

DENSITY ESTIMATES FOR DEGENERATE DOUBLE-WELL POTENTIALS

SERENA DIPIERRO⁽¹⁾, ALBERTO FARINA⁽²⁾, AND ENRICO VALDINOCI^(1,3,4)

(1) – Dipartimento di Matematica “Federigo Enriques”
Università degli studi di Milano
Via Saldini 50, I-20133 Milano (Italy)

(2) – LAMFA, CNRS UMR 7352
Faculté des Sciences
Université de Picardie Jules Verne
33 rue Saint Leu, 80039 Amiens CEDEX 1 (France)

(3) – School of Mathematics and Statistics
University of Melbourne
Grattan Street, Parkville, VIC-3010 Melbourne (Australia)

(4) – Istituto di Matematica Applicata e Tecnologie Informatiche
Via Ferrata 1, 27100 Pavia (Italy)

E-mail addresses: `serena.dipierro@unimi.it`, `alberto.farina@u-picardie.fr`, `enrico@mat.uniroma3.it`

ABSTRACT. We consider a general energy functional for phase coexistence models, which comprises the case of Banach norms in the gradient term plus a double-well potential.

We establish density estimates for Q -minima. Namely, the state parameters close to both phases are proved to occupy a considerable portion of the ambient space.

From this, we obtain the uniform convergence of the level sets to the limit interface in the sense of Hausdorff distance.

The main novelty of these results lies in the fact that we do not assume the double-well potential to be non-degenerate in the vicinity of the minima.

As far as we know, these types of density results for degenerate potentials are new even for minimizers and even in the case of semilinear equations, but our approach can comprise at the same time quasilinear equations, Q -minima and general energy functionals.

1. INTRODUCTION

Phase coexistence models study the separation interfaces between regions corresponding to different values of a suitable state parameter. A typical model, introduced by J. D. van der Waals [Row79] and developed, under various perspectives, by S. M. Allen, J. W. Cahn and J. E. Hilliard [CH58, AC72], and (in the vectorial setting) by V. L. Ginzburg, L. D. Landau and L. P. Pitaevskii [GP58, Lan67], considers a domain $\Omega \subseteq \mathbb{R}^n$ and a state parameter $u : \Omega \rightarrow [-1, 1]$. In such model, the “pure phases” correspond to the values of the state parameters 1 and -1 and the phase separation is driven by the minimization of an energy functional.

The typical energy functional taken into account is the superposition of a potential energy induced by a “double-well” function W , which tries to force the system into the pure phases, and an interaction energy (e.g. a gradient penalization) which avoids the production of unnecessary

2010 *Mathematics Subject Classification.* 35J61, 35J62, 35J70, 35J75 35J15, 35J20, 82B26.

Key words and phrases. Allen-Cahn equation, Cahn-Hilliard equation, phase coexistence models, non-degeneracy assumptions.

interfaces. More precisely, the potential energy is often taken of the form

$$\mathcal{P}_\Omega(u) := \int_\Omega W(u(x)) \, dx$$

with $W \geq 0$ and $W(\tau) > 0 = W(-1) = W(1)$ for any $\tau \in (-1, 1)$, and a natural candidate for the interaction energy is the Dirichlet form

$$\mathcal{I}_\Omega(u) := \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx.$$

In this setting, the total energy becomes

$$(1.1) \quad \mathcal{I}_\Omega(u) + \mathcal{P}_\Omega(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx + \int_\Omega W(u(x)) \, dx$$

and the critical points are solutions of

$$(1.2) \quad \Delta u(x) = W'(u(x)) \quad \text{for any } x \in \Omega.$$

Probably, the most commonly studied case is that in which $W(\tau) = \frac{|1-\tau^2|^2}{4}$: in this case, formula (1.2) reduces to the so-called Allen-Cahn equation

$$\Delta u(x) + u(x) - u^3(x) = 0 \quad \text{for any } x \in \Omega.$$

A natural problem in phase separation models is then to “describe the picture seen from afar”: namely, one can expect that, at a large scale, the two phases tend to “separate” one from the other, with “the least possible interface”.

Till now, two main approaches have been adopted to rigorously describe this phase separation. The first method relies on the theory of Γ -convergence, and aims at identifying a limit functional and preserving the notion of minimizers. The second method is based on density estimates, that is on measuring which portion of the domain the two phases occupy, and gives as a byproduct the convergence of the level sets in the Hausdorff distance.

Very roughly speaking, both the methods of Γ -convergence and density estimates consider the scaled problem obtained by a spatial dilation of a minimizer. Namely, if u is a minimizer of the energy functional and $\varepsilon \in (0, 1)$ is a small parameter, one considers

$$u_\varepsilon(x) := u\left(\frac{x}{\varepsilon}\right).$$

One of the results of the Γ -convergence theory, as established in [MM77] and greatly extended in [Bou90, OS91], is that, as $\varepsilon \searrow 0$, the function u_ε converges (up to a subsequence) in $L^1_{\text{loc}}(\mathbb{R}^n)$ to a function u_0 which takes values only in 1 and -1 (i.e. the state parameters of u_0 are only pure phases); remarkably, if one defines $E := \{u_0 = 1\}$, then ∂E is a minimal surface, namely E is a local minimizer of the perimeter functional. In addition, a suitable rescaling of the original energy functional possesses appropriate convergence properties to the perimeter functional, and this convergence is compatible with energy minimization (see also [Bra02] for a general introduction to this topic).

The second approach, based on density estimates, has been introduced in [CC95], and several extensions also to inhomogeneous, singular and degenerate equations were performed in [Val04, PV05a, PV05b]. A general approach to density estimates was also presented in [FV08], and the vector-valued case has been treated in [AF15]. See also the recent monograph [AFS] for more exhaustive discussions on density estimates for phase transitions.

The basic idea of the density estimates is to consider a point, say the origin, which belongs to the interface, say for concreteness $u(0) = 0$, and measure the proportion of the sets $\{u > 1/2\}$

and $\{u < -1/2\}$ in a large ball. In this configuration, one aims at showing that the Lebesgue measure of $B_r \cap \{u \in [-1/2, 1/2]\}$ is bounded from above by $O(r^{n-1})$, while the Lebesgue measures of $B_r \cap \{u > 1/2\}$ and $B_r \cap \{u < -1/2\}$ are bounded from below by $O(r^n)$ for large r . That is, state parameters close to the pure phases occupy a considerable portion of the space on a large scale, while the interface becomes relatively negligible in measure.

A consequence of these density estimates is also that the level sets of u_ε converge locally in the Hausdorff distance to ∂E : more precisely, for any $\delta, R > 0$, there exists $\varepsilon_0(\delta, R)$ such that if $\varepsilon \in (0, \varepsilon_0(\delta, R))$ then

$$\{u_\varepsilon \in (-1/2, 1/2)\} \cap B_R \subseteq \bigcup_{x \in \partial E} B_\delta(x).$$

It is clear that the Γ -convergence and the density estimates approaches share a common interest in the large-scale phase separation regimes, but address this problem with different methods and obtaining different results. In addition, an important technical difference between these two approaches arises in the main assumptions on the double-well potential. Indeed, while the assumptions on the double-well potential for the Γ -convergence results are very mild and do not involve any quantitative hypothesis, the density estimates usually assume a non-degenerate growth from the minima of the potential. Roughly speaking, the potential W is supposed to have a “sufficiently strong” growth from ± 1 . For instance, in the setting of (1.1), one can consider the case in which $W(\tau) \sim |1 - \tau^2|^m$, with $m \in (0, 2]$ (or even $W(\tau) = \chi_{(-1,1)}(\tau)$), but the case $m > 2$ has never been considered in the literature to the best of our knowledge. The analytic counterpart of this range of m is that the known density estimates till now rely on the non-degeneracy condition $W''(\pm 1) \neq 0$.

The case $W''(\pm 1) = 0$ is conceptually more complicated since, from the physical point of view, if W is “too flat”, then the system might develop “approximative or intermediate phases”. The extreme case of this would be, for instance, the case in which $W(\tau) > 0 = W(\sigma)$ for any $\tau \in (-a, a)$ and any $\sigma \in [-1, -a] \cup [a, 1]$, with $a \in (0, 1)$: in this situation, minimizers would prefer the state parameters $\pm a$ rather than ± 1 and the state parameters with value in $(a, 1]$ (as well as in $[-1, -a)$) become empty, thus violating the density estimates.

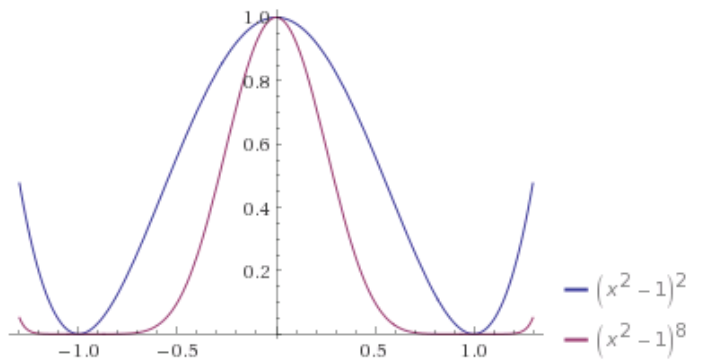


FIGURE 1. *Non-degenerate versus degenerate potentials.*

Interestingly, non-degeneracy assumptions on the potentials also play an important role in the asymptotic theory of critical points in terms of the limiting varifold, see in particular Assumption A on page 51 of [HT00] (moreover, a different growth from the potential wells can also create different phenomena in other settings, see e.g. [DMFL15] for the case of singular potentials).

The goal of this paper is to provide density estimates for degenerate potentials, namely when $W''(\pm 1) = 0$, under weaker growth assumptions. The method of the proof is very general, it also applies to quasiminima, and we can deal with an energy functional, which is possibly

non-quadratic in the gradient term (and so corresponding also to quasilinear PDEs), that we now introduce in details.

Given $n \in \mathbb{N}$, $n \geq 1$, and a bounded, open set $\Omega \subset \mathbb{R}^n$, we consider an energy functional of the form

$$\mathcal{E}_\Omega(u) := \int_\Omega E(x, u(x), \nabla u(x)) dx.$$

We suppose that, for any $x \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, it holds that

$$(1.3) \quad E(x, \tau, \xi) \geq \lambda \left(|\xi|^p + |\tau + 1|^m \chi_{[-\infty, \theta]}(\tau) \right),$$

$$(1.4) \quad \text{and} \quad E(x, \tau, \xi) \leq \Lambda \left(|\xi|^p + |\tau + 1|^m \right),$$

for some $\theta \in (-1, 1)$, $\lambda \in (0, 1]$, $\Lambda \in [1, +\infty)$, $p \in (1, +\infty)$ and $m \in (p, +\infty)$.

Roughly speaking, conditions (1.3) and (1.4) mean that the energy density under consideration is bounded from both sides by a “gradient term of L^p -type” and, in the vicinity of a fixed phase (say, the phase corresponding to -1), by “a well which grows like a power m ” (when $p = 2$, the degeneracy of the potential reflects¹ into the fact that $m > 2$, see Figure 1).

Given $Q \in [1, +\infty)$, we say that $u \in W^{1,p}(\Omega)$ is a Q -minimum in Ω if, for any open and bounded set $\Omega' \Subset \Omega$, we have that $\mathcal{E}_{\Omega'}(u) < +\infty$ and

$$(1.5) \quad \mathcal{E}_{\Omega'}(u) \leq Q \mathcal{E}_{\Omega'}(v),$$

for any $v \in W^{1,p}(\Omega')$ with $u - v \in W_0^{1,p}(\Omega')$.

Of course, when $Q = 1$, the notion in (1.5) boils down to local minimality. In the framework of (1.5) (and denoting, as customary, the n -dimensional Lebesgue measure by \mathcal{L}^n), our main result is the following:

Theorem 1.1. *Let $u \in W^{1,\infty}(\Omega, [-1, 1])$ be a Q -minimum in Ω , with*

$$(1.6) \quad \mathcal{L}^n(B_{r_o} \cap \{u > \theta\}) \geq c_o$$

for some $r_o, c_o > 0$, with $B_{r_o} \subset \Omega$.

Assume that

$$(1.7) \quad \frac{pm}{m-p} > n.$$

Then, there exist $\hat{r}, \hat{c} > 0$ such that for any $r \geq \hat{r}$ for which $B_r \subset \Omega$ it holds that

$$(1.8) \quad \int_{B_r \cap \{u \leq \theta\}} |u(x) + 1|^m dx + \mathcal{L}^n(B_r \cap \{u > \theta\}) \geq \hat{c} r^n.$$

Remark 1.2. The non-degenerate case of Theorem 1.1 corresponds to the ranges $p = 2$ and $m \in (0, 2]$ or, more generally, $p > 1$ and $m \in (0, p]$, and it has been dealt with in [FV08].

Remark 1.3. To the best of our knowledge, the degenerate potential case of Theorem 1.1 is new even when $p = 2$ and for the case of minimizers (instead of quasiminima). It is also new even for energy functionals of the type

$$\frac{1}{2} \int_\Omega |\nabla u(x)|^2 dx + \int_\Omega |1 - u^2(x)|^m dx,$$

¹As a technical observation, we observe that we will focus our analysis on each separate well of the potential. In this way, our result in Theorem 1.1 also applies to double well-potentials in which only one of the two wells possesses a growth condition as in (1.7).

with

$$m \in \begin{cases} (2, +\infty) & \text{if } n \in \{1, 2\}, \\ \left(2, \frac{2n}{n-2}\right) & \text{if } n \geq 3. \end{cases}$$

Remark 1.4. Condition (1.7) is always satisfied when $n = 1$.

Remark 1.5. Condition (1.7) is always satisfied when $n \leq p$.

Remark 1.6. It is interesting to observe that condition (1.7) is a requirement relating the critical Sobolev exponent of $W^{1,p}(\mathbb{R}^m)$ with the dimension of the ambient space.

Interestingly (assuming m and n larger than p and recalling Remark 1.2 for the case $m \leq p$ and Remark 1.5 for the case $n \leq p$), we point out that condition (1.7) can also be written as

$$\frac{np}{n-p} > m,$$

which means that the Sobolev exponent of $W^{1,p}(\mathbb{R}^n)$ is larger than m (i.e., the exponent m in (1.7) is assumed to be “subcritical”).

Remark 1.7. We think that it is an interesting open problem to check whether or not condition (1.7) is sharp.

Remark 1.8. We also think that it is a very interesting open problem to establish whether or not the asymptotic theory of critical points in terms of the limiting varifold established in [HT00] remains valid in the case of degenerate potentials.

From Theorem 1.1, we obtain the following general density estimate:

Corollary 1.9. *Assume that for any $\mu \in (-1, 1)$ there exists $\lambda_\mu > 0$ such that, for any $x \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$,*

$$(1.9) \quad E(x, \tau, \xi) \geq \lambda_\mu (\tau + 1)^m \chi_{(-\infty, \mu]}(\tau).$$

Let $u \in W^{1,\infty}(\Omega, [-1, 1])$ be a Q -minimum in Ω . Then, for any $r > 1$ for which $B_r \subset \Omega$ it holds that

$$(1.10) \quad \mathcal{E}_{B_r}(u) \leq Cr^{n-1},$$

for some $C > 0$, possibly depending on $\lambda, \Lambda, \theta, n, m, p, Q$ and $\|u\|_{W^{1,\infty}(\Omega)}$.

Moreover, if

$$(1.11) \quad \mathcal{L}^n(B_{r_o} \cap \{u > \vartheta\}) \geq c_o$$

for some $\vartheta \in (-1, 1)$ and $r_o, c_o > 0$, with $B_{r_o} \subset \Omega$, and

$$\frac{pm}{m-p} > n,$$

then, for any $\Theta \in (-1, 1)$ there exist $\hat{r}, \hat{c} > 0$, possibly depending on $\lambda, \Lambda, \theta, \vartheta, \Theta, n, m, p, Q, r_o, c_o$ and $\|u\|_{W^{1,\infty}(\Omega)}$, such that for any $r \geq \hat{r}$ for which $B_r \subset \Omega$ it holds that

$$(1.12) \quad \mathcal{L}^n(B_r \cap \{u > \Theta\}) \geq \hat{c}r^n.$$

As a consequence of Corollary 1.9 we also have a convergence result in the Hausdorff distance:

Corollary 1.10. *Assume that*

$$(1.13) \quad E(x, \tau, \xi) \geq \lambda_* (|\xi|^p + |1 - \tau^2|^m),$$

$$(1.14) \quad \text{and} \quad E(x, \tau, \xi) \leq \Lambda_* (|\xi|^p + |1 - \tau^2|^m),$$

for some $\lambda_* \in (0, 1]$, $\Lambda_* \in [1, +\infty)$, $p \in (1, +\infty)$ and $m \in (p, +\infty)$. Suppose that

$$\frac{pm}{m-p} > n.$$

Consider an infinitesimal sequence of ε 's and let $u_\varepsilon \in W^{1,\infty}(\Omega, [-1, 1])$ be a sequence of Q -minima in Ω for the functional

$$\mathcal{E}_\Omega^\varepsilon(v) := \frac{1}{\varepsilon} \int_\Omega E(x, v(x), \varepsilon \nabla v(x)) dx,$$

with

$$(1.15) \quad \sup_\varepsilon \varepsilon \|\nabla u_\varepsilon\|_{L^\infty(\Omega)} < +\infty$$

and

$$(1.16) \quad \sup_\varepsilon \mathcal{E}_\Omega^\varepsilon(u_\varepsilon) < +\infty.$$

Then, there exists $E \subseteq \Omega$ such that, up to a subsequence, u_ε converges to $\chi_E - \chi_{\mathbb{R}^n \setminus E}$ in $L^1_{\text{loc}}(\Omega)$.

Furthermore, given any $\Theta \in (0, 1)$, the set $\{|u_\varepsilon| \leq \Theta\}$ converges to ∂E in the Hausdorff distance locally uniformly in Ω , namely for any $\delta, R > 0$, for which $B_R \Subset \Omega$, there exists $\varepsilon_0(\delta, R, \Theta)$ such that if $\varepsilon \in (0, \varepsilon_0(\delta, R, \Theta))$ then

$$(1.17) \quad \{|u_\varepsilon| \leq \Theta\} \cap B_R \subseteq \bigcup_{x \in \partial E} B_\delta(x).$$

We notice that conditions (1.13) and (1.14) are just a rephrasing of (1.3) and (1.4) that take into account the growth from both the wells of the potential that induces the two phases 1 and -1 .

Next section is devoted to the proof of the main result of Theorem 1.1. Then, the simple proofs of Corollaries 1.9 and 1.10 will be given in Sections 3 and 4, respectively.

2. PROOF OF THEOREM 1.1

We fix an additional parameter $T \in (0, +\infty)$, which will be suitably chosen, possibly also in dependence of $\lambda, \Lambda, \theta, n, m, p, Q, r_o$ and c_o (recall the structural constants in (1.3), (1.4) and (1.6)).

The proof of Theorem 1.1 is based on an iteration argument. To this end, for any $k \in \mathbb{N}$ and $x \in B_{(k+1)T}$, we set

$$(2.1) \quad v_k(x) := \frac{2}{(1 + (k+1)T - |x|)^{\frac{p}{m-p}}} - 1.$$

From now on, $C > 0$ will be a constant, which possibly depends on the structural constants $\lambda, \Lambda, \theta, n, m$ and p in (1.3) and (1.4), and also on the fixed Q, r_o and c_o of Theorem 1.1, but it

is independent of T , and can freely vary from line to line (later on, T will be supposed to be sufficiently large with respect to C). In this way, we have that, for any $x \in B_{(k+1)T}$,

$$(2.2) \quad |\nabla v_k(x)|^p = C \left(\frac{1}{(1 + (k+1)T - |x|)^{\frac{m}{m-p}}} \right)^p = C (v_k(x) + 1)^m.$$

From this and (1.4), letting

$$(2.3) \quad \Omega_k := B_{(k+1)T} \cap \{u > v_k\},$$

we obtain that

$$(2.4) \quad \begin{aligned} \mathcal{E}_{\Omega_k}(v_k) &= \int_{\Omega_k} E(x, v_k(x), \nabla v_k(x)) dx \\ &\leq C \int_{\Omega_k} |\nabla v_k(x)|^p + |v_k(x) + 1|^m dx \\ &\leq C \int_{\Omega_k} |v_k(x) + 1|^m dx. \end{aligned}$$

Notice now that

$$(2.5) \quad v_k = 1 \geq u \text{ on } \partial B_{(k+1)T},$$

and therefore, by the definition of Ω_k in (2.3) and that of v_k in (2.1), we have that $v_k = u$ on $\partial\Omega_k$. Hence, the Q -minimality of u (recall (1.5)) implies that

$$\mathcal{E}_{\Omega_k}(u) \leq Q \mathcal{E}_{\Omega_k}(v_k).$$

Using this, (1.3) and (2.4), it follows that

$$(2.6) \quad \int_{\Omega_k} |\nabla u(x)|^p + |u(x) + 1|^m \chi_{(-\infty, \theta]}(u(x)) dx \leq C \int_{\Omega_k} |v_k(x) + 1|^m dx.$$

On the other hand, by Young's Inequality (with exponents p and $\frac{p}{p-1}$) and Coarea Formula, for any Lipschitz function w we have that

$$(2.7) \quad \begin{aligned} &C \int_{\Omega_k} |\nabla w(x)|^p + |w(x) + 1|^m \chi_{(-\infty, \theta]}(w(x)) dx \\ &\geq \int_{\Omega_k} |\nabla w(x)| |w(x) + 1|^{\frac{m(p-1)}{p}} \chi_{(-\infty, \theta]}(w(x)) dx \\ &= \int_{\mathbb{R}} \left[\int_{\{w=t\}} \chi_{\Omega_k}(x) |w(x) + 1|^{\frac{m(p-1)}{p}} \chi_{(-\infty, \theta]}(w(x)) d\mathcal{H}^{n-1}(x) \right] dt \\ &= \int_{-\infty}^{\theta} \mathcal{H}^{n-1}(\Omega_k \cap \{w = t\}) |t + 1|^{\frac{m(p-1)}{p}} dt. \end{aligned}$$

Moreover, from (2.1),

$$\sup_{x \in B_{kT}} v_k(x) \leq \frac{2}{(1+T)^{\frac{p}{m-p}}} - 1.$$

In particular, choosing T conveniently large, it holds that

$$(2.8) \quad v_k < \frac{\theta - 1}{2} \quad \text{in } B_{kT}.$$

As a consequence, we have that

$$B_{kT} \cap \{u > \theta\} \subseteq B_{kT} \cap \{u > t > v_k\}$$

for any $t \in [\frac{\theta-1}{2}, \theta]$.

In particular, by (2.3),

$$(2.9) \quad B_{kT} \cap \{u > \theta\} \subseteq B_{(k+1)T} \cap \{u > t > v_k\} = \Omega_k \cap \{u > t > v_k\}.$$

for any $t \in [\frac{\theta-1}{2}, \theta]$.

In addition, recalling (2.5), we have that $(\partial B_{(k+1)T}) \cap \{u > t > v_k\} \subseteq \{v_k = 1 \geq u\} \cap \{u > v_k\}$ and so

$$(\partial B_{(k+1)T}) \cap \{u > t > v_k\} = \emptyset.$$

Consequently,

$$(2.10) \quad \begin{aligned} \partial(\Omega_k \cap \{u > t > v_k\}) &= \partial(B_{(k+1)T} \cap \{u > t\} \cap \{t > v_k\}) \\ &\subseteq (B_{(k+1)T} \cap (\partial\{u > t\}) \cap \{t > v_k\}) \cup (B_{(k+1)T} \cap \{u > t\} \cap (\partial\{t > v_k\})) \\ &= (B_{(k+1)T} \cap \{u = t > v_k\}) \cup (B_{(k+1)T} \cap \{u > t = v_k\}) \\ &= (\Omega_k \cap \{u = t > v_k\}) \cup (\Omega_k \cap \{u > t = v_k\}) \\ &\subseteq (\Omega_k \cap \{u = t\}) \cup (\Omega_k \cap \{v_k = t\}). \end{aligned}$$

Now we observe that, by Isoperimetric Inequality,

$$\mathcal{L}^n(\Omega_k \cap \{u > t > v_k\})^{\frac{n-1}{n}} \leq C \mathcal{H}^{n-1}(\partial(\Omega_k \cap \{u > t > v_k\})).$$

This and (2.10) imply that

$$(2.11) \quad \mathcal{L}^n(\Omega_k \cap \{u > t > v_k\})^{\frac{n-1}{n}} \leq C \left(\mathcal{H}^{n-1}(\Omega_k \cap \{u = t\}) + \mathcal{H}^{n-1}(\Omega_k \cap \{v_k = t\}) \right).$$

Accordingly, using first (2.11), and then (2.7), with $w := u$ and $w := v_k$, we conclude that

$$\begin{aligned} &\int_{(\theta-1)/2}^{\theta} \mathcal{L}^n(\Omega_k \cap \{u > t > v_k\})^{\frac{n-1}{n}} dt \\ &\leq C \int_{(\theta-1)/2}^{\theta} \mathcal{L}^n(\Omega_k \cap \{u > t > v_k\})^{\frac{n-1}{n}} |t+1|^{\frac{m(p-1)}{p}} dt \\ &\leq C \int_{-\infty}^{\theta} \mathcal{L}^n(\Omega_k \cap \{u > t > v_k\})^{\frac{n-1}{n}} |t+1|^{\frac{m(p-1)}{p}} dt \\ &\leq C \left(\int_{-\infty}^{\theta} \mathcal{H}^{n-1}(\Omega_k \cap \{u = t\}) |t+1|^{\frac{m(p-1)}{p}} dt + \int_{-\infty}^{\theta} \mathcal{H}^{n-1}(\Omega_k \cap \{v_k = t\}) |t+1|^{\frac{m(p-1)}{p}} dt \right) \\ &\leq C \left(\int_{\Omega_k} |\nabla u(x)|^p + |u(x)+1|^m \chi_{(-\infty, \theta]}(u(x)) dx + \int_{\Omega_k} |\nabla v_k(x)|^p + |v_k(x)+1|^m \chi_{(-\infty, \theta]}(v_k(x)) dx \right). \end{aligned}$$

This and (2.9) imply that

$$\begin{aligned} &C \left(\int_{\Omega_k} |\nabla u(x)|^p + |u(x)+1|^m \chi_{(-\infty, \theta]}(u(x)) dx + \int_{\Omega_k} |\nabla v_k(x)|^p + |v_k(x)+1|^m \chi_{(-\infty, \theta]}(v_k(x)) dx \right) \\ &\geq \mathcal{L}^n(B_{kT} \cap \{u > \theta\})^{\frac{n-1}{n}}. \end{aligned}$$

Consequently, recalling (2.2) and (2.6), we find that

$$(2.12) \quad \mathcal{L}^n(B_{kT} \cap \{u > \theta\})^{\frac{n-1}{n}} \leq C \int_{\Omega_k} |v_k(x)+1|^m dx.$$

Furthermore, exploiting (2.6), we see that

$$\begin{aligned} \int_{\Omega_k \cap \{u \leq \theta\}} |u(x) + 1|^m dx &= \int_{\Omega_k} |u(x) + 1|^m \chi_{(-\infty, \theta]}(u(x)) dx \\ &\leq \int_{\Omega_k} |\nabla u(x)|^p + |u(x) + 1|^m \chi_{(-\infty, \theta]}(u(x)) dx \leq C \int_{\Omega_k} |v_k(x) + 1|^m dx. \end{aligned}$$

As a consequence of this, and recalling (2.3) and (2.8), we have

$$\begin{aligned} &\int_{B_{kT} \cap \{u \leq \theta\}} |u(x) + 1|^m dx \\ &= \int_{B_{kT} \cap \{\theta \geq u > v_k\}} |u(x) + 1|^m dx + \int_{B_{kT} \cap \{u \leq \min\{v_k, \theta\}\}} |u(x) + 1|^m dx \\ &\leq \int_{B_{kT} \cap \{\theta \geq u > v_k\}} |u(x) + 1|^m dx + \int_{B_{kT} \cap \{u \leq v_k\}} (u(x) + 1)^m dx \\ (2.13) \quad &\leq \int_{\Omega_k \cap \{u \leq \theta\}} |u(x) + 1|^m dx + \int_{B_{kT} \cap \{u \leq v_k\}} (v_k(x) + 1)^m dx \\ &\leq C \int_{\Omega_k} |v_k(x) + 1|^m dx + \int_{B_{kT}} |v_k(x) + 1|^m dx \\ &\leq C \left(\int_{\Omega_k \setminus B_{kT}} |v_k(x) + 1|^m dx + \int_{B_{kT}} |v_k(x) + 1|^m dx \right). \end{aligned}$$

Now we define

$$\begin{aligned} (2.14) \quad \alpha_k &:= \int_{B_{kT}} |v_k(x) + 1|^m dx \\ \text{and } \beta_k &:= \int_{\Omega_k \setminus B_{kT}} |v_k(x) + 1|^m dx = \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{u > v_k\}} |v_k(x) + 1|^m dx. \end{aligned}$$

With this, we can rewrite (2.12) and (2.13) as

$$\begin{aligned} (2.15) \quad &\mathcal{L}^n(B_{kT} \cap \{u > \theta\})^{\frac{n-1}{n}} \leq C(\alpha_k + \beta_k) \\ \text{and } &\int_{B_{kT} \cap \{u \leq \theta\}} |u(x) + 1|^m dx \leq C(\alpha_k + \beta_k). \end{aligned}$$

We remark that, in view of (2.1) and (2.14),

$$\begin{aligned} (2.16) \quad \alpha_k &\leq \int_{B_{kT}} \frac{C}{(1 + (k+1)T - |x|)^{\frac{pm}{m-p}}} dx \\ &\leq C \int_0^{kT} \frac{\rho^{n-1}}{(1 + (k+1)T - \rho)^{\frac{pm}{m-p}}} d\rho \\ &\leq C k^{n-1} T^{m-1} \int_0^{kT} \frac{d\rho}{((k+1)T - \rho)^{\frac{pm}{m-p}}} \\ &= C k^{n-1} T^{m-1} \left[\frac{1}{T^{\frac{(p-1)m+p}{m-p}}} - \frac{1}{((k+1)T)^{\frac{(p-1)m+p}{m-p}}} \right] \\ &\leq C k^{n-1} T^{m - \frac{pm}{m-p}}. \end{aligned}$$

On the other hand, from (2.14) and the fact that $v_k(x) \in [-1, 1]$ in $B_{(k+1)T}$,

$$\begin{aligned} \beta_k &= \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{\theta \geq u > v_k\}} (v_k(x) + 1)^m dx + \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{u > \max\{v_k, \theta\}\}} (v_k(x) + 1)^m dx \\ &\leq \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{\theta \geq u > v_k\}} (u(x) + 1)^m dx + \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{u > \max\{v_k, \theta\}\}} 2^m dx \\ &\leq \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{\theta \geq u > v_k\}} |u(x) + 1|^m dx + C \mathcal{L}^n((B_{(k+1)T} \setminus B_{kT}) \cap \{u > \theta\}). \end{aligned}$$

Now we insert this information and (2.16) into (2.15) and we obtain that

$$\begin{aligned} (2.17) \quad & \mathcal{L}^n(B_{kT} \cap \{u > \theta\})^{\frac{n-1}{n}} + \int_{B_{kT} \cap \{u \leq \theta\}} |u(x) + 1|^m dx \\ & \leq C \left(k^{n-1} T^{n - \frac{pm}{m-p}} + \int_{(B_{(k+1)T} \setminus B_{kT}) \cap \{u \leq \theta\}} |u(x) + 1|^m dx + \mathcal{L}^n((B_{(k+1)T} \setminus B_{kT}) \cap \{u > \theta\}) \right). \end{aligned}$$

It is now convenient to introduce the following quantities (somehow reminiscent of “area” and “volume” terms):

$$\begin{aligned} \mathcal{A}_r &:= \int_{B_{rT} \cap \{u \leq \theta\}} |u(x) + 1|^m dx \\ \text{and} \quad \mathcal{V}_r &:= \mathcal{L}^n(B_{rT} \cap \{u > \theta\}). \end{aligned}$$

With this notation, we can rewrite (2.17) as

$$(2.18) \quad \mathcal{V}_k^{\frac{n-1}{n}} + \mathcal{A}_k \leq C \left(k^{n-1} T^{n - \frac{pm}{m-p}} + (\mathcal{A}_{k+1} - \mathcal{A}_k) + (\mathcal{V}_{k+1} - \mathcal{V}_k) \right).$$

Now we let ε be the structural constant² in Lemma 12 of [FV08]. Exploiting (1.7), for T large enough we have that $T^{n - \frac{pm}{m-p}} \leq \varepsilon$. This and (2.18), together with (1.6), imply the hypotheses of Lemma 12 in [FV08], which in turn implies that

$$\mathcal{A}_k + \mathcal{V}_k \geq ck^n,$$

for some constant $c > 0$, and this proves (1.8). \square

3. PROOF OF COROLLARY 1.9

The proof of (1.10) is standard (for instance, one can repeat verbatim the proof of Lemma 10 in [FV08]).

The proof of (1.12) is also a simple consequence of (1.10) and Theorem 1.1: we provide full details for the facility of the reader. We define $\theta_\star := \min\{\theta, \vartheta\}$. Since $\theta_\star \in (-1, \theta]$, the structural assumptions in (1.3) and (1.4) hold true with θ replaced by θ_\star . In addition, since $\theta_\star \leq \vartheta$ we have that $\{u > \vartheta\} \subseteq \{u > \theta_\star\}$ and thus, by (1.11),

$$\mathcal{L}^n(B_{r_o} \cap \{u > \theta_\star\}) \geq \mathcal{L}^n(B_{r_o} \cap \{u > \vartheta\}) \geq c_o.$$

²We take this opportunity to amend a typo in [FV08]: the left-hand side of (61) there should be “ $\mathcal{V}_k^{(n-1)/n} + \mathcal{A}_k$ ” instead of “ $\mathcal{V}_{k+1}^{(n-1)/n} + \mathcal{A}_{k+1}$ ”.

These considerations imply that we can exploit Theorem 1.1 with θ_* replacing θ . In this way, from (1.8) we deduce that, for large r ,

$$(3.1) \quad \int_{B_r \cap \{u \leq \theta_*\}} |u(x) + 1|^m dx + \mathcal{L}^n(B_r \cap \{u > \theta_*\}) \geq \hat{c} r^n.$$

On the other hand, by (1.10) and (1.9), for large r we have that

$$\begin{aligned} \frac{Cr^{n-1}}{\lambda_\Theta} &\geq \frac{1}{\lambda_\Theta} \mathcal{E}_{B_r}(u) \\ &\geq \int_{B_r} |u(x) + 1|^m \chi_{(-\infty, \Theta]}(u(x)) dx \\ &\geq \int_{B_r \cap \{u \leq \theta_*\}} |u(x) + 1|^m dx + \int_{B_r \cap \{\theta_* < u \leq \Theta\}} |u(x) + 1|^m dx \\ &\geq \int_{B_r \cap \{u \leq \theta_*\}} |u(x) + 1|^m dx + (1 + \theta_*)^m \mathcal{L}^n(B_r \cap \{\theta_* < u \leq \Theta\}). \end{aligned}$$

That is

$$\int_{B_r \cap \{u \leq \theta_*\}} |u(x) + 1|^m dx + \mathcal{L}^n(B_r \cap \{\theta_* < u \leq \Theta\}) \leq \tilde{C} r^{n-1},$$

with $\tilde{C} > 0$ also depending on Θ , θ and ϑ .

Hence, recalling (3.1), for large r we have that

$$\begin{aligned} \hat{c} r^n &\leq \int_{B_r \cap \{u \leq \theta_*\}} |u(x) + 1|^m dx + \mathcal{L}^n(B_r \cap \{\theta_* < u \leq \Theta\}) + \mathcal{L}^n(B_r \cap \{u > \Theta\}) \\ &\leq \tilde{C} r^{n-1} + \mathcal{L}^n(B_r \cap \{u > \Theta\}), \end{aligned}$$

which implies (1.12) when r is sufficiently large. \square

4. PROOF OF COROLLARY 1.10

The arguments presented for this proof are standard (see e.g. [MM77, CC95, PV05a, PV05b]). Since the functional considered here is very general, we provide the technical details of the proof for the sake of completeness. We start with the convergence of u_ε in $L^1_{\text{loc}}(\Omega)$ (this is indeed a general argument, which does not use the power-like growth from the potential well, but only a bound from below with a positive and continuous function in $(-1, 1)$ vanishing in $\{1, -1\}$). We set $E_o(\tau) := |1 - \tau^2|^m$ and

$$F_o(\tau) := \int_0^\tau \left(E_o(\sigma)\right)^{\frac{p-1}{p}} d\sigma.$$

Notice that F_o is strictly monotone and so we can denote by F_o^{-1} the inverse function of F_o . Let also $U_\varepsilon(x) := F_o(u_\varepsilon(x))$. Notice that

$$\nabla U_\varepsilon(x) = \left(E_o(u_\varepsilon(x))\right)^{\frac{p-1}{p}} \nabla u_\varepsilon(x) = \left(\frac{1}{\varepsilon} E_o(u_\varepsilon(x))\right)^{\frac{p-1}{p}} \left(\varepsilon^{\frac{p-1}{p}} \nabla u_\varepsilon(x)\right)$$

and therefore, by Young's Inequality,

$$|\nabla U_\varepsilon(x)| \leq \frac{p-1}{p\varepsilon} E_o(u_\varepsilon(x)) + \frac{\varepsilon^{p-1}}{p} |\nabla u_\varepsilon(x)|^p.$$

We rewrite this inequality as

$$c_\star |\nabla U_\varepsilon(x)| \leq \frac{\lambda_\star}{\varepsilon} \left(\varepsilon^p |\nabla u_\varepsilon(x)|^p + E_o(u_\varepsilon(x)) \right),$$

for a suitable $c_\star > 0$. From this, (1.13) and (1.16), we conclude that

$$\begin{aligned} +\infty &> \sup_\varepsilon \mathcal{E}_\Omega^\varepsilon(u_\varepsilon) \\ &\geq \sup_\varepsilon \frac{\lambda_\star}{\varepsilon} \int_\Omega \left(\varepsilon^p |\nabla u_\varepsilon(x)|^p + E_o(u_\varepsilon(x)) \right) dx \\ &\geq c_\star \sup_\varepsilon \int_\Omega |\nabla U_\varepsilon(x)| dx. \end{aligned}$$

Accordingly, by the Rellich-Kondrachov Theorem (used here in $W^{1,1}(\Omega)$), up to a subsequence we may suppose that U_ε converges to some U_0 in $L^1_{\text{loc}}(\Omega)$ and a.e. in Ω . Then, we set $u_0(x) := F_o^{-1}(U_0(x))$ and we have that, a.e. $x \in \Omega$,

$$\lim_{\varepsilon \searrow 0} u_\varepsilon(x) = \lim_{\varepsilon \searrow 0} F_o^{-1}(U_\varepsilon(x)) = F_o^{-1}(U_0(x)) = u_0(x).$$

Since $|u_\varepsilon| \leq 1$, this and the Dominated Convergence Theorem imply that u_ε converges to u_0 in $L^1_{\text{loc}}(\Omega)$, as desired.

Now we show that

$$(4.1) \quad u_0 \text{ takes values only in } \{1, -1\}.$$

To this end, we use again (1.13) and (1.16), together with Fatou's Lemma, and we see that

$$\begin{aligned} 0 &= \lim_{\varepsilon \searrow 0} \varepsilon \mathcal{E}_\Omega^\varepsilon(u_\varepsilon) \\ &= \lim_{\varepsilon \searrow 0} \int_\Omega E(x, u_\varepsilon(x), \varepsilon \nabla u_\varepsilon(x)) dx \\ &\geq \lambda_\star \lim_{\varepsilon \searrow 0} \int_\Omega |1 - u_\varepsilon^2(x)|^m dx \\ &\geq \lambda_\star \int_\Omega |1 - u_0^2(x)|^m dx. \end{aligned}$$

Consequently, we deduce that $u_0^2 = 1$ a.e. in Ω , which establishes (4.1).

From (4.1), we have that u_ε converges to $\chi_E - \chi_{\mathbb{R}^n \setminus E}$ in $L^1_{\text{loc}}(\Omega)$, with $E := \{x \in \Omega \text{ s.t. } u_0(x) = 1\}$. Now, we prove (1.17). To achieve this aim, we argue by contradiction and we suppose that there exist $\Theta \in (0, 1)$, $\delta, R > 0$ and sequences $\varepsilon_k \searrow 0$ and $x_k \in \{|u_{\varepsilon_k}| \leq \Theta\} \cap B_R$ with $B_\delta(x_k) \cap (\partial E) = \emptyset$. We suppose that $B_\delta(x_k) \subseteq \mathbb{R}^n \setminus E$ (the case $B_\delta(x_k) \subset E$ is similar). Then, we have that

$$(4.2) \quad 0 = \lim_{k \rightarrow +\infty} \int_{B_\delta(x_k)} |u_{\varepsilon_k}(x) - \chi_E(x) + \chi_{\mathbb{R}^n \setminus E}(x)| dx = \lim_{k \rightarrow +\infty} \int_{B_\delta(x_k)} |u_{\varepsilon_k}(x) + 1| dx.$$

Now we define $w_k(x) := u_{\varepsilon_k}(x_k + \varepsilon_k x)$ and

$$\Omega_k := \left\{ \frac{x - x_k}{\varepsilon_k}, \quad x \in \Omega \right\}.$$

From (1.15),

$$(4.3) \quad \sup_{k \in \mathbb{N}} \|\nabla w_k\|_{L^\infty(\Omega_k)} = \sup_{k \in \mathbb{N}} \varepsilon_k \|\nabla u_{\varepsilon_k}\|_{L^\infty(\Omega)} < +\infty.$$

We also set

$$\mathcal{F}_{k,\Omega_k}(v) := \int_{\Omega_k} E(\varepsilon_k x, v(x), \nabla v(x)) dx.$$

We remark that w_k is a Q -minimum for \mathcal{F}_{k,Ω_k} . Moreover, we have that $w_k(0) = u_{\varepsilon_k}(x_k) \in [-\Theta, \Theta]$. This and (4.3) imply that, fixed $\Theta' \in (\Theta, 1)$, we have that $|w_k| \leq \Theta'$ in B_{r_o} , for some $r_o > 0$ (in particular, condition (1.11) is fulfilled here with $\vartheta := -\Theta'$ and $c_o := \mathcal{L}^n(B_{r_o})$).

Therefore, we are in the position of exploiting Corollary 1.9 with the function w_k and deducing from (1.12) (used here with $\Theta := 1/2$) that, if r is sufficiently large,

$$\mathcal{L}^n(B_r \cap \{w_k > 1/2\}) \geq \hat{c} r^n,$$

for some $\hat{c} > 0$. Hence, scaling back,

$$\mathcal{L}^n(B_{\varepsilon_k r}(x_k) \cap \{u_{\varepsilon_k} > 1/2\}) \geq \hat{c} (\varepsilon_k r)^n.$$

So, we can choose $r := \delta/\varepsilon_k$ and plug the latter estimate into (4.2). In this way, we obtain that

$$0 \geq \lim_{k \rightarrow +\infty} \int_{B_\delta(x_k) \cap \{u_{\varepsilon_k} > 1/2\}} |u_{\varepsilon_k}(x) + 1| dx \geq \frac{3}{2} \mathcal{L}^n(B_\delta(x_k) \cap \{u_{\varepsilon_k} > 1/2\}) \geq \frac{3\hat{c}}{2} \delta^n.$$

This is a contradiction, and the proof of (1.17) is complete. \square

REFERENCES

- [AF15] Nicholas D. Alikakos and Giorgio Fusco, *Density estimates for vector minimizers and applications*, Discrete Contin. Dyn. Syst. **35** (2015), no. 12, 5631–5663, DOI 10.3934/dcds.2015.35.5631. MR3393249
- [AFS] Nicholas D. Alikakos, Giorgio Fusco, and Panayotis Smyrnelis, *Elliptic Systems of Phase Transition Type*.
- [AC72] S. M. Allen and J. W. Cahn, *Ground State Structures in Ordered Binary Alloys with Second Neighbor Interactions*, Acta Met. **20** (1972), no. 423.
- [Bou90] Guy Bouchitté, *Singular perturbations of variational problems arising from a two-phase transition model*, Appl. Math. Optim. **21** (1990), no. 3, 289–314, DOI 10.1007/BF01445167. MR1036589
- [Bra02] Andrea Braides, *Γ -convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, vol. 22, Oxford University Press, Oxford, 2002. MR1968440
- [CC95] Luis A. Caffarelli and Antonio Córdoba, *Uniform convergence of a singular perturbation problem*, Comm. Pure Appl. Math. **48** (1995), no. 1, 1–12, DOI 10.1002/cpa.3160480101. MR1310848
- [CH58] J. W. Cahn and J. E. Hilliard, *Free energy of a nonuniform system. I. Interfacial free energy*, J. Chem. Phys. **28** (1958), no. 258.
- [DMFL15] Gianni Dal Maso, Irene Fonseca, and Giovanni Leoni, *Second order asymptotic development for the anisotropic Cahn-Hilliard functional*, Calc. Var. Partial Differential Equations **54** (2015), no. 1, 1119–1145, DOI 10.1007/s00526-015-0819-0. MR3385194
- [FV08] Alberto Farina and Enrico Valdinoci, *Geometry of quasiminimal phase transitions*, Calc. Var. Partial Differential Equations **33** (2008), no. 1, 1–35, DOI 10.1007/s00526-007-0146-1. MR2413100
- [GP58] V. L. Ginzburg and L. P. Pitaevskiĭ, *On the theory of superfluidity*, Soviet Physics. JETP **34** (7) (1958), 858–861 (1240–1245 *Ž. Eksper. Teoret. Fiz.*) MR0105929
- [HT00] John E. Hutchinson and Yoshihiro Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. Partial Differential Equations **10** (2000), no. 1, 49–84, DOI 10.1007/PL00013453. MR1803974
- [Lan67] L. D. Landau, *Collected papers of L. D. Landau*, Edited and with an introduction by D. ter Haar. Second printing, Gordon and Breach Science Publishers, New York-London-Paris, 1967. MR0237287
- [MM77] Luciano Modica and Stefano Mortola, *Un esempio di Γ^- -convergenza*, Boll. Un. Mat. Ital. B (5) **14** (1977), no. 1, 285–299 (Italian, with English summary). MR0445362
- [OS91] Nicholas C. Owen and Peter Sternberg, *Nonconvex variational problems with anisotropic perturbations*, Nonlinear Anal. **16** (1991), no. 7-8, 705–719, DOI 10.1016/0362-546X(91)90177-3. MR1097327

- [PV05a] Arshak Petrosyan and Enrico Valdinoci, *Geometric properties of Bernoulli-type minimizers*, *Interfaces Free Bound.* **7** (2005), no. 1, 55–77, DOI 10.4171/IFB/113. MR2126143
- [PV05b] ———, *Density estimates for a degenerate/singular phase-transition model*, *SIAM J. Math. Anal.* **36** (2005), no. 4, 1057–1079, DOI 10.1137/S0036141003437678. MR2139200
- [Row79] J. S. Rowlinson, *Translation of J. D. van der Waals’ “The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density”*, *J. Statist. Phys.* **20** (1979), no. 2, 197–244, DOI 10.1007/BF01011513. MR523642
- [Val04] Enrico Valdinoci, *Plane-like minimizers in periodic media: jet flows and Ginzburg-Landau-type functionals*, *J. Reine Angew. Math.* **574** (2004), 147–185, DOI 10.1515/crll.2004.068. MR2099113