d', d'')$ and $(\bar{d}_n, d', d'')A'(\bar{c}_n, c', c'')$ so condition (2) for A' is fulfilled for every k. □

Definition 5. *A formula $\varphi(x)$ is invariant with respect to asimulations iff for any Σ' such that $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, any pointed Σ'-models (M, a) and (N, b), if there exists an $\langle (M, a), (N, b) \rangle$-asimulation A and $M, a \models \varphi(x)$, then $N, b \models \varphi(x)$.*

Example 4. *Consider again models M and N from Example 1. Since $\exists y(R(x, y) \wedge P(y))$ is true at (M, a), but not at (N, d), the fact that binary relation B from Example 3 is an $\langle (M, a), (N, d) \rangle$-asimulation means that this formula is not invariant with respect to asimulations.*

Corollary 2. *If $\varphi(x)$ is equivalent to a standard x-translation of an intuitionistic formula, then $\varphi(x)$ is invariant with respect to asimulations.*

Proof. Let $\varphi(x)$ be equivalent to a standard x-translation of an intuitionistic formula, let A be an $\langle (M, a), (N, b) \rangle$-asimulation and let A' be defined as in Lemma 5. Then by this Lemma A' is an $\langle (M, a), (N, b) \rangle_k$-asimulation for every k. So if we have $M, a \models \varphi(x)$, but not $N, b \models \varphi(x)$, then $\varphi(x)$ is not invariant with respect to k-asimulations for any k, which is in contradiction with Theorem 2. □

In what follows we will also need some notions and facts from model theory of modal propositional logic. Thus, standard modal x-translation $Tr(m, x)$ of a modal propositional formula m in first-order logic is defined by the following induction on the complexity of modal propositional formula:

$$Tr(p_n, x) = P_n(x);$$
$$Tr(m \wedge m', x) = Tr(m, x) \wedge Tr(m', x);$$
$$Tr(\neg m, x) = \neg Tr(m, x);$$
$$Tr(\Box m, x) = \forall y(R(x, y) \to Tr(m, y)).$$

Another important idea is the notion of bisimulation:

Definition 6. *Let Σ' be a predicate vocabulary such that $\Sigma' \subseteq \Sigma$, $R^2 \in \Sigma'$, and (M, a), (N, b) be pointed Σ'-models. Then a binary relation $E \subseteq D(M) \times D(N)$ is a $\langle(M, a), (N, b)\rangle$-bisimulation iff aEb and for any $a' \in M$, $b' \in N$, whenever $a'Eb'$, the following conditions hold:*

$$\forall P \in \Sigma'(M, a' \models P(x) \Leftrightarrow N, b' \models P(x)); \quad (48)$$
$$(a'' \in D(M) \wedge a'R^M a'') \Rightarrow \exists b'' \in D(N)(b'R^N b'' \wedge a''Eb''); \quad (49)$$
$$(b'' \in D(N) \wedge b'R^N b'') \Rightarrow \exists a'' \in D(M)(a'R^M a'' \wedge a''Eb''). \quad (50)$$

Definition 7. *A formula $\varphi(x)$ is invariant with respect to bisimulations iff for any Σ' such that $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, any pointed Σ'-models (M, a) and (N, b), and any $\langle(M, a), (N, b)\rangle$-bisimulation it is true that*

$$M, a \models \varphi(x) \Rightarrow N, b \models \varphi(x).$$

The concept of standard modal translation and that of bisimulation invariance are tied together by Van Benthem's famous modal characterization theorem:

Theorem 3. *A formula $\varphi(x)$ is invariant with respect to bisimulations iff it is equivalent to a standard modal x-translation of a modal propositional formula.*

Its proof can be found, for example, in [Blackburn et al. 2001, Theorem 2.68, pp. 103–104]. It is easy to see that our main result below (Theorem 5) is in an analogy with Van Benthem's characterization theorem for intuitionistic propositional logic both in its formulation and in methods of proof employed.

Lemma 6. *Let $\varphi(x)$ be a formula invariant with respect to asimulations. Then:*

1. *$\varphi(x)$ is invariant with respect to bisimulations.*

2. *$\neg\varphi(x)$ is invariant with respect to bisimulations.*

Proof. (1) Let $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, let (M, a), (N, b) be pointed Σ'-models and let E be an $\langle(M, a), (N, b)\rangle$-bisimulation such that $M, a \models \varphi(x)$ but not $N, b \models \varphi(x)$. Then define A as $E \cup E^{-1}$. It is easy to verify that A is an $\langle(M, a), (N, b)\rangle$-asimulation: we obviously have aAb, and condition (46) is fulfilled.

To verify (47), assume that $a'Ab'$. Then either $a' \in D(M) \wedge b' \in D(N)$ or $a' \in D(N) \wedge b' \in D(M)$. So in the former case, by Definition 6 and our definition of A, we must have $a'Eb'$, while in the latter case we must have $b'Ea'$. Therefore, in the former case, if $b'R^Nb''$ we apply condition (50) and choose $a'' \in D(M)$ such that $a'R^Ma'' \wedge a''Eb''$, and so, by definition of A, we have both $a''Ab''$ and $b''Aa''$. In the latter case, if $b'R^Mb''$ we apply condition (49) and choose $a'' \in D(N)$ such that $a'R^Na'' \wedge b''Ea''$, and so, again by definition of A, we have both $b''Aa''$ and $a''Ab''$. Thus A is an $\langle(M,a),(N,b)\rangle$-asimulation and $\varphi(x)$ is not invariant with respect to asimulations, contrary to our assumption. The first statement of the lemma is proved.

(2) Let $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, let (M,a), (N,b) be pointed Σ'-models and let E be an $\langle(M,a),(N,b)\rangle$-bisimulation such that $N,b \models \varphi(x)$ but not $M,a \models \varphi(x)$. Again, define A as $E \cup E^{-1}$. In the previous paragraph it was established that A verifies conditions (46) and (47). But since aEb, we also have bAa and so A is in fact an $\langle(N,b),(M,a)\rangle$-asimulation, which contradicts our assumption that $\varphi(x)$ is invariant with respect to asimulations. □

Definition 8. *A model M is called m-saturated iff for any $a \in D(M)$ and for any set $\Theta(x)$ of standard modal x-translations of modal propositional formulas it is true that*

$$[\forall(\Theta'(x) \subseteq \Theta(x))(\Theta'(x) \text{ is finite} \Rightarrow \exists b \in D(M)(aR^Mb \wedge M,b \models \Theta'(x)))] \Rightarrow$$
$$\Rightarrow \exists c \in D(M)(aR^Mc \wedge M,c \models \Theta(x)).$$

Let $\Sigma' \subseteq \Sigma$. In what follows we adopt the following notation for the fact that for any x all Σ'-formulas that are standard x-translations of intuitionistic formulas true at (M,a), are also true at (N,b):

$$(M,a) \leq_{\Sigma'} (N,b).$$

Lemma 7. *Let $\Sigma' \subseteq \Sigma$, let M, N be m-saturated Σ'-models and let $(M,a) \leq_{\Sigma'} (N,b)$. Then relation $\leq_{\Sigma'}$ is an $\langle(M,a),(N,b)\rangle$-asimulation.*

Proof. It is obvious that aAb, and since for any unary predicate letter P and variable x formula $P(x)$ is a standard x-translation of an atomic intuitionistic formula, condition (46) is trivially satisfied for $\leq_{\Sigma'}$. To verify condition (47), choose any $\alpha,\beta \in \{M,N\}$, and $a' \in D(\alpha)$, $b',b'' \in D(\beta)$ such that $(\alpha,a') \leq_{\Sigma'} (\beta,b')$ and $b'R^\beta b''$. Then choose any variable x and consider the following two sets:

$$\Gamma = \{\, i \mid ST(i,x) \text{ is a } \Sigma'\text{-formula, and } \beta,b'',\models ST(i,x)\,\};$$
$$\Delta = \{\, i \mid ST(i,x) \text{ is a } \Sigma'\text{-formula, and } \beta,b'',\models \neg ST(i,x)\,\}.$$

We have by the choice of Γ, Δ that for every finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ the formula $ST(\bigwedge(\Gamma') \to \bigvee(\Delta'),x)$ is disproved by b'' for (β,b'). So, by our premise that $(\alpha,a') \leq_{\Sigma'} (\beta,b')$, the standard translation of every such implication must be false at (α,a') as well. This means that every finite subset of the set

$$\{\, ST(i,x) \mid i \in \Gamma\,\} \cup \{\, \neg ST(i,x) \mid i \in \Delta\,\}$$

is true at some $a'' \in D(\alpha)$ such that $a'R^\alpha a''$. (We set $\Delta' = \{\, ST(\bot,x)\,\}$ if the finite set in question has an empty intersection with Δ and $\Gamma' = \{\, ST(\bot \to \bot,x)\,\}$ if it has an empty intersection with Γ.) But by Corollary 2 and Lemma 6 every formula in the set under consideration is invariant with respect to bisimulations and hence equivalent to a standard modal x-translation of a modal propositional formula. Therefore, by m-saturation of both M and N there must be an $a'' \in D(\alpha)$ such that $a'R^\alpha a''$ and

$$\alpha, a'' \models \{\, ST(i,x) \mid i \in \Gamma \,\} \cup \{\, \neg ST(i,x) \mid i \in \Delta \,\}.$$

By choice of Γ and Δ and by the independence of truth at a pointed model from the choice of a single free variable in a formula we will have both $(\alpha, a'') \leq_{\Sigma'} (\beta, b'')$ and $(\beta, b'') \leq_{\Sigma'} (\alpha, a'')$ and so condition (47) is also verified. □

Lemma 8. *Let $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$ and let M be a Σ'-model. Then there is a Σ'-model N such that N is an extension of M, N is m-saturated and there is a map $f : D(M) \to D(N)$ such that for any formula $\varphi(x)$ which is invariant with respect to bisimulations and any $a \in D(M)$ it is true that*

$$M, a \models \varphi(x) \Leftrightarrow N, f(a) \models \varphi(x).$$

Proof. Let N be an ultrafilter extension of M and let $f(a)$ be the principal ultrafilter generated by a for any $a \in D(M)$. Then our lemma follows from Propositions 2.59 and 2.61 of [Blackburn et al. 2001, pp. 96-97] and Theorem 3. □

We are prepared now to state and prove our main result.

Theorem 4. *Let $\varphi(x)$ be invariant with respect to asimulations. Then $\varphi(x)$ is equivalent to a standard x-translation of an intuitionistic formula.*

Proof. We may assume that $\varphi(x)$ is satisfiable, for $\bot$ is clearly invariant with respect to asimulations and $\bot \leftrightarrow ST(\bot, x)$ is a valid formula. In what follows we will write $IC(\varphi(x))$ for the set of Σ_φ-formulas that are standard x-translations of intuitionistic formulas following from $\varphi(x)$. For any pointed Σ_φ-model (M, a) we will denote the set of Σ_φ-formulas that are standard x-translations of intuitionistic formulas true at (M, a), or *intuitionistic* Σ_φ*-theory* of (M, a) by $IT_\varphi(M, a)$. It is obvious that for any pointed Σ_φ-models (M, a) and (N, b) we will have $(M, a) \leq_{\Sigma_\varphi} (N, b)$ if and only if $IT_\varphi(M, a) \subseteq IT_\varphi(N, b)$.

Our strategy will be to show that $IC(\varphi(x)) \models \varphi(x)$. Once this is done we will apply compactness of first-order logic and conclude that $\varphi(x)$ is equivalent to a finite conjunction of standard x-translations of intuitionistic formulas and hence to a standard x-translation of the corresponding intuitionistic conjunction.

To show this, take any pointed Σ_φ-model (M, a) such that $M, a \models IC(\varphi(x))$. Such a model exists, because $\varphi(x)$ is satisfiable and $IC(\varphi(x))$ will be satisfied in any pointed model satisfying $\varphi(x)$. Then we can also choose a pointed Σ_φ-model (N, b) such that $N, b \models \varphi(x)$ and $IT_\varphi(N, b) \subseteq IT_\varphi(M, a)$.

For suppose otherwise. Then for any pointed Σ_φ-model (N, b) such that $N, b \models \varphi(x)$ we can choose an intuitionistic formula $i_{(N,b)}$ such that $ST(i_{(N,b)}, x)$ is a Σ_φ-formula true at (N, b) but not at (M, a). Then consider the set

$$S = \{\, \varphi(x) \,\} \cup \{\, \neg ST(i_{(N,b)}, x) \mid N, b \models \varphi(x) \,\}$$

Let $\{\, \varphi(x), \neg ST(i_{(N_1,b_1)}, x) \ldots, \neg ST(i_{(N_u,b_u)}, x) \,\}$ be a finite subset of this set. If this set is unsatisfiable, then we must have $\varphi(x) \models ST(i_{(N_1,b_1)}, x) \vee \ldots \vee ST(i_{(N_u,b_u)}, x)$, but then we will also have $(ST(i_{(N_1,b_1)}, x) \vee \ldots \vee ST(i_{(N_u,b_u)}, x)) \in IC(\varphi(x)) \subseteq IT_\varphi(M, a)$, and hence $(ST(i_{(N_1,b_1)}, x) \vee \ldots \vee ST(i_{(N_u,b_u)}, x))$ will be true at (M, a). But then at least one of $ST(i_{(N_1,b_1)}, x) \ldots, ST(i_{(N_u,b_u)}, x)$ must also be true at (M, a), which contradicts the choice of these formulas. Therefore, every finite subset of S is satisfiable, and by compactness S itself is satisfiable as well. But then take any pointed Σ_φ-model (N', b') of S and this will be a model for which we will have both $N', b' \models ST(i_{(N',b')}, x)$ by choice of $i_{(N',b')}$ and $N', b' \models \neg ST(i_{(N',b')}, x)$ by the satisfaction of S, a contradiction.

Therefore, we will assume in the following that (M, a), (N, b) are pointed Σ_φ-models, $M, a \models IC(\varphi(x))$, $N, b \models \varphi(x)$, and $IT_\varphi(N, b) \subseteq IT_\varphi(M, a)$. Then, according to Lemma 8, consider m-saturated models M', N' that are extensions of M and N, respectively, and maps $f : D(M) \to D(M')$ and $g : D(N) \to D(N')$ such that for any Σ_φ-formula $\chi(x)$ which is invariant with respect to bisimulations and for any $a' \in M$ and $b' \in N$ we have

$$M, a' \models \chi(x) \Leftrightarrow M', f(a') \models \chi(x); N, b' \models \chi(x) \Leftrightarrow N', g(b') \models \chi(x)$$

By our assumption, $\varphi(x)$ is invariant with respect to asimulations and so, by Lemma 6 we get:

$$M, a \models \varphi(x) \Leftrightarrow M', f(a) \models \varphi(x) \tag{51}$$

$$N', g(b) \models \varphi(x) \tag{52}$$

Any standard x-translation of an intuitionistic formula is also, by Corollary 2, invariant with respect to asimulations. Therefore, we have

$$IT_\varphi(N', g(b)) = IT_\varphi(N, b) \subseteq IT_\varphi(M, a) = IT_\varphi(M', f(a)).$$

But then we have $(N', g(b)) \leq_{\Sigma_\varphi} (M', f(a))$, and by m-saturation of M', N' and Lemma 7 the relation $\leq_{\Sigma_\varphi}$ is an $\langle (N', g(b)), (M', f(a)) \rangle$-asimulation. But then by (52) and asimulation invariance of $\varphi(x)$ we get $M', f(a) \models \varphi(x)$, and further, by (51) we conclude that $M, a \models \varphi(x)$. Therefore, $\varphi(x)$ in fact follows from $IC(\varphi(x))$. □

The following theorem is an immediate consequence of Corollary 2 and Theorem 4:

Theorem 5. *A formula $\varphi(x)$ is invariant with respect to asimulations iff it is equivalent to a standard x-translation of an intuitionistic formula.*

Theorem 5 stated above establishes a criterion for the equivalence of first-order formula to a standard translation of intuitionistic formula on arbitrary first-order models. But, unlike in the case of modal propositional logic, some of these models will not be intended models for intuitionistic logic. Therefore it would be interesting to look for the criterion of equivalence of first-order formula to a standard translation of intuitionistic formula on 'intuitionistic' subclass of first-order models. As the class of intended models of intuitionistic propositional logic constitutes a first-order definable subclass of first-order models in general, we can show that such a criterion is provided by invariance with respect to asimulations on the models from this subclass using but a slight modification of our proof for Theorems 4 and 5.

To tighten up on terminology, we introduce the following definitions:

Definition 9. *Let $\Sigma' \subseteq \Sigma$. Then Σ'-model M is intuitionistic, iff R^M is transitive and reflexive, and it is true that*

$$\forall(P \in \Sigma')\forall(a, b \in D(M))(aR^M b \wedge M, a \models P(x) \Rightarrow M, b \models P(x)).$$

The notion of intuitionistic model naturally leads to the following semantic definitions:

Definition 10. *1. Γ is intuitionistically satisfiable iff Γ is satisfied in some intuitionistic model.*

2. φ is an intuitionistic consequence of Γ ($\Gamma \models_i \varphi$) iff $\Gamma \cup \{ \neg\varphi \}$ is intuitionistically unsatisfiable.

3. φ is intuitionistically equivalent to ψ iff both $\psi \models_i \varphi$ and $\varphi \models_i \psi$.

For $\Sigma' \subseteq \Sigma$ let $Int(\Sigma')$ be the following set of formulas

$$\{ \forall y R(y,y), \forall yzw((R(y,z) \wedge R(z,w)) \to R(y,w)) \} \cup \{ \forall yz((P(y) \wedge R(y,z)) \to P(z)) \mid P \in \Sigma' \}.$$

It is clear that for any set Γ of Σ'-formulas and for any Σ'-formula φ, Γ is intuitionistically satisfiable iff $\Gamma \cup Int(\Sigma')$ is satisfiable, and $\Gamma \models_i \varphi$ iff $\Gamma \cup Int(\Sigma') \models \varphi$.

Definition 11. *A formula $\varphi(x)$ is intuitionistically invariant with respect to asimulations iff for any Σ' such that $\Sigma_\varphi \subseteq \Sigma' \subseteq \Sigma$, any pointed intuitionistic Σ'-models (M,a) and (N,b), if there exists an $\langle (M,a),(N,b) \rangle$-asimulation A and $M,a \models \varphi(x)$, then $N,b \models \varphi(x)$.*

Example 5. *Formula $\exists y(R(x,y) \wedge P(y))$ is not intuitionistically invariant with respect to asimulations. However, our argument from Example 4 does not show this, because models considered in this example are not intuitionistic. To prove the absence of intuitionistic invariance with respect to asimulations, consider two $\{ R^2, P^1 \}$-models M_1 and N_1 such that $D(M_1) = \{ a,b,c \}$, $R^{M_1} = \{ (a,a),(a,b),(a,c),(b,b),(c,c) \}$, $P^{M_1} = \{ c \}$, and $D(N_1) = \{ d,e \}$, $R^{N_1} = \{ (d,d),(d,e),(e,e) \}$, $P^{N_1} = \varnothing$. These are intuitionistic models. Then binary relation $C = \{ (a,d),(b,d),(d,b),(b,e),(e,b) \}$ is an $\langle (M_1,a),(N_1,d) \rangle_k$-asimulation. It remains to note that the formula under consideration is true at (M_1,a) but not at (N_1,d).*

Now for the criterion of equivalence on the restricted class of intuitionistic models:

Theorem 6. *Let $\varphi(x)$ be intuitionistically invariant with respect to asimulations. Then $\varphi(x)$ is intuitionistically equivalent to a standard x-translation of an intuitionistic formula.*

Proof. We may assume that $\varphi(x)$ is intuitionistically satisfiable, otherwise $\varphi(x)$ is intuitionistically equivalent to $ST(\bot,x)$ and we are done. In what follows we will write $IntC(\varphi(x))$ for the set of Σ_φ-formulas that are standard x-translations of intuitionistic formulas intuitionistically following from $\varphi(x)$.

Our strategy will be to show that $IntC(\varphi(x)) \models_i \varphi(x)$. Once this is done we will conclude that

$$Int(\Sigma_\varphi) \cup IntC(\varphi(x)) \models \varphi(x).$$

Then we apply compactness of first-order logic and conclude that $\varphi(x)$ is equivalent to a finite conjunction $\psi_1(x) \wedge \ldots \wedge \psi_n(x)$ of formulas from this set. But it follows then that $\varphi(x)$ is intuitionistically equivalent to the conjunction of the set $IntC(\varphi(x)) \cap \{ \psi_1(x) \ldots, \psi_n(x) \}$. In fact, by our choice of $IntC(\varphi(x))$ we have

$$\varphi(x) \models_i \bigwedge(IntC(\varphi(x)) \cap \{ \psi_1(x) \ldots, \psi_n(x) \}),$$

And by our choice of $\psi_1(x)\dots,\psi_n(x)$ we have

$$Int(\Sigma_\varphi)\cup(IntC(\varphi(x))\cap\{\,\psi_1(x)\dots,\psi_n(x)\,\})\models\varphi(x)$$

and hence

$$IntC(\varphi(x))\cap\{\,\psi_1(x)\dots,\psi_n(x)\,\}\models_i\varphi(x).$$

To show that $IntC(\varphi(x))\models_i\varphi(x)$, take any pointed intuitionistic Σ_φ-model (M,a) such that $M,a\models IntC(\varphi(x))$. Such a model exists, because $\varphi(x)$ is intuitionistically satisfiable and $IntC(\varphi(x))$ will be intuitionistically satisfied in any pointed intuitionistic model satisfying $\varphi(x)$. Then we can also choose a pointed intuitionistic Σ_φ-model (N,b) such that $N,b\models\varphi(x)$ and $IT_\varphi(N,b)\subseteq IT_\varphi(M,a)$.

For suppose otherwise. Then for any pointed intuitionistic Σ_φ-model (N,b) such that $N,b\models\varphi(x)$ we can choose an intuitionistic formula $i_{(N,b)}$ such that $ST(i_{(N,b)},x)$ is a Σ_φ-formula true at (N,b) but not at (M,a). Then consider the set

$$S=\{\,\varphi(x)\,\}\cup\{\,\neg ST(i_{(N,b)},x)\mid N\text{ is intuitionistic, }N,b\models\varphi(x)\,\}$$

Let $\{\,\varphi(x),\neg ST(i_{(N_1,b_1)},x)\dots,\neg ST(i_{(N_u,b_u)},x)\,\}$ be a finite subset of this set. If this set is intuitionistically unsatisfiable, then we must have

$$\varphi(x)\models_i ST(i_{(N_1,b_1)},x)\vee\ldots\vee ST(i_{(N_u,b_u)},x),$$

but then we will also have

$$(ST(i_{(N_1,b_1)},x)\vee\ldots\vee ST(i_{(N_u,b_u)},x))\in IntC(\varphi(x))\subseteq IT_\varphi(M,a),$$

and hence $(ST(i_{(N_1,b_1)},x)\vee\ldots\vee ST(i_{(N_u,b_u)},x))$ will be true at (M,a). But then at least one of $ST(i_{(N_1,b_1)},x)\dots,ST(i_{(N_u,b_u)},x)$ must also be true at (M,a), which contradicts the choice of these formulas. Therefore, every finite subset of S is intuitionistically satisfiable. But then every finite subset of the set $S\cup Int(\Sigma_\varphi)$ is satisfiable as well. By compactness of first-order logic $S\cup Int(\Sigma_\varphi)$ is satisfiable, hence S is satisfiable intuitionistically. But then take any pointed intuitionistic Σ_φ-model (N',b') of S and this will be a model for which we will have both $N',b'\models ST(i_{(N',b')},x)$ by choice of $i_{(N',b')}$ and $N',b'\models\neg ST(i_{(N',b')},x)$ by the satisfaction of S, a contradiction.

Therefore, for any given pointed intuitionistic Σ_φ-model (M,a) of $IntC(\varphi(x))$ we can choose a pointed intuitionistic Σ_φ-model (N,b) such that $N,b\models\varphi(x)$ and $IT_\varphi(N,b)\subseteq IT_\varphi(M,a)$. Then, reasoning exactly as in the proof of Theorem 4, we conclude that $M,a\models\varphi(x)$. Therefore, $\varphi(x)$ in fact intuitionistically follows from $IntC(\varphi(x))$. □

Theorem 7. *A formula $\varphi(x)$ is intuitionistically invariant with respect to asimulations iff it is intuitionistically equivalent to a standard x-translation of an intuitionistic formula.*

Proof. From left to right our theorem follows from Theorem 6. In the other direction, assume that $\varphi(x)$ is intuitionistically equivalent to $ST(i,x)$ and assume that for some Σ' such that $\Sigma_\varphi\subseteq\Sigma'\subseteq\Sigma$, some pointed intuitionistic Σ'-models (M,a) and (N,b), and some $\langle(M,a),(N,b)\rangle$-asimulation A we have $M,a\models\varphi(x)$. Then, by Corollary 2 we have $N,b\models ST(i,x)$, but since $ST(i,x)$ is intuitionistically equivalent to $\varphi(x)$ and N is an intuitionistic model, we also have $N,b\models\varphi(x)$. Therefore, $\varphi(x)$ is intuitionistically invariant with respect to asimulations. □

4 Conclusion and further research

Theorems 2, 5, and 7 proved above show that the general idea of asimulation for intuitionistic propositional logic is a faithful analogue of the idea of bisimulation for modal propositional logic in many important respects.

As for the future research, it is natural to concentrate on extending the above results onto the level of intuitionistic predicate logic in order to obtain theorems analogous to Theorem 21 of [Van Benthem 2010, p. 124]. In fact, we already obtained a proof of a 'parametrized' version of such result by extending techniques employed in the section 2 to cover the predicate case. We hope to publish this result in some of our future papers.

References

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