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Good Vibrations from Good Linearizations

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STRUCTURED POLYNOMIAL EIGENVALUE PROBLEMS: GOOD VIBRATIONS FROM GOOD LINEARIZATIONS[*]

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Abstract. Many applications give rise to nonlinear eigenvalue problems with an underlying structured matrix polynomial. In this paper several useful classes of structured polynomials (e.g., palindromic, even, odd) are identified and the relationships between them explored. A special class of linearizations which reflect the structure of these polynomials, and therefore preserve symmetries in their spectra, is introduced and investigated. We analyze the existence and uniqueness of such linearizations and show how they may be systematically constructed.

Key words. nonlinear eigenvalue problem, palindromic matrix polynomial, even matrix polynomial, odd matrix polynomial, Cayley transformation, structured linearization, preservation of eigenvalue symmetry

AMS subject classifications. 65F15, 15A18, 15A57, 93B60

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1. Introduction. We consider $n \times n$ matrix polynomials of the form

$$(1.1) \qquad P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i, \quad A_0, \dots, A_k \in \mathbb{F}^{n \times n}, \ A_k \neq 0,$$

where $\mathbb{F}$ denotes the field $\mathbb{R}$ or $\mathbb{C}$. The numerical solution of the associated polynomial eigenvalue problem $P(\lambda)x = 0$ is one of the most important tasks in the vibration analysis of buildings, machines, and vehicles [11], [22], [35]. In many applications, several of which are summarized in [27], the coefficient matrices have a further structure which reflects the properties of the underlying physical model, and it is important that numerical methods respect this structure.

Our main motivation stems from a project with the company SFE GmbH in Berlin which investigates rail traffic noise caused by high speed trains [16], [17]. The eigenvalue problem that arises in this project from the vibration analysis of rail tracks has the form

$$(1.2) \qquad (\lambda^2 A + \lambda B + A^T)x = 0,$$

where A, B are complex square matrices with B complex symmetric and A singular. The impact of the theory developed in this paper on the solution of this particular eigenvalue problem will be discussed further in section 4. (See also the article in [20].)

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Observe that the matrix polynomial in (1.2) has the property that reversing the order of the coefficient matrices, followed by taking their transpose, leads back to the original matrix polynomial. By analogy with linguistic palindromes, of which

<div align="center">"sex at noon taxes"</div>

is perhaps a less well-known example,[1] we say such matrix polynomials are *T-palindromic*.

Quadratic real and complex T-palindromic eigenvalue problems also arise in the mathematical modeling and numerical simulation of the behavior of periodic surface acoustic wave (SAW) filters [36], whereas the computation of the Crawford number [15] associated with the perturbation analysis of symmetric generalized eigenvalue problems produces a quadratic $*$-palindromic eigenvalue problem, where $*$ stands for conjugate transpose. Higher order matrix polynomials with a $*$-palindromic structure arise in the optimal control of higher order difference equations [27].

A related class of structured eigenvalue problems arises in the study of corner singularities in anisotropic elastic materials [3], [4], [25], [33] and gyroscopic systems [35]. Here the problem is of the form

$$(1.3) \qquad P(\lambda)v = (\lambda^2 M + \lambda G + K)v = 0,$$

with large and sparse coefficients $M = M^T$, $G = -G^T$, $K = K^T$ in $\mathbb{R}^{n \times n}$. The matrix polynomial in (1.3) is reminiscent of an even function: replacing λ with $-\lambda$, followed by taking the transpose, leads back to the original matrix polynomial. We therefore say such matrix polynomials are *T-even*. Higher order $*$-even eigenvalue problems arise in the linear quadratic optimal control problem for higher order systems of ordinary differential equations [34]. Under different nomenclature, even matrix polynomials have recently received much attention [3], [5], [33], [34].

The classical approach to investigating or numerically solving polynomial eigenvalue problems is *linearization*, in which the given polynomial (1.1) is transformed into a $kn \times kn$ matrix pencil $L(\lambda) = \lambda X + Y$ that satisfies

$$(1.4) \qquad E(\lambda)L(\lambda)F(\lambda) = \begin{bmatrix} P(\lambda) & 0 \\ 0 & I_{(k-1)n} \end{bmatrix},$$

where $E(\lambda)$ and $F(\lambda)$ are *unimodular matrix polynomials* [11]. (A matrix polynomial is *unimodular* if its determinant is a nonzero constant, independent of λ.) Standard methods for linear eigenvalue problems as in [2], [26], [30] can then be applied.

The companion forms [11] provide the standard examples of linearizations for a matrix polynomial $P(\lambda)$ as in (1.1). Let $X_1 = X_2 = \text{diag}(A_k, I_n, \ldots, I_n)$,

$$Y_1 = \begin{bmatrix} A_{k-1} & A_{k-2} & \ldots & A_0 \\ -I_n & 0 & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}, \quad \text{and} \quad Y_2 = \begin{bmatrix} A_{k-1} & -I_n & & 0 \\ A_{k-2} & 0 & \ddots & \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \ldots & 0 \end{bmatrix}.$$

Then $C_1(\lambda) = \lambda X_1 + Y_1$ and $C_2(\lambda) = \lambda X_2 + Y_2$ are, respectively, the *first* and *second companion forms* for $P(\lambda)$. Unfortunately, since these companion linearizations do not reflect the structure present in palindromic or even matrix polynomials, the

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[1]Invented by the mathematician Peter Hilton in 1947 for his thesis advisor J.H.C. Whitehead. It is probable, Hilton says, that this palindrome may have been known before 1947. When Whitehead lamented its brevity, Hilton responded by crafting the palindromic masterpiece "Doc, note, I dissent. A fast never prevents a fatness. I diet on cod" [18], [19].

corresponding linearized pencil can usually only be treated with methods for general matrix pencils. In a finite precision environment, a numerical method that ignores the structure may produce physically meaningless results [35], e.g., may lose symmetries in the spectrum. Therefore, it is important to construct linearizations that reflect the structure of the given matrix polynomial, and then develop numerical methods for the corresponding linear eigenvalue problem that properly address these structures as well. The latter topic has been an important recent area of research; see, e.g., [5], [6], [7], [30], [32], and the references therein.

In this paper we show that the pencil spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$, developed in [28] by generalizing the first and second companion forms, are rich enough to include subspaces of pencils that reflect palindromic, even, or odd structure of a matrix polynomial P. Extending the notion of *Cayley transformation* to matrix polynomials, we show in section 2.2 how this transformation connects (anti-)palindromic and odd/even structures. Section 3 is devoted to the introduction and analysis of structured linearizations for the various structured matrix polynomials under consideration. The general linearization approach of [28] is summarized and then exploited to obtain the main results of this paper: identification of structured pencils in $\mathbb{L}_1(P)$, a constructive method for generating them, and necessary and sufficient conditions for these pencils to be linearizations, thereby correctly retaining information on eigenvalues and eigenvectors of the original matrix polynomial. These results are then used to identify situations when existence of structure-preserving linearizations is not guaranteed.

Finally, in section 4 we elucidate the subtitle "good vibrations from good linearizations" by discussing the impact of the theory developed in this paper on the palindromic eigenvalue problem (1.2) arising in the vibration analysis of rail tracks.

2. Basic structures, spectral properties, and Cayley transformations. In this section we formally define the structured polynomials that are studied in this paper, show how the structure of a polynomial is reflected in its spectra, and establish connections between the various classes of structured polynomials by extending the classical definition of Cayley transformations to matrix polynomials. For conciseness, the symbol $\star$ is used as an abbreviation for transpose T in the real case and for either T or conjugate transpose $*$ in the complex case.

DEFINITION 2.1. *Let* $Q(\lambda) = \sum_{i=0}^{k} \lambda^i B_i$, *where* $B_0, \ldots, B_k \in \mathbb{F}^{m \times n}$, *be a matrix polynomial of degree* k, *that is,* $B_k \neq 0$. *Then we define the* adjoint $Q^\star(\lambda)$ *and the* reversal $\operatorname{rev} Q(\lambda)$ *of* $Q(\lambda)$, *respectively, by*

$$(2.1) \qquad Q^\star(\lambda) := \sum_{i=0}^{k} \lambda^i B_i^\star \quad and \quad \operatorname{rev} Q(\lambda) := \lambda^k Q(1/\lambda) = \sum_{i=0}^{k} \lambda^{k-i} B_i.$$

If $\deg(Q(\lambda))$ denotes the *degree* of the matrix polynomial $Q(\lambda)$, then, in general, $\deg(\operatorname{rev} Q(\lambda)) \leq \deg(Q(\lambda))$ and $\operatorname{rev}\big(Q_1(\lambda) \cdot Q_2(\lambda)\big) = \operatorname{rev} Q_1(\lambda) \cdot \operatorname{rev} Q_2(\lambda)$, whenever the product $Q_1(\lambda) \cdot Q_2(\lambda)$ is defined. Using (2.1), the various structured matrix polynomials under consideration are defined in Table 2.1.

For a scalar polynomial $p(x)$, T-palindromic is the same as palindromic (i.e., $\operatorname{rev} p(x) = p(x)$), while $*$-palindromic is equivalent to conjugate-palindromic (i.e., $\operatorname{rev} \overline{p}(x) = p(x)$). Analogous simplifications occur in the scalar polynomial case for all other structures defined in Table 2.1.

Two matrices that play an important role in our investigation are the $k \times k$ reverse identity R_k in the context of palindromic structures, and the $k \times k$ diagonal matrix Σ_k of alternating signs in the context of even/odd structures (the subscript k will be

TABLE 2.1
Definitions of basic structures.

palindromic	$\operatorname{rev} P(\lambda) = P(\lambda)$	anti-palindromic	$\operatorname{rev} P(\lambda) = -P(\lambda)$
$\star$-palindromic	$\operatorname{rev} P^\star(\lambda) = P(\lambda)$	$\star$-anti-palindromic	$\operatorname{rev} P^\star(\lambda) = -P(\lambda)$
even	$P(-\lambda) = P(\lambda)$	odd	$P(-\lambda) = -P(\lambda)$
$\star$-even	$P^\star(-\lambda) = P(\lambda)$	$\star$-odd	$P^\star(-\lambda) = -P(\lambda)$

suppressed whenever it is clear from the context):

$$(2.2) \quad R := R_k := \begin{bmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{bmatrix}_{k \times k} \quad \text{and} \quad \Sigma := \Sigma_k := \begin{bmatrix} (-1)^{k-1} & & 0 \\ & \ddots & \\ 0 & & (-1)^0 \end{bmatrix}.$$

2.1. Spectral symmetry. A distinguishing feature of the structured matrix polynomials in Table 2.1 is the special symmetry properties of their spectra, described in the following result.

THEOREM 2.2. *Let* $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, $A_k \neq 0$, *be a regular matrix polynomial that has one of the structures listed in Table* 2.1. *Then the spectrum of* $P(\lambda)$ *has the pairing depicted in Table* 2.2. *Moreover, the algebraic, geometric, and partial multiplicities of the two eigenvalues in each such pair are equal. (Here, we allow* $\lambda = 0$ *and interpret* $1/\lambda$ *as the eigenvalue* ∞.)

TABLE 2.2
Spectral symmetries.

Structure of $P(\lambda)$	Eigenvalue pairing
(anti-)palindromic, T-(anti-)palindromic $*$-palindromic, $*$-anti-palindromic	$(\lambda, 1/\lambda)$ $(\lambda, 1/\overline{\lambda})$
even, odd, T-even, T-odd $*$-even, $*$-odd	$(\lambda, -\lambda)$ $(\lambda, -\overline{\lambda})$

Proof. We first recall some well-known facts [8], [10], [11] about the companion forms $C_1(\lambda)$ and $C_2(\lambda)$ of a regular matrix polynomial $P(\lambda)$:

- $P(\lambda)$ and $C_1(\lambda)$ have the same eigenvalues (including ∞), with the same algebraic, geometric, and partial multiplicities.
- $C_1(\lambda)$ and $C_2(\lambda)$ are always *strictly equivalent*, i.e., there exist nonsingular *constant* matrices E and F such that $C_1(\lambda) = E \cdot C_2(\lambda) \cdot F$.
- Strictly equivalent pencils have the same eigenvalues (including ∞), with the same algebraic, geometric, and partial multiplicities.

With these facts in hand, we first consider the case when $P(\lambda)$ is $\star$-palindromic or $\star$-anti-palindromic, so that $\operatorname{rev} P^\star(\lambda) = \chi_P P(\lambda)$ for $\chi_P = \pm 1$. Our strategy is to show that $C_1(\lambda)$ is strictly equivalent to $\operatorname{rev} C_1^\star(\lambda)$, from which the desired eigenvalue pairing and equality of multiplicities then follow. Using the nonsingular matrix

$$T := \begin{bmatrix} \chi_P I & \chi_P A_{k-1} & \cdots & \chi_P A_1 \\ 0 & 0 & & -I \\ \vdots & & \cdot^{\cdot^{\cdot}} & \\ 0 & -I & & 0 \end{bmatrix},$$

we first show that $C_1(\lambda)$ is strictly equivalent to rev $C_2^\star(\lambda)$:

$$T \cdot C_1(\lambda) \cdot (R_k \otimes I_n) = T \cdot \left(\lambda \begin{bmatrix} 0 & & & A_k \\ & & I & \\ & \ddots & & \\ I & & & 0 \end{bmatrix} + \begin{bmatrix} A_0 & A_1 & \cdots & A_{k-1} \\ 0 & 0 & & -I \\ \vdots & & \ddots & \\ 0 & -I & & 0 \end{bmatrix} \right)$$

$$= \lambda \begin{bmatrix} \chi_P A_1 & \cdots & \chi_P A_{k-1} & \chi_P A_k \\ -I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I & 0 \end{bmatrix} + \begin{bmatrix} \chi_P A_0 & & & 0 \\ & I & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix}$$

$$= \lambda \begin{bmatrix} A_{k-1} & -I & & 0 \\ \vdots & & \ddots & \\ A_1 & 0 & & -I \\ A_0 & 0 & \cdots & 0 \end{bmatrix}^\star + \begin{bmatrix} A_k & & & 0 \\ & I & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix}^\star$$

$$= \text{rev } C_2^\star(\lambda).$$

But rev $C_2^\star(\lambda)$ is always strictly equivalent to rev $C_1^\star(\lambda)$, since $C_1(\lambda)$ and $C_2(\lambda)$ are. This completes the proof for this case.

For the case of palindromic or anti-palindromic matrix polynomials, i.e., polynomials $P(\lambda)$ satisfying rev $P(\lambda) = \chi_P P(\lambda)$, an analogous computation shows that

$$T \cdot C_1(\lambda) \cdot (R_k \otimes I_n) = \text{rev } C_1(\lambda),$$

i.e., $C_1(\lambda)$ is strictly equivalent to rev $C_1(\lambda)$, which again implies the desired eigenvalue pairing and equality of multiplicities.

Next assume that $P(\lambda)$ is $\star$-even or $\star$-odd, so $P^\star(-\lambda) = \varepsilon_P P(\lambda)$ for $\varepsilon_P = \pm 1$. We show that $C_1(\lambda)$ is strictly equivalent to $C_1^\star(-\lambda)$, from which the desired pairing of eigenvalues and equality of multiplicities follows. We begin by observing that $C_1(\lambda)$ is strictly equivalent to $C_2^\star(-\lambda)$:

$$\big(\text{diag}(\varepsilon_P, -\Sigma_{k-1}) \otimes I_n \big) \cdot C_1(\lambda) \cdot (\Sigma_k \otimes I_n)$$

$$= \lambda \begin{bmatrix} \varepsilon_P(-1)^{k-1}A_k & & & 0 \\ & -I & & \\ & & \ddots & \\ 0 & & & -I \end{bmatrix} + \begin{bmatrix} \varepsilon_P(-1)^{k-1}A_{k-1} & \cdots & \varepsilon_P(-1)^1 A_1 & \varepsilon_P A_0 \\ -I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & -I & 0 \end{bmatrix}$$

$$= -\lambda \begin{bmatrix} A_k & & & 0 \\ & I & & \\ & & \ddots & \\ 0 & & & I \end{bmatrix}^\star + \begin{bmatrix} A_{k-1} & -I & & 0 \\ \vdots & & \ddots & \\ A_1 & 0 & & -I \\ A_0 & 0 & \cdots & 0 \end{bmatrix}^\star = C_2^\star(-\lambda).$$

The strict equivalence of $C_2^\star(-\lambda)$ and $C_1^\star(-\lambda)$ now follows from that of $C_2(\lambda)$ and $C_1(\lambda)$, and the proof for this case is complete.

For even or odd polynomials, that is, when $P(-\lambda) = \varepsilon_P P(\lambda)$, an analogous computation

$$\big(\text{diag}(\varepsilon_P, -\Sigma_{k-1}) \otimes I_n \big) \cdot C_1(\lambda) \cdot (\Sigma_k \otimes I_n) = C_1(-\lambda)$$

shows that $C_1(\lambda)$ is strictly equivalent to $C_1(-\lambda)$, which implies the desired eigenvalue pairing and equality of multiplicities. $\square$

If the coefficient matrices of P are real, then the eigenvalues of a $\star$-even or $\star$-odd matrix polynomial occur in quadruples $(\lambda, \overline{\lambda}, -\lambda, -\overline{\lambda})$. This property has sometimes been referred to as "Hamiltonian spectral symmetry" because the eigenvalues of real Hamiltonian matrices have such symmetry [31], [34]. However, this feature is not confined to Hamiltonian matrices, but is shared by matrices in Lie algebras associated with any real scalar product. Similarly, the eigenvalues of real $\star$-palindromic and $\star$-anti-palindromic matrix polynomials occur in quadruples $(\lambda, \overline{\lambda}, 1/\lambda, 1/\overline{\lambda})$, a property sometimes referred to as "symplectic spectral symmetry" because real symplectic matrices exhibit this behavior. But once again, this type of eigenvalue symmetry is an instance of a more general phenomenon associated with matrices in the Lie group of any real scalar product, such as the real pseudo-orthogonal (Lorentz) groups. See [1], [7], [24], [31] for detailed coverage of Hamiltonian and symplectic matrices, and see [12], [29] for properties of matrices in the Lie algebra or Lie group of more general scalar products.

Remark 2.3. In Definition 2.1 we could have defined the adjoint of an $n \times n$ matrix polynomial with respect to the adjoint of a more general scalar product, rather than restricting $\star$ to being just transpose or conjugate transpose. For example, with any nonsingular matrix M we can define a bilinear scalar product $\langle x, y \rangle := x^T M y$ and denote the adjoint of a matrix $A \in \mathbb{F}^{n \times n}$ with respect to this scalar product by $A^\star = M^{-1} A^T M$. (For a sesquilinear scalar product $\langle x, y \rangle := x^* M y$, its corresponding adjoint is $A^\star = M^{-1} A^* M$.) Then for an $n \times n$ matrix polynomial $P(\lambda)$, the definition of the corresponding adjoint $P^\star(\lambda)$ is formally identical to Definition 2.1; the structures in Table 2.1 also make sense as written with $\star$ denoting the adjoint of a general scalar product. Well-known examples of this are the skew-Hamiltonian/Hamiltonian pencils [33], which are $\star$-odd with respect to the symplectic form defined by $M = J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix} \right]$.

However, if the matrix M defining a bilinear scalar product satisfies $M^T = \varepsilon M$ for $\varepsilon = \pm 1$ (or $M^* = \varepsilon M$, $|\varepsilon| = 1$, $\varepsilon \in \mathbb{C}$, in the sesquilinear case), then not much is gained by this apparent extra generality. Note this includes all the standard examples, which are either symmetric or skew-symmetric bilinear forms or Hermitian sesquilinear forms. In the bilinear case, we have

$$P(\lambda) \text{ is } \star\text{-palindromic} \Leftrightarrow \operatorname{rev} P^\star(\lambda) = \operatorname{rev} \left(M^{-1} P^T(\lambda) M \right) = P(\lambda)$$
$$\Leftrightarrow \operatorname{rev} \left(M P(\lambda) \right)^T = \operatorname{rev} \left(P^T(\lambda) M^T \right) = \varepsilon M P(\lambda)$$
$$\Leftrightarrow M P(\lambda) \text{ is } T\text{-palindromic or } T\text{-anti-palindromic},$$

depending on the sign of ε. A similar argument shows that $\star$-evenness or $\star$-oddness of $P(\lambda)$ is equivalent to T-evenness or T-oddness of $M P(\lambda)$. Analogous results also hold in the sesquilinear case when $M^* = \varepsilon M$. Thus for any of the standard scalar products with adjoint $\star$, the $\star$-structures in Table 2.1 can be reduced to either the $\star = T$ or $\star = *$ case; in particular this implies that the eigenvalue pairing results of Theorem 2.2 extend to these more general $\star$-structures. Note this reduction shows that the skew-Hamiltonian/Hamiltonian pencils mentioned above are equivalent to T-even or $*$-even pencils.

2.2. Cayley transformations of matrix polynomials. It is well known that the *Cayley transformation* and its generalization to matrix pencils [24], [32] relates Hamiltonian structure to symplectic structure for both matrices and pencils. By extending the classical definition of this transformation to matrix polynomials, we now develop analogous relationships between (anti-)palindromic and odd/even matrix polynomials, and their $\star$-variants.

Our choice of definition is motivated by the following observation: the only Möbius transformations of the complex plane that map reciprocal pairs $(\mu, 1/\mu)$ to plus/minus pairs $(\lambda, -\lambda)$ are $\alpha\left(\frac{\mu-1}{\mu+1}\right)$ and $\beta\left(\frac{1+\mu}{1-\mu}\right)$, where α, $\beta \in \mathbb{C}$ are nonzero constants. When $\alpha = \beta = 1$, these transformations also map conjugate reciprocal pairs $(\mu, 1/\overline{\mu})$ to conjugate plus/minus pairs $(\lambda, -\overline{\lambda})$. Combining this with Theorem 2.2, we see that the Möbius transformations $\frac{\mu-1}{\mu+1}$, $\frac{1+\mu}{1-\mu}$ translate the spectral symmetries of (anti-)palindromic matrix polynomials and their $\star$-variants to those of odd/even matrix polynomials and their $\star$-variants. Consequently, it is reasonable to anticipate that Cayley transformations modeled on these particular Möbius transformations might have an analogous effect on structure at the level of matrix polynomials. These observations therefore lead us to adopt the following definition as the natural extension, given our context, of the Cayley transformation to matrix polynomials.

DEFINITION 2.4. *Let $P(\lambda)$ be a matrix polynomial of degree k as in (1.1). Then the* Cayley transformation *of $P(\lambda)$ with pole at -1 or $+1$, respectively, is the matrix polynomial*

$$(2.3) \quad \mathcal{C}_{-1}(P)(\mu) := (\mu+1)^k P\left(\frac{\mu-1}{\mu+1}\right), \quad \text{resp.}, \quad \mathcal{C}_{+1}(P)(\mu) := (1-\mu)^k P\left(\frac{1+\mu}{1-\mu}\right).$$

When viewed as maps on the space of $n \times n$ matrix polynomials of degree $k \geq 1$, the Cayley transformations in (2.3) can be shown by a direct calculation to be inverses of each other, up to a scaling factor.

PROPOSITION 2.5. *For any $n \times n$ matrix polynomial P of degree $k \geq 1$, we have $\mathcal{C}_{+1}(\mathcal{C}_{-1}(P)) = \mathcal{C}_{-1}(\mathcal{C}_{+1}(P)) = 2^k \cdot P$.*

The next lemma gives some straightforward observations that are helpful in relating the structure in a matrix polynomial to that in its Cayley transformations.

LEMMA 2.6. *Let P be a matrix polynomial of degree $k \geq 1$. Then*

$$(2.4) \qquad \left(\mathcal{C}_{-1}(P)\right)^\star(\mu) = \mathcal{C}_{-1}(P^\star)(\mu), \qquad \left(\mathcal{C}_{+1}(P)\right)^\star(\mu) = \mathcal{C}_{+1}(P^\star)(\mu),$$

$$(2.5a) \qquad \mathrm{rev}\left(\mathcal{C}_{-1}(P)\right)^\star(\mu) = (\mu+1)^k P^\star\left(-\frac{\mu-1}{\mu+1}\right), \quad \mu \neq -1,$$

$$(2.5b) \qquad \mathrm{rev}\left(\mathcal{C}_{+1}(P)\right)^\star(\mu) = (-1)^k (1-\mu)^k P^\star\left(-\frac{1+\mu}{1-\mu}\right), \quad \mu \neq 1.$$

Proof. The proof of (2.4) is straightforward. We only prove (2.5b); the proof of (2.5a) is similar. Since $\mathcal{C}_{+1}(P)$, and hence $\mathcal{C}_{+1}(P)^\star$, are matrix polynomials of degree k,

$$\mathrm{rev}\left(\mathcal{C}_{+1}(P)\right)^\star(\mu) = \mu^k \left(\mathcal{C}_{+1}(P)\right)^\star\left(\frac{1}{\mu}\right) = \mu^k \mathcal{C}_{+1}(P^\star)\left(\frac{1}{\mu}\right) \qquad \text{by (2.4), (2.1)}$$

$$= \mu^k (1 - 1/\mu)^k P^\star\left(\frac{1+1/\mu}{1-1/\mu}\right) \qquad\qquad \text{by (2.3)}$$

and (2.5b) is now immediate. $\quad\square$

THEOREM 2.7. *Let $P(\lambda)$ be a matrix polynomial of degree $k \geq 1$. Then the correspondence between structure in $P(\lambda)$ and in its Cayley transformations is as stated in Table* 2.3.

Proof. Since the proofs of the equivalences are similar, we establish only one of them. We show that $P(\lambda)$ is $\star$-even if and only if $\mathcal{C}_{+1}(P)(\mu)$ is $\star$-palindromic when k is even and $\star$-anti-palindromic when k is odd. Now $P(\lambda)$ being $\star$-even is equivalent,

TABLE 2.3
Cayley transformations of structured matrix polynomials.

$P(\lambda)$	$\mathcal{C}_{-1}(P)(\mu)$		$\mathcal{C}_{+1}(P)(\mu)$	
	k even	k odd	k even	k odd
palindromic $\star$-palindromic	even $\star$-even	odd $\star$-odd	even $\star$-even	
anti-palindromic $\star$-anti-palindromic	odd $\star$-odd	even $\star$-even	odd $\star$-odd	
even $\star$-even	palindromic $\star$-palindromic		palindromic $\star$-palindromic	anti-palindromic $\star$-anti-palindromic
odd $\star$-odd	anti-palindromic $\star$-anti-palindromic		anti-palindromic $\star$-anti-palindromic	palindromic $\star$-palindromic

by definition, to $P^{\star}(-\lambda) = P(\lambda)$ for all λ. Setting $\lambda = \frac{1+\mu}{1-\mu}$ and multiplying by $(1-\mu)^k$ yields

$$P(\lambda) \text{ is } \star\text{-even} \iff (1-\mu)^k P^{\star}\left(-\frac{1+\mu}{1-\mu}\right) = (1-\mu)^k P\left(\frac{1+\mu}{1-\mu}\right) \quad \text{for all } \mu \neq 1$$

$$\iff (-1)^k \text{rev}\,(\mathcal{C}_{+1}(P))^{\star}(\mu) = \mathcal{C}_{+1}(P)(\mu) \quad \text{by Lemma 2.6,}$$

from which the desired result follows. $\square$

Observe that the results in Table 2.3 are consistent with $\mathcal{C}_{-1}(P)$ and $\mathcal{C}_{+1}(P)$ being essentially inverses of each other (Proposition 2.5).

Theorem 2.7 establishes a relationship between $\star$-(anti-)palindromic and $\star$-even/odd matrix polynomials via the Cayley transformation. Since $\star$-even/odd matrix polynomials can be interpreted as generalizations of Hamiltonian matrices [33], [34], and since it is well known that Hamiltonian matrices and symplectic matrices are related via the Cayley transformation [31], $\star$-(anti-)palindromic matrix polynomials can be thought of as generalizations of symplectic matrices.

3. Structured linearizations. As sources of structured linearizations for the structured polynomials listed in Table 2.1, we consider the vector spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$, introduced in [28]. We establish the existence of structured pencils in these spaces, show how they can be explicitly constructed, and give necessary and sufficient conditions for them to be linearizations of the given matrix polynomial P.

3.1. Vector spaces of potential linearizations. The vector spaces $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ consist of pencils that generalize the first and second companion forms $C_1(\lambda)$ and $C_2(\lambda)$ of $P(\lambda)$, respectively:

(3.1) $\quad \mathbb{L}_1(P) := \left\{ L(\lambda) = \lambda X + Y \ : \ L(\lambda) \cdot (\Lambda \otimes I_n) = v \otimes P(\lambda), \, v \in \mathbb{F}^k \right\},$

(3.2) $\quad \mathbb{L}_2(P) := \left\{ L(\lambda) = \lambda X + Y \ : \ (\Lambda^T \otimes I_n) \cdot L(\lambda) = w^T \otimes P(\lambda), \, w \in \mathbb{F}^k \right\},$

where $\Lambda = \begin{bmatrix} \lambda^{k-1} & \lambda^{k-2} & \ldots & \lambda & 1 \end{bmatrix}^T$, and $\otimes$ denotes the Kronecker product. A direct calculation shows that

$$C_1(\lambda) \cdot (\Lambda \otimes I_n) = e_1 \otimes P(\lambda) \quad \text{and} \quad (\Lambda^T \otimes I_n) \cdot C_2(\lambda) = e_1^T \otimes P(\lambda),$$

so $C_1(\lambda) \in \mathbb{L}_1(P)$ and $C_2(\lambda) \in \mathbb{L}_2(P)$ for any $P(\lambda)$. The vector v in (3.1) is called the *right ansatz vector* of $L(\lambda) \in \mathbb{L}_1(P)$ because $L(\lambda)$ is multiplied on the right by $\Lambda \otimes I_n$ to give $v \otimes P(\lambda)$. Analogously, the vector w in (3.2) is called the *left ansatz vector* of $L(\lambda) \in \mathbb{L}_2(P)$.

The pencil spaces $\mathbb{L}_j(P)$ were designed with the aim of providing an arena of potential linearizations that is fertile enough to contain those that reflect additional structures in P, but small enough that these linearizations still share salient features of the companion forms $C_j(\lambda)$. First, when $P(\lambda)$ is regular, the mild hypothesis of a pencil in $\mathbb{L}_j(P)$ being regular is sufficient to guarantee that it is indeed a linearization for P. In fact, as shown in [28], regularity makes these pencils strong linearizations for $P(\lambda)$, i.e., rev $L(\lambda)$ is also a linearization for rev $P(\lambda)$. This ensures that the Jordan structures of both the finite and infinite eigenvalues of P are always faithfully reflected in L, just as is done by the companion forms. Without this extra property of being a strong linearization, any Jordan structure compatible with the algebraic multiplicity of the infinite eigenvalue of $P(\lambda)$ can be realized by some linearization [23]. Second, eigenvectors of $P(\lambda)$ are easily recoverable from those of $L(\lambda)$. Indeed, the definition of $\mathbb{L}_1(P)$ implies that $L(\lambda) \cdot (\Lambda \otimes x) = v \otimes (P(\lambda)x)$ for all $x \in \mathbb{F}^n$. Thus, whenever x is a right eigenvector of $P(\lambda)$ associated with an eigenvalue λ, then $\Lambda \otimes x$ is a right eigenvector of $L(\lambda)$ associated with λ. Similar observations hold for $L(\lambda) \in \mathbb{L}_2(P)$ and left eigenvectors. Finally, when $P(\lambda)$ is regular, almost all pencils in $\mathbb{L}_j(\lambda)$ are regular, and thus strong linearizations for $P(\lambda)$—the ones that are not from a closed nowhere dense set of measure zero [28].

3.1.1. Shifted sums. The *column-shifted sum* and *row-shifted sum* are convenient tools that readily allow one to construct pencils in $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$, respectively. They also enable one to easily test when a given pencil is in $\mathbb{L}_j(P)$.

DEFINITION 3.1 (shifted sums). *Let $X = (X_{ij})$ and $Y = (Y_{ij})$ be block $k \times k$ matrices in $\mathbb{F}^{kn \times kn}$ with blocks $X_{ij}, Y_{ij} \in \mathbb{F}^{n \times n}$. Then the* column shifted sum $X \boxplus\!\!\rightarrow Y$ *and* row shifted sum $X \boxplus\!\!\downarrow Y$ *are defined to be*

$$X \boxplus\!\!\rightarrow Y := \begin{bmatrix} X_{11} & \ldots & X_{1k} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ X_{k1} & \ldots & X_{kk} & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_{11} & \ldots & Y_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & Y_{k1} & \ldots & Y_{kk} \end{bmatrix} \in \mathbb{F}^{kn \times k(n+1)},$$

$$X \boxplus\!\!\downarrow Y := \begin{bmatrix} X_{11} & \cdots & X_{1m} \\ \vdots & \ddots & \vdots \\ X_{m1} & \ldots & X_{mm} \\ 0 & \ldots & 0 \end{bmatrix} + \begin{bmatrix} 0 & \ldots & 0 \\ Y_{11} & \ldots & Y_{1m} \\ \vdots & \ddots & \vdots \\ Y_{m1} & \ldots & Y_{mm} \end{bmatrix} \in \mathbb{F}^{k(n+1) \times kn}.$$

With $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$, and $L(\lambda) = \lambda X + Y$, a straightforward calculation with the shifted sums now reveals the equivalences

(3.3)
$$\begin{array}{c} L(\lambda) \in \mathbb{L}_1(P) \\ \text{with right ansatz vector } v \end{array} \iff X \boxplus\!\!\rightarrow Y = v \otimes [A_k \; A_{k-1} \; \ldots \; A_0],$$

(3.4)
$$\begin{array}{c} L(\lambda) \in \mathbb{L}_2(P) \\ \text{with left ansatz vector } w \end{array} \iff X \boxplus\!\!\downarrow Y = w^T \otimes \begin{bmatrix} A_k \\ \vdots \\ A_0 \end{bmatrix}.$$

3.2. Building T-palindromic pencils in $\mathbb{L}_1(P)$. For the moment, let us focus our attention on $\mathbb{L}_1(P)$ and try to construct a T-palindromic pencil in $\mathbb{L}_1(P)$ for a

matrix polynomial $P(\lambda)$ that is T-palindromic. We begin with the simplest nontrivial example.

Example 3.2. Consider the T-palindromic matrix polynomial $\lambda^2 A + \lambda B + A^T$, where $B = B^T$ and $A \neq 0$. We try to construct a T-palindromic pencil $L(\lambda) \in \mathbb{L}_1(P)$ with a nonzero right ansatz vector $v = [v_1, v_2]^T \in \mathbb{F}^2$. This means that $L(\lambda)$ must be of the form

$$L(\lambda) = \lambda Z + Z^T =: \lambda \begin{bmatrix} D & E \\ F & G \end{bmatrix} + \begin{bmatrix} D^T & F^T \\ E^T & G^T \end{bmatrix}, \quad D, E, F, G \in \mathbb{F}^{n \times n}.$$

Since $L(\lambda) \in \mathbb{L}_1(P)$, the equivalence given by (3.3) implies that

$$Z \boxplus Z^T = \begin{bmatrix} D & E + D^T & F^T \\ F & G + E^T & G^T \end{bmatrix} = \begin{bmatrix} v_1 A & v_1 B & v_1 A^T \\ v_2 A & v_2 B & v_2 A^T \end{bmatrix}.$$

Equating corresponding blocks in the first and last columns, we obtain $D = v_1 A$, $F = v_2 A = v_1 A$, and $G = v_2 A$. This forces $v_1 = v_2$, since $A \neq 0$ by assumption. From either block of the middle column, we see that $E = v_1(B - A^T)$; with this choice for E all the equations are consistent, thus yielding

$$(3.5) \qquad L(\lambda) = \lambda Z + Z^T = v_1 \left(\lambda \begin{bmatrix} A & B - A^T \\ A & A \end{bmatrix} + \begin{bmatrix} A^T & A^T \\ B - A & A^T \end{bmatrix} \right).$$

This example illustrates three important properties: (1) the choice of right ansatz vectors v for which $L(\lambda) \in \mathbb{L}_1(P)$ is T-palindromic is restricted; (2) once one of these restricted right ansatz vectors v is chosen, a T-palindromic pencil $L(\lambda) \in \mathbb{L}_1(P)$ is uniquely determined; (3) interchanging the first and second block rows of $L(\lambda)$, i.e., premultiplying by $R_2 \otimes I$, yields the pencil

$$(R_2 \otimes I)L(\lambda) = v_1 \left(\lambda \begin{bmatrix} A & A \\ A & B - A^T \end{bmatrix} + \begin{bmatrix} B - A & A^T \\ A^T & A^T \end{bmatrix} \right),$$

which the column- and row-shifted sums easily confirm to be a pencil in the *double ansatz space* $\mathbb{DL}(P) := \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$ with left and right ansatz vector $v = [v_1, v_1]^T$. These three observations turn out to be true in general for T-palindromic matrix polynomials P and T-palindromic pencils in $\mathbb{L}_1(P)$.

THEOREM 3.3. *Let $P(\lambda)$ be a T-palindromic matrix polynomial and suppose $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v. Then the pencil $L(\lambda)$ is T-palindromic if and only if $Rv = v$ and $(R \otimes I)L(\lambda) \in \mathbb{DL}(P)$ with right and left ansatz vector Rv, where R is the reverse identity as in (2.2). Moreover, for any $v \in \mathbb{F}^k$ satisfying $Rv = v$ there exists a unique pencil $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v and T-palindromic structure.*

The proof of this theorem is deferred to the next section, where it is subsumed under the even more general result stated in Theorem 3.5.

The double ansatz space $\mathbb{DL}(P)$ was introduced in [28] as a natural space in which to look for pencils that enjoy both the right and left eigenvector recovery properties. This feature was successfully exploited in [14] to find linearizations with optimally conditioned eigenvalues. Now Example 3.2 suggests that $\mathbb{DL}(P)$ could also play an important role in the search for structured linearizations.

3.3. Existence of structured pencils in $\mathbb{L}_1(P)$. For a $\star$-(anti-)palindromic or $\star$-even/odd polynomial it is natural to seek a linearization with the same structure

as P. From the point of view of numerical analysis, however, one of the most important reasons for using a structure-preserving method is to preserve spectral symmetries. But we see in Table 2.2 that for each structure under consideration there is also an "anti" version of that structure with the same spectral symmetry. Thus it makes sense to try to linearize a structured polynomial with an "anti-structured" pencil as well as with a structured one; so in this section we also characterize the pencils in $\mathbb{L}_1(P)$ having the "anti-structure" of P.

Before turning to the main results of this section, we draw the reader's attention to two key properties of $\mathbb{DL}(P)$ that will be systematically used in their proofs. Recall that the left and right ansatz vectors of the double ansatz pencil $(R_2 \otimes I)L(\lambda)$ in Example 3.2 coincide. This is, in fact, a property shared by all pencils in $\mathbb{DL}(P)$, thus leading to the notion of a single *ansatz vector* instead of separate left/right ansatz vectors for these pencils. Furthermore, every pencil in $\mathbb{DL}(P)$ is uniquely determined by its ansatz vector.

THEOREM 3.4 (see [13], [28]). *Let* $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ *be a (not necessarily regular) matrix polynomial with coefficients in* $\mathbb{F}^{n \times n}$ *and* $A_k \neq 0$. *Then for vectors* $v, w \in \mathbb{F}^k$ *there exists a* $kn \times kn$ *matrix pencil* $L(\lambda) \in \mathbb{DL}(P)$ *with right ansatz vector* w *and left ansatz vector* v *if and only if* $v = w$. *Moreover, the pencil* $L(\lambda) \in \mathbb{DL}(P)$ *is uniquely determined by* v.

We now extend the result of Theorem 3.3 to $\star$-(anti-)palindromic structures, showing that there is only a restricted class of admissible right ansatz vectors v that can support a structured or "anti-structured" pencil in $\mathbb{L}_1(P)$. In each case the restrictions on the vector v can be concisely described using the reverse identity $R = R_k$ as defined in (2.2).

THEOREM 3.5. *Suppose the matrix polynomial* $P(\lambda)$ *is* $\star$-*palindromic or* $\star$-*anti-palindromic. Then for pencils* $L(\lambda) \in \mathbb{L}_1(P)$ *with right ansatz vector* v, *conditions* (i) *and* (ii) *in Table* 3.1 *are equivalent. Moreover, for any* $v \in \mathbb{F}^k$ *satisfying one of the admissibility conditions for* v *in* (ii), *there exists a unique pencil* $L(\lambda) \in \mathbb{L}_1(P)$ *with right ansatz vector* v *and the corresponding structure in* (i).

TABLE 3.1

Structure of $P(\lambda)$	Equivalent conditions	
	(i) $L(\lambda)$ is	(ii) $(R \otimes I)L(\lambda) \in \mathbb{DL}(P)$ with ansatz vector Rv and
T-palindromic	T-palindromic	$Rv = v$
	T-anti-palindromic	$Rv = -v$
T-anti-palindromic	T-palindromic	$Rv = -v$
	T-anti-palindromic	$Rv = v$
$*$-palindromic	$*$-palindromic	$Rv = \overline{v}$
	$*$-anti-palindromic	$Rv = -\overline{v}$
$*$-anti-palindromic	$*$-palindromic	$Rv = -\overline{v}$
	$*$-anti-palindromic	$Rv = \overline{v}$

Proof. We consider all eight cases simultaneously. Let $P(\lambda)$ be $\star$-palindromic or $\star$-anti-palindromic, so that $\operatorname{rev} P^\star(\lambda) = \chi_{_P} P(\lambda)$ for $\chi_{_P} = \pm 1$.

(i) $\Rightarrow$ (ii):   By (i), $\operatorname{rev} L^\star(\lambda) = \chi_{_L} L(\lambda)$ for $\chi_{_L} = \pm 1$. Since $L(\lambda) \in \mathbb{L}_1(P)$, we have

$$(3.6) \qquad L(\lambda)(\varLambda \otimes I) = v \otimes P(\lambda).$$

Taking the reversal of both sides of (3.6), and noting that $R\varLambda = \operatorname{rev} \varLambda$, we have

$$\operatorname{rev} L(\lambda)(R \otimes I)(\varLambda \otimes I) = \operatorname{rev} L(\lambda)\big((\operatorname{rev} \varLambda) \otimes I\big) = v \otimes \operatorname{rev} P(\lambda).$$

Now applying the adjoint $\star$ to both sides, we obtain

$$(\varLambda^\star \otimes I)(R \otimes I) \operatorname{rev} L^\star(\lambda^\star) = v^\star \otimes \operatorname{rev} P^\star(\lambda^\star),$$

or equivalently

$$(3.7) \qquad \big(\varLambda^\star \otimes I\big)(R \otimes I)L(\lambda^\star) = (\chi_{_P}\chi_{_L} v^\star) \otimes P(\lambda^\star),$$

since $L(\lambda)$ and $P(\lambda)$ are either $\star$-palindromic or $\star$-anti-palindromic. Then using the fact that (3.7) is an identity, we replace $\lambda^\star$ with λ to obtain

$$(\varLambda^T \otimes I)(R \otimes I)L(\lambda) = (\chi_{_P}\chi_{_L} v^\star) \otimes P(\lambda),$$

thus showing $(R \otimes I)L(\lambda)$ to be in $\mathbb{L}_2(P)$ with left ansatz vector $w = \chi_{_P}\chi_{_L}(v^\star)^T$. On the other hand, multiplying (3.6) on the left by $R \otimes I$ yields

$$(R \otimes I)L(\lambda)(\varLambda \otimes I) = (Rv) \otimes P(\lambda),$$

so $(R \otimes I)L(\lambda)$ is also in $\mathbb{L}_1(P)$ with right ansatz vector Rv. Thus $(R \otimes I)L(\lambda)$ is in $\mathbb{DL}(P) = \mathbb{L}_1(P) \cap \mathbb{L}_2(P)$, and from Theorem 3.4 the equality of right and left ansatz vectors implies that $Rv = \chi_{_P}\chi_{_L}(v^\star)^T$. All eight variants of condition (ii) now follow by noting that $(v^*)^T = \overline{v}$ and $(v^T)^T = v$.

(ii) $\Rightarrow$ (i):   Since $(R \otimes I)L(\lambda)$ is in $\mathbb{DL}(P)$ with ansatz vector Rv, we have

$$(3.8) \qquad\qquad (R \otimes I)L(\lambda)(\varLambda \otimes I) \;=\; (Rv) \otimes P(\lambda),$$
$$(3.9) \;\; \big((\varLambda^T R) \otimes I\big)L(\lambda) \;=\; (\varLambda^T \otimes I)(R \otimes I)L(\lambda) \;=\; (Rv)^T \otimes P(\lambda).$$

Applying the adjoint $\star$ to both ends of (3.9) gives

$$L^\star(\lambda^\star)\big((R(\varLambda^T)^\star) \otimes I\big) = R(v^T)^\star \otimes P^\star(\lambda^\star),$$

or equivalently

$$(3.10) \qquad L^\star(\lambda)\big((R\varLambda) \otimes I\big) = R(v^T)^\star \otimes P^\star(\lambda).$$

Note that all cases of condition (ii) may be expressed in the form $R(v^T)^\star = \varepsilon\chi_{_P} v$, where $\varepsilon = \pm 1$. Then taking the reversal of both sides in (3.10) and using $R\varLambda = \operatorname{rev} \varLambda$, we obtain

$$\operatorname{rev} L^\star(\lambda)(\varLambda \otimes I) = (\varepsilon\chi_{_P} v) \otimes \operatorname{rev} P^\star(\lambda) = (\varepsilon v) \otimes P(\lambda),$$

and after multiplying by $\varepsilon(R \otimes I)$,

$$\varepsilon(R \otimes I) \operatorname{rev} L^\star(\lambda)(\varLambda \otimes I) = (Rv) \otimes P(\lambda).$$

Thus we see that the pencil $\varepsilon(R \otimes I)\operatorname{rev} L^\star(\lambda)$ is in $\mathbb{L}_1(P)$ with right ansatz vector Rv. Starting over again from identity (3.8) and taking the adjoint $\star$ of both sides, we obtain by analogous reasoning that

$$
(R \otimes I)L(\lambda)(\Lambda \otimes I) = (Rv) \otimes P(\lambda)
$$
$$
\iff (\Lambda^T \otimes I)L^\star(\lambda)(R \otimes I) = (v^\star R) \otimes P^\star(\lambda) = \big(v^\star \otimes P^\star(\lambda)\big)(R \otimes I)
$$
$$
\iff (\Lambda^T \otimes I)L^\star(\lambda) = v^\star \otimes P^\star(\lambda)
$$
$$
\iff (\operatorname{rev} \Lambda^T \otimes I)\operatorname{rev} L^\star(\lambda) = v^\star \otimes \operatorname{rev} P^\star(\lambda)
$$
$$
\iff (\Lambda^T R \otimes I)\operatorname{rev} L^\star(\lambda) = (\varepsilon \chi_P Rv)^T \otimes \operatorname{rev} P^\star(\lambda) = (\varepsilon Rv)^T \otimes P(\lambda)
$$
$$
\iff (\Lambda^T \otimes I)\big(\varepsilon(R \otimes I)\operatorname{rev} L^\star(\lambda)\big) = (Rv)^T \otimes P(\lambda).
$$

Thus the pencil $\varepsilon(R \otimes I)\operatorname{rev} L^\star(\lambda)$ is also in $\mathbb{L}_2(P)$ with left ansatz vector Rv, and hence in $\mathbb{DL}(P)$ with ansatz vector Rv. But $(R \otimes I)L(\lambda) \in \mathbb{DL}(P)$ with exactly the same ansatz vector Rv, and so the uniqueness property of Theorem 3.4 implies that

$$
\varepsilon(R \otimes I)\operatorname{rev} L^\star(\lambda) = (R \otimes I)L(\lambda),
$$

or equivalently $\varepsilon \operatorname{rev} L^\star(\lambda) = L(\lambda)$. Hence $L(\lambda)$ is $\star$-palindromic or $\star$-anti-palindromic, depending on the parameter ε, which implies all the variants of condition (i) in Table 3.2.

Finally, the existence and uniqueness of a structured pencil $L(\lambda)$ corresponding to any admissible right ansatz vector v follow directly from the existence and uniqueness in Theorem 3.4 of the $\mathbb{DL}(P)$-pencil $(R \otimes I)L(\lambda)$ for the ansatz vector Rv. $\square$

We next present the analogue of Theorem 3.5 for $\star$-even and $\star$-odd polynomials. Here $\Sigma = \Sigma_k$ as defined in (2.2) helps to concisely describe the restriction on the ansatz vector v.

THEOREM 3.6. *Suppose the matrix polynomial $P(\lambda)$ is $\star$-even or $\star$-odd. Then for pencils $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v, conditions* (i) *and* (ii) *in Table* 3.2 *are equivalent. Moreover, for any $v \in \mathbb{F}^k$ satisfying one of the admissibility conditions for v in* (ii), *there exists a unique pencil $L(\lambda) \in \mathbb{L}_1(P)$ with right ansatz vector v and the corresponding structure in* (i).

TABLE 3.2

Structure of $P(\lambda)$	Equivalent conditions	
	(i) $L(\lambda)$ is	(ii) $(\Sigma \otimes I)L(\lambda) \in \mathbb{DL}(P)$ with ansatz vector Σv and
T-even	T-even	$\Sigma v = v$
	T-odd	$\Sigma v = -v$
T-odd	T-even	$\Sigma v = -v$
	T-odd	$\Sigma v = v$
$*$-even	$*$-even	$\Sigma v = \overline{v}$
	$*$-odd	$\Sigma v = -\overline{v}$
$*$-odd	$*$-even	$\Sigma v = -\overline{v}$
	$*$-odd	$\Sigma v = \overline{v}$

Proof. Follow the strategy, mutatis mutandis, of the proof of Theorem 3.5: replace multiplications by $R \otimes I$ with multiplications by $\Sigma \otimes I$, and replace taking reversals with the substitution $\lambda \mapsto -\lambda$. Observe that for the vector Λ, this substitution is equivalent to premultiplication by Σ, since $\Sigma\Lambda = [(-\lambda)^{k-1}, \dots, -\lambda, 1]^T$. □

Thus we see that the ansatz vectors of structured pencils closely reflect the structure of the pencil itself. This pleasing fact influences both the existence and the construction of structured linearizations, as we will see in the following sections.

3.4. Construction of structured pencils. As we have seen in Theorems 3.5 and 3.6, pencils in $\mathbb{L}_1(P)$ with one of the $\star$-structures listed in Table 2.1 are strongly related to pencils in $\mathbb{DL}(P)$. This observation leads to the following procedure for the construction of potential structured linearizations:

(1) Choose a right ansatz vector $v \in \mathbb{F}^k$ that is admissible for the desired type of $\star$-structure.

(2) Construct the unique $\widetilde{L}(\lambda) \in \mathbb{DL}(P)$ with ansatz vector $w = Rv$ or $w = \Sigma v$, according to the desired structure.

(3) Premultiply $\widetilde{L}(\lambda)$ by $R^{-1} \otimes I$ or $\Sigma^{-1} \otimes I$ to obtain a structured pencil in $\mathbb{L}_1(P)$ with right ansatz vector v.

All that remains is to show how to carry out step (2). This can be done concretely and explicitly using the following canonical basis for $\mathbb{DL}(P)$ derived in [13]. Given a matrix polynomial $P(\lambda) = \sum_{i=0}^k \lambda^i A_i$, consider for $j = 0, \dots, k$ the block diagonal matrices $X_j = \mathrm{diag}(\mathcal{L}_j, -\mathcal{U}_{k-j})$, whose diagonal blocks are the block $j \times j$ block-Hankel matrices

$$\mathcal{L}_j = \begin{bmatrix} & & & A_k \\ & & \cdot^{\cdot^{\cdot}} & A_{k-1} \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ A_k & A_{k-1} & \dots & A_{k-j+1} \end{bmatrix} \quad \text{and} \quad \mathcal{U}_j = \begin{bmatrix} A_{j-1} & \dots & A_1 & A_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ A_1 & \cdot^{\cdot^{\cdot}} & & \\ A_0 & & & \end{bmatrix}.$$

Observe that $\mathcal{L}_j, \mathcal{U}_j \in \mathbb{F}^{jn \times jn}$, with the convention that they are empty when $j = 0$. Thus each X_j is a block $k \times k$ matrix in $\mathbb{F}^{kn \times kn}$. As an illustration we give the complete list of matrices X_0, X_1, X_2, X_3 for $k = 3$:

$$\begin{bmatrix} -A_2 & -A_1 & -A_0 \\ -A_1 & -A_0 & 0 \\ -A_0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} A_3 & 0 & 0 \\ 0 & -A_1 & -A_0 \\ 0 & -A_0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & A_3 & 0 \\ A_3 & A_2 & 0 \\ 0 & 0 & -A_0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & A_3 \\ 0 & A_3 & A_2 \\ A_3 & A_2 & A_1 \end{bmatrix}.$$

Matrices of this type have appeared in the literature; see, e.g., [9], [22], and for the scalar ($n = 1$) case see [21]. One can easily compute the shifted sums

$$X_j \boxright (-X_{j-1}) = e_j \otimes \begin{bmatrix} A_k & \dots & A_0 \end{bmatrix} \quad \text{and} \quad X_j \boxdown (-X_{j-1}) = e_j^T \otimes \begin{bmatrix} A_k \\ \vdots \\ A_0 \end{bmatrix},$$

thus verifying by (3.3) and (3.4) that the pencil $\lambda X_j - X_{j-1}$ is in $\mathbb{DL}(P)$ with ansatz vector e_j for $j = 1, \dots, k$. Consequently the set $\{\lambda X_j - X_{j-1} : j = 1, \dots, k\}$ constitutes the natural or canonical basis for $\mathbb{DL}(P)$. A pencil $\lambda X + Y$ in $\mathbb{DL}(P)$ with ansatz vector $w = [w_1, \dots, w_k]^T$ can now be uniquely expressed as a linear combination

$$(3.11) \qquad \lambda X + Y = \sum_{j=1}^k w_j (\lambda X_j - X_{j-1}) = \lambda \sum_{j=1}^k w_j X_j - \sum_{j=1}^k w_j X_{j-1}.$$

Note that there are alternative procedures for the construction of pencils from $\mathbb{DL}(P)$; an explicit formula, for example, is given in [28], while a recursive method using the shifted sum is presented in [27].

3.5. Which structured pencils are linearizations? Recall from section 3.1 that when $P(\lambda)$ is regular, then any regular pencil in $\mathbb{L}_1(P)$ is a (strong) linearization for P. Although there is a systematic procedure [28] for determining the regularity of a pencil $L(\lambda) \in \mathbb{L}_1(P)$, there is, in general, no connection between this regularity and the right ansatz vector of $L(\lambda)$. By contrast, for pencils in $\mathbb{DL}(P)$ there is a criterion that characterizes regularity directly in terms of their ansatz vectors, which gives these pencils an important advantage. Let $v = [v_1, v_2, \ldots, v_k]^T$ be the ansatz vector of $L(\lambda) \in \mathbb{DL}(P)$ and define the associated v-*polynomial* to be the scalar polynomial

$$\mathsf{p}(x\,;v) := v_1 x^{k-1} + v_2 x^{k-2} + \cdots + v_{k-1}x + v_k.$$

By convention, we say that ∞ is a root of $\mathsf{p}(x\,;v)$ if $v_1 = 0$. Then regularity of $L(\lambda) \in \mathbb{DL}(P)$ can be expressed in terms of the roots of this v-polynomial and the eigenvalues of P, as follows.

THEOREM 3.7 (eigenvalue exclusion theorem [28]). *Suppose that $P(\lambda)$ is a regular matrix polynomial and $L(\lambda)$ is in $\mathbb{DL}(P)$ with nonzero ansatz vector v. Then $L(\lambda)$ is regular and thus a (strong) linearization for $P(\lambda)$ if and only if no root of the v-polynomial $\mathsf{p}(x\,;v)$ is an eigenvalue of $P(\lambda)$.*

Note that in Theorem 3.7 we include ∞ as one of the possible roots of $\mathsf{p}(x\,;v)$ or eigenvalues of P. We can now quickly deduce the following theorem.

THEOREM 3.8 (structured linearization theorem). *Suppose the regular matrix polynomial $P(\lambda)$ and the nonzero pencil $L(\lambda) \in \mathbb{L}_1(P)$ have one of the 16 combinations of $\star$-structure considered in Tables 3.1 and 3.2. Let v be the nonzero right ansatz vector of $L(\lambda)$, and let*

$$w = \begin{cases} Rv & \textit{if } P \textit{ is } \star\textit{-palindromic or } \star\textit{-anti-palindromic}, \\ \Sigma v & \textit{if } P \textit{ is } \star\textit{-even or } \star\textit{-odd}. \end{cases}$$

Then $L(\lambda)$ is a (strong) linearization for $P(\lambda)$ if and only if no root of the v-polynomial $\mathsf{p}(x\,;w)$ is an eigenvalue of $P(\lambda)$.

Proof. For all eight structure combinations in Table 3.1, it was shown in Theorem 3.5 that $(R \otimes I)L(\lambda)$ is in $\mathbb{DL}(P)$ with ansatz vector Rv. Similarly, for the eight even/odd structure combinations in Table 3.2 it was shown in Theorem 3.6 that $(\Sigma \otimes I)L(\lambda)$ is in $\mathbb{DL}(P)$ with ansatz vector Σv. Since $L(\lambda)$ is a linearization for $P(\lambda)$ if and only if $(R \otimes I)L(\lambda)$ or $(\Sigma \otimes I)L(\lambda)$ is, the desired result follows immediately from the eigenvalue exclusion theorem. □

We illustrate the implications of Theorem 3.8 with an example.

Example 3.9. Suppose the T-palindromic polynomial $P(\lambda) = \lambda^2 A + \lambda B + A^T$ from Example 3.2 is regular. Theorem 3.3 restricts the admissible ansatz vectors $v \in \mathbb{F}^2$ of a T-palindromic pencil $L(\lambda) \in \mathbb{L}_1(P)$ to those that satisfy $Rv = v$, or equivalently, $v = (v_1, v_1)^T$. We see from Theorem 3.8 that such an $L(\lambda)$ will be a strong linearization for $P(\lambda)$ if and only if none of the roots of the v-polynomial $\mathsf{p}(x\,;Rv) = v_1 x + v_1$ are eigenvalues of $P(\lambda)$, that is, if and only if -1 is not an eigenvalue of $P(\lambda)$. On the other hand, a T-anti-palindromic pencil $\widetilde{L}(\lambda) \in \mathbb{L}_1(P)$ will be a linearization for P if and only if $\lambda = 1$ is not an eigenvalue of $P(\lambda)$. This is because every admissible ansatz vector for $\widetilde{L}(\lambda)$ is constrained by Theorem 3.5 to be of the form $\widetilde{v} = [\, v_1, -v_1 \,]^T$, forcing $\mathsf{p}(x\,;R\widetilde{v}) = -v_1 x + v_1$, with only $+1$ as a root.

This example also illustrates another way in which structure influences the players in our story: when P is T-palindromic, any ansatz vector admissible for a T-(anti-)palindromic pencil in $\mathbb{L}_1(P)$ has components that read the same forwards or backwards (up to sign). This in turn forces the corresponding v-polynomial to be (anti-)palindromic. Theorems 3.5 and 3.6 imply that analogous parallels in structure hold for other combinations of $\star$-structures in P and L and the relevant v-polynomial $\mathsf{p}(x\,;Rv)$ or $\mathsf{p}(x\,;\Sigma v)$; for convenience these are listed together in Table 3.3.

TABLE 3.3
Parallelism of structures.

$P(\lambda)$	$L(\lambda) \in \mathbb{L}_1(P)$	v-polynomial	$P(\lambda)$	$L(\lambda) \in \mathbb{L}_1(P)$	v-poly.
$\star$-palindromic	$\star$-palindromic	$\star$-palindromic	$\star$-even	$\star$-even	$\star$-even
	$\star$-anti-palindromic	$\star$-anti-palindromic		$\star$-odd	$\star$-odd
$\star$-anti-palindromic	$\star$-palindromic	$\star$-anti-palindromic	$\star$-odd	$\star$-even	$\star$-odd
	$\star$-anti-palindromic	$\star$-palindromic		$\star$-odd	$\star$-even

3.6. When pairings degenerate. The parallel of structures between matrix polynomial, $\mathbb{L}_1(P)$-pencil, and v-polynomial (see Table 3.3) is aesthetically very pleasing: structure in a v-polynomial forces a pairing of its roots, as in Theorem 2.2, which is always of the *same qualitative type* as the eigenvalue pairing present in the original structured matrix polynomial. However, it turns out that this root pairing can sometimes be an obstruction to the existence of any structured linearization in $\mathbb{L}_1(P)$ at all.

Using an argument based mainly on the very simple form of admissible ansatz vectors when $k = 2$, we saw in Example 3.9 that a quadratic T-palindromic matrix polynomial having both 1 and -1 as eigenvalues cannot have a structured linearization in $\mathbb{L}_1(P)$: the presence of -1 in the spectrum precludes the existence of a T-palindromic linearization, while the eigenvalue 1 excludes T-anti-palindromic linearizations. We now show that this difficulty is actually a consequence of root pairing, and therefore can occur for higher degree polynomials.

When $P(\lambda)$ has even degree, all its ansatz vectors have even length, and hence the corresponding v-polynomials all have an odd number of roots (counting multiplicities and including ∞). Root pairing then forces at least one root of every v-polynomial to lie in a subset of $\mathbb{C}$ where this pairing "degenerates." This means that for any T-(anti-)palindromic matrix polynomial $P(\lambda)$ of even degree, *every* v-polynomial of a T-(anti-)palindromic pencil in $\mathbb{L}_1(P)$ has at least one root belonging to $\{-1, +1\}$. It follows that any such $P(\lambda)$ having both $+1$ and -1 as eigenvalues can have neither a T-palindromic nor a T-anti-palindromic linearization in $\mathbb{L}_1(P)$. For T-even/odd matrix polynomials $P(\lambda)$ of even degree, every relevant v-polynomial has a root belonging to $\{0, \infty\}$; thus if the spectrum of $P(\lambda)$ includes both 0 and ∞, then P cannot have a T-even or T-odd linearization in $\mathbb{L}_1(P)$.

When no structured linearization for $P(\lambda)$ exists in $\mathbb{L}_1(P)$, it is natural to ask whether $P(\lambda)$ has a structured linearization that is *not* in $\mathbb{L}_1(P)$, or perhaps has no structured linearizations at all. The next examples show that both alternatives may occur.

Example 3.10. Consider the T-palindromic polynomial $P(\lambda) = \lambda^2 + 2\lambda + 1$. Then the only eigenvalue of $P(\lambda)$ is -1, so by the observation in Example 3.9 we see that $P(\lambda)$ cannot have any T-palindromic linearization in $\mathbb{L}_1(P)$. But does $P(\lambda)$ have a

T-palindromic linearization $L(\lambda)$ which is not in $\mathbb{L}_1(P)$? Consider the general 2×2 T-palindromic pencil

$$(3.12) \quad L(\lambda) = \lambda Z + Z^T = \lambda \begin{bmatrix} w & x \\ y & z \end{bmatrix} + \begin{bmatrix} w & y \\ x & z \end{bmatrix} = \begin{bmatrix} w(\lambda+1) & \lambda x + y \\ \lambda y + x & z(\lambda+1) \end{bmatrix},$$

and suppose it is a linearization for P. Since the only eigenvalue $\lambda = -1$ of $P(\lambda)$ has geometric multiplicity 1, the same must be true for $L(\lambda)$, that is, rank $L(-1) = 1$. But inserting $\lambda = -1$ in (3.12), we obtain a matrix that does not have rank 1 for any values of w, x, y, z. Thus $P(\lambda)$ does not have any T-palindromic linearization. However, $P(\lambda)$ does have a T-anti-palindromic linearization $\widetilde{L}(\lambda)$ in $\mathbb{L}_1(P)$ because it does not have the eigenvalue $+1$. Choosing $\widetilde{v} = (1, -1)^T$ as the right ansatz vector and following the procedure in section 3.4 yields the structured linearization

$$\widetilde{L}(\lambda) = \lambda \widetilde{Z} - \widetilde{Z}^T = \lambda \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \in \mathbb{L}_1(P).$$

Example 3.11. Consider the T-palindromic matrix polynomial

$$P(\lambda) = \lambda^2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Since $\det P(\lambda) = (\lambda^2 - 1)^2$, this polynomial $P(\lambda)$ has $+1$ and -1 as eigenvalues, each with algebraic multiplicity 2. Thus $P(\lambda)$ has neither a T-palindromic nor a T-anti-palindromic linearization in $\mathbb{L}_1(P)$. However, it is possible to construct a T-palindromic linearization for $P(\lambda)$ that is not in $\mathbb{L}_1(P)$. Starting with the first companion form $C_1(\lambda)$, one can verify that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot C_1(\lambda) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

is a T-palindromic linearization for $P(\lambda)$. Using shifted sums, it can easily be verified that this linearization is in neither $\mathbb{L}_1(P)$ nor $\mathbb{L}_2(P)$.

Example 3.12. Consider the scalar matrix polynomial $P(\lambda) = \lambda^2 - 1$ which is T-anti-palindromic and has the roots ± 1. Again, the presence of these eigenvalues precludes the existence of either a T-palindromic or T-anti-palindromic linearization in $\mathbb{L}_1(P)$. But even more is true. It turns out that $P(\lambda)$ does not have any T-palindromic or T-anti-palindromic linearization at all. Indeed, suppose that $L_\varepsilon(\lambda) = \lambda Z + \varepsilon Z^T$ was a linearization for $P(\lambda)$, where $\varepsilon = \pm 1$; that is, $L_\varepsilon(\lambda)$ is T-palindromic if $\varepsilon = 1$ and T-anti-palindromic if $\varepsilon = -1$. Since $P(\lambda)$ does not have the eigenvalue ∞, neither does $L(\lambda)$, and so Z must be invertible. Thus $L_\varepsilon(\lambda)$ is strictly equivalent to the pencil $\lambda I + \varepsilon Z^{-1}Z^T$. But this being a linearization for $P(\lambda)$ forces the matrix $\varepsilon Z^{-1}Z^T$ to have the simple eigenvalues $+1$ and -1, and hence $\det \varepsilon Z^{-1}Z^T = -1$. However, we also see that

$$\det \varepsilon Z^{-1}Z^T = \varepsilon^2 \frac{1}{\det Z} \det Z = 1,$$

which is a contradiction. Hence $P(\lambda)$ has neither a T-palindromic linearization nor a T-anti-palindromic linearization.

One possibility for circumventing the difficulties associated with the eigenvalues ± 1 is to first deflate them in a structure-preserving manner, using a procedure that

works directly on the original matrix polynomial. Since the resulting matrix polynomial $P(\lambda)$ will not have these troublesome eigenvalues, a structured linearization in $\mathbb{L}_1(P)$ can then be constructed. Such structure-preserving deflation strategies are currently under investigation.

The situation is quite different for $*$-(anti-)palindromic and $*$-even/odd matrix polynomials, because now the set where pairing degenerates is the entire unit circle in $\mathbb{C}$, or the imaginary axis (including ∞), respectively. The contrast between having a continuum versus a finite set in which the root pairing degenerates makes a crucial difference in our ability to guarantee the existence of structured linearizations in $\mathbb{L}_1(P)$. Indeed, consider a regular $*$-palindromic matrix polynomial $P(\lambda)$ of degree k. Then the v-polynomial $\mathsf{p}(x\,;Rv)$ corresponding to an admissible ansatz vector is again $*$-palindromic, with $k-1$ roots occurring in pairs $(\lambda, 1/\bar{\lambda})$ by Theorem 2.2. Thus if k is even, at least one root of $\mathsf{p}(x\,;Rv)$ must lie on the unit circle. But since the spectrum of $P(\lambda)$ is a finite set, it is always possible to choose v so that all the roots of $\mathsf{p}(x\,;Rv)$ avoid the spectrum of $P(\lambda)$. Example 3.13 illustrates the case $k=2$.

Example 3.13. Consider a regular matrix polynomial $P(\lambda) = \lambda^2 A + \lambda B + A^*$ which is $*$-palindromic, that is, $B = B^*$. Choose ζ on the unit circle in $\mathbb{C}$ such that ζ is not an eigenvalue of $P(\lambda)$. Now choose $z \in \mathbb{C}$ so that $\zeta = -z/\bar{z}$. Then $v = (z, \bar{z})^T$ satisfies $Rv = \bar{v}$, and the associated v-polynomial $\mathsf{p}(x\,;Rv) = \bar{z}x + z$ has ζ as its only root. Therefore the $*$-palindromic pencil

$$L(\lambda) = \lambda \begin{bmatrix} zA & zB - \bar{z}A^* \\ \bar{z}A & zA \end{bmatrix} + \begin{bmatrix} \bar{z}A^* & zA^* \\ \bar{z}B - zA & \bar{z}A^* \end{bmatrix} \in \mathbb{L}_1(P)$$

with right ansatz vector v is a (strong) linearization for $P(\lambda)$ by Theorem 3.8.

The observations made in this section have parallels for $\star$-even/odd structures. A list of structured linearizations in $\mathbb{L}_1(P)$ for $\star$-(anti-)palindromic and $\star$-even/odd matrix polynomials of degree $k = 2, 3$ is compiled in Tables 3.4, 3.5, and 3.6.

3.7. The missing structures. So far in section 3 our attention has been focused on finding structured linearizations only for the eight $\star$-structures in Table 2.1. But what about "purely" palindromic, anti-palindromic, even, and odd matrix polynomials? Why have they been excluded from consideration? It turns out that these structures cannot be linearized in a structure-preserving way. For example, consider a regular palindromic polynomial $P(\lambda)$ of degree $k \geq 2$. By [11, Theorem 1.7] a pencil can be a linearization for $P(\lambda)$ only if the geometric multiplicity of each eigenvalue of the pencil is less than or equal to n. On the other hand, any palindromic linearization has the form $L(\lambda) = \lambda Z + Z$, and thus must have the eigenvalue -1 with geometric multiplicity kn. Analogous arguments exclude structure-preserving linearizations for anti-palindromic, even, and odd polynomials.

TABLE 3.4

Structured linearizations for $\lambda^2 A + \lambda B + C$. Except for the parameters $r \in \mathbb{R}$ and $z \in \mathbb{C}$, the linearizations are unique up to a (suitable) scalar factor. The last column lists the roots of the v-polynomial $\mathsf{p}(x\,;Mv)$ corresponding to $M = R$ or $M = \Sigma$, respectively.

Structure of $P(\lambda)$	Structure of $L(\lambda)$	v	$L(\lambda)$ with ansatz vector v	Root of $\mathsf{p}(x; Mv)$
T-palin-dromic	T-palin-dromic	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} A & B-C \\ A & A \end{bmatrix} + \begin{bmatrix} C & C \\ B-A & C \end{bmatrix}$	-1
$B = B^T$ $C = A^T$	T-anti-palin-dromic	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\lambda \begin{bmatrix} A & B+C \\ -A & A \end{bmatrix} + \begin{bmatrix} -C & C \\ -B-A & -C \end{bmatrix}$	1
T-anti-palin-dromic	T-palin-dromic	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$	$\lambda \begin{bmatrix} A & B+C \\ -A & A \end{bmatrix} + \begin{bmatrix} -C & C \\ -B-A & -C \end{bmatrix}$	1
$B = -B^T$ $C = -A^T$	T-anti-palin-dromic	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} A & B-C \\ A & A \end{bmatrix} + \begin{bmatrix} C & C \\ B-A & C \end{bmatrix}$	-1
$*$-palin-dromic	$*$-palin-dromic	$\begin{bmatrix} z \\ \bar{z} \end{bmatrix}$	$\lambda \begin{bmatrix} zA & zB-\bar{z}C \\ \bar{z}A & zA \end{bmatrix} + \begin{bmatrix} \bar{z}C & zC \\ \bar{z}B-zA & \bar{z}C \end{bmatrix}$	$-z/\bar{z}$
$B = B^*$ $C = A^*$	$*$-anti-palin-dromic	$\begin{bmatrix} z \\ -\bar{z} \end{bmatrix}$	$\lambda \begin{bmatrix} zA & zB+\bar{z}C \\ -\bar{z}A & zA \end{bmatrix} + \begin{bmatrix} -\bar{z}C & zC \\ -\bar{z}B-zA & -\bar{z}C \end{bmatrix}$	$z/\bar{z}$
T-even	T-even	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & -A \\ A & B \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$	∞
$A = A^T$ $B = -B^T$ $C = C^T$	T-odd	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\lambda \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} B & C \\ -C & 0 \end{bmatrix}$	0
T-odd	T-even	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\lambda \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} B & C \\ -C & 0 \end{bmatrix}$	0
$A = -A^T$ $B = B^T$ $C = -C^T$	T-odd	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & -A \\ A & B \end{bmatrix} + \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}$	∞
$*$-even	$*$-even	$\begin{bmatrix} i \\ r \end{bmatrix}$	$\lambda \begin{bmatrix} iA & -rA \\ rA & rB+iC \end{bmatrix} + \begin{bmatrix} rA+iB & iC \\ -iC & rC \end{bmatrix}$	$-ir$
$A = A^*$ $B = -B^*$ $C = C^*$	$*$-odd	$\begin{bmatrix} r \\ i \end{bmatrix}$	$\lambda \begin{bmatrix} rA & -iA \\ iA & iB+rC \end{bmatrix} + \begin{bmatrix} iA+rB & rC \\ -rC & iC \end{bmatrix}$	$\dfrac{i}{r}$

TABLE 3.5

*★-palindromic linearizations for the ★-palindromic matrix polynomial $\lambda^3 A + \lambda^2 B + \lambda B^\star + A^\star$. The last column lists the roots of the v-polynomial $\mathsf{p}(x; Rv)$ corresponding to Rv. All ★-palindromic linearizations in $\mathbb{L}_1(P)$ for this matrix polynomial are linear combinations of the first two linearizations in the case ★ = T, and are real linear combinations of the first three linearizations in the case ★ = *. A specific example is given by the fourth linearization.*

v	$L(\lambda)$ with right ansatz vector v	Roots of $\mathsf{p}(x; Rv)$
$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & 0 & -A^\star \\ A & B & 0 \\ 0 & A & 0 \end{bmatrix} + \begin{bmatrix} 0 & A^\star & 0 \\ 0 & B^\star & A^\star \\ -A & 0 & 0 \end{bmatrix}$	$0, \infty$
$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} A & B - A^\star & B^\star \\ 0 & A - B^\star & B - A^\star \\ A & 0 & A \end{bmatrix} + \begin{bmatrix} A^\star & 0 & A^\star \\ B^\star - A & A^\star - B & 0 \\ B & B^\star - A & A^\star \end{bmatrix}$	$i, -i$
$\begin{bmatrix} i \\ 0 \\ -i \end{bmatrix}$	$\lambda \begin{bmatrix} iA & iB + iA^* & iB^* \\ 0 & iA + iB^* & iB + iA^* \\ -iA & 0 & iA \end{bmatrix} + \begin{bmatrix} -iA^* & 0 & iA^* \\ -iB^* - iA & -iA^* - iB & 0 \\ -iB & -iB^* - iA & -iA^* \end{bmatrix}$	$1, -1$
$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} A & B - A^\star & B^\star - A^\star \\ A & B + A - B^\star & B - A^\star \\ A & A & A \end{bmatrix} + \begin{bmatrix} A^\star & A^\star & A^\star \\ B^\star - A & B^\star + A^\star - B & A^\star \\ B - A & B^\star - A & A^\star \end{bmatrix}$	$\dfrac{-1 \pm i\sqrt{3}}{2}$

TABLE 3.6

*★-even linearizations for the ★-even matrix polynomial $P(\lambda) = \lambda^3 A + \lambda^2 B + \lambda C + D$, where $A = -A^\star$, $B = B^\star$, $C = -C^\star$, $D = D^\star$. The last column lists the roots of the v-polynomial $\mathsf{p}(x; \Sigma v)$ corresponding to Σv. All ★-even linearizations in $\mathbb{L}_1(P)$ for this matrix polynomial are linear combinations of the first two linearizations in the case ★ = T, and are real linear combinations of the first three linearizations in the case ★ = *. A specific example is given by the fourth linearization.*

v	$L(\lambda)$ with right ansatz vector v	Roots of $\mathsf{p}(x; \Sigma v)$
$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & 0 & A \\ 0 & -A & -B \\ A & B & C \end{bmatrix} + \begin{bmatrix} 0 & -A & 0 \\ A & B & 0 \\ 0 & 0 & D \end{bmatrix}$	∞
$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\lambda \begin{bmatrix} A & 0 & 0 \\ 0 & C & D \\ 0 & -D & 0 \end{bmatrix} + \begin{bmatrix} B & C & D \\ -C & -D & 0 \\ D & 0 & 0 \end{bmatrix}$	0
$\begin{bmatrix} 0 \\ i \\ 0 \end{bmatrix}$	$\lambda \begin{bmatrix} 0 & -iA & 0 \\ iA & iB & 0 \\ 0 & 0 & iD \end{bmatrix} + \begin{bmatrix} iA & 0 & 0 \\ 0 & iC & iD \\ 0 & -iD & 0 \end{bmatrix}$	$0, \infty$
$\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$	$\lambda \begin{bmatrix} A & 0 & 4A \\ 0 & C - 4A & D - 4B \\ 4A & 4B - D & 4C \end{bmatrix} + \begin{bmatrix} B & C - 4A & D \\ 4A - C & 4B - D & 0 \\ D & 0 & 4D \end{bmatrix}$	$2i, -2i$

4. Good vibrations from good linearizations. As an illustration of the importance of structure preservation in practical problems, we indicate how the techniques developed in this paper have had a significant impact on computations in an eigenvalue problem occurring in the vibration analysis of rail tracks under excitation arising from high speed trains. This eigenvalue problem has the form

$$(4.1) \qquad \left(\kappa A(\omega) + B(\omega) + \tfrac{1}{\kappa} A(\omega)^T\right) x = 0,$$

where A, B are large, sparse, parameter-dependent, complex square matrices with B complex symmetric and A highly singular. For details of the derivation of this model, see [16] and [17]. The parameter ω is the excitation frequency and the eigenvalue problem has to be solved over a wide frequency range of $\omega = 0$–5,000 Hz. Clearly, for any fixed value of ω, multiplying (4.1) by κ leads to the T-palindromic eigenvalue problem introduced in (1.2). In addition to the presence of a large number of zero and infinite eigenvalues caused by the rank deficiency of A, the finite nonzero eigenvalues cover a wide range of magnitudes that increases as the finite element discretization is made finer. The eigenvalues of the problem under consideration range from 10^{15} to 10^{-15}, thereby making this a very challenging numerical problem.

Attempts at solving this problem with the QZ-algorithm without respecting its structure resulted in computed eigenvalues with no correct digits even in quadruple precision arithmetic. Furthermore, the symmetry of the spectrum with respect to the unit circle was highly perturbed [16].

As an alternative, in [16], [17] a T-palindromic linearization for the eigenvalue problem (4.1) was used. Based on this linearization, the infinite and zero eigenvalues of the resulting T-palindromic pencil could be deflated in a structure-preserving way. The resulting smaller T-palindromic problem was then solved via different methods, resulting in eigenvalues with good accuracy in double precision arithmetic; i.e., the computed eigenvalues were accurate to within the range of the discretization error of the underlying finite element discretization. Thus physically useful eigenvalues were determined, with no modification in the mathematical model or in the discretization scheme. Only the numerical linear algebra was changed, to methods based on the new structure-preserving linearization techniques described in this paper.

Thus we see that the computation of "good vibrations" (i.e., accurate eigenvalues and eigenvectors) requires the use of "good linearizations" (i.e., linearizations that reflect the structure of the original polynomial).

5. Conclusions. The numerical solution of structured nonlinear eigenvalue problems is an important component of many applications. Building on the work in [28], we have developed a theory that provides criteria for the existence of strong linearizations that reflect $\star$-even/odd or $\star$-(anti-)palindromic structure of a matrix polynomial, and have presented a systematic method to construct such linearizations. As shown in [16], [17], numerical methods based on these structured linearizations are expected to be more effective in computing accurate eigenvalues in practical applications.

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