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NUMERICAL ALGORITHMS FOR TIME-FRACTIONAL SUBDIFFUSION EQUATION WITH SECOND-ORDER ACCURACY*

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Abstract. This article aims to fill in the gap of the second-order accurate schemes for the time-fractional subdiffusion equation with unconditional stability. Two fully discrete schemes are first proposed for the time-fractional subdiffusion equation with space discretized by finite element and time discretized by the fractional linear multistep methods. These two methods are unconditionally stable with maximum global convergence order of $O(\tau + h^{r+1})$ in the L^2 norm, where τ and h are the step sizes in time and space, respectively, and r is the degree of the piecewise polynomial space. The average convergence rates for the two methods in time are also investigated, which shows that the average convergence rates of the two methods are $O(\tau^{1.5} + h^{r+1})$. Furthermore, two improved algorithms are constrcted, they are also unconditionally stable and convergent of order $O(\tau^2 + h^{r+1})$. Numerical examples are provided to verify the theoretical analysis. The comparisons between the present algorithms and the existing ones are included, which show that our numerical algorithms exhibit better performances than the known ones.

Key words. Finite element method, fractional linear multistep method, fractional derivative, subdiffusion, unconditional stability, convergence.

AMS subject classifications. 26A33, 65M06, 65M12, 65M15, 35R11

1. Introduction. In last few decades, fractional calculus has attracted great interests of many researchers. Fractional integral and derivatives are used more and more by scientists and engineers to simulate many phenomena in physics, material science, control, biology, signal processing, finance, etc., see for example [1, 2, 19, 22, 25, 27, 33, 35, 45]. In physics, fractional derivatives are used to model anomalous diffusion (i.e., subdiffusion and superdiffusion), where particles spread in a power-law manner [27].

This paper deals with the following time-fractional subdiffusion equation [27]

$$\begin{cases} {}_CD_{0,t}^{\beta}u = \mu\, \partial_x^2 u + f(x,t), \quad (x,t) \in I{\times}(0,T], I = (a,b), T > 0, \\ u(x,0) = \phi_0(x), \quad x \in I, \\ u = 0, \quad (x,t) \in \partial I \times (0,T], \end{cases} \tag{1.1}$$

where $0 < \beta < 1$, $\mu > 0$ and ${}_CD_{0,t}^{\beta}$ is the βth-order Caputo derivative operator defined by

$${}_CD_{0,t}^{\beta}u(x,t) = D_{0,t}^{-(1-\beta)}\left[\partial_t u(x,t)\right] = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-s)^{-\beta}\partial_s u(x,s)\,\mathrm{d}s, \tag{1.2}$$

in which $D_{0,t}^{-\beta}$ is the fractional integral operator defined by [33]

$$D_{0,t}^{-\beta}u(x,t) = {}_{RL}D_{0,t}^{-\beta}u(x,t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(x,s)\,\mathrm{d}s, \quad \beta > 0. \tag{1.3}$$

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Another commonly used fractional derivative is the Riemann-Liouville derivative. The βth-order Riemann-Liouville derivative operator ${}_{RL}D_{0,t}^{\beta}$ is defined by [33]

$$ {}_{RL}D_{0,t}^{\beta}u(x,t) = \partial_t\left[D_{0,t}^{-(1-\beta)}u(x,t)\right] = \frac{1}{\Gamma(1-\beta)}\frac{\partial}{\partial t}\left[\int_0^t (t-s)^{-\beta}u(x,s)\,\mathrm{d}s\right], \quad \beta\in(0,1). $$

If $u(x,t)$ is suitably smooth in time, then we have the following relationship [33]

$$ {}_{RL}D_{0,t}^{\beta}[u(x,t)-u(x,0)] = {}_{C}D_{0,t}^{\beta}u(x,t). $$

Till now, there have been many techniques to solve the fractional differential equations (FDEs). The analytical methods cover the Fourier transform method, the Laplace transform method, the Mellin transform method, and the Green function method, and so on [33]. In real applications, analytical methods can not work well on most of FDEs due to the nonlocality and complexity of the fractional differential operators. Hence, it is of great importance to seek the efficient and reliable numerical techniques to solve the FDEs. Nowadays, the numerical methods include finite difference methods (FDMs) [3, 7, 8, 9, 10, 20, 22, 26, 44, 41, 52], finite element methods (FEMs) [13, 30, 37, 42, 48, 51], and spectral methods [6, 21, 23]. There are also other numerical techniques such as the matrix approach, matrix transfer method, spine collocation method, etc., see for example [11, 28, 31, 32, 34, 38, 40, 43].

Up to now, there have been some literatures carried out on the numerical simulations of the subdiffusion equation of the form (1.1). The classical L1 method is often used to discretize the time fractional derivative of (1.1), which is convergent of order $(2-\beta)$. When $\beta\to 1$, the L1 method has first-order accuracy, which is not satisfactory. Readers can refer to some published papers as [16, 17, 18, 23, 36, 39, 49] for more detailed information. In [46], the fractional linear multistep methods were adopted to discretize the temporal of (1.1) with space discretized by finite element. Recently, Zhang et al. [50] proposed a new time discretization to solve (1.1), in which the time fractional derivative was approximated on the nonuniform grids that can be seen as a generalization of the classical L1 method. To the best of authors' knowledge, there are very few numerical works to solve (1.1) with second-order accuracy in time. This paper aims to construct unconditionally stable numerical methods to solve (1.1), which have second-order accuracy in time.

In this paper, we first construct two kinds of time discretization approaches to solve the subdiffusion equation (1.1) with the spatial discretization performed by the finite element and the time approximated by the fractional linear multistep methods. We give rigorous stability and convergence analysis for the established methods, which shows that the two methods are unconditionally stable with first-order accuracy in time. In some special cases, i.e., the analytical solution $u(x,t)$ is suitably smooth with $\partial_t u(x,0)=0$, the global second-order can be insured. Numerical experiments show that the two methods even have second-order accuracy for any smooth solutions since the local truncation errors have second-order accuracy when time level increases. Hence, we study the average convergence rates for the two methods, which shows that the average convergence rates of the two methods in time are of order 1.5! Then we propose two improved algorithms such that the global convergence rates in time are of order 2. It is shown that the two improved algorithms are also unconditionally stable. Even if the analytical solutions are not smooth enough, the present methods can also show second-order accuracy in some cases. The optimal error estimates in space are obtained for all the algorithms in the present paper. Numerical examples are presented to verify the theoretical analysis. Comparisons are made between the derived algorithms in this paper and the existing ones [14, 16, 17, 36, 47, 49], which show that our algorithms show better performances in the numerical experiments.

The remainder of this paper is outlined as follows. In Section 2, some necessary notations and lemmas are introduced. In Section 3, two fully discrete finite element methods for the subdiffusion equation (1.1) are first established, the stability and error estimate are given. Afterwards, the two improved algorithms are constructed with stability and convergence included. The numerical results are presented in Section 4, and the conclusion is included in the last section.

2. Preliminaries. In this section, we introduce some notations and lemmas that are needed in the following sections.

Let $I = (a, b)$ be a finite domain, and denote by $(\cdot, \cdot)$ the inner product defined on the space $L^2(I)$ with the L^2 norm $\|\cdot\|$ and the maximum norm $\|\cdot\|_\infty$. Denote $H^r(I)$ and $H_0^r(I)$ as the commonly used Sobolev spaces with the norm $\|\cdot\|_r$ and semi-norm $|\cdot|_r$, respectively. Define $\mathbb{P}_r(I)$ as the space of polynomials defined on I with the degree no greater than r, $r \in Z^+$. Let S_h be a uniform partition of I, which is given by

$$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b, \quad N \in Z^+.$$

Denote by $h = (b-a)/N = x_i - x_{i-1}$ and $I_i = [x_{i-1}, x_i]$ for $i = 1, 2, ..., N$. We define the finite element space X_h^r as the set of piecewise polynomials with degree at most $r\,(r \geq 1)$ on the mesh S_h, which can be expressed by

$$X_h^r = \{v : v|_{I_i} \in \mathbb{P}_r(I_i), v \in C(I)\}.$$

Introduce the piecewise interpolation operator $I_h : C(\bar{I}) \to X_h^r$ as

$$I_h u\Big|_{I_i} = \sum_{k=0}^{r} u(x_k^i) F_k^i(x), \quad u \in C(\bar{I}),$$

where $F_k^i(x)$ are Lagrangian basis functions defined by

$$F_k^i(x) = \prod_{l=0, l\neq k}^{r} \frac{x - x_l^i}{x_k^i - x_l^i}, \quad i = 1, 2, ..., N,$$

and $\{x_k^i, k = 0, 1, ..., r\}$ are the interpolation nodes on the interval I_i with $x_0^i = x_{i-1}$ and $x_r^i = x_i$.

Define φ^i $(i = 0, 1, ..., N)$ and φ_k^i $(k = 1, 2, ..., r-1; i = 1, 2, ..., N)$ as

$$\varphi_k^i(x) = \begin{cases} F_k^i(x), & x \in [x_{i-1}, x_i], \quad k = 1, 2, ..., r-1, i = 1, ..., N, \\ 0, & \text{others}, \end{cases}$$

$$\varphi^i(x) = \begin{cases} F_r^i(x), & x \in [x_{i-1}, x_i], \quad i = 1, ..., N-1, \\ F_0^{i+1}(x), & x \in [x_i, x_{i+1}], \quad i = 1, ..., N-1, \\ 0, & \text{others}, \end{cases}$$

$$\varphi^0(x) = \begin{cases} F_0^1(x), & x \in [x_0, x_1], \\ 0, & \text{others}, \end{cases}$$

$$\varphi^N(x) = \begin{cases} F_r^N(x), & x \in [x_{N-1}, x_N], \\ 0, & \text{others}. \end{cases}$$

Let $X_{h0}^r = X_h^r \cap H_0^1(I)$. Then the spaces X_{h0}^r and X_h^r can be expressed as

$$X_{h0}^r = \text{span}\left\{\varphi_k^i, k = 1, 2, ..., r-1, i = 1, 2, ..., N\right\} \cup \left\{\varphi^i, i = 1, 2, ..., N-1\right\},$$
$$X_h^r = \text{span}\left\{\varphi_k^i, k = 1, 2, ..., r-1, i = 1, 2, ..., N\right\} \cup \left\{\varphi^i, i = 0, 1, ..., N\right\}.$$

The basis functions $\{\varphi_k^i\} \cup \{\varphi^i\}$ will be used in the numerical simulation with grid points $x_k^i = x_0^i + kh/r,\ k = 0, 1, ..., r.$

The orthogonal projection operator $\Pi_h^{1,0} : H_0^1(I) \to X_{h0}^r$ is defined as

$$(\partial_x(u - \Pi_h^{1,0}u), \partial_x v) = 0, \quad u \in H_0^1(I), \forall v \in X_{h0}^r. \tag{2.1}$$

Next, we introduce the properties of the projector $\Pi_h^{1,0}$ and interpolation operator I_h that will be used later on.

LEMMA 2.1 ([5]). *Let $m, r \in Z^+,\ r \geq 1$, and $u \in H^m(I) \cap H_0^1(I)$. If $1 \leq m \leq r + 1$, then there exists a positive constant C independent of h, such that*

$$\|u - \Pi_h^{1,0}u\|_l \leq Ch^{m-l}\|u\|_m, \quad l = 0, 1.$$

LEMMA 2.2 (see p. 108 in [4]). *Let $m, r \in Z^+,\ r \geq 1$, and $u \in H^m(I)$. If $0 \leq m \leq r + 1$, then there exists a positive constant C independent of h, such that*

$$\|u - I_h u\| \leq Ch^m\|u\|_m.$$

3. The schemes. In this section, we first present the time discretization for (1.1). Then, the fully discrete schemes with space approximated by the finite element are given. At last, we prove the stability and convergence.

3.1. Time discretization. Let τ be the time step size and n_T be a positive integer with $\tau = T/n_T$ and $t_n = n\tau$ for $n = 0, 1, ..., n_T$. For the function $u(x, t) \in C([0, T]; L^2(I))$, denote by $u^n = u^n(\cdot) = u(\cdot, t_n)$.

We use the fractional linear multistep methods (FLMMs) developed in [24] by Lubich to discretize the time fractional derivative of (1.1). The pth-order FLMMs for $D_{0,t}^{-\beta}u(t)$ is given by

$$D_{0,t}^{-\beta}u(t)\Big|_{t=t_n} = \tau^\beta \sum_{k=0}^{n} \omega_{n-k}^{(\beta)} u(t_k) + \tau^\beta \sum_{k=0}^{s} w_{n,k}^{(\beta)} u(t_k) + O(\tau^p), \quad \beta > 0, \tag{3.1}$$

where $\{\omega_k^{(\beta)}\}$ can be the coefficients of the Taylor expansions of the following generating functions

$$w^{(\beta)}(z) = \left[\sum_{j=1}^{p} \frac{1}{j}(1 - z)^j\right]^{-\beta}, \quad p = 1, 2, ..., 6, \tag{3.2}$$

$$w^{(\beta)}(z) = (1 - z)^{-\beta}\left[\gamma_0 + \gamma_1(1 - z) + \gamma_2(1 - z)^2 + ... + \gamma_{p-1}(1 - z)^{p-1}\right], \tag{3.3}$$

$$w^{(\beta)}(z) = \left(\frac{1}{2}\frac{1 + z}{1 - z}\right)^\beta, \tag{3.4}$$

in which $\{\gamma_k\}$ in (3.3) satisfy the following relation

$$\left(\frac{\ln z}{z - 1}\right)^{-\beta} = \sum_{k=0}^{\infty} \gamma_k(1 - z)^k, \quad \gamma_0 = 1, \gamma_1 = -\frac{\beta}{2}.$$

The starting weights $\{w_{n,k}^{(\beta)}\}$ are chosen such that the asymptotic behavior of the function $u(t)$ near the origin ($t = 0$) are taken into account [12]. One way to determine $\{w_{n,k}^{(\beta)}\}$ for the sufficiently smooth function $u(t)$ is given as follows [12, 24]

$$\sum_{k=1}^{p-1} \omega_{n,k}^{(\beta)} k^q = \frac{\Gamma(q+1)}{\Gamma(q+\beta+1)} n^{q+\beta} - \sum_{k=1}^{n} \omega_{n-k}^{(\beta)} k^q, \quad q = 0, 1, \cdots, p-1. \tag{3.5}$$

The FLMM (3.1) has second-order accuracy if the generating function (3.4) is used. In the following, we will use the FLMMs based the generating function (3.3) with $p = 2$ or (3.4) to discretize the time discretization of (1.1). For simplicity, we denote by

$$w_1^{(\beta)}(z) = \left(\frac{1}{2}\frac{1+z}{1-z}\right)^{\beta}, \tag{3.6}$$

$$w_2^{(\beta)}(z) = (1-z)^{-\beta}\left[1 - \frac{\beta}{2}(1-z)\right]. \tag{3.7}$$

We first consider the following fractional ordinary differential equation (FODE)

$$ {}_C D_{0,t}^{\beta} y(t) = \mu y(t) + g(t), \quad y(0) = y_0, \quad 0 < \beta < 1. \tag{3.8}$$

The above FODE is equivalent to the following Volterra integral equation

$$y(t) - y_0 = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} (\mu y(s) + g(s))\, \mathrm{d}s = \mu D_{0,t}^{-\beta} y(t) + D_{0,t}^{-\beta} c(t) \tag{3.9}$$

in the sense that a continuous function is a solution of (3.8) if and only if it is a solution of (3.9), see Lemma 2.3 in [13]

Before discretizing (3.9), we introduce three lemmas.

LEMMA 3.1 ([24, 46]). *If* $y(t) = t^{\nu-1}, \nu > 0$*, then*

$$\left[D_{0,t}^{-\beta} y(t)\right]_{t=t_n} = \tau^{\beta} \sum_{k=0}^{n} \omega_{n-k}^{(\beta)} y(t_k) + O(t_n^{\nu-1+\beta-p}\tau^p) + O(t_n^{\beta-1}\tau^{\nu}),$$

where $\omega_k^{(\beta)}$ *can be the coefficients of the Taylor series of the generating functions defined as* (3.2)–(3.4)*, and* $p = 2$ *if* (3.4) *is used.*

LEMMA 3.2 ([46]). *Denote by*

$$y_n = \tau^{\beta} \sum_{k=0}^{n} \omega_{n-k}^{(\beta)} G(t_k, y_k), \tag{3.10}$$

where $\{\omega_k^{(\beta)}\}$ *are the coefficients of Taylor expansions of the generating functions* $w^{(\beta)}(z)$ *defined by Eq.* (3.2)*, Eq.* (3.3)*, or Eq.* (3.4)*. Then, Eq.* (3.10) *is equivalent to the following form*

$$\sum_{k=0}^{n} \alpha_k y_{n-k} = \tau^{\beta} \sum_{k=0}^{n} \theta_{n-k} G(t_k, y_k) \tag{3.11}$$

where α_k *and* θ_k *are the coefficients of Taylor expansions of* $\alpha(z)$ *and* $\theta(z)$ *satisfying* $w^{(\beta)}(z) = \theta(z)/\alpha(z)$.

LEMMA 3.3. *Suppose that* $0 < \beta < 1$. *Let* $\{\alpha_k\}$ *be the coefficients Taylor expansions of the generating function* $\alpha(z) = (1-z)^{\beta}$, *i.e.,* $\alpha_k = (-1)^k\binom{\beta}{k}$. *Then*

$$\sum_{k=1}^{n}\alpha_{n-k}k^{\gamma-1} = O(n^{\gamma-1-\beta}) + O(n^{-\beta-1}), \quad \gamma \in \mathbb{R}, \quad \gamma \neq 0,-1,-2,\cdots.$$

Proof. See Lemma 3.5 and the last line on page 713 in [24], which ends the proof. □

Now, we are in a position to discretize (3.9). In order to obtain the desired discretization, we rewrite (3.9) into the the following form

$$y(t) - y_0 = \mu D_{0,t}^{-\beta}y(t) + D_{0,t}^{-\beta}g(t) = \mu D_{0,t}^{-\beta}\left(y(t)-y(0)\right) + \frac{\mu t^{\beta}}{\Gamma(1+\beta)}y_0 + D_{0,t}^{-\beta}g(t). \tag{3.12}$$

Let $t = t_n$ in (3.12). Then we have

$$y(t_n) - y_0 = \mu\left[D_{0,t}^{-\beta}\left(y(t)-y_0\right)\right]_{t=t_n} + \frac{\mu t_n^{\beta}}{\Gamma(1+\beta)}y_0 + \left[D_{0,t}^{-\beta}g(t)\right]_{t=t_n}. \tag{3.13}$$

If $y(t)$ is smooth enough, then $y(t) - y_0$ can be expressed as $y(t) - y_0 = y'(0)t + D_{0,t}^{-2}y''(t)$. Therefore, by Lemma 3.1 and Theorem 2.4 in [24], we can have the following discretization for $\left[D_{0,t}^{-\beta}\left(y(t)-y_0\right)\right]_{t=t_n}$

$$\left[D_{0,t}^{-\beta}\left(y(t)-y_0\right)\right]_{t=t_n} = \tau^{\beta}\sum_{k=0}^{n}\omega_{n-k}^{(\beta)}(y(t_k)-y_0) + \tilde{R}_1^n + \tilde{R}_2^n, \tag{3.14}$$

where $\omega_k^{(\beta)}$ are the coefficients of Taylor expansions of the generating function (3.6) or (3.7), and $\tilde{R}_1^n = O(t_n^{\beta-1}\tau^2) = y'(0)\tau^{1+\beta}n^{\beta-1}(r_0 + O(n^{-1}))$, $\tilde{R}_2^n = O(t_n^{\beta}\tau^2) = c_n t_n^{\beta}\tau^2$, r_0 is a constant only dependent on β [24] and c_n is bounded.

Let $\tilde{R}^n = \tilde{R}_1^n + \tilde{R}_2^n$. Then, Eq. (3.13) has the following discretization

$$y(t_n) - y_0 = \mu\tau^{\beta}\sum_{k=0}^{n}\omega_{n-k}^{(\beta)}(y(t_k)-y_0) + \frac{\mu t_n^{\beta}}{\Gamma(1+\beta)}y_0 + \left[D_{0,t}^{-\beta}g(t)\right]_{t=t_n} + \tilde{R}^n. \tag{3.15}$$

Next, we discuss the equivalent form of (3.15). From Lemma 3.2, we can obtain the equivalent form of (3.15) as

$$\sum_{k=0}^{n}\alpha_{n-k}\left[(y(t_k)-y_0) - \left(\frac{\mu t_k^{\beta}}{\Gamma(1+\beta)}y_0 + + \left[D_{0,t}^{-\beta}g(t)\right]_{t=t_k} + \tilde{R}^k\right)\right] = \mu\tau^{\beta}\sum_{k=0}^{n}\theta_{n-k}(y(t_k)-y_0). \tag{3.16}$$

Rewriting (3.16) into the following form

$$\begin{aligned}\frac{1}{\tau^{\beta}}\sum_{k=0}^{n}\alpha_{n-k}\left(y(t_k)-y_0\right) =&\mu\sum_{k=0}^{n}\theta_{n-k}y(t_k) + \mu\left[\frac{1}{\Gamma(1+\beta)}\sum_{k=0}^{n}\alpha_{n-k}k^{\beta} - \sum_{k=0}^{n}\theta_k\right]y_0\\ &+ \frac{1}{\tau^{\beta}}\sum_{k=0}^{n}\alpha_{n-k}\left[D_{0,t}^{-\beta}g(t)\right]_{t=t_k} + y'(0)\tau^2\left[\frac{1}{\tau^{\beta}}\sum_{k=0}^{n}\alpha_{n-k}t_k^{\beta-1}r^k\right] + O(\tau^2)\\ =&\mu\sum_{k=0}^{n}\theta_{n-k}y(t_k) + \mu B_n y_0 + \frac{1}{\tau^{\beta}}G^n + R^n,\end{aligned} \tag{3.17}$$

where $R^n = y'(0)\tau^2\left[\frac{1}{\tau^\beta}\sum_{k=0}^n \alpha_{n-k}t_k^{\beta-1}r^k\right] + \tau^2\left[\frac{1}{\tau^\beta}\sum_{k=0}^n \alpha_{n-k}c_k t_k^\beta\right]$ and

$$B_n = \frac{1}{\Gamma(1+\beta)}\sum_{k=0}^n \alpha_{n-k}k^\beta - \sum_{k=0}^n \theta_k, \qquad G^n = \sum_{k=0}^n \alpha_{n-k}\left[D_{0,t}^{-\beta}g(t)\right]_{t=t_k}. \tag{3.18}$$

Let y_k be the approximate solution to $y(t_k)$. From (3.17), we obtain the numerical scheme for (3.8) as follows

$$\frac{1}{\tau^\beta}\sum_{k=0}^n \alpha_{n-k}\left(y_k - y_0\right) = \mu\sum_{k=0}^n \theta_{n-k}y_k + \mu B_n y_0 + \frac{1}{\tau^\beta}G^n. \tag{3.19}$$

Choosing different α_k and θ_k in (3.19) leads to different schemes. The two ways for the choices of α_k and θ_k in (3.19) are given follows:

- Use the generating function (3.6), where $\alpha(z)$ and $\theta(z)$ in Lemma 3.2 can be chosen as $\alpha(z) = (1-z)^\beta = \sum_{k=0}^\infty \alpha_k z^k = \sum_{k=0}^\infty (-1)^k\binom{\beta}{k}z^k$, $\theta(z) = \frac{(1+z)^\beta}{2^\beta} = \sum_{k=0}^\infty \theta_k z^k = \frac{1}{2^\beta}\sum_{k=0}^\infty \binom{\beta}{k}z^k$.
- Use the generating function (3.7), where $\alpha(z)$ and $\theta(z)$ in Lemma 3.2 can be chosen as $\alpha(z) = (1-z)^\beta = \sum_{k=0}^\infty \alpha_k z^k = \sum_{k=0}^\infty (-1)^k\binom{\beta}{k}z^k$, $\theta(z) = \sum_{k=0}^\infty \theta_k z^k = (1-\frac{\beta}{2}) + \frac{\beta}{2}z$.

Next, we analyse the truncation error $R^n = y'(0)\tau^2\left[\frac{1}{\tau^\beta}\sum_{k=0}^n \alpha_{n-k}t_k^{\beta-1}r^k\right] + O(\tau^2)$ defined in (3.17) when the generating function (3.6) or (3.7) is used. In both cases, we have $\alpha_k = (-1)^k\binom{\beta}{k}$ in (3.19). We can obtain a bound of the truncation error R^n of (3.19) as follows

$$|R^n| = \left|\tau\sum_{k=1}^n \alpha_{n-k}k^{\beta-1}(y'(0)r_0 + O(k^{-1})) + O(\tau^2)\right| \le C\left(|y'(0)|n^{-1}\tau + \tau^2\right), \tag{3.20}$$

where we have used $\frac{1}{\tau^\beta}\sum\limits_{k=0}^n \alpha_{n-k}c_k t_k^\beta \tau^2 = O(\tau^2)$ and Lemma 3.3.

Now, we are in a position to present the time discretization for (1.1). For simplicity, we introduce the following notations

$$D^{(\beta)}u^n = \frac{1}{\tau^\beta}\sum_{k=0}^n \omega_k(u^{n-k} - u^0) = \frac{1}{\tau^\beta}\left[\sum_{k=0}^n \omega_k u^{n-k} - b_n u^0\right], \tag{3.21}$$

$$L_1^{(\beta)}u^n = \frac{1}{2^\beta}\sum_{k=0}^n \omega_k(-1)^k u^{n-k}, \tag{3.22}$$

$$L_2^{(\beta)}u^n = \left(1 - \frac{\beta}{2}\right)u^n + \frac{\beta}{2}u^{n-1}, \tag{3.23}$$

where ω_k and b_n are defined by

$$\omega_k = (-1)^k\binom{\beta}{k} = \frac{\Gamma(k-\beta)}{\Gamma(-\beta)\Gamma(k+1)}, \tag{3.24}$$

$$b_n = \sum_{k=0}^n \omega_k = \frac{\Gamma(n+1-\beta)}{\Gamma(1-\beta)\Gamma(n+1)}, \quad n \ge 0. \tag{3.25}$$

Assume that $u(x,t)$ is sufficiently smooth in time. From (3.19), we can obtain the two approaches to the time discretization of the subdiffusion equation (1.1) as follows.

- **Time discretization I:** Applying the time discretization (3.19) with the generating function (3.6) to subdiffusion equation (1.1) yields

$$D^{(\beta)}u^n = \mu L_1^{(\beta)}(\partial_x^2 u^n) + \mu B_n^{(1)}\partial_x^2 u^0 + \frac{1}{\tau^\beta}F^n + R^n, \tag{3.26}$$

where $D^{(\beta)}$ and $L_1^{(\beta)}$ are defined by (3.21) and (3.22), respectively, $B_n^{(1)}$ and F^n are defined by

$$B_n^{(1)} = \frac{1}{\Gamma(1+\beta)}\sum_{k=0}^{n}\omega_{n-k}k^\beta - \sum_{k=0}^{n}(-1)^k\omega_k, \tag{3.27}$$

and

$$F^n = \sum_{k=0}^{n}\omega_{n-k}\left[D_{0,t}^{-\beta}f(x,t)\right]_{t=t_k}, \tag{3.28}$$

respectively, and R^n is the discretization error in time satisfying $|R^n| \le C(n^{-1}\tau + \tau^2)$.
- **Time discretization II:** Applying the time discretization (3.19) with the generating function defined by (3.7) to subdiffusion equation (1.1) leads to

$$D^{(\beta)}u^n = \mu L_2^{(\beta)}(\partial_x^2 u^n) + \mu B_n^{(2)}\partial_x^2 u^0 + \frac{1}{\tau^\beta}F^n + R^n, \tag{3.29}$$

where $D^{(\beta)}$, $L_2^{(\beta)}$, and F^n are defined by (3.21),(3.23), and (3.28), respectively, $B_n^{(2)}$ is defined by

$$B_n^{(2)} = \frac{1}{\Gamma(1+\beta)}\sum_{k=0}^{n}\omega_{n-k}k^\beta - 1, \tag{3.30}$$

and R^n is the truncation error in time discretization satisfying $|R^n| \le C(n^{-1}\tau + \tau^2)$.

REMARK 3.1. *From* (3.20)*, one obtains* $R^n = O(\tau^2)$ *in* (3.26) *or* (3.29) *for* $\partial_t u(x,0) = 0$.

Next, we present two fully discrete approximations for equation (1.1). From the time discretization (3.26) and (3.29), we present the corresponding fully discrete approximations for (1.1) as follows.

- **Scheme I:** Find $u_h^n \in X_{h0}^r$ for $n = 1, 2, ..., n_T - 1$, such that

$$\begin{cases} (D^{(\beta)}u_h^n, v) = -\mu(L_1^{(\beta)}\partial_x u_h^n, \partial_x v) - \mu B_n^{(1)}(\partial_x u_h^0, \partial_x v) + \dfrac{1}{\tau^\beta}(I_h F^n, v), \quad \forall v \in X_{h0}^r, \\ u_h^0 = \Pi_h^{1,0}\phi_0, \end{cases} \tag{3.31}$$

where $D^{(\beta)}$, $L_1^{(\beta)}$, $B_n^{(1)}$, and F^n are defined by (3.21), (3.22), (3.27), and (3.28), respectively.
- **Scheme II:** Find $u_h^n \in X_{h0}^r$ for $n = 1, 2, ..., n_T - 1$, such that

$$\begin{cases} (D^{(\beta)}u_h^n, v) = -\mu(L_2^{(\beta)}\partial_x u_h^n, \partial_x v) - \mu B_n^{(2)}(\partial_x u_h^0, \partial_x v) + \dfrac{1}{\tau^\beta}(I_h F^n, v) \quad \forall v \in X_{h0}^r, \\ u_h^0 = \Pi_h^{1,0}\phi_0, \end{cases} \tag{3.32}$$

where $D^{(\beta)}$, $L_2^{(\beta)}$, $B_n^{(2)}$, and F^n are defined by (3.21), (3.23), (3.30), and (3.28), respectively.

Calculation of F^n: In (3.31) and (3.32), we do not illustrate how to calculate $\left[D_{0,t}^{-\beta} f(x,t)\right]_{t=t_k}$ in F^n. In the stability and convergence analysis, and the numerical simulations, $\left[D_{0,t}^{-\beta} f(x,t)\right]_{t=t_n}$ is approximated by the following second-order formula

$$\begin{aligned}\left[D_{0,t}^{-\beta} f(x,t)\right]_{t=t_n} =&\left[D_{0,t}^{-\beta}\left(f(x,t)-f(x,0)\right)\right]_{t=t_n} + \frac{t_n^{\beta}}{\Gamma(1+\beta)} f(x,0)\\ =&\tau^{\beta}\sum_{k=0}^{n}\omega_{n-k}^{(\beta)}(f(x,t_k)-f(x,t_0)) + \tau^{\beta} w_{n,1}^{(\beta)}(f(x,t_1)-f(x,t_0))\\ &+\frac{t_n^{\beta}}{\Gamma(1+\beta)} f(x,0) + R^n,\end{aligned} \tag{3.33}$$

where $\{\omega_k^{(\beta)}\}$ are the coefficients of the Taylor expansions of the generating function (3.7). The coefficients $\{w_{n,1}^{(\beta)}\}$ are chosen such that (3.33) is exact for $f(x,t)-f(x,0) = t^{1-\beta}\tilde{f}(x)$. Hence, one has

$$w_{n,1}^{(\beta)} = \frac{\Gamma(q+1)}{\Gamma(q+\beta+1)} n^{q+\beta} - \sum_{k=1}^{n}\omega_{n-k}^{(\beta)} k^{q}, \quad q = 1-\beta. \tag{3.34}$$

If $u(x,t)$ is sufficiently smooth in time, then $f(x,t)-f(x,0)$ has the form $f(x,t)-f(x,0) = (c_0 t^{1-\beta} + c_1 t)\tilde{f}(x,t)$, $\tilde{f}(x,t)$ is sufficiently smooth in time. So q and R^n in (3.34) is chosen as $q = 1-\beta$ and $R^n = O(t_n^{\beta-1}\tau^2)$. From [24], $w_{n,1}^{(\beta)}$ satisfies $w_{n,1}^{(\beta)} = O(n^{\beta-1})$.

3.2. Stability and convergence. This subsection deals with the stability and convergence for the schemes (3.31) and (3.32). Next, we introduce a lemma.

LEMMA 3.4 ([15]). *Let $\{\omega_k\}$ be given by* (3.24)*. Then we have*

$$\begin{aligned}&\omega_0 = 1,\ \omega_n < 0,\ |\omega_{n+1}| < |\omega_n|, \quad n = 1,2,...;\\ &\omega_0 = -\sum_{k=1}^{\infty}\omega_k > -\sum_{k=1}^{n}\omega_k > 0, \quad n = 1,2,...;\\ &b_{n-1} = \sum_{k=0}^{n-1}\omega_k = \frac{\Gamma(n-\beta)}{\Gamma(1-\beta)\Gamma(n)} = \frac{n^{-\beta}}{\Gamma(1-\beta)} + O(n^{-1-\beta}), \quad n = 1,2,\cdots.\end{aligned} \tag{3.35}$$

Furthermore, $b_n - b_{n-1} = \omega_n < 0$ for $n > 0$, i.e., $b_n < b_{n-1}$.

Before analysing the stability, we give a bound for $B_n^{(1)}$ and $B_n^{(2)}$ defined in (3.31) and (3.32), which will be used in the stability analysis. From Lemma 3.1, we obtain

$$d_n = \left[D_{0,t}^{-\beta} y(t)\right]_{t=t_n} = \tau^{\beta}\sum_{k=0}^{n}\omega_{n-k}^{(\beta)} y(t_k) + \tau t_n^{\beta-1} r^n$$

when $y(t) = 1$, where $\{\omega_k^{(\beta)}\}$ are the coefficients of the Taylor series of the generating function defined by (3.6) or (3.7), and r^n is bounded satisfying $r^n = r_0 + O(n^{-1})$.

By Lemma 3.2 (or see Eq. (3.16)), we have

$$\sum_{k=0}^{n}\alpha_{n-k}\left(d_k - r^k t_k^{\beta-1}\tau\right) = \tau^{\beta}\sum_{k=0}^{n}\theta_{n-k} y(t_k). \tag{3.36}$$

Inserting $y(t_k) = 1$ and $d_n = \left[D_{0,t}^{-\beta} y(t)\right]_{t=t_n} = \frac{t_n^{\beta}}{\Gamma(1+\beta)} = \frac{n^{\beta}\tau^{\beta}}{\Gamma(1+\beta)}$ into (3.36) yields

$$\frac{1}{\Gamma(1+\beta)}\sum_{k=0}^{n}\alpha_{n-k} k^{\beta} - \sum_{k=0}^{n}\theta_{n-k} = \sum_{k=0}^{n}\alpha_{n-k} k^{\beta-1} r^k = O(n^{-1}), \tag{3.37}$$

where Lemma 3.3 is used. So we have

$$|B_n^{(i)}| \le Cn^{-1}, \quad i = 1,2, \quad n > 0, \tag{3.38}$$

where C is a positive constant independent of n and τ.

For convenience, we define the norms $|||\cdot|||_1$ and $|||\cdot|||_2$ as

$$|||u|||_1 = \left(||u||^2 + \mu\tau^{\beta}(1/2)^{\beta}||\partial_x u||^2\right)^{1/2}, \quad |||u|||_2 = \left(||u||^2 + \mu\tau^{\beta}(1-\beta/2)||\partial_x u||^2\right)^{1/2}.$$

Now, we have the following theorem.

Theorem 3.5. *Suppose that u_h^k for $k = 1,2,...,n_T$ is the solution of* (3.31). *Then, there exists positive constants C_1 independent of n, h, τ and T, and C_2 independent of n, h and τ such that*

$$|||u_h^n|||_1^2 \le |||u_h^0|||_1^2 + 2||u_h^0||^2 + C_1\tau^{\beta}||\partial_x u_h^0||^2 + C_2 \max_{0\le t\le T}||f(t)||^2. \tag{3.39}$$

Inequality (3.39) *means that the method* (3.31) *is unconditionally stable.*

Proof. We prove (3.39) by using the mathematical induction method. Letting $v = u_h^n$ in (3.31) yields

$$(D^{(\beta)}u_h^n, u_h^n) = -\mu(L_1^{(\beta)}\partial_x u_h^n, \partial_x u_h^n) - \mu B_n^{(1)}(\partial_x u_h^0, \partial_x u_h^n) + \frac{1}{\tau^{\beta}}(I_h F^n, u_h^n). \tag{3.40}$$

Using the property $b_n - b_{n-1} = \omega_n$ (see Lemma 3.4), we rewrite (3.40) as

$$\begin{aligned}|||u_h^n|||_1^2 =&(u_h^n, u_h^n) + \mu(\tau/2)^{\beta}(\partial_x u_h^n, \partial_x u_h^n)\\ =&\sum_{k=1}^{n}(b_{k-1} - b_k)\left[(u_h^{n-k}, u_h^n) + \mu(\tau/2)^{\beta}(-1)^k(\partial_x u_h^{n-k}, \partial_x u_h^n)\right]\\ &+ b_n(u_h^0, u_h^n) - \mu\tau^{\beta}B_n^{(1)}(\partial_x u_h^0, \partial_x u_h^n) + (I_h F^n, u_h^n).\end{aligned} \tag{3.41}$$

Using (3.41), $b_n - b_{n-1} \le 0$, and the Cauchy-Schwartz inequality yields

$$\begin{aligned}|||u_h^n|||_1^2 \le& \frac{1}{2}\sum_{k=1}^{n}(b_{k-1} - b_k)\left[||u_h^{n-k}||^2 + ||u_h^n||^2 + \mu(\tau/2)^{\beta}(||\partial_x u_h^{n-k}||^2 + ||\partial_x u_h^n||^2)\right]\\ &+ b_n||u_h^0||^2 + \frac{b_n}{4}||u_h^n||^2 + \frac{\tau^{2\beta}}{b_n}||I_h F^n||^2 + \frac{b_n}{4}||u_h^n||^2 + \mu B_n^{(1)}\tau^{\beta}\left(\epsilon||\partial_x u_h^n||^2 + \frac{1}{4\epsilon}||\partial_x u_h^0||^2\right)\\ =&\frac{1}{2}|||u_h^n|||_1^2 + \frac{1}{2}\sum_{k=1}^{n}(b_{k-1} - b_k)|||u_h^{n-k}|||_1^2 + \frac{1}{b_n}||I_h F^n||^2 + b_n||u_h^0||^2\\ &- \frac{1}{2}b_n\mu(\tau/2)^{\beta}||\partial_x u_h^n||^2 + \epsilon\mu B_n^{(1)}\tau^{\beta}||\partial_x u_h^n||^2 + \frac{\mu B_n^{(1)}\tau^{\beta}}{4\epsilon}||\partial_x u_h^0||^2,\end{aligned} \tag{3.42}$$

where ϵ is a suitable positive constant independent of n and τ, satisfying

$$-\frac{1}{2}b_n(1/2)^{\beta} + \epsilon B_n^{(1)} \le 0.$$

Such an ϵ exists, which can be deduced from Lemma 3.4 and (3.38). From Lemma 3.4, we have $1/b_n \le C_{\beta}n^{\beta}$, C_{β} is only dependent on β. Hence, we have from (3.42)

$$|||u_h^n|||_1^2 \le \sum_{k=1}^{n}(b_{k-1} - b_k)|||u_h^{n-k}|||_1^2 + \frac{2}{b_n}||I_h F^n||^2 + 2b_n||u_h^0||^2 + C\tau^{\beta}b_n||\partial_x u_h^0||^2, \tag{3.43}$$

where C is a positive constant independent of n, h, and τ. Noticing that

$$\frac{\tau^{2\beta}}{b_n} = b_n \frac{1}{b_n^2} \frac{T^{2\beta}}{n_T^{2\beta}} \leq b_n C_\beta^2 T^{2\beta} \left(\frac{n}{n_T}\right)^{2\beta} \leq (C_\beta T^\beta)^2 b_n. \tag{3.44}$$

Hence, we have from and (3.44)

$$\frac{2}{b_n}\|I_h F^n\|^2 \leq \frac{C\tau^{2\beta}}{b_n} \max_{0\leq t\leq t_n} \|f(t)\|^2 \leq C_2 b_n \max_{0\leq t\leq t_n} \|f(t)\|^2, \tag{3.45}$$

where we have used the relation $\|I_h F^n\|^2 \leq \tilde{C}_1\|F^n\|^2 \leq \tilde{C}_2\tau^{2\beta}\max_{0\leq t\leq t_n}\|f(t)\|^2$. Combining (3.43) and (3.45) yields

$$|||u_h^n|||_1^2 \leq \sum_{k=1}^{n}(b_{k-1}-b_k)|||u_h^{n-k}|||_1^2 + 2b_n\|u_h^0\|^2 + C_1 b_n\tau^\beta\|\partial_x u_h^0\|^2 + C_3 b_n \max_{0\leq t\leq t_n}\|f(t)\|^2, \tag{3.46}$$

Denote by

$$E = |||u_h^0|||_1^2 + 2\|u_h^0\|^2 + C_4\tau^\beta\|\partial_x u_h^0\|^2 + C_3 \max_{0\leq t\leq T}\|f(t)\|^2.$$

Then we have from (3.46)

$$|||u_h^n|||_1^2 \leq \sum_{k=1}^{n}(b_{k-1}-b_k)|||u_h^{n-k}|||_1^2 + b_n E. \tag{3.47}$$

Setting $n = 0$ in (3.47), and noticing that $|||u_h^0|||_1^2 \leq E$, one has

$$|||u_h^1|||_1^2 \leq (1-b_1)|||u_h^0|||_1^2 + b_1 E \leq (1-b_1)E + b_1 E = E. \tag{3.48}$$

Hence, the inequality (3.39) holds for $n = 1$. Suppose that the inequality (3.39) holds for $0 \leq n \leq m-1$, i.e. $|||u_h^n|||_1^2 \leq E\,(0 \leq n \leq m-1)$. Next, we just need to prove that the inequality (3.39) still holds for $n = m$.

Letting $v = u_h^m$ in (3.40) yields (3.47) with $n = m$. Considering the inequality (3.47) with $n = m$ and using the assumption $|||u_h^n|||_1^2 \leq E$ for $0 \leq n \leq m-1$, one has

$$|||u_h^m|||_1^2 \leq \sum_{k=1}^{m}(b_{k-1}-b_k)|||u_h^{m-k}|||_1^2 + b_m E \leq \sum_{k=1}^{m}(b_{k-1}-b_k)E + b_m E = E, \tag{3.49}$$

which means that (3.39) holds for $n = m$. Hence, (3.39) is true for any $0 \leq n \leq n_T$, which ends the proof. □

Next, we consider the convergence analysis for the scheme (3.31). Denote by $u_* = \Pi_h^{1,0}u$, $e = u_* - u_h$, and $\eta = u - u_*$. Noticing that $(\partial_x\eta, \partial_x v) = 0$ from (2.1), we obtain the error equation for (3.31) below

$$(D^{(\beta)}e^n, v) = -\mu(L_1^{(\beta)}\partial_x e^n, \partial_x v) - \mu B_n^{(1)}(\partial_x e^0, \partial_x v) + (R^n, v), \quad \forall v \in X_{h0}^r, \tag{3.50}$$

where $R^n = R_1^n + R_2^n + R_3^n$, and

$$|R_1^n| \leq Cn^{-\beta-1}\tau, \quad R_2^n = -D^{(\beta)}\eta^n, \quad R_3^n = F^n - \Pi_h^{1,0}F^n. \tag{3.51}$$

By Theorem 3.5, we obtain the following convergence theorem.

THEOREM 3.6. *Suppose that $r \geq 1$, u and u_h^n $(1 \leq n \leq n_T)$ are the solutions to* (1.1) *and* (3.31)*, respectively. If $m \geq r+1$, $u \in C^2(0,T;H^m(I)\cap H_0^1(I))$, $f \in C(0,T;H^m(I))$ and $\phi_0 \in H^m(I)$, then there exists a positive constant C independent of n, h and τ, such that*

$$\|u_h^n - u(t_n)\| \leq C(\tau + h^{r+1}). \tag{3.52}$$

Proof. According to Theorem 3.5, we only need to estimate

$$|||e^0|||_1^2 + 2\|e^0\|^2 + C_1\|\partial_x e^0\|^2 + C_2 \max_{0\leq k\leq n_T}\left\{\|R_1^k\|^2 + \|R_2^k\|^2 + \|R_3^k\|^2\right\}$$

to get an error bound. By (3.51), Lemmas 2.1 and 2.2, we can get the error following bounds

$$\|R_1^n\| \leq C\tau, \qquad \|R_2^n\| = \frac{1}{\tau^\beta}\|\sum_{j=0}^{n}\omega_j(\eta^{n-j} - \eta^0)\| \leq Ch^{r+1},$$

$$\|R_3^n\| = \|F^n - \Pi_h^{1,0}F^n\| = \|\sum_{k=0}^{n}\omega_{n-k}\left[D_{0,t}^{-\beta}(I - I_h)f(x,t)\right]_{t=t_k}\| \leq Ch^{r+1}.$$

For the initial errors e^0, we have $e^0 = 0$. Hence, one derives

$$\|e^n\| \leq |||e^n|||_1 \leq C(\tau + h^{r+1}). \tag{3.53}$$

By using Lemma 2.1 again, one has

$$\begin{aligned}\|u_h^n - u(t_n)\| &= \|u_h^n - \Pi_h^{1,0}u^n + \Pi_h^{1,0}u(t_n) - u(t_n)\|\\ &\leq \|e^n\| + \|\Pi_h^{1,0}u(t_n) - u(t_n)\|\\ &\leq C(\tau + h^{r+1}).\end{aligned} \tag{3.54}$$

The proof is completed. □

Theorem 3.6 shows that the Scheme I (3.31) has first-order global accuracy in time for all time levels n. From (3.20), we find that the local truncation error of (3.31) in time is $O(\tau^2)$ when n is sufficiently large. Although the errors at the first several time levels (n is small) are a little larger, the influences of these errors caused on the following time levels are weaker and weaker such that they can be ignored when n is large enough. Hence, we can predict that the convergence rate will be better when n is big enough, which is verified by the numerical experiments in Section 4. In the following, we study the average error in time that considers all the errors on each time levels. It shows that the average error exhibits much better convergence rate. We first introduce a lemma.

LEMMA 3.7. *Let b_n be defined as in* (3.35)*. For any $\mathbf{G} = \{G^1, G^2, G^3, ...\}$ and q, where $G^j = G^j(x)$ and $q = q(x)$ are real-valued functions defined on I. Then we have*

$$\begin{aligned}A^k(\mathbf{G},q) &= \frac{1}{\tau^{\beta-1}}\sum_{n=1}^{k}\left[b_0(G^n,G^n) - \sum_{j=1}^{n-1}(b_{n-j-1} - b_{n-j})(G^j,G^n) - b_{n-1}(q,G^n)\right]\\ &\geq \frac{1}{2}\frac{1}{\tau^{\beta-1}}\left[\sum_{n=1}^{k}b_{k-n}\|G^n\|^2 - \|q\|^2\sum_{n=1}^{k}b_{n-1}\right]\\ &\geq \tau C t_k^{-\beta}\sum_{n=1}^{k}\|G^n\|^2 - Ct_k^{1-\beta}\|q\|^2,\end{aligned} \tag{3.55}$$

where C is a positive constants only dependent on β.

Proof. From Lemma 3.4, one has $b_{n-1} \le b_n$. Using Cauchy-Schwartz inequality yields

$$
\begin{aligned}
A^k(\mathbf{G}, q) \ge & \frac{1}{\tau^{\beta-1}} \frac{1}{2} \sum_{n=1}^{k} \Big[2b_0 \|G^n\|^2 - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j})(\|G^j\|^2 + \|G^n\|^2) - b_{n-1}(\|q\|^2 + \|G^n\|^2) \Big] \\
= & \frac{1}{\tau^{\beta-1}} \frac{1}{2} \sum_{n=1}^{k} \Big(b_0 \|G^n\|^2 - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j}) \|G^j\|^2 - b_{n-1} \|q\|^2 \Big) \\
= & \frac{1}{2} \frac{1}{\tau^{\beta-1}} \left(\sum_{n=1}^{k} b_{k-n} \|G^n\|^2 - \|q\|^2 \sum_{n=1}^{k} b_{n-1} \right) \\
\ge & \tau C t_k^{-\beta} \sum_{n=1}^{k} \|G^n\|^2 - C t_k^{1-\beta} \|q\|^2,
\end{aligned}
\tag{3.56}
$$

where we have used $\sum_{n=1}^{k} b_{n-1} = O(k^{1-\beta})$ and $b_n \ge C_0 \tau^\beta$ from Lemma 3.4, C_0 is only dependent on β. The proof is completed. □

REMARK 3.2.

(1) *If b_{n-1} in (3.55) is replaced by B_n, $|B_n| \le C b_{n-1}, C > 0$, then (3.55) still holds.*

(2) *If the coefficients $(b_{n-j-1} - b_{n-j})$ in (3.55) are replaced by $(-1)^{\sigma(j)}(b_{n-j-1} - b_{n-j})$, where $\sigma(j)$ is chosen randomly as 0 or 1, then (3.55) still holds.*

THEOREM 3.8. *Suppose that $r \ge 1$, u and u_h^n $(1 \le n \le n_T)$ are the solutions to (1.1) and (3.31), respectively. If $m \ge r+1$, $u \in C^2(0, T; H^m(I) \cap H_0^1(I))$, $f \in C(0, T; H^m(I))$ and $\phi_0 \in H^m(I)$, then there exists a positive constant C independent of k, h and τ, such that*

$$
\sqrt{\tau \sum_{n=1}^{k} \|u_h^n - u(t_n)\|^2} \le C(\tau^{1.5} + h^{r+1}). \tag{3.57}
$$

Proof. From (3.50) and noticing that $e^0 = 0$, we obtain the error equation as

$$
\begin{aligned}
& \frac{1}{\tau^\beta} \Big[b_0(e^n, v) - \sum_{j=1}^{n-1} (b_{n-j-1} - b_{n-j})(e^j, v) \Big] \\
& + \frac{\mu}{2^\beta} \Big[b_0(\partial_x e^n, \partial_x v) + \sum_{j=1}^{n-1} (-1)^{n-j} (b_{n-j-1} - b_{n-j})(\partial_x e^j, \partial_x v) \Big] = (R^n, v).
\end{aligned}
\tag{3.58}
$$

Letting $v = e^n$ in (3.58), summing up n from 1 to k, and using Lemma 3.7 and Remark 3.2 yield

$$
\begin{aligned}
& \frac{1}{2} \frac{1}{\tau^\beta} \sum_{n=1}^{k} b_{k-n} \|e^n\|^2 + \frac{1}{2} \frac{\mu}{2^\beta} \sum_{n=1}^{k} b_{k-n} \|\partial_x e^n\|^2 \\
\le & \sum_{n=1}^{k} (R^n, e^n) \le \sum_{n=1}^{k} \Big(\frac{1}{4} \frac{b_{k-n}}{\tau^\beta} \|e^n\|^2 + \frac{\tau^\beta}{b_{k-n}} \|R^n\|^2 \Big).
\end{aligned}
\tag{3.59}
$$

Hence, one obtains

$$
\sum_{n=1}^{k} b_{k-n} \left(\|e^n\|^2 + \tau^\beta 2^{1-\beta} \mu \|\partial_x e^n\|^2 \right) \le 4\tau^\beta \sum_{n=1}^{k} \frac{\tau^\beta}{b_{k-n}} \|R^n\|^2 \le C\tau^\beta \sum_{n=1}^{k} \|R^n\|^2. \tag{3.60}
$$

Noticing that $\tau^\beta \le Cb_n$ from Lemma 3.4, one has

$$\begin{aligned}\tau\sum_{n=1}^{k}\|e^n\|^2 &\le \tau\sum_{n=1}^{k}(\|e^n\|^2+\tau^\beta 2^{1-\beta}\mu\|\partial_x e^n\|^2) \le C\tau\sum_{n=1}^{k}\|R^n\|^2\\ &\le C\tau\sum_{n=1}^{k}(\|R_1^n\|^2+\|R_2^n\|^2+\|R_3^n\|^2) \le C\left(\tau\sum_{n=1}^{k}\|R_1^n\|^2+h^{2(r+1)}\right)\\ &\le C\left(\tau^3\sum_{n=1}^{k}n^{-2}+\tau^4+h^{2(r+1)}\right)\le C(\tau^3+h^{2(r+1)}),\end{aligned} \tag{3.61}$$

where $\|R_1^n\| \le C(n^{-1}\tau + \tau^2)$, $\|R_2^n\| \le Ch^{r+1}$, and $\|R_3^n\| \le Ch^{r+1}$ have been used. Similarly to (3.54), we obtain (3.57), which completes the proof. □

Similarly to Theorem 3.5, we can immediately obtain the stability for the scheme (3.32).

THEOREM 3.9. *Suppose that u_h^n for $n = 1, 2, ..., n_T$ are solutions to* (3.32). *Then, there exists positive constants C_1 independent of n, h, τ and T, and C_2 independent of n, h and τ such that*

$$|||u_h^n|||_2^2 \le |||u_h^0|||_2^2 + 2\|u_h^0\|^2 + C_1\|\partial_x u_h^0\|^2 + C_2 \max_{0\le k\le n_T}\|f^k\|^2. \tag{3.62}$$

The inequality (3.62) *means that the Scheme II* (3.32) *is unconditionally stable.*

By Theorems 3.8 and 3.9, one can obtains the error estimate for (3.32) as follows.

THEOREM 3.10. *Suppose that $r \ge 1$, u and u_h^n $(1 \le n \le n_T)$ are the solutions to* (1.1) *and* (3.32)*, respectively. If $m \ge r+1$, $u \in C^2(0, T; H^m(I))$, $f \in C(0, T; H^m(I)\cap H_0^1(I))$ and $\phi_0 \in H^m(I)$, then there exists a positive constant C independent of k, h and τ, such that*

$$\|u_h^k - u(t_k)\| \le C(\tau + h^{r+1}), \tag{3.63}$$

and

$$\sqrt{\tau\sum_{n=1}^{k}\|u_h^n - u(t_n)\|^2} \le C(\tau^{1.5} + h^{r+1}). \tag{3.64}$$

3.3. Improved algorithms. This subsection presents two improved time discretization techniques such that the derived algorithms have global second-order accuracy in time.

We still consider the discretization of (3.13), and we assume $y(t)$ has the from $y(t)-y(0) = t^\alpha\tilde{y}(t), \alpha > 0$, $\tilde{y}(t)$ is suitably smooth. By Lemma 3.1, we can discretize $\left[D_{0,t}^{-\beta}(y(t)-y_0)\right]_{t=t_n}$ in (3.13) with the following approach

$$\left[D_{0,t}^{-\beta}(y(t)-y_0)\right]_{t=t_n} = \tau^\beta\sum_{k=0}^{n}\omega_{n-k}^{(\beta)}(y(t_k)-y_0) + \tau^\beta w_{n,1}^{(\beta)}(y(t_1)-y(0)) + O(t_n^{\alpha+\beta}\tau^2), \tag{3.65}$$

where $\omega_k^{(\beta)}$ are the coefficients of Taylor expansions of the generating function (3.6) or (3.7), and $w_{n,1}^{(\beta)}$ is chosen such that (3.65) is exact for $y(t) - y(0) = t^\alpha$, which is given below

$$w_{n,1}^{(\beta)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)}n^{\alpha+\beta} - \sum_{k=1}^{n}\omega_{n-k}^{(\beta)}k^\alpha = O(n^{\beta-1}). \tag{3.66}$$

Repeating the processes (3.15)–(3.16), we obtain the following discretization

$$\frac{1}{\tau^{\beta}}\sum_{k=0}^{n}\alpha_{n-k}\left(y(t_k)-y_0\right)=\mu\sum_{k=0}^{n}\theta_{n-k}y(t_k)+\mu B_n y_0+\mu C_n(y(t_1)-y(0))+\frac{1}{\tau^{\beta}}F^n+R^n, \tag{3.67}$$

where $R^n=O(\tau^2)$, B_n and F_n are defined by (3.18), and C_n is given by

$$C_n=\sum_{k=0}^{n}\alpha_{n-k}w_{k,1}^{(\beta)}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)}\sum_{k=0}^{n}\alpha_{n-k}k^{\alpha+\beta}-\sum_{k=1}^{n}\theta_{n-k}k^{\alpha}=O(n^{\alpha-\beta-1}), \tag{3.68}$$

where we have used $w_{n,1}^{(\beta)}=O(n^{\beta-1})$ and Lemma 3.3.

Suppose that $u(x,t)$ is sufficiently smooth, i.e., $\alpha=1$ in (3.67). From (3.67), we obtain two improved time discretization for (1.1), which are similar to (3.26) and (3.29) that are listed below.

- **Improved time discretization I(α):** Applying the time discretization (3.67) with the generating function (3.6) to subdiffusion equation (1.1) yields

$$D^{(\beta)}u^n=\mu L_1^{(\beta)}(\partial_x^2u^n)+\mu B_n^{(1)}\partial_x^2u^0+\mu C_n^{(1)}\partial_x^2(u^1-u^0)+\frac{1}{\tau^{\beta}}F^n+R^n, \tag{3.69}$$

where $D^{(\beta)}$, $L_1^{(\beta)}$, $B_n^{(1)}$, and F^n are defined by (3.21), (3.22), (3.27), and (3.28), respectively, and R^n is the truncation error in time discretization satisfying $R^n=O(\tau^2)$ when $\alpha=1$, and $C_n^{(1)}$ is given by

$$C_n^{(1)}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)}\sum_{k=0}^{n}\omega_{n-k}k^{\beta+\alpha}-\sum_{k=1}^{n}\frac{(-1)^k}{2^{\beta}}\omega_{n-k}k^{\alpha}. \tag{3.70}$$

- **Improved time discretization II(α):** Applying the time discretization (3.67) with the generating function defined by (3.7) to subdiffusion equation (1.1) leads to

$$D^{(\beta)}u^n=\mu L_2^{(\beta)}(\partial_x^2u^n)+\mu B_n^{(2)}\partial_x^2u^0+\mu C_n^{(2)}\partial_x^2(u^1-u^0)+\frac{1}{\tau^{\beta}}F^n+R^n, \tag{3.71}$$

where $D^{(\beta)}$, $L_1^{(\beta)}$, $B_n^{(1)}$, and F^n are defined by (3.21), (3.23), (3.30), and (3.28), respectively, R^n is the truncation error in time discretization satisfying $R^n=O(\tau^2)$ when $\alpha=1$, and $C_n^{(2)}$ is given by

$$C_n^{(2)}=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+\beta)}\sum_{k=0}^{n}\omega_{n-k}k^{\beta+\alpha}-\left[(1-\frac{\beta}{2})n^{\alpha}+\frac{\beta}{2}(n-1)^{\alpha}\right]. \tag{3.72}$$

Improved time discretization I(α) means that (3.69) is exact for $u(x,t)=t^{\alpha}$, as is for Improved time discretization II(α) in (3.71).

From (3.69) and (3.71), we obtain the two fully improved algorithms below.

- **Improved scheme I(α):** Find $u_h^n\in X_h^r$ for $n=1,2,...,n_T-1$, such that

$$\begin{cases}(D^{(\beta)}u_h^n,v)=-\mu(L_1^{(\beta)}\partial_xu_h^n,\partial_xv)-\mu B_n^{(1)}(\partial_xu_h^0,\partial_xv)-\mu C_n^{(1)}(\partial_xu_h^1-\partial_xu_h^0,\partial_xv)\\ \qquad\qquad+\dfrac{1}{\tau^{\beta}}(I_hF^n,v),\quad \forall v\in X_{h0}^r,\\ u_h^0=\Pi_h^{1,0}\phi_0,\end{cases} \tag{3.73}$$

where $D^{(\beta)}$, $L_1^{(\beta)}$, $B_n^{(1)}$, F^n, and $C_n^{(1)}$ are defined by (3.21), (3.22), (3.27), (3.28), and (3.70), respectively.

- **Improved scheme II(α):** Find $u_h^n \in X_h^r$ for $n = 1, 2, ..., n_T - 1$, such that

$$\begin{cases} (D^{(\beta)} u_h^n, v) = -\mu(L_2^{(\beta)} \partial_x u_h^n, \partial_x v) - \mu B_n^{(2)}(\partial_x u_h^0, \partial_x v) - \mu C_n^{(2)}(\partial_x u_h^1 - \partial_x u_h^0, \partial_x v) \\ \qquad\qquad + \dfrac{1}{\tau^\beta}(I_h F^n, v) \quad \forall v \in X_{h0}^r, \\ u_h^0 = \Pi_h^{1,0}\phi_0, \end{cases} \tag{3.74}$$

where $D^{(\beta)}$, $L_2^{(\beta)}$, $B_n^{(2)}$, F^n, and $C_n^{(2)}$ are defined by (3.21), (3.23), (3.30), (3.28), and (3.72), respectively.

If $\alpha = 1$, then we denote the Improved scheme I(α) and Improved scheme II(α) as Improved scheme I and Improved scheme II, respectively.

From (3.68), we know that $C_n^{(i)}$ also has the property as $B_n^{(i)}$, i.e., $|C_n^{(i)}| \le Cn^{-1}$ when $u(x, t)$ is sufficiently smooth in time. Hence, the stability and convergence analysis of the improved schemes (3.73) and (3.74) are very similar to those of (3.31) and (3.32), we just list them below.

THEOREM 3.11. *Suppose that u_h^k for $k = 1, 2, ..., n_T$ is the solution of* (3.73) *or* (3.74) *. Then, there exists positive constants C_1 independent of n, h, τ and T, and C_2 independent of n, h and τ such that*

$$|||u_h^n|||_1^2 \le C_1 |||u_h^0|||_1^2 + C_2 \max_{0 \le t \le T} \|f(t)\|^2.$$

The above inequality means that the method (3.73) *or* (3.74) *is unconditionally stable.*

THEOREM 3.12. *Suppose that $r \ge 1$, u and u_h^n $(1 \le n \le n_T)$ are the solutions to* (1.1) *and* (3.73) *(or* (3.74)*), respectively. If $m \ge r + 1$, $u \in C^2(0, T; H^m(I))$, $f \in C(0, T; H^m(I) \cap H_0^1(I))$ and $\phi_0 \in H^m(I)$, then there exists a positive constant C independent of k, h and τ, such that*

$$\|u_h^k - u(t_k)\| \le C(\tau^2 + h^{r+1}).$$

4. Numerical examples. In this section, we present several numerical examples. For convenience, we use the interpolation operator I_h to replace the projectors $\Pi_h^{1,0}$ and Π_h^1 for the computation. We first numerically verify the error estimates and the convergence orders of the methods Scheme I (see Eq. (3.31)), Scheme II (see Eq. (3.32)), Improved scheme I (see Eq. (3.73)) and Improved scheme II (see Eq. (3.74)).

EXAMPLE 4.1. *Consider the following subdiffusion equation [17, 23]*

$$\begin{cases} {}_C D_{0,t}^\beta u = \partial_x^2 u + f(x, t), & (x, t) \in (0, 1) \times (0, 1], \\ u(x, 0) = 2\sin(2\pi x), & x \in [0, 1], \\ u(0, t) = u(1, t) = 0, & t \in (0, 1]. \end{cases} \tag{4.1}$$

Choose a suitable right hand side function f such that the exact solution to (4.3) *is*

$$u = (t^{2+\beta} + t + 2)\sin(2\pi x).$$

Denote $\varepsilon^n(\tau, h) = u_h^n - u^n$ as the error equation at time level n. The convergence orders in time and space in the sense of the L^2 norm are defined as

$$\text{order} = \begin{cases} \log(\|\varepsilon^n(\tau_1, h)\| / \|\varepsilon^n(\tau_2, h)\|)/\log(\tau_1/\tau_2), & \text{in time}, \\ \log(\|\varepsilon^n(\tau, h_1)\| / \|\varepsilon^n(\tau, h_2)\|)/\log(h_1/h_2), & \text{in space}, \end{cases} \tag{4.2}$$

where τ, τ_1, τ_2 ($\tau_1 \neq \tau_2$) and h, h_1, h_2 ($h_1 \neq h_2$) are the time and space step sizes, respectively.

The cubic element ($r = 3$) is used in this example, the space and time steps sizes are chosen as $h = 1/1000$ and $\tau = 1/32, 1/64, 1/128, 1/256, 1/512$.

We first check the global maximum L^2 error $\max_{0\le n\le n_T} \|u_h^n - u^n\|$, the average L^2 error $(\tau \sum_{n=0}^{n_T} \|u_h^n - u^n\|^2)^{1/2}$, and the L^2 error $\|u_h^n - u^n\|$ at $n = n_T$, which are shown in Tables 4.1–4.3. From Table 4.1, we find that the Scheme I and II show about first-order accuracy in time for $\beta = 0.1, 0.5$. When $\beta = 0.9$, the Scheme I and II show much better results than theoretical analysis. Obviously, the Improved scheme I and II show the expected convergence rates, even better than expected. Table 4.2 gives the average L^2 errors, which shows that the four algorithms yields the desired convergence rates even better than anticipated. Table 4.3 display the L^2 error at $t = 1$. Obviously, the Scheme I and II show about second-order accuracy in time, the Improved scheme I and II show second-order accuracy as expected. Briefly, the Scheme I and II show better numerical results than theoretical analysis here, and the Improved scheme I and II show about second-order accuracy as expected.

Next, we compare the present FEMs Scheme I, Scheme II, Improved scheme I, and Improved scheme II with the FEM in [17], where time was discretized by the L1 method, we denote it by L1FEM. The L1FEM has convergence order of $O(\tau^{2-\beta} + h^{r+1})$. We choose the same parameters in the computation, the results are shown in Table 4.4. Obviously, the present methods show better performances than the L1FEM, especially when β increases. It is easy to verify that the present four algorithms show second-order experimental accuracy and the L1FEM shows $(2-\beta)$th-order experimental accuracy, which are inline with the theoretical analysis.

Table 4.1
The global maximum L^2 errors $\max_{0\le n\le n_T} \|u_h^n - u^n\|$ for Example 4.1, $N = 1000, r = 3$.

Methods	$1/\tau$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.9$	order
Scheme I (3.31)	32	4.7141e-4		1.0490e-3		1.3175e-4	
	64	2.4645e-4	0.9357	5.2706e-4	0.9930	6.7038e-5	0.9747
	128	1.2553e-4	0.9733	2.4738e-4	1.0912	2.3457e-5	1.5150
	256	6.3191e-5	0.9902	1.1150e-4	1.1498	7.2003e-6	1.7039
	512	3.1637e-5	0.9981	4.8600e-5	1.1980	2.0743e-6	1.7954
Scheme II (3.32)	32	7.4577e-5		6.8243e-5		1.4795e-4	
	64	5.0606e-5	0.5594	1.5696e-5	2.1203	2.9862e-5	2.3088
	128	2.8356e-5	0.8356	8.3982e-6	0.9022	7.3010e-6	2.0322
	256	1.4864e-5	0.9319	4.7785e-6	0.8135	1.9438e-6	1.9092
	512	7.5776e-6	0.9720	2.2498e-6	1.0867	5.2847e-7	1.8790
Improved Scheme I (3.73)	32	5.2941e-5		1.2396e-4		9.3790e-5	
	64	1.2332e-5	2.1020	2.1236e-5	2.5453	1.2378e-5	2.9217
	128	2.8724e-6	2.1021	5.1989e-6	2.0302	3.1014e-6	1.9968
	256	6.6894e-7	2.1023	1.3003e-6	1.9994	7.7612e-7	1.9986
	512	1.5581e-7	2.1021	3.2515e-7	1.9996	1.9403e-7	2.0000
Improved Scheme II (3.74)	32	5.2941e-5		1.2396e-4		9.3790e-5	
	64	1.2332e-5	2.1020	2.1236e-5	2.5453	1.1039e-5	3.0869
	128	2.8726e-6	2.1020	3.6080e-6	2.5573	1.3108e-6	3.0741
	256	6.6894e-7	2.1024	6.0779e-7	2.5695	1.5970e-7	3.0370
	512	1.5571e-7	2.1030	1.0149e-7	2.5823	3.1009e-8	2.3646

Example 4.2. *Consider the following subdiffusion equation [16]*

$$\begin{cases} {}_C D_{0,t}^{\beta} u = \partial_x^2 u + f(x,t), & (x,t) \in (0,1)\times(0,1], \\ u(x,0) = \exp(x), & x \in (0,1), \\ u(0,t) = t^{1+\beta}, \quad u(1,t) = t^{1+\beta}\exp(1) & t \in (0,1], \end{cases} \tag{4.3}$$

TABLE 4.2
The average L^2 errors $(\tau \sum_{n=0}^{n_T} \|u_h^n - u^n\|^2)^{1/2}$ for Example 4.1, $N = 1000, r = 3$..

Methods	$1/\tau$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.9$	order
Scheme I (3.31)	32	1.1498e-4		3.3381e-4		4.2705e-5	
	64	4.1847e-5	1.4581	1.0495e-4	1.6693	1.1796e-5	1.8561
	128	1.4962e-5	1.4838	3.1468e-5	1.7378	2.9836e-6	1.9832
	256	5.3064e-6	1.4955	9.2010e-6	1.7740	7.3814e-7	2.0151
	512	1.8747e-6	1.5011	2.6420e-6	1.8002	1.8227e-7	2.0178
Scheme II (3.32)	32	1.8257e-5		1.5479e-5		2.7191e-5	
	64	8.0767e-6	1.1766	3.0432e-6	2.3466	4.3535e-6	2.6429
	128	3.1225e-6	1.3710	1.2367e-6	1.2991	9.0489e-7	2.2664
	256	1.1462e-6	1.4458	4.5034e-7	1.4574	2.1165e-7	2.0960
	512	4.1152e-7	1.4779	1.5235e-7	1.5636	5.1627e-8	2.0355
Improved Scheme I (3.73)	32	1.7649e-5		6.3719e-5		3.5855e-5	
	64	4.2297e-6	2.0610	1.5076e-5	2.0795	8.0732e-6	2.1509
	128	1.0439e-6	2.0186	3.7252e-6	2.0169	1.9916e-6	2.0193
	256	2.6030e-7	2.0037	9.2916e-7	2.0033	4.9700e-7	2.0026
	512	6.5029e-8	2.0010	2.3217e-7	2.0007	1.2415e-7	2.0011
Improved Scheme II (3.74)	32	9.8452e-6		2.2325e-5		1.8170e-5	
	64	1.6221e-6	2.6015	2.8235e-6	2.9831	2.3677e-6	2.9400
	128	2.6828e-7	2.5961	4.0332e-7	2.8075	4.9677e-7	2.2528
	256	4.4794e-8	2.5824	7.3597e-8	2.4542	1.2155e-7	2.0311
	512	7.6500e-9	2.5498	1.6522e-8	2.1553	3.0315e-8	2.0034

TABLE 4.3
The L^2 errors $\|u_h^n - u^n\|$ at $n = n_T(t = 1)$ for Example 4.1, $N = 1000, r = 3$.

Methods	$1/\tau$	$\alpha = 0.1$	order	$\alpha = 0.5$	order	$\alpha = 0.9$	order
Scheme I (3.31)	32	2.3997e-6		1.5308e-5		4.8896e-5	
	64	9.3699e-7	1.3567	1.1924e-5	0.3604	1.2355e-5	1.9847
	128	3.2532e-7	1.5262	4.2184e-6	1.4991	3.1005e-6	1.9945
	256	1.0460e-7	1.6370	1.1979e-6	1.8162	7.7624e-7	1.9979
	512	3.2047e-8	1.7066	3.1472e-7	1.9284	1.9409e-7	1.9998
Scheme II (3.32)	32	3.1243e-6		4.6797e-6		7.9556e-6	
	64	8.1059e-7	1.9465	1.1587e-6	2.0139	1.9910e-6	1.9985
	128	1.9968e-7	2.0212	2.8364e-7	2.0304	4.9787e-7	1.9997
	256	4.8185e-8	2.0511	6.9455e-8	2.0299	1.2441e-7	2.0007
	512	1.1575e-8	2.0575	1.7069e-8	2.0247	3.0986e-8	2.0054
Improved Scheme I (3.73)	32	1.8316e-5		8.4548e-5		4.9286e-5	
	64	4.5797e-6	1.9998	2.0821e-5	2.0217	1.2378e-5	1.9934
	128	1.1438e-6	2.0014	5.1989e-6	2.0018	3.1014e-6	1.9968
	256	2.8575e-7	2.0010	1.3003e-6	1.9994	7.7612e-7	1.9986
	512	7.1316e-8	2.0025	3.2515e-7	1.9996	1.9403e-7	2.0000
Improved Scheme II (3.74)	32	1.3694e-7		3.9832e-6		7.9564e-6	
	64	1.1867e-7	0.2066	1.0307e-6	1.9504	1.9917e-6	1.9981
	128	3.8589e-8	1.6206	2.5996e-7	1.9872	4.9811e-7	1.9995
	256	1.0583e-8	1.8665	6.5024e-8	1.9992	1.2448e-7	2.0006
	512	2.7949e-9	1.9208	1.6227e-8	2.0026	3.1005e-8	2.0053

where $0 < \beta < 1$, *and*

$$f(x,t) = \left(\Gamma(2+\beta)t - t^{\beta+1}\right)\exp(x).$$

The exact solution of (4.1) *is* $u = t^{1+\beta}\exp(x)$.

In this example, the L^∞ error on the grid points $\{x_i\}$ at $t = t_n$ is defined as

$$\varepsilon_\infty(\tau, h, t_n) = \max_{0\le i\le N} |u(x_i, t_n) - u_h^n(x_i)|.$$

We mainly compare the numerical results obtained by the cubic element in Scheme I,

TABLE 4.4
Comparison of the L^2 errors $\|u_h^n - u^n\|$ at $n = n_T(t = 1)$ for Example 4.1, $N = 1000, r = 3$.

β	$1/\tau$	Scheme I	Scheme II	Improved Scheme I	Improved Scheme II	L1FEM [17]
0.4	32	2.6133e-5	4.2566e-6	7.3488e-5	2.9381e-6	4.0588e-5
	64	1.3567e-6	1.0444e-6	1.7955e-5	7.8792e-7	1.3740e-5
	128	1.2998e-6	2.5085e-7	4.4601e-6	2.0074e-7	4.6194e-6
	256	6.6074e-7	6.0246e-8	1.1137e-6	5.0390e-8	1.5450e-6
	512	2.1850e-7	1.4488e-8	2.7828e-7	1.2532e-8	5.1494e-7
0.6	32	5.9934e-5	5.3346e-6	8.9378e-5	5.0233e-6	1.5276e-4
	64	1.9695e-5	1.3308e-6	2.2267e-5	1.2773e-6	5.8654e-5
	128	5.3544e-6	3.2966e-7	5.5749e-6	3.2041e-7	2.2413e-5
	256	1.3765e-6	8.1818e-8	1.3950e-6	8.0201e-8	8.5394e-6
	512	3.4737e-7	2.0098e-8	3.4871e-7	1.9809e-8	3.2467e-6
0.8	32	7.2323e-5	7.0592e-6	7.4360e-5	7.0381e-6	5.1564e-4
	64	1.8542e-5	1.7682e-6	1.8673e-5	1.7661e-6	2.2617e-4
	128	4.6721e-6	4.4172e-7	4.6794e-6	4.4170e-7	9.8860e-5
	256	1.1713e-6	1.1048e-7	1.1712e-6	1.1054e-7	4.3132e-5
	512	2.9302e-7	2.7603e-8	2.9286e-7	2.7632e-8	1.8799e-5

TABLE 4.5
Comparison of the L^∞ errors at $t = 1$ for Example 4.2.

β	$1/\tau$	N	Scheme I	Scheme II	Improved Scheme I	Improved Scheme II	L1C[16]
0.25	4	4	7.9798e-04	8.6898e-05	2.3006e-04	1.2659e-04	7.2822e-04
	64	8	2.7301e-06	2.8640e-07	6.5009e-07	1.3719e-07	6.0359e-06
	1028	16	6.8502e-09	8.3677e-09	8.5961e-09	8.3113e-09	3.8460e-08
	8	8	2.1647e-04	1.9277e-05	5.2969e-05	2.2607e-05	2.2328e-04
	128	16	5.6815e-07	4.9021e-08	1.2734e-07	2.5393e-08	1.8709e-06
	2048	32	9.7178e-10	5.3225e-10	7.2499e-10	5.2555e-10	1.4338e-08
0.75	4	4	7.9340e-04	1.1024e-05	1.2806e-03	1.4880e-04	1.7300e-03
	64	8	1.3147e-07	1.2932e-07	1.6305e-06	1.2356e-07	5.3739e-04
	1028	16	8.3028e-09	8.3022e-09	1.1579e-08	8.3023e-09	1.6874e-05
	8	8	1.0239e-04	7.6851e-07	1.3003e-04	2.0395e-05	7.2000e-03
	128	16	3.1043e-08	8.2769e-09	3.9629e-07	7.9173e-09	2.2716e-04
	2048	32	5.2618e-10	5.2606e-10	1.8712e-09	5.2615e-10	7.1085e-06

Scheme II, Improved Scheme I, and Improved Scheme II with the compact finite difference method in [16] with time discretized by the L1 method (L1C). Table 4.5 displays the L^∞ errors at $t = 1$. Clearly, the present algorithms yields better numerical results.

EXAMPLE 4.3. *Consider the following subdiffusion equation [30]*

$$\begin{cases} \partial_t u = {}_{RL}D_{0,t}^{1-\beta}(\partial_x^2 u) + f(x,t), & (x,t) \in (0,1)\times(0,1], \\ u(0,t) = u(1,t) = 0, & t \in (0,1], \end{cases} \tag{4.4}$$

where $0 < \beta < 1$. Choosing the suitable initial datum $u(x,0) = u_0$ and source term $f(x,t)$ such that the exact solution is

$$u(x,t) = \left(t^\beta - \frac{\Gamma(\beta+1)}{\pi^2}\right)\sin(\pi x).$$

In this example, we first convert the equation (4.4) into the form as (1.1). Then we solve the converted equation by using Improved Scheme I(α) (IS-I(α)) and Improved Scheme II(α) (IS-I(α)) with $\alpha = \beta$ in the computation. Let $\beta = 0.5$ and $h^4 = N^{-4} = \tau^3$ as those in [30], so we have $\alpha = 0.5$. We use the linear element in the computation as that in [30], the maximum

TABLE 4.6
Comparison of the maximum L^2 errors $\max_{0\le n\le n_T}\|u_h^n - u^n\|$ for Example 4.3, $r = 1, \beta = 0.5$.

N	IS-I(0.5)	order	IS-II(0.5)	order	γ=1	order	γ=1.5	order	γ=2	order
27	6.62e-4		6.62e-4		9.33e-3		1.63e-3		5.42e-4	
64	1.18e-4	1.50	1.18e-4	1.50	3.89e-3	0.76	3.50e-4	1.34	9.64e-5	1.50
125	3.09e-5	1.50	3.09e-5	1.50	1.87e-3	0.82	9.91e-5	1.41	2.53e-5	1.50
216	1.04e-5	1.50	1.04e-5	1.50	9.96e-4	0.86	3.44e-5	1.45	8.46e-6	1.50

TABLE 4.7
The maximum L^2 errors $\max_{0\le n\le n_T}\|u_h^n - u^n\|$ for Example 4.3, $r = 3, \beta = 0.5$.

N	$1/\tau$	IS-I(0.2)	IS-II(0.2)	IS-I(0.5)	IS-II(0.5)	IS-I(0.8)	IS-II(0.8)
27	81	8.12e-09	8.12e-09	8.13e-09	8.15e-09	8.20e-09	8.10e-09
64	256	2.57e-10	2.57e-10	2.58e-10	2.58e-10	2.61e-10	2.57e-10
125	625	1.77e-11	1.78e-11	1.80e-11	1.84e-11	1.83e-11	1.83e-11
216	1296	6.63e-12	5.05e-12	9.74e-12	1.12e-11	4.48e-12	1.05e-11
27	100	8.12e-09	8.12e-09	8.16e-09	8.15e-09	8.10e-09	8.10e-09
64	100	2.57e-10	2.57e-10	2.58e-10	2.58e-10	2.57e-10	2.57e-10
125	100	1.78e-11	1.77e-11	1.81e-11	1.78e-11	1.79e-11	1.79e-11
216	100	3.95e-12	8.24e-12	1.37e-11	1.11e-11	8.76e-12	8.76e-12
64	10	2.57e-10	2.57e-10	2.60e-10	2.58e-10	2.67e-10	2.57e-10
64	20	2.57e-10	2.57e-10	2.59e-10	2.58e-10	2.65e-10	2.57e-10
64	40	2.57e-10	2.57e-10	2.59e-10	2.58e-10	2.64e-10	2.57e-10
64	80	2.57e-10	2.57e-10	2.58e-10	2.58e-10	2.63e-10	2.57e-10

L^2 errors are shown in Table 4.6. Obviously, the present methods IS-I(0.5) and IS-II(0.5) show a little better results than the cases $\gamma = 1$ and $\gamma = 1.5$ in [30], and have almost similar results for the case $\gamma = 2$ in [30].

Since the methods IS-I(α) and IS-II(α) are exact in time for this example when $\alpha = \beta$. So they should show better performances when the space accuracy are improved. We use the cubic element in the improved algorithms IS-I(β) and IS-II(β) for different β ($\beta = 0.2, 0.5, 0.8$) and time steps, the results are shown in Table 4.7. Obviously, the satisfactory numerical results are displayed. One also finds that the error does not dependent on the time steps sizes, which is inline with the theoretical analysis.

5. Conclusion. In this paper, we first propose two fully discrete FEMs for the subdiffusion equation (1.1) with the time discretized by the fractional linear multistep methods. We give the strict stability and convergence analysis, which shows that both methods are unconditionally stable, and the global convergence orders in time are at least 1 for the suitably smooth solutions. We also explore the average error estimates, which covers all the truncation errors in the fully discrete schemes. It is shown that the average convergence rates are 1.5 for both schemes! Then, we propose two improved algorithms with global second-order accuracy for the smooth enough solutions. The two improved algorithms are also unconditionally stable. Even if the exact solution is not smooth enough, the present methods can attain second-order accuracy in temporal for some special cases. To the best knowledge of the authors, there are few works on the numerical methods with convergence order of 2 with unconditional stability for the subdiffusion equation (1.1), while the numerical methods with $(2-\beta)$th-order accuracy can be found in several papers, see for instance [14, 16, 17, 23]. One can also see [20, 29, 46] for the corresponding works.

We present enough numerical experiments to verify the theoretical analysis, and the comparisons with other methods are also given, which exhibit better accuracy than many of the existing numerical methods. Obviously, the present methods can be easily extended to the corresponding two- and three-dimensional problems. The stability and convergence analysis are very similar to those given here.

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