Travelling waves for a bistable reaction-diffusion equation with delay

S. Trofimchuk[1], **V. Volpert**[2,3,4]

[1] Instituto de Matematica y Fisica, Universidad de Talca, Casilla 747, Talca, Chile
[2] Institut Camille Jordan, UMR 5208 CNRS, University Lyon 1, 69622 Villeurbanne, France
[3] INRIA Team Dracula, INRIA Lyon La Doua, 69603 Villeurbanne, France
[4] Peoples Friendship University of Russia, ul. Miklukho-Maklaya 6, Moscow, 117198 Russia

Abstract. The paper is devoted to a reaction-diffusion equation with delay arising in modelling the immune response. We prove the existence of travelling waves in the bistable case using the Leray-Schauder method. In difference with the previous works, we do not assume here quasi-monotonicity of the delayed reaction term.

Key words: reaction-diffusion equation, time delay, bistable case, wave existence

1 Introduction

In this work we study the existence of travelling waves for the reaction-diffusion equation with delay:

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + kv(1-v) - f(v_\tau)v. \tag{1.1}$$

Here $v = v(x,t)$, $v_\tau = v(x, t-\tau)$, the function $f(v_\tau)$ will be specified below. This equation models the spreading of viral infection in tissues such as spleen or lymph nodes (see [7]). The first term in the right-hand side of this equation describes virus diffusion, the second term its production and the last term its elimination by the immune cells. The parameter D is the diffusion coefficient (or diffusivity) and k stands for the replication rate constant. In the sequel, without loss of generality, we can assume that $D = k = 1$. The parameterised function $f(v_\tau)$ (where v_τ is the concentration of virus some time τ before) characterises the virus induced clonal expansion of T cells, i.e. the number and function of these cells upon their maturation during some time τ. In this work we will suppose that the function $f(w)$ satisfies the following conditions implied by its biological meaning (Figure 1):

$$f(w) > 0 \ \text{ for } \ 0 \le w < 1, \ \ f(1) = 0, \ \ f'(1) > -1, \tag{1.2}$$

$$f(0) > 1, \ f'(0) > 0, \ f(w) > 1 \ \text{ for } \ 0 \le w < w_*, \tag{1.3}$$

for some $w_* \in (0,1)$. Furthermore,

$$\text{equation } f(w) = 1 - w \text{ has a single solution } w_0 \text{ for } 0 < w < 1; \quad f'(w_0) < -1. \tag{1.4}$$

Hence $f(w) > 1 - w$ for $0 \le w < w_0$ and $f(w) < 1 - w$ for $w_0 < w < 1$. Under these

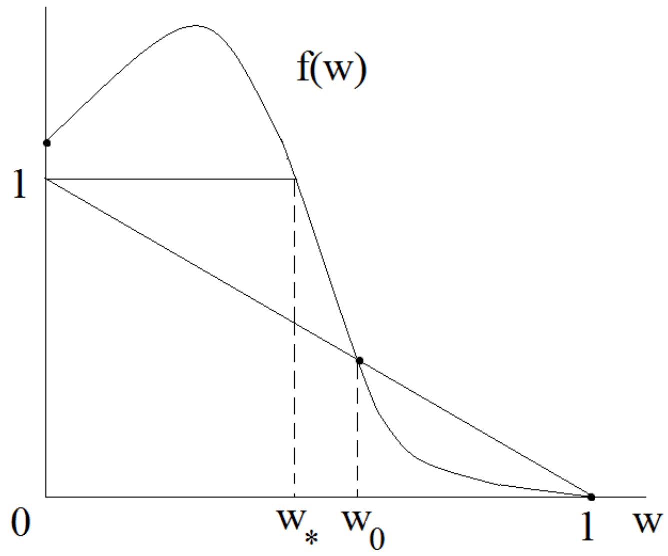

Figure 1: The typical form of the function $f(v)$. It is growing for small v and decreasing for large v.

conditions, the function $F(w) = w(1 - w - f(w))$ has three zeros: $w = 0$, $w = w_0$, $w = 1$. Moreover $F'(0) < 0$, $F'(1) < 0$. In the other words, we consider so-called bistable travelling waves. We recall here that travelling wave is a solution of equation (1.1) having the form $v(x,t) = w(x - ct)$, where the constant c is the wave speed. Clearly, the wave profile $w(z)$ satisfies the relation

$$w'' + cw' + w(1 - w - f(w(z + c\tau))) = 0. \tag{1.5}$$

Equation (1.1) is a quasi-linear functional reaction-diffusion equation. The basic concepts of the general theory of these equations were developed in [14, 23]. In this respect, the delayed reaction-diffusion equations whose reaction term g is either of logistic type (i.e. g is as in (1.1), when v is not separated multiplicatively from v_τ) or of the Mackey-Glass type (when v is separated multiplicatively from v_τ, i.e. $g = -kv + b(v_\tau)$) are between the most studied ones, e.g. cf. [8, 18]. It is an interesting point of discussion whether the Mackey-Glass type models reflect more adequately the biological reality than the logistic models, e.g. see [19, p. 56-58] and especially [12, Section 1.1] for further details. Importantly, in certain relevant situations both models exhibit similar types of qualitative behaviour of solutions. It is also worth noting that the investigation of delayed logistic models is more difficult and technically involved than the studies of the Mackey-Glass type systems, precisely because of the multiplicative non-separateness of v and v_τ. For example, so far no analytical results on the existence and uniqueness of bistable waves in delayed equations which include model (1.1) with *non-monotone nonlinear* response f were available in the literature, cf. [1, 9, 13, 15, 16, 17, 24].

In order to understand what kind of results can be expected in the delayed case, first we recall the main existence assertion [22] about bistable waves in the classical reaction-diffusion equation without delay ($\tau = 0$):

$$w'' + cw' + F(w) = 0, \quad F(0) = F(1) = 0, \; F'(0) < 0, \; F'(1) < 0. \tag{1.6}$$

In this case, for a unique value of c, there exists a unique (up to translation) monotonically decreasing solution of problem (1.6) on the whole axis with the limits

$$w(-\infty) = 1, \quad w(\infty) = 0. \tag{1.7}$$

Furthermore, in the delayed and monotone case (i.e. when $\tau > 0$ and $f(w)$ is decreasing), existence of solutions for problem (1.5), (1.7) was proved in [24, Theorem 5.8]. In such a case, equation (1.1) admits the maximum and comparison principles, and existence of solutions can be studied using these conventional techniques. See also recent work by Fang and Zhao for a similar result [15, Theorem 6.4] obtained in an abstract setting of monotone bistable semiflows. However, the properties of equation (1.5) change seriously if the function $f(w)$ is not monotonically decreasing. In this case, a first attempt to tackle the aforementioned existence and uniqueness problem was recently made in [7] . The idea of [7] was to consider a discontinuous piece-wise constant approximation instead of the original continuous function f, in this way simplifying the model (1.1) and allowing the use of the phase plane method.

In any event, the approaches of [7, 15, 24] are not applicable in the case of continuous non-monotone f, and the question about the existence of travelling waves remains open.

In this paper, we answer this question affirmatively, by proposing a different approach based on the construction of the topological degree for an elliptic operator considered on a subset containing only bistable waves monotonically decreasing on the whole real axis. Hence, basically we are going to use the Leray-Schauder method not for all solutions of (1.5), (1.7) but only for monotonically decreasing ones. It appears that during a continuous deformation monotone waves are separated from the non-monotone waves in the sense that the norm of their difference is uniformly bounded from below by a positive constant, cf. [8, 18]. This property allows us to construct a domain in the function space which contains all monotone waves and which does not contain any non-monotone solution of (1.5), (1.7). We prove that the value of the topological degree of the corresponding operator is different from zero in this domain. Then the existence of solutions to (1.5), (1.7) follows. This procedure was successfully applied in [22] to some classes of reaction-diffusion systems while the recent works [2, 3, 4, 21] suggested that it can also be extended to the framework of the theory of reaction-diffusion equations with delay and bistable nonlinearity. The key ingredient of this method is the construction of topological degree for elliptic boundary problems similar to (1.5), (1.7). Essentially this construction was proposed in [22] and recently developed further in [21, Chapter 11] with [2, 3, 4]. It is worth to note that we apply the Leray-Schauder method in a rather direct fashion, without the use of truncation argument, cf. [1, 5]. In any case, in the theory of functional reaction-diffusion equation, the wave profile equations (as (1.5) or equivalent integral equations) are usually solved either through the iteration procedure or by means of the Schauder fixed point theorem. These approaches lead to restrictive monotonicity assumptions on the delayed term (in some situations, squeezing technique allows to weaken them and consider non-monotone delayed terms as well). In this way, the application of the Leray-Shauder method, which is less demanding in regard to the shape and smoothness properties of the delayed nonlinearity, seems to be an interesting new possibility in this area of research.

Finally, we say a few words about the organisation of the paper. In the next section, we analyse briefly the basic properties of nonlinear operators required for the degree construction: by [21], these operators should be proper and their Frechet derivatives augmented by the term $-\lambda I$ should be Fredholm operators with zero index for all $\lambda \geq 0$. Section 3 is devoted to separation of monotone waves from non-monotone ones while in Section 4 we establish a priori estimates of waves. Finally, in Section 5 we prove our main result in this paper, Theorem 5.1: it says that if C^4-smooth f satisfies (1.2), (1.3), (1.4) and $f'(w) < 0$ for all $w \in [f^{-1}(1), 1]$, then at least one monotone bistable wave for problem (1.5), (1.7) exists for every fixed delay $\tau \geq 0$.

2 Operators and topological degree

Fredholm property of associated linear operators. Let E be the Hölder space $C^{2+\alpha}(\mathbb{R})$ which consists of functions continuous in $\mathbb{R}$ together with their second derivatives and the second derivative satisfies the Hölder condition with the exponent $\alpha \in (0,1)$. Recall that the Hölder semi-norm of some function $p : \mathbb{R} \to \mathbb{R}$ is defined as

$$[p]_\alpha := \sup_{x \neq y} \frac{|p(x) - p(y)|}{|x - y|^\alpha}.$$

In this section, we will use several times the following obvious estimate of $[p]_\alpha$ for $p \in C^1(\mathbb{R})$:

$$[p]_\alpha \leq \max\{2|p|_\infty, |p'|_\infty\}. \tag{2.1}$$

Here we use the standard notation $|p|_\infty = \sup_{x\in\mathbb{R}} |p(x)|$ for bounded functions $f : \mathbb{R} \to \mathbb{R}$.

Similarly, F will denote the space of Hölder continuous functions with the same exponent α. The norms in spaces E and F are given by the formulas $\|p\|_E := |p|_\infty + |p'|_\infty + |p''|_\infty + [p'']_\alpha$ and $\|p\|_F := |p|_\infty + [p]_\alpha$, respectively. We also will consider weighted spaces E_μ and F_μ defined as follows: $u \in E_\mu$ is and only if $u\mu \in E$, $\mu(x) = 1 + x^2$. Clearly, E_μ is a Banach space with the norm $\|f\|_{E_\mu} = \|\mu f\|_E$. The space F_μ is defined similarly. The choice of the weight function is not unique. It can be any positive function with polynomial growth at infinity.

Now, for some fixed real parameter h, consider the linear operator $L : E \to F$,

$$Lu = u'' + a(x)u' + b(x)u + d(x)u_h,$$

where $u_h(x) = u(x+h)$. We assume that the coefficients $a(x), b(x)$ and $d(x)$ belong to the space E. Assuming, further, that the coefficients have limits at infinity,

$$a(x) \to a_\pm, \quad b(x) \to b_\pm, \quad d(x) \to d_\pm, \quad x \to \pm\infty,$$

we can introduce the limiting operators

$$L_\pm u = u'' + a_\pm u' + b_\pm u + d_\pm u_h.$$

Let us recall that the operator L is called normally solvable if its image is closed. This definition is equivalent to the condition that the equation $Lu = f$ is solvable if and only if f is orthogonal to all functionals from some closed subspace of the dual space F^*.

Condition NS. The operator L is said to satisfy Condition NS if equations $L_\pm u = 0$ do not have nonzero bounded solutions.

Proposition 2.1. *The operator L is normally solvable with a finite dimensional kernel if and only if Condition NS is satisfied.*

Proof. The demonstration of this proposition is similar to the proof of Theorem 2.2 in [2] (sufficiency) and of Theorem 2.3 in [3] (necessity), and therefore it is omitted here. □

If we substitute $\exp(i\xi)$ in the equations $L_\pm u = 0$, then we obtain

$$-\xi^2 + a_\pm i\xi + b_\pm + d_\pm e^{i\xi h} = 0. \tag{2.2}$$

Condition NS is satisfied if and only if these equations do not have solutions for any real ξ [20]. Similarly, equations

$$L_\pm u = \lambda u \tag{2.3}$$

with $u = \exp(i\xi)$ are equivalent to

$$\lambda = -\xi^2 + a_\pm i\xi + b_\pm + d_\pm e^{i\xi h}, \quad \xi \in \mathbb{R}. \tag{2.4}$$

Theorem 2.2. *If the curves $\lambda(\xi)$ given by (2.4) are in the open left-half plane of the complex plane for all real ξ, then the operator $L - \lambda : E \to F$ with $\lambda \geq 0$ satisfies the Fredholm property, and its index equals* 0*.*

Proof. Note that the operator $L - \lambda$ is normally solvable with a finite dimensional kernel for all real $\lambda \geq 0$. Therefore its index is constant for such values of λ. On the other hand, the operator $L - \lambda$ is invertible for λ sufficiently large (to see the latter, it suffices to transform equation $(L - \lambda)u = f$ into equivalent integral equation and then apply Banach contraction principle for all large $\lambda > 0$). Therefore its index is 0. □

Finally, recall that the set of complex numbers λ for which the operator $L - \lambda$ does not satisfy the Fredholm property is called the essential spectrum of the operator L. It is known that a) the essential spectrum of L coincides with the union of two curves given by (2.4) and that b) polynomial weight does not change the essential spectrum, cf. [21, Chapter 5]. Thus Theorem 2.2 remains valid if we replace in its statement action $L : E \to F$ with $L : E_\mu \to F_\mu$.

Nonlinear operators. As we have already mentioned in the introductory section, in this work we study the existence of solutions of the problem

$$w'' + cw' + w(1 - w - f(w(x + c\tau))) = 0, \quad w(-\infty) = 1,\ w(\infty) = 0, \tag{2.5}$$

where c is an unknown constant which should be chosen to provide the existence of solution. The function $f(w)$ is supposed to be C^4-smooth and have uniformly bounded derivatives. In the case of equation (2.5), the limiting operators $L_\pm$ introduced in the previous subsection have the following forms:

$$(L_+u)(x) = u''(x)+cu'(x)+(1-f(0))u(x), \quad (L_-u)(x) = u''(x)+cu'(x)-u(x)-f'(1)u(x+c\tau).$$

Since $f(0) > 1$, the operator L_+ clearly satisfies the assumption NS. The same property holds for the operator L_- (considered with arbitrary τ) because of the inequalities $-1 < f'(1) \le 0$. Indeed, it is immediate to see that the characteristic equation $z^2+cz-1 = f'(1)e^{c\tau z}$ associated with L_-, can not have pure imaginary solutions. Furthermore, the respective curves

$$\lambda_+(\xi) = -\xi^2 + ic\xi + 1 - f(0); \quad \lambda_-(\xi) = -\xi^2 + ic\xi - 1 - f'(1)e^{ic\tau\xi}$$

satisfy, for all $\xi \in \mathbb{R}$, the inequalities

$$\Re\lambda_+(\xi) = -\xi^2 + 1 - f(0) < 0; \quad \Re\lambda_-(\xi) = -\xi^2 - 1 - f'(1)\cos(c\tau\xi) < 0.$$

All the above implies that, in case of equation (2.5), Condition NS, Proposition 2.1 and Theorem 2.2 can be used without restrictions.

In order to introduce the nonlinear operator corresponding to problem (2.5), we set $w(x) = u(x)+\psi(x)$, where $\psi(x)$ is an infinitely differentiable non-increasing function, $\psi(x) \equiv 1$ for $x \le 0$, $\psi(x) \equiv 0$ for $x \ge 1$. Then we consider the nonlinear operator

$$A_\tau(u) = (u + \psi)'' + c(u)(u + \psi)' + (u + \psi)(1 - u - \psi - f(u(x + c(u)\tau) + \psi(x + c(u)\tau))),$$

$$c(u) = \ln\sqrt{\int_{\mathbb{R}}(u(s) + \psi(s))^2 \min\{e^s, 1\}ds} =: 1/2\ln\rho(u),$$

acting from the space $E_\mu \times \mathbb{R}$ ($u \in E_\mu, \tau \in \mathbb{R}$) into the space F_μ. The main purpose of introducing functional $c(u)$ in the definition of $A_\tau(u)$ instead of considering c as a real parameter (as in (2.5)) is twofold: first, this obliges all solutions of the equation $A_\tau(u) = 0$, $\tau \in [0, \tau_*]$, belong to an open bounded set $D \subset E_\mu$; second, for $\tau = 0$, it allows to calculate topological degree $\gamma(A_0, D)$ by removing zero eigenvalues of some associated linear operator.

Since $\gamma(A_0, D)$ was already found in [22, Chapter 3, §3.2]: $\gamma(A_0, D) = 1$, we will use $c(u)$ only to obtain necessary a priori estimates of the wave solutions.

It is easy to see that functional $c : E_\mu \to \mathbb{R}$ is C^1-smooth and

$$c'(u)h(x) = \frac{1}{\rho(u)} \int_{\mathbb{R}} (u(s) + \psi(s))h(s) \min\{e^s, 1\} ds.$$

Therefore, to prove that the operator $A_\tau(u)$ depends C^1-smoothly on u, τ it suffices to establish that the nonlinear part of $A_\tau(u)$, i.e. $\mathfrak{N}(u, \tau) :=$

$$F(u+\psi, u(\cdot + c(u)\tau) + \psi(\cdot + c(u)\tau)) := (u+\psi)(1 - u - \psi - f(u(\cdot + c(u)\tau) + \psi(\cdot + c(u)\tau))),$$

is continuously differentiable. It is worth to mention that the above formula for $\mathfrak{N}(u, \tau)$ contains state-depending shifts of arguments (i.e. expressions like $u(x+c(u)\tau)$) and therefore the differentiability question for $\mathfrak{N}(u, \tau)$ should be handled with certain care, e.g. see [10, Section 3]. Let us show, for example, the existence of the Fréchet derivative $D_u\mathfrak{N}$. We claim that $[D_u\mathfrak{N}(u, \tau)]h(x) =$

$$F_1(P_0(x))h(x) + F_2(P_0(x)) \left(h(x + c(u)\tau) + \tau[u'(x + c(u)\tau) + \psi'(x + c(u)\tau)]c'(u)h\right),$$

where $F_j = F_j(v_1, v_2)$ denotes the partial derivative of $F(v_1, v_2)$ with respect to v_j:

$$F_j(P_0(x)) = F_j(u(x) + \psi(x), u(x + c(u)\tau) + \psi(x + c(u)\tau)),$$

$$P_s(x) = (u(x)+sh(x)+\psi(x), u(x+c(u+sh)\tau)+sh(x+c(u+sh)\tau)+\psi(x+c(u+sh)\tau)) \in \mathbb{R}^2.$$

Indeed, a straightforward computation shows that

$$\mathfrak{N}(u + h, \tau)(x) - \mathfrak{N}(u, \tau)(x) - [D_u\mathfrak{N}(u, \tau)]h(x) = R_1(u, h)(x)h(x) + R_2(u, h)(x),$$

where $R_1(u, h)(x) = \int_0^1 (F_1(P_s(x)) - F_1(P_0(x)))ds$,

$$R_2(u, h)(x) = \int_0^1 (F_2(P_s(x)) - F_2(P_0(x)))\Lambda(x, s)ds + \int_0^1 F_2(P_0(x))(\Lambda(x, s) - \Lambda(x, 0))ds,$$

$$\Lambda(x, s) = \tau(u'(x + c(u + sh)\tau) + sh'(x + c(u + sh)\tau) + \psi'(x + c(u + sh)\tau))(c'(u + sh)h)+ \\ h(x + c(u + sh)\tau).$$

Since $f \in C^4(\mathbb{R})$, $u \in E_\mu$, we find that, for some positive k_u and δ_u depending only on u,

$$|R_1(u, h)|_\infty \le k_u|h|_\infty, \ |\mu R_2(u, h)|_\infty \le k_u\|h\|_{E_\mu}|h|_\infty, \quad \text{for all} \quad \|h\|_{E_\mu} \le \delta_u;$$

$$|R_1'(u, h)|_\infty \le k_u(|h|_\infty + |h'|_\infty), \ |\mu R_2'(u, h)|_\infty \le k_u\|h\|^{1+\alpha}_{E_\mu}, \quad \text{for all} \quad \|h\|_{E_\mu} \le \delta_u.$$

In view of (2.1), the above estimates imply that $\|R_1(u, h)h + R_2(u, h)\|_{F_\mu} = O(\|h\|^{1+\alpha}_{E_\mu})$. This assures the differentiability of $\mathfrak{N}(u, \tau)$ with respect to u. A similar reasoning also shows that $D_u\mathfrak{N}$ depends continuously on u, τ in the operator norm. Finally,

$$[D_uA_\tau(u)]h(x) = h''(x) + c(u)h'(x) + (c'(u)h)u'(x) + [D_u\mathfrak{N}(u, \tau)]h(x) =$$

$$h''(x) + a(x)h'(x) + b(x)h(x) + d(x)h(x + c(u)\tau) + (Kh)(x) = (Lh)(x) + (Kh)(x),$$

where $a(x) = c(u),\ b(x) = F_1(P_0(x)),\ d(x) = F_2(P_0(x))$ and $K : E_\mu \to F_\mu$ defined by

$$(Kh)(x) = (c'(u)h)\,[u'(x) + \tau u'(x + c(u)\tau) + \tau\psi'(x + c(u)\tau)]$$

is one-dimensional linear operator. Since finite dimensional perturbations of the Fredholm operator does not change its index, we obtain the following version of Theorem 2.2:

Theorem 2.3. *If the curves $\lambda(\xi)$ given by (2.4) are in the open left-half plane of the complex plane for all real ξ, then the operator $D_uA_\tau - \lambda : E_\mu \to F_\mu$ with $\lambda \geq 0$ satisfies the Fredholm property, and its index equals 0. Moreover, there exists $C_0, \lambda_0 > 0$ such that*

$$\|(D_uA_\tau - \lambda)^{-1}\| \leq C_0 \text{ for all } \lambda > \lambda_0,\ \tau \in [0, \tau_*]. \tag{2.6}$$

Proof. *In view of the above said, we only have to prove inequality (2.6). We have that*

$$(D_uA_\tau)h(x) - \lambda^2 h(x) = h''(x) - \lambda^2 h(x) + a(x)h'(x) + b(x)h(x) + d(x)h(x + c\tau) + \phi(x)l(h),$$

where linear functional $l(h) := c'(u)h,\ l : E_\mu \to F_\mu$, is continuous and $\phi, \phi' \in F_\mu$. Set $v = \mu h$, then

$$h''(x) - \lambda^2 h(x) + a(x)h'(x) + b(x)h(x) + d(x)h(x + c\tau) + \phi(x)l(h) = g(x)$$

if and only if

$$v''(x) - \lambda^2 v(x) + a_1(x)v'(x) + b_1(x)v(x) + d_1(x)v(x + c\tau) + \phi_1(x)l_1(v) = g_1(x), \tag{2.7}$$

where $g_1(x) = \mu(x)g(x),\ \phi_1(x) = \mu(x)\phi(x),\ l_1(v) = l(v/\mu)$ and

$$a_1(x) = a(x) - \frac{2\mu'(x)}{\mu(x)},\ b_1(x) = b(x) + \frac{2(\mu'(x))^2}{(\mu(x))^2} - \frac{\mu''(x)}{\mu(x)} - a(x)\frac{\mu'(x)}{\mu(x)}, d_1(x) = d(x)\frac{\mu(x)}{\mu(x + c\tau)}.$$

Each bounded solution of equation (2.7) should satisfy the integral equation

$$v = \frac{1}{2\lambda}(M_+Tv - M_+g_1), \tag{2.8}$$

where

$$M_\pm w(t) = \left(\pm\int_{-\infty}^{t} e^{-\lambda(t-s)}w(s)ds + \int_{t}^{+\infty} e^{\lambda(t-s)}w(s)ds\right),$$

$$T(x, v(x), v'(x)) = a_1(x)v'(x) + b_1(x)v(x) + d_1(x)v(x + c\tau) + \phi_1(x)l_1(v).$$

After differentiating (2.8), we find that

$$v' = \frac{1}{2}(M_-Tv - M_-g_1). \tag{2.9}$$

Consider $M : C(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ defined by

$$M(v, w) = \frac{1}{2}(\frac{1}{\lambda}M_+Tv, M_-Tv),$$

it is immediate to see that $\|M\| \le K\lambda^{-1}$, where $K = 1 + |a_1|_\infty + |b_1|_\infty + |d_1|_\infty + |\phi_1|_\infty\|l_1\|$ does not depend on τ. Therefore system (2.8), (2.9) has a unique solution $(v, v') \in C(\mathbb{R}, \mathbb{R}^2)$ once $\lambda > K = \lambda_*$. Moreover,

$$|(v, v')|_\infty \le 2|(M_+g_1, M_-g_1)|_\infty \le \frac{4}{\lambda}|g_1|_\infty \text{ for all } \lambda > 2\lambda_*.$$

Using this inequality in equation (2.8) again, we can improve the estimate for $|v|_\infty$ as follows: $|v|_\infty \le K_1\lambda^{-2}|g_1|_\infty$, $[v]_\alpha \le \lambda^{-2}([g_1]_\alpha + [Tv]_\alpha)$, $\lambda > 2\lambda_*$. Then (2.7) yields that $|v''|_\infty \le K_2|g_1|_\infty$ for all $\lambda > 2\lambda_*$. Therefore, using (2.1), we also find that $[Tv]_\alpha \le K_3[g_1]_\infty$. Hence, $[v]_\alpha \le K_4\lambda^{-2}\|g\|_{F_\mu}$ for all $\lambda > 2\lambda_*$. Here $K_j, j = 1, 2, 3, 4$, stand for some universal constants. In this way,

$$[v'']_\alpha \le \lambda^2[v]_\alpha + [a_1v']_\alpha + [b_1v]_\alpha + [d_1v]_\alpha + [\phi_1]_\alpha|l_1(v)| + [g_1]_\alpha \le K_5\|g\|_{F_\mu}, \ \lambda > 2\lambda_*,$$

so that

$$\|(D_uA_\tau - \lambda)^{-1}g\|_{E_\mu} = \|h\|_{E_\mu} = \|v\|_E \le K_5\|g\|_{F_\mu}, \ \lambda > 2\lambda_*.$$

The latter estimate completes the proof of Theorem 2.3. □

The nonlinear operator A_τ has another useful property: it is a proper operator in the sense that the inverse image of compact sets is compact in any bounded closed set:

Theorem 2.4. *Let the condition NS be satisfied for $D_{u_0}A_{\tau_0}$. Assume $A_{\tau_n}(u_n) = g_n$ for converging sequences of elements $g_n \in F_\mu$, $\tau_n \ge 0$. If, in addition, sequence $\{u_n\}$ is bounded in E_μ, then it has a subsequence converging in E_μ.*

Proof. Suppose that $|u_n|_{E_\mu} \le K$. Since the inclusion $E_\mu \subset C^2(\mathbb{R})$ is compact, there exists a subsequence u_{n_j} converging in $C^2(\mathbb{R})$ to some element u_0. Clearly, $v_0 = \mu u_0 \in C^{2,\alpha}$ and $|v_0|_E \le K$. Without loss of generality, we can assume that $u_n \to u_0$, then $c_n := c(u_n) \to c_0 = c(u_0)$ and also $v_n := \mu u_n \to v_0$ uniformly on compact subsets of $\mathbb{R}$. The same type of convergence holds for the first and second derivates of v_n. To prove that $v_n := \mu u_n \to v_0$ uniformly on $\mathbb{R}$, we consider the following relation:

$$\mu\left[A_{\tau_n}\left(\frac{v_n}{\mu}\right) - A_{\tau_0}\left(\frac{v_0}{\mu}\right)\right] = \mu(g_n - g_0). \tag{2.10}$$

Suppose for a moment that $V_n = v_n - v_0$ does not converge, uniformly on $\mathbb{R}$, to the zero function. Then there exist positive ϵ_0 and some sequence x_n such that $|V_n(x_n)| \ge \epsilon_0$ for all n. Since sequence $\{x_n\}$ can not be bounded, we may suppose first that $x_n \to +\infty$. Now, since $W_n(x) = V_n(x + x_n)$ satisfies $|W_n|_E \le 2K$, without restricting generality, we can assume that $W_n(x)$, together with their first and second derivatives, converges uniformly on compact subsets of $\mathbb{R}$. Let $W_0(x)$ be the limit function for $\{W_n(x)\}$, then $|W_0(x)| \leq 2K$, $|W_0'(x)| \leq 2K$, $x \in \mathbb{R}$, and $|W_0(0)| \geq \epsilon_0$. After taking limit $n \to +\infty$ in (2.10), since $\psi(x+x_n) \to 0$ uniformly on compact sets, it is easy to find that

$$W_0''(x) + c_0 W_0'(x) + (1 - f(0))W_0(x) = 0,$$

where $(1 - f(0))W_0(x) = \lim_{n\to+\infty} R_n(x + x_n)$ and

$$R_n(x) = (v_n(x) + \mu(x)\psi(x))(1 - u_n(x) - \psi(x) - f(u_n(x + c_n\tau_n) + \psi(x + c_n\tau_n)))-$$

$$(v_0(x) + \mu(x)\psi(x))(1 - u_0(x) - \psi(x) - f(u_0(x + c_0\tau_0) + \psi(x + c_0\tau_0))).$$

Since the only bounded solution of the latter differential equation is $W_0(x) \equiv 0$, we arrive to a contradiction (recall that $W_0(0) \neq 0$).

Now, if $x_n \to -\infty$, then $\psi(x + x_n) \to 1$ uniformly on compact sets and therefore

$$\lim_{n\to+\infty} R_n(x + x_n) = -W_0(x)(1 + f(1))-$$

$$\lim_{n\to+\infty} \tilde{W}_n(x) \left(\frac{f(u_n(x + x_n + c_n\tau_n) + 1)) - f(u_0(x + x_n + c_0\tau_0) + 1))}{u_n(x + x_n + c_n\tau_n) - u_0(x + x_n + c_0\tau_0)} \right) =$$

$$-W_0(x)(1 + f(1)) - W_0(x + c_0\tau_0)f'(1),$$

where $\lim \tilde{W}_n(x) = W_0(x + c_0\tau_0)$ because of

$$\tilde{W}_n(x) := \mu(x + x_n)(u_n(x + x_n + c_n\tau_n) - u_0(x + x_n + c_0\tau_0)) =$$

$$W_n(x + c_n\tau_n) + \mu(x + x_n)(u_0(x + x_n + c_n\tau_n) - u_0(x + x_n + c_0\tau_0)) =$$

$$W_n(x + c_n\tau_n) + \mu(x + x_n)u_0'(x + x_n + \theta_n)(c_n\tau_n - c_0\tau_0).$$

Observe here that $u_n(x + x_n + c_n\tau_n), u_0(x + x_n + c_0\tau_0) \to 0$ as $n \to +\infty$.

Hence, $W_0(x)$ satisfies the functional differential equation

$$W_0''(x) + c_0 W_0'(x) - W_0(x)(1 + f(1)) - W_0(x + c_0\tau_0)f'(1) = 0,$$

which in view of condition NS can have only zero bounded solution. The obtained contradiction shows that $W_n(x)$ converges to 0 uniformly on $\mathbb{R}$ (so that $v_n = \mu u_n \to v_0 = \mu u_0$ uniformly on $\mathbb{R}$).

Similarly, if $V_n'(x)$ does not converge uniformly on $\mathbb{R}$ to 0, then $|V_n'(x_n)| \geq \epsilon_0 > 0$ for some ϵ_0 and $\{x_n\}$. Considering $W_n(x) = V_n(x + x_n)$, we obtain from the the previous part of the proof that $W_n(x) \to 0$, $n \to +\infty$ (uniformly on $\mathbb{R}$). Therefore, since $W_n'(x)$ converges, uniformly on compact sets, to some continuous function $W_*(x)$, we conclude that $W_*(x) \equiv 0$. Clearly, this contradicts to the inequalities $|W_n'(0)| \geq \epsilon_0$, $n = 1, 2, \dots$ Hence, we can conclude that $V_n'(x)$ converges to 0 uniformly on $\mathbb{R}$. The proof of the uniform (on $\mathbb{R}$) convergence $V_n''(x) \to 0$ is based on the same argument. Finally, in order to estimate the Hölder semi-norm $[V_n'']_\alpha$, it suffices to use the following equivalent form of (2.10):

$$V_n''(x) = \mu(x)(g_n(x) - g_0(x)) + V_n'(x)\left(\frac{2\mu'(x)}{\mu(x)} - c_0\right) + (c_0 - c_n)\left(\psi'(x)\mu(x) - v_n\frac{\mu'(x)}{\mu(x)} + v_n'(x)\right) +$$

$$V_n(x)\left(\frac{\mu''(x)}{\mu(x)}-\frac{2(\mu'(x))^2}{\mu^2(x)}+2\psi(x)-1+c_0\frac{\mu'(x)}{\mu(x)}-u_0(x)-u_n(x)+f(u_0(x+c_0\tau_0)+\psi(x+c_0\tau_0))\right)$$
$$+(\mu(x)+v_n(x))\left(f(u_n(x+c_n\tau_n)+\psi(x+c_n\tau_n))-f(u_0(x+c_0\tau_0)+\psi(x+c_0\tau_0))\right).$$

It is easy to see from this representation (directly leading to a Schauder type interior estimate) that $\lim_{n\to+\infty}[V_n'']_\alpha=0$. For instance, we can estimate the Hölder semi-norm of the third line in the above expression by using inequality (2.1) and the uniform on $\mathbb{R}$ convergences

$$\mu(x)\left((u_n(x+c_n\tau_n)+\psi(x+c_n\tau_n))-(u_0(x+c_0\tau_0)+\psi(x+c_0\tau_0))\right)\to 0,\ n\to+\infty,$$

$$\mu(x)\left((u_n'(x+c_n\tau_n)+\psi'(x+c_n\tau_n))-(u_0'(x+c_0\tau_0)+\psi'(x+c_0\tau_0))\right)\to 0,\ n\to+\infty.$$

This completes the proof of Theorem 2.4. □

Theorems 2.3 and 2.4 show that, according to the abstract theory developed in [21, §3.3 of Chapter 11], the Leray-Schauder type topological degree can defined for the operator A_τ. Indeed, in terms of [21], Theorems 2.3 and 2.4 imply that, for a fixed $\tau\geq 0$, the operator A_τ belongs to the class **F** while the family of operators A_τ with $\tau\in[0,\tau_0]$ belongs to the class **H**.

3 Monotonicity of solutions

In this section, we will consider the wave profile equation

$$w''+cw'+w(1-w-f(w(x+c\tau)))=0 \tag{3.1}$$

with the boundary conditions

$$w(-\infty)=1,\quad w(+\infty)=0. \tag{3.2}$$

Lemma 3.1. *Suppose that $c\geq 0$, $f'(w)<0$ for $w_0\leq w<1$. If solution $w(x)$ of problem (3.1), (3.2) satisfies condition $w'(x)\leq 0$ for all $x\in\mathbb{R}$, then $w'(x)<0$, $x\in\mathbb{R}$.*

Proof. Suppose that the assertion of the lemma does not hold and $w'(x_0)=0$ for some x_0. Then $w''(x_0)=0$ and from the equation (3.1) we obtain the equality

$$w(x_0)(1-w(x_0)-f(w(x_0+c\tau)))=0.$$

If $w(x_0)=0$, then by virtue of the uniqueness of solution $w(x)\equiv 0$, and we obtain a contradiction with (3.2). Hence

$$1-w(x_0)=f(w(x_0+c\tau)). \tag{3.3}$$

Since $c\tau>0$, then $w(x_0+c\tau)\leq w(x_0)$. Set $w_2=w(x_0+c\tau)$. Then $w_2\leq w(x_0)$. From (3.3) we get

$$1-w_2\geq f(w_2). \tag{3.4}$$

Suppose that $w(x_0)<w_0$. Since $1<f(0)$, that is $1-w<f(w)$ for $w=0$, then by virtue of (3.4), equation $f(w)=1-w$ has a solution in the interval $0<w\leq w_2$ ($\leq w(x_0)<w_0$). This conclusion contradicts the assumption on the function $f(w)$. Hence $w(x_0)\geq w_0$.

Next, we show that $w_2\geq w_0$. Indeed, suppose that $w_2<w_0$. Then (3.4) contradicts the assumption that $1-w<f(w)$ for $0\leq w<w_0$. Thus, $f'(w_2)<0$.

Set $u(x) = -w'(x)$. Differentiating equation (3.1), we obtain

$$u'' + cu' + a(x)u + b(x) = 0, \tag{3.5}$$

where

$$a(x) = k(1 - 2w(x) - f(w(x + c\tau))), \quad b(x) = -w(x)f'(w(x + c\tau))u(x + c\tau).$$

Let us recall that $u(x) \geq 0$ for all x, $u(x_0) = 0$, $u'(x_0) = 0$, $f'(w(x_0 + c\tau)) < 0$. Since the function $w(x)$ satisfies (3.2), then $u(x) \not\equiv 0$.

Let I_0 be the maximal interval containing the point $x = x_0$ and such that $u(x) = 0$ for all $x \in I_0$. Similar to the arguments presented above we can verify that $f'(w(x + c\tau)) < 0$ for $x \in I_0$. We take an interval I slightly larger than I_0 such that $f'(w(x+c\tau)) < 0$ and $u(x) \not\equiv 0$ for $x \in I$. If this set is reduced to a single point $x = x_0$, then it is a sufficiently small interval around this point. Since $b(x) \geq 0$ in this interval, $u(x) \geq 0$ and not identically 0, then we obtain a contradiction with the maximum principle for equation (3.5). □

Lemma 3.2. *Suppose that $c \leq 0$, $f'(w) < 0$ for $w_* \leq w \leq w_0$, where $f(w_*) = 1$ and $f(w) > 1$ for $0 \leq w < w_*$. If solution $w(x)$ of problem (3.1), (3.2) satisfies condition $w'(x) \leq 0$ for all $x \in \mathbb{R}$, then $w'(x) < 0$, $x \in \mathbb{R}$.*

Proof. Suppose that the assertion of the lemma does not hold and $w'(x_0) = 0$ for some x_0. Then we obtain equality (3.3). Since $c\tau \leq 0$, then $w(x_0 + c\tau) \geq w(x_0)$. Set $w_2 = w(x_0 + c\tau)$. It follows from (3.3) that

$$1 - w_2 \leq f(w_2).$$

Hence $w_2 \leq w_0$. From (3.3) we get that $f(w(x_0 + c\tau)) \leq 1$. Therefore, $w(x_0 + c\tau) \geq w_*$. Thus, $f'(w(x_0 + c\tau)) < 0$ and we proceed with equation (3.5) as in the proof of Lemma 3.1. □

Lemma 3.3. Let $w_n(x)$ be solutions of problem (3.1), (3.2) for some $\tau_n \in [0, \tau_*]$, $c = c_n$, $|c_n| \leq c_*$ (for some c_*), $n = 1, 2, ...$ Suppose that $w_0'(x) < 0$ for all $x \in \mathbb{R}$ and $w_n(x) \to w_0(x)$ in $C^1(\mathbb{R})$, $c_n \to c_0, \tau_n \to \tau_0$. If $f'(1) > -1$, then there exists $x = x_0$ such that $w_n'(x) < 0$ for $x \leq x_0$ and n sufficiently large.

Proof. Let $\epsilon > 0$ be such that

$$1 - w > f(w),\ 1 - \epsilon \leq w < 1; \quad 1 - w < f(w),\ 1 < w < 1 + \epsilon. \tag{3.6}$$

We choose such x_0 that $w_0(x) > 1 - \epsilon/2$ for all $x \leq x_0$. Then for n sufficiently large

$$1 - \epsilon < w_n(x) < 1 + \epsilon, \ \ x \leq x_0.$$

Denote by M a positive constant such that $|c_n\tau_n| < M$ for all n. Then for all n sufficiently large, $w_n(x) < 1$ and $w_n'(x) < 0$ for $x_0 - M \leq x \leq x_0$.

Suppose that the assertion of the lemma does not hold. We consider two cases: $w_n(x) > 1$ for some $x \leq x_0$ and $w_n(x) \leq 1$ for all $x \leq x_0$. In the first case, since $w_n(x) \to 1$ as $x \to -\infty$, then there is a global maximum of this function for $x \leq x_0$:

$$w_n'(x_n) = 0, \ \ w_n(x) \leq w_n(x_n), \ x \leq x_0, \ \ w_n(x_n) > 1.$$

We have:

$$1 - w_n(x_n) < f(w_n(x_n)) \leq f(w_n(x_n + c_n\tau_n)).$$

Hence $w_n''(x_n) > 0$ and we obtain a contradiction.

Suppose now that $w_n(x) \le 1$ for $x \le x_0$ and all n sufficiently large. If $w_n'(x) \le 0$ for $x \le x_0$ and $w_n'(x_n) = 0$ for some $x_n < x_0$, then we obtain a contradiction with Lemma 3.1. Therefore, if the assertion of this lemma is not satisfied, then the function $w_n(x)$ has a minimum for $x < x_0$. Suppose that there exists the most right minimum x_* of this function for $x \le x_0$. By virtue of the construction above, $x_* < x_0 - M$. Since $w_n''(x_*) \ge 0$, then we conclude from the equation that

$$1 - w_n(x_*) \le f(w_n(x_* + c_n\tau_n)).$$

It follows from condition (3.6) that

$$1 - w_n(x_* + c_n\tau_n) > f(w_n(x_* + c_n\tau_n)).$$

Therefore

$$w_n(x_* + c_n\tau_n) \le w_n(x_*). \tag{3.7}$$

If $c_n < 0$, then there is another minimum $x_{**} < x_*$ of this function, and $w_n(x_{**}) \le w_n(x_*)$. Repeating the same arguments, we will obtain a sequence of minima going to $-\infty$. This contradicts the convergence $w_n(x) \to 1$ as $x \to -\infty$.

If $c_n > 0$, then there is a single maximum x^* of this function in the interval $x_* < x^* < x_0$ since x_* is the most right minimum and $w_n'(x_0) < 0$. Suppose first that $x_* + c_n\tau_n \le x^*$. Then $w_n(x_* + c_n\tau_n) > w_n(x_*)$, and we obtain a contradiction with (3.7). Let now $x_* + c_n\tau_n > x^*$. Since $w_n(x)$ is decreasing for $x > x^*$, then

$$w_n(x^* + c_n\tau_n) < w_n(x_* + c_n\tau_n)$$

and

$$1 - w_n(x^*) < 1 - w_n(x_*) \le f(w_n(x_* + c_n\tau_n)) < f(w_n(x^* + c_n\tau_n)).$$

Therefore, $w_n''(x^*) > 0$ and we obtain a contradiction since x^* is a point of maximum.

If the most right minimum does not exist and there is a sequence of extrema converging to some $\hat{x}$, then it is sufficient to take a minimum sufficiently close to $\hat{x}$ and to repeat similar arguments as above. Let us also note that for $c_n < 0$ it is not necessary to take the most right minimum. Finally, if $c_n = 0$, then we obtain the equation without delay for which the assertion of the lemma is known [22]. □

Lemma 3.4. *Let $w_n(x)$ be solutions of problem (3.1), (3.2) for some $\tau_n \in [0, \tau_*]$, $c = c_n$, $|c_n| \le c_*$ (for some c_*), $n = 1, 2, \ldots$. Suppose that $\tau_n \to \tau_0$, $c_n \to c_0$ and $w_n(x) \to w_0(x)$ in $C^1(\mathbb{R})$, where $w_0(x)$ is a solution of problem (3.1), (3.2) for $\tau = \tau_0$, $c = c_0$. If $w_0'(x) < 0$ for all $x \in \mathbb{R}$, then $w_n'(x) < 0$ for all $x \in \mathbb{R}$ and n sufficiently large.*

Proof. Suppose that the assertion of the lemma does not hold, and there is a sequence x_n such that $w_n'(x_n) = 0$. If this sequence is bounded, then we can choose a convergent subsequence, $x_{n_k} \to x_0$. Then $w_0'(x_0) = 0$, and we obtain a contradiction with the assumption of the lemma.

Consider next the case where $x_n \to \infty$. Let $x = x_*$ be the solution of the equation $w_0(x) = w_* - \epsilon$ for some $\epsilon > 0$ sufficiently small. Let us recall that $f(w) > 1$ for $0 \le w < w_*$. Then for all n sufficiently large and for all $x \ge x_*$, $w_n(x) \le w_*$. Hence

$$f(w_n(x + c\tau)) > 1 \text{ for } x \ge x^* = x_* + c_*\tau_*$$

and for n sufficiently large since

$$x + c\tau \geq x^* + c\tau = (x_* + c_*\tau_*) + c\tau \geq x_*.$$

We use here the assumption that $|c\tau| \leq c_*\tau_*$. Furthermore, $w_n(x^*) \to w_0(x^*) > 0$, $w_n'(x^*) \to w_0'(x^*) < 0$.

Let $x_n > x^*$. If $w_n(x_n) > 0$, then $w_n''(x_n) = -w_n(x_n)(1 - w_n(x_n) - f(w_n(x_n + c\tau))) > 0$. Hence any positive extremum is a minimum, and the function $w_n(x)$ cannot converge to 0 at infinity.

Suppose now that $w_n(x_n) < 0$. Since $w_n(x) \to 0$ as $x \to \infty$, without loss of generality we can assume that x_n is a global minimum of the function $w_n(x)$. Hence $w_n(x_n) \leq w_n(x_n + c\tau)$. Therefore,

$$1 - w_n(x_n) < f(w_n(x_n)) \leq f(w_n(x_n + c\tau)). \tag{3.8}$$

The first inequality in (3.8) holds since $f(w) > 1 - w$ in some neighborhood of $w = 0$ (including small negative w). The second inequality in (3.8) takes place for sufficiently small in absolute value $w_n(x_n)$ and $w_n(x_n+c\tau)$ since $f'(0) > 0$. Thus, $w_n''(x_n) = -w_n(x_n)(1-w_n(x_n)-f(w_n(x_n+c\tau))) < 0$. Therefore x_n is a point of maximum, and it cannot be the global minimum, as supposed.

It remains to note that convergence $x_n \to -\infty$ cannot hold due to Lemma 3.3. □

4 A priori estimates

4.1 Estimate of the wave speed

Let us introduce functions $f_0(w)$ and $f_1(w)$ such that

$$f_0(w) \leq f(w) \leq f_1(w), \quad f_0'(w) \leq 0, \quad f_1'(w) \leq 0, \quad 0 \leq w \leq 1, \tag{4.1}$$

$f_0(0) > 1, f_1(0) > 1$, and equations

$$1 - w = f_0(w), \quad 1 - w = f_1(w)$$

have unique solutions in the interval $0 \leq w < 1$. Then the problem

$$w'' + cw' + w(1 - w - f_0(w)) = 0, \quad w(-\infty) = 1, \quad w(\infty) = 0 \tag{4.2}$$

(without delay) has a unique solution (up to translation in space) $w_0(x)$ for a unique value $c = c_0$. Similarly, the problem

$$w'' + cw' + w(1 - w - f_1(w)) = 0, \quad w(-\infty) = 1, \quad w(\infty) = 0 \tag{4.3}$$

(without delay) has a unique solution (up to translation in space) $w_1(x)$ for a unique value $c = c_1$.

Lemma 4.1. *If there exists a monotonically decreasing solution $w(x)$ of problem (3.1), (3.2) for some $c > 0$, then $c \leq c_0$.*

Proof. Since $c\tau > 0$ and $w(x)$ is a decreasing function, then $w(x + c\tau) < w(x)$. Hence

$$f(w(x + c\tau)) \geq f_0(w(x + c\tau)) > f_0(w(x)). \tag{4.4}$$

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + u(1 - u - f_0(u)) \tag{4.5}$$

(without delay) with the initial condition

$$u(x, 0) = w(x). \tag{4.6}$$

Taking into account equation (3.1) and inequality (4.4), we obtain:

$$w'' + cw' + w(1 - w - f_0(w)) = w'' + cw' + w(1 - w - f(w(x + c\tau)))+$$

$$w(f(w(x + c\tau)) - f_0(w(x))) > 0, \quad x \in \mathbb{R}.$$

Hence $w(x)$ is a lower function, and solution $u(x, t)$ of problem (4.5), (4.6) is monotonically increasing with respect to t for each x.

On the other hand, by virtue of global stability of monotone waves for the bistable equation, $u(x, t) \to w_0(x + (c - c_0)t)$ as $t \to \infty$ uniformly on the whole axis. From this convergence we can conclude that $c \le c_0$. Indeed, if $c > c_0$, then for each x fixed $w_0(x + (c - c_0)t) \to 0$ as $t \to \infty$. However, $u(x, t) \ge w(x)$ for all x and t. This contradiction completes the proof of the lemma. □

Lemma 4.2. *If there exists a monotonically decreasing solution $w(x)$ of problem (3.1), (3.2) for some $c < 0$, then $c \ge c_1$.*

Proof. Since $c\tau < 0$ and $w(x)$ is a decreasing function, then $w(x + c\tau) > w(x)$. Hence

$$f(w(x + c\tau)) \le f_1(w(x + c\tau)) < f_1(w(x)). \tag{4.7}$$

Consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} + u(1 - u - f_1(u)) \tag{4.8}$$

(without delay) with the initial condition

$$u(x, 0) = w(x). \tag{4.9}$$

Taking into account equation (3.1) and inequality (4.7), we obtain:

$$w'' + cw' + w(1 - w - f_1(w)) = w'' + cw' + w(1 - w - f(w(x + c\tau)))+$$

$$w(f(w(x + c\tau)) - f_1(w(x))) < 0, \quad x \in \mathbb{R}.$$

Hence $w(x)$ is an upper function, and solution $u(x, t)$ of problem (4.5), (4.6) is monotonically decreasing with respect to t for each x.

On the other hand, by virtue of global stability of monotone waves for the bistable non-delayed equation, $u(x, t) \to w_1(x+(c-c_1)t)$ as $t \to \infty$ uniformly on the whole axis. From this convergence we can conclude that $c \ge c_1$. Indeed, if $c < c_1$, then for each x fixed $w_0(x+(c-c_1)t) \to 1$ as $t \to \infty$. However, $u(x, t) \le w(x)$ for all x and t. This contradiction completes the proof of the lemma. □

From the last two lemmas we obtain the following estimate for the wave speed:

$$|c| \le c_* := \max\{|c_0|, |c_1|\}. \tag{4.10}$$

We note that $c \ge 0$ if $c_1 \ge 0$ and $c \le 0$ if $c_0 \le 0$.

4.2 Estimates of solutions

In this section, we will use repeatedly the following simple observation:

Lemma 4.3. *Let $w(x)$ be a monotonically decreasing solution of equation (3.1) with the limits $w(-\infty) = w_0, w(+\infty) = 0$ at infinity. Then $c < 0$. If $w(-\infty) = 1, w(+\infty) = w_0$, then $c > 0$.*

Proof. Consider a decreasing wave $w(x)$ connecting w_0 and 0 and suppose that $c \geq 0$. Then $0 \leq w(x + c\tau) \leq w(x) \leq w_0$ and therefore $1 - w(x) \leq f(w(x + c\tau))$ so that

$$w''(x) = -cw'(x) - w(x)(1 - w(x) - f(w(x + c\tau))) \geq 0.$$

Since a convex function cannot connect two final equilibria, we have got a contradiction. Thus $c < 0$.

Similarly, suppose that some decreasing wave $w(x)$ connects 1 and w_0 with the speed $c \leq 0$. Then $1 \geq w(x + c\tau) \geq w(x) \geq w_0$ and therefore $1 - w(x) \geq f(w(x + c\tau))$, see Fig. 1. In this way, we will get a contradiction again:

$$w''(x) = -cw'(x) - w(x)(1 - w(x) - f(w(x + c\tau))) \leq 0.$$

This completes the proof of Lemma 4.3. □

We can now estimate the weighted norm of wave profiles:

Lemma 4.4. *Let $w_\tau(x)$ be a monotonically decreasing solutions of problem (3.1), (3.2) for possibly different values of $\tau \in [0, \tau_*]$ such that $w_\tau(0) \in [\alpha, \beta] \in (0, 1)$ for some fixed α, β and all admissible $\tau \in [0, \tau_*]$. Then there exists a positive constant M independent of τ such that*

$$\sup_{x \in \mathbb{R}} |w'_\tau(x) - \psi'(x)|\mu(x) + \sup_{x \in \mathbb{R}} |w_\tau(x) - \psi(x)|\mu(x) \leq M. \tag{4.11}$$

Proof. Let c_* be as defined in (4.10). Denote by $x_1(\tau)$ the solution of the equation $w_\tau(x) = w_1$ and by $x_2(\tau)$ the solution of the equation $w_\tau(x) = w_2$ where $0 < w_2 < \min\{w_0, \alpha\} \leq \max\{w_0, \beta\} < w_1 < 1$ are some fixed numbers.

Clearly, $x_1(\tau) < 0,\ x_2(\tau) > 0$. Moreover, we claim that the difference $x_1(\tau) - x_2(\tau)$ is uniformly bounded. Suppose that this in not the case and this difference tends to infinity for some convergent sequence of $\tau_n \in [0, \tau_*]$: $\tau_n \to \tau_0$. Set

$$v_n(x) = w_{\tau_n}(x + x_2(\tau_n)).$$

Then $v_n(0) = w_2$, $v_n(x_1(\tau_n) - x_2(\tau_n)) = w_1$. We can choose a locally convergent subsequence from the sequence $v_n(x)$. Denote its limit by $v_0(x)$. Then it is a solution of equation (3.1), $v_0(0) = w_2$, $v_0(-\infty) \leq w_1$ since $x_1(\tau_n) - x_2(\tau_n) \to -\infty$. Then $v_0(-\infty) = w_0$ and therefore $c < 0$ in view of Lemma 4.3.

Similarly, consider the sequence

$$z_n(x) = w_{\tau_n}(x + x_1(\tau_n)).$$

Then $z_n(0) = w_1$, $z_n(x_2(\tau_n) - x_1(\tau_n)) = w_2$. We can choose a locally convergent subsequence from the sequence $z_n(x)$. Let $z_0(x)$ denote its limit. Then it is a solution of equation (3.1), $z_0(0) = w_1$,

$z_0(+\infty) \geq w_2$ since $x_2(\tau_n) - x_1(\tau_n) \to \infty$. Then $z_0(-\infty) = 1$, $z_0(+\infty) = w_0$ and by virtue of Lemma 4.3, $c > 0$.

The obtained contradiction ($0 < c < 0$) shows that functions $x_1(\tau)$ and $x_2(\tau)$ are uniformly bounded. Hence, for some fixed T independent of τ, it holds

$$w_\tau(x) \leq w_2,\ t \geq T - c_*\tau_*, \quad w_\tau(x) \geq w_1,\ t \leq -T + c_*\tau_*, \text{ for all } \ \tau \in [0, \tau_*]. \tag{4.12}$$

A useful consequence of this result is compactness of the set $\mathfrak{T}$ of all 'admissible' delays and speeds:

$$\mathfrak{T} = \{(\tau, c) \in [0, \tau_*] \times [-c_*, c_*] : \text{system } (3.1), (3.2) \text{ has a monotone wave for these } \tau, c\}.$$

Indeed, if $w_{\tau_n}(x)$, $w_{\tau_n}(0) \in [\alpha, \beta]$, is a sequence of monotone bistable waves propagating with speeds c_n, then each converging subsequence of $w_{\tau_n}(x), \tau_n, c_n$, has a limit also satisfying, in view of inequalities (4.12), the boundary conditions (3.2).

We claim that if the parameters $1 - w_1$ and w_2 are sufficiently small then the functions $u_\tau = w_\tau - \psi$, w'_τ admit exponential estimates for $x \leq -T$ and $x \geq T$ which are uniform uniform with respect to $\tau \in [0, \tau_*]$. This means that there exist some positive numbers K_1, γ_1 independent on τ such that

$$|1 - w_\tau(x)| + |w'_\tau(x)| \leq K_1 e^{\gamma_1 x},\ x \leq -T; \quad |w_\tau(x)| + |w'_\tau(x)| \leq K_1 e^{-\gamma_1 x},\ x \geq T. \tag{4.13}$$

These estimates follow from the assumptions $f(0) > 0$, $f'(1) \in (-1, 0]$, and their proof is based on arguments adapted from the exponential dichotomy theory [6, 11]. Since the set of all admissible delays $\mathfrak{T}$ is compact, it suffices to establish (4.13) locally, i.e. to prove that estimate (4.13) is true within sufficiently small neighbourhood $\mathcal{O}$ of each point $(h_0, c_0) \in [0, c_*\tau_*] \times [-c_*, c_*]$ (so that K_1 and γ_1 may depend on (h_0, c_0)). In the appendix, we briefly outline the proof of the first inequality in (4.13) in more difficult case when $c_{\tau_0} \geq 0$ and $c_\tau > 0$. Finally, since functions $w_\tau(x), w'_\tau(x)$ are uniformly (in x and τ) bounded, Lemma 4.4 is an immediate consequence of inequalities (4.13). □

Corollary 4.5. *Assume all the conditions of Lemma 4.4. Then the functions $u_\tau = w_\tau - \psi$ are uniformly (in $\tau \in [0, \tau_*]$) bounded in the norm of $C^{2+\alpha}_\mu(\mathbb{R})$.*

Proof. We have

$$(u + \psi)'' + c(u + \psi)' + w(1 - w - f(w(x + c\tau))) = 0.$$

Set $v = u\mu$ and multiply the last equation by μ. Then we obtain

$$v'' + cv' + g(x, c, \tau) = 0, \tag{4.14}$$

where

$$g(x, c, \tau) = -2u'\mu' - u\mu'' - cu\mu' + \psi''\mu + c\psi'\mu + w(1 - w - f(w(x + c\tau)))\mu.$$

In view of (4.13), this function is uniformly (in τ, c) bounded in the norm of $C^1(\mathbb{R})$. Therefore the norm of v in $C^{2+\alpha}(\mathbb{R})$ is also uniformly bounded due to the Schauder estimate. □

Lemma 4.5. *Let $w_\tau : \mathbb{R} \to (0, 1)$ be monotonically decreasing wave solutions of equation $A_\tau(w_\tau - \psi) = 0$ for possibly different values of $\tau \in [0, \tau_*]$. Then $u_\tau = w_\tau - \psi$ are uniformly bounded in the norm of $C^{2+\alpha}_\mu(\mathbb{R})$.*

Proof. Clearly, $w_\tau(x)$ is a bistable wave of equation (1.5) propagating with the speed $c(u_\tau)$. In view of (4.10), $|c| \leq c_*$ for some $c_* > 0$.

Suppose now that the set $\{u_\tau : \tau \in [0, \tau_*]\}$ is not uniformly bounded in $E_\mu = C^{2+\alpha}_\mu(\mathbb{R})$. Then there exist sequences τ_n and $u_n := u_{\tau_n}$ such that $c(u_n) \to c_\star$ and $\|u_n\|_{E_\mu} \to +\infty$ as $n \to +\infty$. Moreover, without loss of generality, we can assume that $w_n^{(j)}(x) := w_{\tau_n}^{(j)}(x) \to w_\star^{(j)}(x),\ j = 0, 1, 2,$ uniformly on compact subsets of $\mathbb{R}$. Here $w_\star$ denotes some non-increasing bounded solution of (1.5) with $c = c_\star$.

We claim that $w_\star(-\infty) = 1,\ w_\star(+\infty) = 0$. Indeed, if $w_\star(+\infty) > 0$ then, applying the Fatou's lemma, we get the following contradiction:

$$+\infty = c(w_\star - \psi) \leq \liminf c(u_n) = \lim c_n = c_\star < c_*.$$

On the other hand, if $w_\star(-\infty) = 0$ then $w_\star \equiv 0$ so that $w_n'(0) \to 0$ and $w_n(x) \to 0,\ n \to +\infty$, uniformly on each half-line $[s, +\infty)$. In addition, since each $w_n(x)$ satisfies

$$w'' + c_n w' + a_n(x) w = 0, \quad a_n(x) := 1 - w_n(x) - f(w_n(x + c_n\tau_n)), \tag{4.15}$$

where $a_n(x) \to 1 - f(0)$ uniformly on $[0, +\infty)$, we can conclude (e.g. see [6, Proposition 1, p.34]) that, for some positive constant K, γ, it holds

$$|w_n(x)| \leq K e^{-\gamma t}(w_n(0) + |w_n'(0)|),\ t \geq 0, \quad n = 1, 2, \ldots$$

This implies, however, that $c(u_n) \to -\infty$, in view of the Lebesgue's dominated convergence theorem. The obtained contradiction shows that $w_\star(-\infty) \in \{w_0, 1\}$. For a moment, let suppose that $w_\star(-\infty) = w_0$. In this case, for each positive ϵ, the intervals $T_n(\epsilon) := \{x : w_0 - \epsilon < w_n(x) < w_0 + \epsilon\}$ have lengths $d_n(\epsilon)$ converging to $+\infty$ as $n \to +\infty$. Consequently, the intervals $Q_n := \{x : w_0 - \epsilon < w_n(x) \leq (w_0 + 1)/2\}$ have lengths $q_n > d_n$. If x_n denotes the unique solution of equation $w_n(x_n) = (w_0 + 1)/2$, then the sequence of shifted waves $\{w_n(x + x_n)\}$ converges, uniformly on compact subsets of $\mathbb{R}$, to a bounded non-increasing solution $w_l : \mathbb{R} \to [0, 1]$ of equation (1.5) considered with $c = c_\star$. Since $w_l(0) = (w_0 + 1)/2$ and $w_l(x) \geq w_0 - \epsilon$ for all $x \in [0, q_n]$ with $q_n \to +\infty$, we conclude that $w_l(-\infty) = 1$ and $w_l(+\infty) = w_0$. However, the simultaneous existence of non-increasing waves w_l and $w_\star$ of equation (1.5) considered with the same speed $c = c_\star$ contradicts conclusions of Lemma 4.3.

Consequently, $w_\star(-\infty) = 1,\ w_\star(+\infty) = 0$ so that $w_n(0) \to w_\star(0) \in (0, 1),\ n \to +\infty$. This means that $w_n(0) \in [\alpha, \beta] \in (0, 1),\ n = 1, 2, 3\ldots$ for some appropriate fixed α, β. By Corollary 4.5, $u_n,\ n = 1, 2, 3\ldots$ must be uniformly bounded in the norm $C^{2+\alpha}_\mu(\mathbb{R})$ that contradicts our choice of this sequence. □

5 Existence of solutions

Once topological degree is defined and a priori estimates of solutions are obtained, we can use the Leray-Schauder method.

Theorem 5.1. *Suppose that $f \in C^4(\mathbb{R}_+)$ satisfies conditions (1.2), (1.3), (1.4) and, in addition, $f'(w) < 0$ for $w_* \leq w < 1$, where $w_* = f^{-1}(1)$. Then problem (3.1), (3.2) has a monotonically decreasing solution for any $\tau \geq 0$.*

Proof. Fix an arbitrary positive number τ_*. With operator $A_\tau : E_\mu \to F_\mu$ defined in Section 2, we consider all solutions of the equation

$$A_\tau(u) = 0, \quad \tau \in [0, \tau_*], \tag{5.16}$$

such that the functions $w = u + \psi$ are monotonically decreasing. We denote such solutions by u_M. It follows from Lemma 4.5 that $\|u_M\|_{E_\mu} \le K$, where a positive constant K does not depend on solution and on $\tau \in [0, \tau_*]$. If the function $w = u + \psi$ is not monotonically decreasing, then we denote such solutions u_N.

We claim that there exists a positive constant r such that

$$\|u_M - u_N\|_{E_\mu} \ge r \tag{5.17}$$

for any solutions u_M and u_N. Indeed, if this is not the case, then there are two sequences of solutions u_M^i and u_N^i such that

$$\|u_M^i - u_N^i\|_{E_\mu} \to 0, \quad i \to \infty. \tag{5.18}$$

Therefore, since $|u_M^i|_{E_\mu} \le K$ for all i and $A_\tau(u)$ is a proper operator with respect to (u, τ) (see Theorem 2.4), we can choose convergent subsequence of solutions. Without loss of generality we assume that $u_M^i \to u_0$ in E_μ, $\tau_i \to \tau_0$. Then the function $w_0 = u_0 + \psi$ is a solution of problem (3.1), (3.2) for some τ_0 and c. Hence $w_0'(x) \le 0$, and by virtue of Lemmas 3.1, 3.2, we have $w'(x) < 0$ for all $x \in \mathbb{R}$.

Set $w_i = u_N^i + \psi$. It follows from (5.18) that $\|w_i - w_0\|_{C^2} \to 0$ as $i \to \infty$. This convergence contradicts Lemma 3.4 and proves (5.17).

We construct an open ball of the radius $r/2$ around each solution u_M. Since the set of solutions is compact, we can choose a finite subcovering of the set of solutions by the balls. Denote this domain by D. It contains all solutions u_M and it does not contain solutions u_N.

Consider the topological degree $\gamma(A_\tau, D)$. Since $A_\tau(u) \ne 0$ on ∂D, then this degree does not depend on τ. It remains to verify that it is different from 0. Indeed, in the non-delayed case ($\tau = 0$), equation (5.16) has a unique solution $u_0 = w_0 - \psi$, where w_0 is the unique solution of problem (3.1), (3.2) with $\tau = 0$. The value of the degree $\gamma(A_0, D)$ equals the index of this solution which can be found through the eigenvalues of the linearized operator. This computation was already done in [22, Chapter 3, §3.2] where it was shown that $\gamma(A_0, D) = 1$. Hence, $\gamma(A_\tau, D) = 1$ for any $\tau \in [0, \tau_*]$. This proves the existence of solutions u_M of equation (5.16) for each $\tau \in [0, \tau_*]$. □

We note that if we consider the case where $c > 0$ or $c < 0$, the conditions on the function $f(w)$ can be somewhat weakened (see Lemmas 3.1, 3.2).

Appendix

In this section, we briefly outline the proof of the first inequality in (4.13) in the case when $(h, c) := (\tau c_\tau, c)$, $c_\tau > 0$, belongs to some small neighbourhood $\mathcal{O}$ of the point $(h_0, c_{\tau 0})$ with $c_{\tau 0} \ge 0$. In such a case, it is convenient to transform equation (3.1) into usual delayed differential equation by inverting the time: $w(t) = v(-t)$. In this way, instead of system (3.1), (3.2), we obtain

$$v''(x) - cv'(x) + v(x)(1 - v(x) - f(v(x - h))) = 0, \quad v(-\infty) = 0, \;\; v(+\infty) = 1. \tag{5.19}$$

Thus, for $x \ge T$, the function $u(t) = v(t) - 1$ satisfies the following delay differential equation:

$$u''(x) - cu'(x) - (1 + a(x))u(x) - (f'(1) + b(x))u(x - h) = 0, \tag{5.20}$$

where, uniformly with respect to $x \geq T$, it holds

$$a(x) = u(x) = O(|1-w_1|),\ b(x) = -f'(1) + (1+u(x))f'(1+\theta u(x-h)) = O(|1-w_1|),\ \theta \in (0,1).$$

We will extend $a(x), b(x)$ continuously on the whole $\mathbb{R}$ in such a way that $|a|_\infty, |b|_\infty = O(|1-w_1|)$. In the standard way, equation (5.20) generates a semi-flow on the extended phase space $\mathbb{R} \times \mathcal{X}$, where $\mathcal{X} := C \times \mathbb{R}$ and C stands for the space of scalar continuous functions $C[-c_*\tau_*, 0]$ provided with the sup-norm $|\phi|_\infty = \sup\{\phi(s) : s \in [-c_*\tau_*, 0]\}$. In view of the assumption $f'(1) \in (-1, 0]$, the 'ω-limit' equation of (5.20),

$$u''(x) - c_0 u'(x) - u(x) - f'(1)u(x-h_0) = 0, \tag{5.21}$$

has only the trivial bounded solution and therefore it possesses an exponential dichotomy with some projection $P_{h_0}(s) : \mathcal{X} \to \mathcal{X},\ s \in \mathbb{R}$, and positive constants K_0, γ_0. The latter amounts to the following two properties:

- If $(\phi_+, a_+) = P_{h_0}(s)(\psi, b)$, with (ψ, b) being an arbitrary fixed element of $\mathcal{X}$, then the solution $u(t, s, \phi_+, a_+),\ t \geq s$, of the initial value problem $u(s+x) = \phi_+(x)$, $x \in [-c_*\tau_*, 0]$, $u'(s) = a_+$, for equation (5.21) satisfies the inequality
$$|u(t+\cdot, s, \phi_+, a_+)|_\infty + |u'(t, s, \phi_+, a_+)| \leq K_0 e^{-\gamma_0(t-s)}(|b| + |\psi|_\infty), \quad t \geq s.$$
- On the other hand, if $(\phi_-, a_-) = (\psi, b) - P_{h_0}(s)(\psi, b)$ then the solution $u(t, s, \phi_-, a_-)$ of the initial value problem $u(s+x) = \phi_-(x),\ x \in [-c_*\tau_*, 0]$, $u'(s) = a_-$, for equation (5.21) can be extended for all $t \leq s$ and satisfies the inequality
$$|u(t+\cdot, s, \phi_-, a_-)|_\infty + |u'(t, s, \phi_-, a_-)| \leq K_0 e^{\gamma_0(t-s)}(|b| + |\psi|_\infty), \quad t \leq s.$$

Observe that since equation (5.21) has constant coefficients, $P_{h_0}(s)$ is also a constant function, $P_{h_0}(s) \equiv P_0$. We claim that $P_0(0,1) \neq (0,1)$. Indeed, if $P_0(0,1) = (0,1)$, then the solution $\hat{u}(t) := u(t, 0, 0, 1),\ t \geq 0$, of equation (5.21) is exponentially converging to 0 as $t \to +\infty$ while $\hat{u}(s) = 0$ for $s \in [-h_0, 0]$ and $\hat{u}'(0) = 1$. This means that $\hat{u}(t)$ reaches its positive absolute maximum at some leftmost point $t_M > 0$. At this point, $\hat{u}''(t_M) \leq 0$, $\hat{u}'(t_M) = 0$, so that, taking into account the inequality $0 \leq -f'(1) < 1$, we get the following contradiction

$$\hat{u}(t_M) = \hat{u}''(t_M) - c_0\hat{u}'(t_M) - f'(1)\hat{u}(t_M - h_0) \leq -f'(1)\hat{u}(t_M - h_0) < \hat{u}(t_M).$$

Next, the roughness property of the exponential dichotomy guarantees (cf. [11, Theorem 7.6.10]) the existence of small $\delta > 0$ such that equation (5.20) possesses an exponential dichotomy with some projection $P_h(s) : \mathcal{X} \to \mathcal{X},\ s \in \mathbb{R}$, and constants $2K_0, 0.5\gamma_0$ for all non-negative c, h, w_1 such that

$$\max\{|c - c_0|, |h - h_0|, |w_1 - 1|\} < \delta. \tag{5.22}$$

Moreover, $\sup_{s\in\mathbb{R}} |P_0 - P_h(s)| \to 0$ as $\delta \to 0$. In particular, since $(Id - P_0)(0,1) \neq (0,0)$, we can take δ sufficiently small to have $i_* = \inf_{s\in\mathbb{R}} |(Id - P_h(s))(0,1)| > 0$ once (5.22) is satisfied.

Hence, assuming (5.22) and taking arbitrary solution $u(t) < 1 - w_1,\ t \geq T - h,\ u(+\infty) = u'(+\infty) = 0$ of equation (5.20), we can conclude that $P_h(t)(u(t+\cdot), u'(t)) = (u(t+\cdot), u'(t))$ so that, for all $t \geq T$,

$$i_*|u'(t)| \leq |u'(t)||(P_h(t) - Id)(0,1)| = |(Id - P_h(t))(u(t+\cdot), 0)| \leq |P_h(t) - Id||u(t+\cdot)|_\infty.$$

Thus we obtain that, for all c, h satisfying (5.22) there exists some universal constant $C > 0$ such that $|u'(t)| \leq C|u(t+\cdot)|_\infty,\ t \geq T$. This yields the required uniform exponential estimate

$$|u(t+\cdot)|_\infty + |u'(t)| \leq 2K_0 e^{-0.5\gamma_0(t-T)}(|u(T+\cdot)|_\infty + |u'(T)|) \leq$$
$$2(1+C)K_0 e^{-0.5\gamma_0(t-T)}|u(T+\cdot)|_\infty \leq 2(1+C)(1-w_1)K_0 e^{-0.5\gamma_0(t-T)}, \quad t \geq T,$$

which holds for all $c \geq 0, h$ satisfying (5.22).

Acknowledgements

The first author was supported by FONDECYT (Chile), project 1150480. The second author was partially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008).

References

[1] M. Alfaro, J. Coville, G. Raoul, Bistable travelling waves for nonlocal reaction diffusion equations, Discret. Contin. Dyn. Syst. 34 (2014) 1775–1791.

[2] A. Apreutesei, A. Ducrot, V. Volpert. Competition of species with intra-specific competition. Math. Model. Nat. Phenom. 3 (4) (2008) 1–27.

[3] N. Apreutesei, A. Ducrot, V. Volpert, Travelling waves for integro-differential equations in population dynamics, DCDS B 11 (2009) 541–561.

[4] N. Apreutesei, V. Volpert, Properness and Topological Degree for Nonlocal Reaction-Diffusion Operators, Abstract and Applied Analysis, 2011, Art. ID 629692, 21 pp.

[5] H. Berestycki, G. Nadin, B. Perthame, L. Ryzhik, The non-local Fisher-KPP equation: travelling waves and steady states, Nonlinearity 22 (2009) 2813–2844.

[6] W. A. Coppel, Dichotomies in Stability Theory, Lectures Notes in Mathematics, 629, Springer-Verlag, Berlin, 1978.

[7] G. Bocharov, A. Meyerhans, N. Bessonov, S. Trofimchuk, V. Volpert, Spatiotemporal dynamics of virus infection spreading in tissues, PLoS ONE 11(12) (2016): e0168576. doi:10.1371/journal.pone.0168576

[8] A. Gomez, S. Trofimchuk, Global continuation of monotone wavefronts, J. London Math. Soc. 89 (2014) 47–68.

[9] S. Gourley, J. So, J. Wu, Non-locality of R-D equations induced by delay: biological modeling and nonlinear dynamics, J. Math. Sciences **124** (2004) 5119–5153.

[10] F. Hartung, T. Krisztin, H.-O. Walther, and J. Wu, Functional differential equations with state-dependent delay: theory and applications, in Handbook of Differential Equations: Ordinary Differential Equations, volume 3, edited by A. Canada, P. Drbek and A. Fonda, Elsevier, North-Holand, 2006, 435-545.

[11] D. Henry, Geometric theory of semilinear parabolic equations, Lect. Notes Math., 840, Springer, 1981.

[12] H. L. Smith, An introduction to delay. Differential equations with applications to the life sciences, 2011, Springer.

[13] S. Ma, J. Wu, Existence, uniqueness and asymptotic stability of traveling wavefronts in a non-local delayed diffusion equation, J. Dynam. Differential Equations 19 (2007) 391–436.

[14] R. H. Martin, H. L. Smith, Abstract functional differential equations and reaction-diffusion systems, Trans. Amer. Math. Soc. 321 (1990) 1–44.

[15] J. Fang, X.-Q. Zhao, Bistable traveling waves for monotone semiflows with applications, J. Eur. Math. Soc. (JEMS) 17 (2015) 2243–2288.

[16] K. Schaaf, Asymptotic behavior and travelling wave solutions for parabolic functional differential equations, Trans. Amer. Math. Soc. 302 (1987) 587-615.

[17] H. L. Smith, X.-Q. Zhao, Global asymptotic stability of traveling waves in delayed reaction-diffusion equations, SIAM J. Math. Anal., 31 (2000) 514– 534.

[18] E. Trofimchuk, M. Pinto, S. Trofimchuk, Monotone waves for non-monotone and non-local monostable reaction-diffusion equations, J. Differential Equations 261 (2016) 1203–1236.

[19] P. Turchin, Complex Population Dynamics, Princeton University, 2003.

[20] V. Volpert, A. Volpert, J.F. Collet, Topological degree for elliptic operators in unbounded cylinders. Adv. Diff. Eq., 4 (1999) 777–812.

[21] V. Volpert, Elliptic partial differential equations. Volume 1. Fredholm theory of elliptic problems in unbounded domains. Birkhäuser, 2011.

[22] A. Volpert, Vit. Volpert, Vl. Volpert, Traveling wave solutions of parabolic systems. Translation of Mathematical Monographs, Vol. 140, Amer. Math. Society, Providence, 1994.

[23] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, 1996.

[24] Z.-C. Wang, W.-T. Li and S. Ruan, Existence and stability of traveling wave fronts in reaction advection diffusion equations with nonlocal delay, J. Differential Equations 238 (2007), 153-200.