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Effects Of Symmetry On The Structural Controllability Of Neural Networks: A Perspective

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Abstract

The controllability of a dynamical system or network describes whether a given set of control inputs can completely exert influence in order to drive the system towards a desired state. Structural controllability develops the canonical coupling structures in a network that lead to un-controllability, but does not account for the effects of explicit symmetries contained in a network. Recent work has made use of this framework to determine the minimum number and location of the optimal actuators necessary to completely control complex networks. In systems or networks with structural symmetries, group representation theory provides the mechanisms for how the symmetry contained in a network will influence its controllability, and thus affects the placement of these critical actuators, which is a topic of broad interest in science from ecological, biological and man-made networks to engineering systems and design.

I. INTRODUCTION

Controllability is an essential concept to the design of feedback controllers for networked brain systems. For example, non-controllable mathematical models of real systems have subspaces that influence model behavior, but cannot be controlled by an input. Such subspaces can be difficult to determine in complex nonlinear brain networks. Recent advances in the theory of network control [1], [2], demonstrate how a structural controllability [3] framework can be used to search over the nodes of a network to find the minimum number and location of the optimal [2] control points to obtain complete influence over a given network. Since almost all of the present theory was developed for networks without symmetries, here we present the detailed group representation framework to re-envision structural controllability to include systems possessing group symmetries, which complements and expands Lin's seminal theorems on structural controllability [3].

II. STURCTURAL CONTROLLABILITY

Lin [3] utilized the notion of structure in a dynamic system to define two canonical situations where a system would be un-controllable, then proved how two basic controllable structures could be used to evaluate the controllability of a network by determining if they spanned the network.

Recall that for an arbitrary linear time invariant (LTI) system

$$\dot{x}(t)=Ax(t)+Bu(t), \quad (1)$$

with state variables $x(t) \in \mathbb{R}^n$, system matrix $A \in \mathbb{R}^{n \times n}$, input matrix $B \in \mathbb{R}^{n \times p}$ and control input $u(t) \in \mathbb{R}^{p \times n}$, the controllability matrix is defined as

$$Q=[B \;\; AB \;\; A^2B \;\; \cdots \;\; A^{(n-1)}B], \quad (2)$$

and when Q is full rank (rank(Q) = n), the system defined by the pair (A, B) is fully controllable. Lin supposed that the nonzero parameters of a real-world system (A, B) are only generally known within some measurement error, and accordingly, postulated that any specific system that was uncontrollable due to the choice of parameters was arbitrarily close to a parameter set that would render the system fully controllable. From this view of a system, the position of the zeros in (A, B) are assumed to be fixed while the remaining system parameters are arbitrary. The structure of the system (1) is hence defined by the zeros in (A, B), which we will denote:

Definition 1: The set of matrix entries α_{ij} in the pair (A, B) that are equal to zero are called the structure of the pair, and are defined as

$$\Psi(A,B)\colon \{\alpha_{ij}\colon\alpha_{ij}=0\} \quad (3)$$

where $1 \le i \le n$, $1 \le j \le n+p$, and where any two systems with equal Ψ indicates both have zero entries in the same positions.

Now, structural controllability [3] states for a controllable pair (A', B') with structure $\Psi(A', B')$,

$$\Psi(A',B')=\Psi(A'',B'') \Rightarrow (A'',B'') \text{ is S.C.}; \quad (4)$$

i.e. if a pair (A', B') is controllable, any other pair (A'', B'') with the same structure as (A', B') is therefore structurally controllable (S.C.). The major assumption of this prior work lies in the invariance of the arbitrary parameters (non-zero entires) in determining structural controllability, which could actually lead to a system being uncontrollable due to the specific non-zero entries in (A, B).

A. Un-controllable Structures: Dilation and Isolation

The first un-controllable canonical structure, is defined in [3] as a *dilation*, and shown in Fig. 1a. It is easy to determine upon inspection of the system equations in (5) that the controllability matrix Q, composed of the pair (A, B) from (2) will never be full rank for $n = 3$ with two columns of zeros, thus even control input into all three nodes cannot independently control V_1, V_2, and V_3.

$$A = \begin{bmatrix} 0 & \alpha_{12} & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & \alpha_{32} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_{14} \\ \alpha_{24} \\ \alpha_{34} \end{bmatrix} \quad (5)$$

Likewise, the second un-controllable structure – called an *isolation*, is exemplified in Fig. 1b; indeed by inspection of the system graph in the figure, it is readily apparent that the control input to node V_3 can only influence that node since there are no edges directed from V_3 to either of the other two nodes, thus the control influence of this node is isolated and controllability is lost when controlling only this isolated node.

B. Controllable Structures: Bud, Stem and Cactus

In addition to the un-controllable canonical structures, [3] defined two structures called a *bud* and a *stem*, which are always structurally controllable. The first of the two, called a *bud* is shown in Fig. 2a, and the pair for (A, B) takes the form,

$$A = \begin{bmatrix} 0 & \alpha_1 & 0 & \cdots & 0 \\ 0 & 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{n-1} \\ \alpha_{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \alpha_n \end{bmatrix} \quad (6)$$

where it is straightforward to determine by inspection that the controllability matrix Q of this pair (A, B) is full rank for any $n \in \mathbb{N}$. This elementary cycle is controllable from any node in the structure, and when taken in combination with the second controllable structure – a directed path called a *stem* in Fig. 2b – forms a more complex structure that is completely controllable. Inherited from the *stem* structure, the *cactus* is also completely controllable from the foremost upstream node in the chain as shown in Fig. 3. Lin proved how a network that is spanned by a *cactus* is always structurally controllable [3], and these structural definitions were the basis for finding the minimum required control inputs to completely control a linear network in [1], [2].

III. GROUP REPRESENTATION THEORY

For linear systems containing group symmetries, Rubin and Meadows [4] used a similarity transform T to change the coordinates of the n-dimensional system (1) to an orthogonal basis defined by the group action of the symmetry on $\mathbb{R}^n$ (called the symmetry basis). Furthermore, Ref. [4] demonstrated how group representation theory [5] is used to construct the symmetry basis for a symmetric group from the irreducible representations of the group which transforms the system matrix A into block diagonal form. In some cases the type of symmetry would cause the network to be non-controllable due to symmetries (termed NCS), evident by inspection of the structure of the transformed system. Likewise, this same type of similiarity transformation was shown to define the subspaces of a network that synchronize [6], which also has deep implications for neural networks [7]. Essentially, this transformation may reveal a new structure containing dilations or isolated nodes in which the transformed system is actually not controllable on the basis of that structure.

Structural controllability [3] did not explicitly cover symmetry, so for any structurally controllable pair (A, B) that contains no dilations or isolated nodes, the presence of symmetry could still cause the network to be NCS (as shown in [4]), as the act of transforming the network to the symmetry basis would redefine the structure to one that is un-controllable. These two theorems together paint a more complete picture of controllability than either alone as shown in [8], where both are used in concert to explain and understand why certain neural networks were not controllable from particular inputs. Including symmetry constraints makes structural controllability a more general concept, as it does not depend on the explicit non-zero entries of the system pair (A, B) (necessary, but not sufficient), while a network that has the NCS property possesses specific sets of the non-zero entries in (A, B) that define the symmetry contained by the system.

A. The Symmetry Group and Basis

Symmetry present in a network is defined by the set of symmetry operations R that transform the system into itself. Formally, this set of network permutations forms an algebraic group – called the symmetric group on q elements, from which matrix representations of the group elements $D(R)$ can be constructed from monomial matrices (those with only one non-zero entry per column that describe how each operation R permutes the state variables of the network). For example, a network with $\mathbf{S}_2$ symmetry will have the form given in Fig. 4. The $\mathbf{S}_2$ symmetric group contains two elements or symmetry operations $R : \{E, \sigma_y\}$, the identity E, and a reflection across the y axis σ_y – swap x with z. The cardinality g of the set of symmetry operations R for the symmetric group gives the order of the symmetric group $\mathbf{S}_q$ as $g = q!$. For the network in Fig. 4, $n = 3$, $q = 2$, $g = 2$, and we can construct matrix representations $D(R)$ of the elements of $\mathbf{S}_2$ as,

$$E=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \sigma_y=\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad (7)$$

where these matrix representations carry the group action of $\mathbf{S}_2$ to the linear vector space on $\mathbb{R}^3$ for the system pair:

$$A=\begin{bmatrix}\alpha_{11} & \alpha_{12} & 0\\ 0 & \alpha_{22} & 0\\ 0 & \alpha_{32} & \alpha_{33}\end{bmatrix},\quad B=\begin{bmatrix}0\\ \alpha_{24}\\ 0\end{bmatrix} \qquad (8)$$

defined in (1).

In [4] the similarity transformation that takes the system into the coordinates of the symmetry basis is defined from the irreducible representations $\Gamma^{(p)}(R)$ of the group symmetry present in the network. The number of irreducible representations $p = 1 \ldots k$ can be determined from the character of the representation $\chi(R)$. Defined by taking the trace of the matrix representations

$$\chi(R)=\mathrm{Tr}[D(R)], \qquad (9)$$

the value of the character determines the conjugacy class for each group element R and the total number of conjugacy classes (distinct character values) equals the number of irreducible representations. The conjugacy class of each group element refers to the type of symmetry operation of that group element, so that rotations, reflections, etc. have the same conjugacy class, and matrix representations of the group elements $D(R)$ have the same trace. Computing the character (9) of the matrix representations $D(R)$ in (7) for each group element in $\mathbf{S}_2$ yields $\chi(E) = 3$, and $\chi(\sigma_y) = 1$, indicating two conjugacy classes which implies there are two irreducible representations of $\mathbf{S}_2$ on $\mathbb{R}^3$. Furthermore, the irreducible representations form an orthogonal basis in the g-dimensional space of the group, and following from Schur's lemma on orthogonality [5], the dimensionality theorem provides the relation between the number of irreducible representations and their dimensionality as

$$\sum_{p=1}^{k} l_p^2 = g, \qquad (10)$$

where the sum is taken over all irreducible representations of the group k, and l_p is the dimension of the pth irreducible representation. Thus, using the character of the representations (9) to find the total number of irreducible representations along with the dimensionality equation in (10), we can find all of the possible irreducible representations that span the group [5]. So for our network in Fig. 4 with $\mathbf{S}_2$ symmetry and 2 irreducible representations, the dimensionality equation (10) allows us to conclude that their are a total of 2, 1-dimensional irreducible representations of $\mathbf{S}_2$.

The 1-dimensional irreducible representations of $\mathbf{S}_2$ can be found from the identity representation (every group element represented by 1) and the alternating representation A_n, defined by $A_n = \det[D(R)]$ which yields the table of irreducible representations $\Gamma^{(p)}$, with $p = \{1, 2\}$ from (10):

R	E	σ_2
$\Gamma^{(1)}(R)$	1	1
$\Gamma^{(2)}(R)$	1	-1

(11)

Finally, the similarity transform T that takes the system (1) into the coordinates of the symmetry basis is defined in [4] as a projection of the irreducible representations onto the space of the system (1) on $\mathbb{R}^n$:

$$G_i^{(p)}=\sum_R \Gamma^{(p)}(R)_{ii}^* D(R) \quad (12)$$

where $G_i^{(p)}$ generates basis vectors on $\mathbb{R}^n$ from the pth irreducible representation $\Gamma^{(p)}$, * indicates the complex conjugate, and i indicates the (i, i)-th entry of the irreducible representation for $i = 1 \dots l_p$. Once $G_i^{(p)}$ is computed for each p and i, the similarity transform is constructed from the normalized linearly independent vectors that span $\mathbb{R}^n$. For our example network in Fig. 4, we can compute G for each irreducible representation,

$$G_1^{(1)}=\Gamma^{(1)}(E)_{11}^* D(E)+\Gamma^{(1)}(\sigma_y)_{11}^* D(\sigma_y)=1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+1\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}1&0&1\\0&2&0\\1&0&1\end{bmatrix}, \quad (13)$$

where each linearly independent column of G forms a column of T. After normalizing we have

$$\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}0\\2\\0\end{bmatrix}\xrightarrow[\text{normalize}]{}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}\tau_{11}&\tau_{21}\\\tau_{12}&\tau_{22}\\\tau_{13}&\tau_{23}\end{bmatrix}, \quad (14)$$

which defines the first and second columns of T. Computing G for the second irreducible representation we have,

$$G_1^{(2)}=\Gamma^{(2)}(E)_{11}^{*}D(E)+\Gamma^{(2)}(\sigma_y)_{11}^{*}D(\sigma_y)=1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-1\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}1&0&-1\\0&0&0\\-1&0&1\end{bmatrix},\qquad(15)$$

which after normalization yields the final column of T

$$\begin{bmatrix}1\\0\\-1\end{bmatrix}\xrightarrow[\text{normalize}]{}\begin{bmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\tau_{31}\\\tau_{32}\\\tau_{33}\end{bmatrix}.\qquad(16)$$

and the similarity transformation matrix T is given as

$$T=\begin{bmatrix}\frac{1}{\sqrt{2}}&0&\frac{1}{\sqrt{2}}\\0&1&0\\\frac{1}{\sqrt{2}}&0&-\frac{1}{\sqrt{2}}\end{bmatrix}.\qquad(17)$$

B. Symmetry in the Structure

As mentioned previously in Sec. III-A, when a system or network contains an explicit group symmetry the generic entries of the pair (A, B) become constrained, and in fact once the pair (A, B) is transformed by T into the coordinates of the symmetry basis, the structure of the transformed pair $(\hat{A}, \hat{B})$ could be different from (A, B) which could also alter whether or not (A, B) is structurally controllable. As an example, take the network in Fig. 4 which has the pair (8). The unconstrained network is easily shown to be structurally controllable as the controllability matrix Q computed from (8) is full rank for this choice of the non-zero entries. However, if we constrain $\alpha_{11} = \alpha_{33}$ and $\alpha_{12} = \alpha_{32}$, the network has $\mathbf{S}_2$ symmetry and transforming (A, B) in (8) into the symmetry basis we have, $\hat{A}=T^{\dagger}AT$, $\hat{B}=T^{\dagger}B$ which yields

$$\hat{A}=\begin{bmatrix}\alpha'_{11}&\alpha'_{12}&0\\0&\alpha'_{22}&0\\0&0&\alpha'_{33}\end{bmatrix},\ \hat{B}=\begin{bmatrix}0\\\alpha'_{24}\\0\end{bmatrix}\qquad(18)$$

where † is the complex conjugate transpose, and the transformed pair $(\hat{A}, \hat{B})$ has a new structure in Fig. 6 that is easily seen to contain an isolated node and therefore not structurally controllable as well as NCS.

While this result exemplifies how symmetry in a network can alter its apparent structure, let us also demonstrate how certain symmetries leave the network structure invariant. Consider again Lin's *bud* structure for $n = 3$ in Fig. 5 with the pair

$$A=\begin{bmatrix} 0 & \alpha_{12} & 0 \\ 0 & 0 & \alpha_{23} \\ \alpha_{31} & 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 0 \\ 0 \\ \alpha_{34} \end{bmatrix} \tag{19}$$

For $\mathbf{C}_3$ symmetry on $\mathbb{R}^3$ we have matrix representations $D(R)$ constructed as

$$C_3=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C_3^2=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{20}$$

and the identity element E as before in (7). Next, the table of irreducible representations of $\mathbf{C}_3$ cyclic symmetry is given in [8] as

R	E	C_3	C_3^2
$\Gamma^{(1)}(R)$	1	1	1
$\Gamma^{(2)}(R)$	1	ω	ω^2
$\Gamma^{(3)}(R)$	1	ω^2	ω

(21)

where $\omega = e\frac{2\pi i}{3}$, C_3 is a rotation by $2\pi/3$ and C_3^2 by $4\pi/3$. Now generating the symmetry basis via (12) we have for the first irreducible representation,

$$
\begin{aligned}
G_1^{(1)} =& \Gamma^{(1)}(E)_{11}^* D(E) \\
&+ \Gamma^{(1)}(C_3)_{11}^* D(C_3) \ldots \\
&+ \Gamma^{(1)}(C_3^2)_{11}^* D(C_3^2) \\
&= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&+ 1 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&+ 1 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= 1 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{normalize}]{} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \\
&= \begin{bmatrix} \tau_{11} \\ \tau_{21} \\ \tau_{31} \end{bmatrix}.
\end{aligned}
\tag{22}
$$

Next compute G for the second irreducible representation

$$
\begin{aligned}
G_1^{(2)} =& \Gamma^{(2)}(E)_{11}^* D(E) \\
&+ \Gamma^{(2)}(C_3)_{11}^* D(C_3) \ldots \\
&+ \Gamma^{(2)}(C_3^2)_{11}^* D(C_3^2) \\
&= 1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&+ \omega \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
&+ \omega^2 \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{bmatrix} \xrightarrow[\text{normalize}]{} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{\omega^2}{\sqrt{3}} \\ \frac{\omega}{\sqrt{3}} \end{bmatrix} \\
&= \begin{bmatrix} \tau_{12} \\ \tau_{22} \\ \tau_{32} \end{bmatrix},
\end{aligned}
\tag{23}
$$

and lastly for the third irreducible representation

$$
\begin{aligned}
G_1^{(3)}=&\Gamma^{(3)}(E)_{11}^{*}D(E)\\
&+\Gamma^{(3)}(C_3)_{11}^{*}D(C_3)\dots\\
&+\Gamma^{(3)}(C_3^2)_{11}^{*}D(C_3^2)\\
&=1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\\
&+\omega^2\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\\
&+\omega\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\\
&=\begin{bmatrix}1&\omega^2&\omega\\\omega&1&\omega^2\\\omega^2&\omega&1\end{bmatrix}\xrightarrow[\text{normalize}]{}\begin{bmatrix}\frac{1}{\sqrt{3}}\\\frac{\omega}{\sqrt{3}}\\\frac{\omega^2}{\sqrt{3}}\end{bmatrix}\\
&=\begin{bmatrix}\tau_{13}\\\tau_{23}\\\tau_{33}\end{bmatrix}.
\end{aligned}
\tag{24}
$$

Thus our similarity transformation T for $\mathbf{C}_3$ symmetry is constructed as

$$
T=\frac{1}{\sqrt{3}}\begin{bmatrix}1&1&1\\1&e^{\frac{4\pi i}{3}}&e^{\frac{2\pi i}{3}}\\1&e^{\frac{2\pi i}{3}}&e^{\frac{4\pi i}{3}}\end{bmatrix},
\tag{25}
$$

and the transformed system $(\hat{A},\hat{B})$ defined by $\hat{A}=T^{\dagger}AT$, $\hat{B}=T^{\dagger}B$ becomes:

$$
\hat{A}=\begin{bmatrix}\alpha'_{11}&0&0\\0&\alpha'_{22}&0\\0&0&\alpha'_{33}\end{bmatrix},\quad \hat{B}=\begin{bmatrix}\alpha'_{14}\\\alpha'_{24}\\\alpha'_{34}\end{bmatrix}
\tag{26}
$$

which is readily seen to be fully controllable, and hence structurally controllable in the presence of $\mathbf{C}_3$ symmetry, which agrees exactly with the presentation of the *bud* structure in [3].

IV. DISCUSSION

Here we have shown the details of the application of group representation theory in order to determine the effects of symmetry on structurally controllable networks. While [3] identified the basic canonical structures that define how a control input can reach through all nodes of a generic network, the connection between structural controllability and networks containing symmetries was established in [8] and demonstrated here. Additionally, it is worth noting that the *dilation* un-controllable structure in Fig. 1a from [3] is a peculiar one in that the downstream nodes V_1 and V_3 are absent any self-connections, which cause degeneracy and hence un-controllability in the form of two columns of zeros in the system pair (A, B) as shown in (5). The importance of these self-connections in defining the structural controllability is given treatment in [9], and including the intuition that the controllability of nonlinear networks depends on the system trajectory [8], calls into question the utility of the *dilation* canonical structure in determining structural controllability for real world networks – which have nodal dynamics at each node.

Group elements that belong to the same conjugacy class have the same character, which categorically describes the type of symmetry operations and provides further insight into how certain types of symmetry influence controllability (degeneracy) of a network. For example, a class of group elements that all commute with one another (called Abelian) are the rotational symmetry operations and these are defined by:

$$A^n = E, \quad (27)$$

where n is the order of the generating element, and E is the identity. The intuition here comes from the fact that symmetry operations that commute do not introduce degeneracy into the network, hence if the group symmetry is comprised of only rotations (and the identity element which is just $A^1 = E$) as in Lin's "bud" structure, or the example $\mathbf{C}_3$ symmetry network in Fig. 5, no degeneracy is introduced by the symmetry to the network, which will allow controllability. Furthermore, if the group symmetry of the network contains non-commuting elements (symmetry operations), then a degeneracy may be inherent in the network causing it to be NCS, as in the example $\mathbf{S}_2$ symmetry network in Fig. 4, which contains a non-commuting reflection resulting in un-controllability.

Structural controllability is indeed affected by certain types of symmetries present in the network which manifests as change in the structure of the transformed system, while other types of symmetries leave the structure invariant. The link between symmetries and synchrony has been established in previous work [10], and the fact that synchronous network activity in living neural networks suggests such symmetries are inherent to brains. These results offer new insights into the strategic structural control design of the class of networks containing symmetries, and demonstrates the utility of group representation theory as applied to such networks which extends to any natural or man-made network.

Acknowledgments

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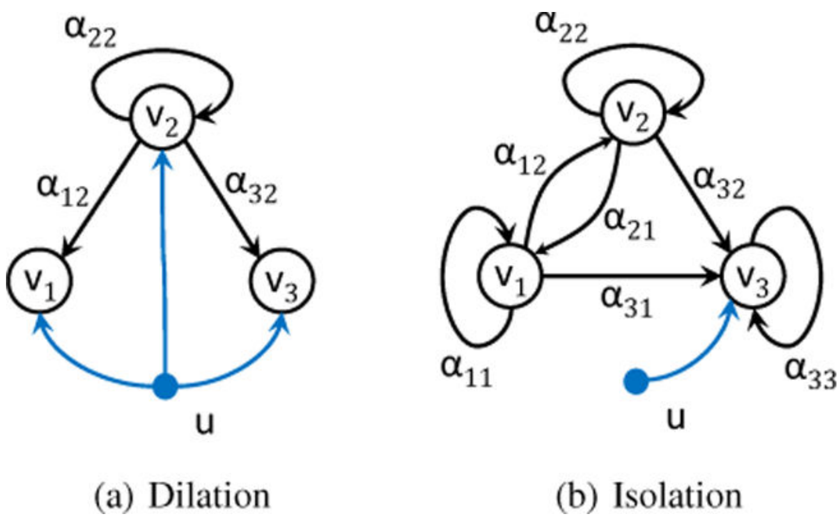

Fig. 1.

The two canonical un-controllable structures in [3], with control input *u*.

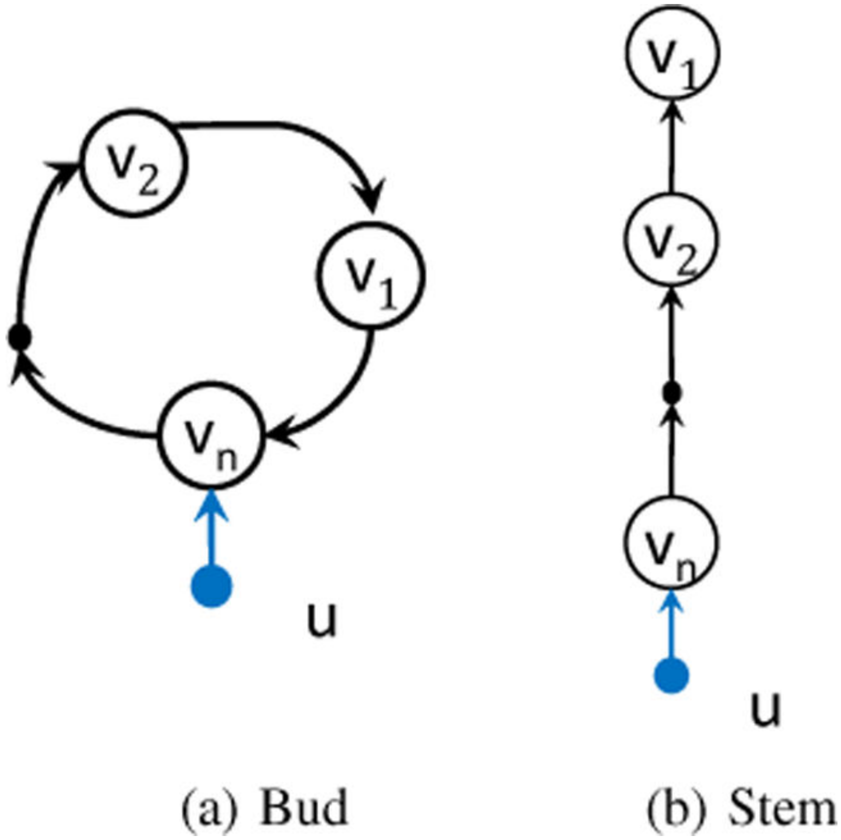

Fig. 2.
The two always controllable canonic structures in [3].

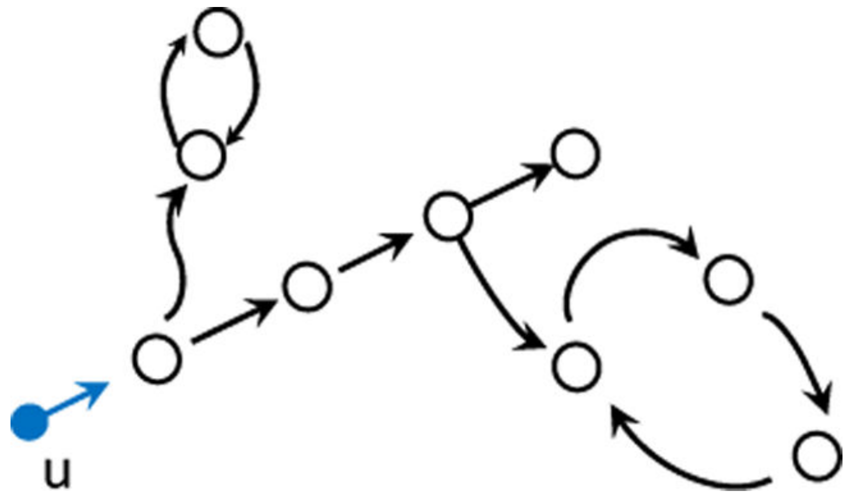

Fig. 3.
The cactus is the union of bud and stem structures, which determines the structurally controllability of a network that is spanned by a cactus [3].

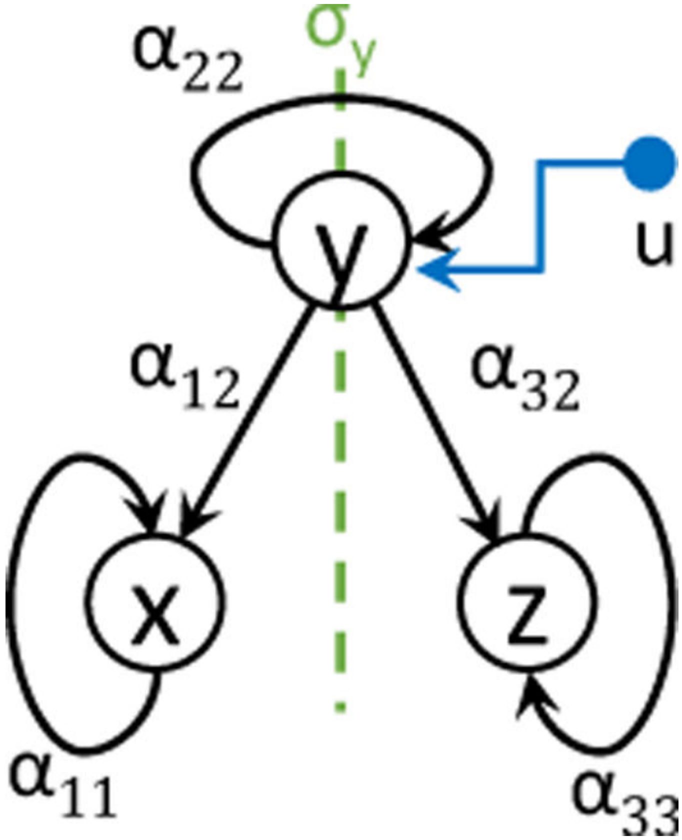

Fig. 4.
An example 3-node network with $\mathbf{S}_2$ symmetry when $\alpha_{12} = \alpha_{32}$, and $\alpha_{11} = \alpha_{33}$.

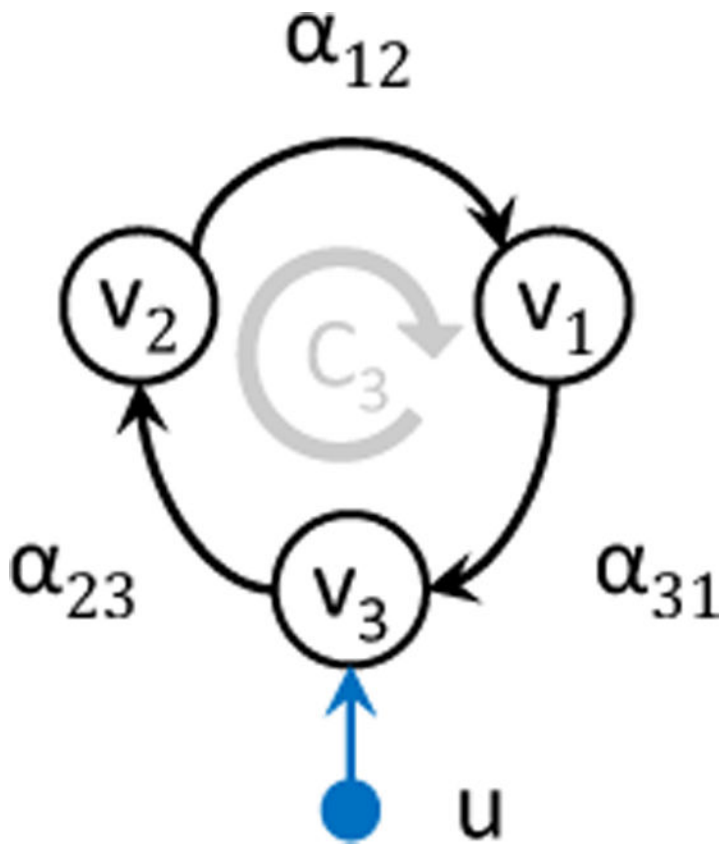

Fig. 5.

An example 3-node *bud* network which is structurally controllable and has $\mathbf{C}_3$ symmetry when $\alpha_{12} = \alpha_{23} = \alpha_{31}$.

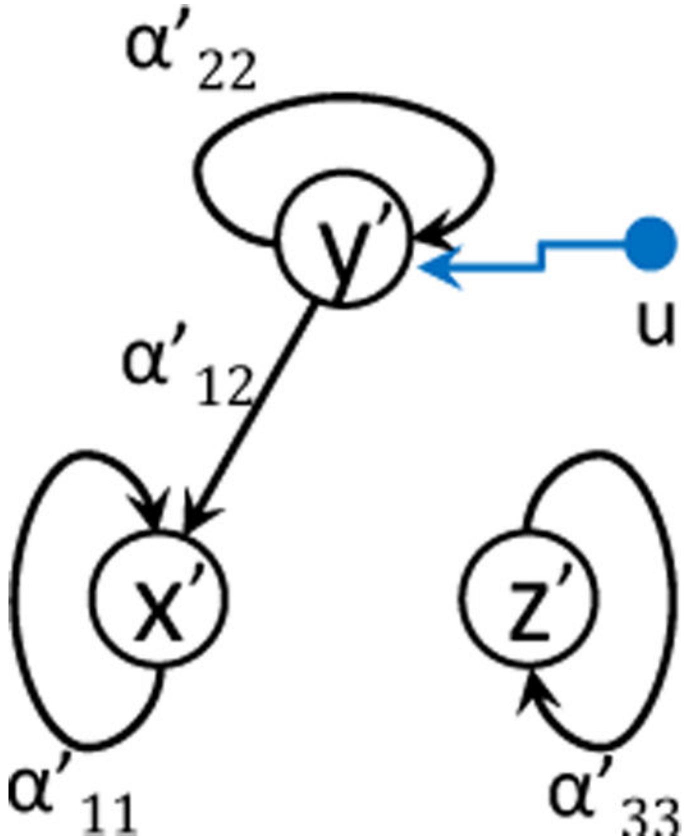

Fig. 6.
The network in Fig. 4 transformed into a new structure defined by the $\mathbf{S}_2$ symmetry basis.