

# Cobham's Theorem and Automaticity

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## Abstract

We make certain bounds in Krebs' proof of Cobham's theorem explicit and obtain corresponding upper bounds on the length of a common prefix of an aperiodic  $a$ -automatic sequence and an aperiodic  $b$ -automatic sequence, where  $a$  and  $b$  are multiplicatively independent. We also show that an automatic sequence cannot have arbitrarily large factors in common with a Sturmian sequence.

## 1 Introduction

This paper is concerned with the following question: Given a  $b$ -automatic sequence  $f$  and a sequence  $g$  from some other family of sequences  $\mathcal{G}$ , how similar can  $f$  and  $g$  be? By “similar” we could mean several things:

1.  $f$  and  $g$  are identical;
2.  $f$  and  $g$  have a long common prefix;
3.  $f$  and  $g$  have a factor of length  $n$  in common for infinitely many  $n$ ;
4.  $f$  and  $g$  have the same set of factors of length  $n$  for all sufficiently large  $n$ ;
5.  $f$  and  $g$  agree on a set of positions of density 1.

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When  $\mathcal{G}$  is the family of  $a$ -automatic sequences, where  $a$  and  $b$  are *multiplicatively independent* ( $a$  and  $b$  are not powers of the same integer), then we have some answers. Notably, Cobham’s theorem [6] states that  $f$  and  $g$  can be identical only if  $f$  and  $g$  are ultimately periodic. Recently, Krebs [8] has given a very short and elegant proof of Cobham’s theorem. Much of what we do in the first part of this paper is based on this proof of Cobham’s theorem. We also note that Byszewski and Konieczny [4] generalized Cobham’s theorem by showing that if  $f$  and  $g$  coincide on a set of positions of density 1, then they are periodic on a set of positions of density 1.

One of the main results of this paper concerns the “long common prefix” measure of similarity. In particular we give explicit bounds (in terms of the number of states of the automata generating the sequences) on how long  $f$  and  $g$  can agree before they are forced to agree forever. As an example of a result of this type, consider the following generalization of the Fine–Wilf theorem [10, Theorem 2.3.5]: If  $f \in w\{w, x\}^\omega$  and  $g \in x\{w, x\}^\omega$  ( $w$  and  $x$  are finite words) agree on a prefix of length  $|w| + |x| - \gcd(|w|, |x|)$ , then  $f = g$ . (Here the notation  $\{w, x\}^\omega$  denotes the set of infinite words of the form  $U_1U_2U_2\cdots$ , where each  $U_i \in \{w, x\}$ .) In our setting, where  $f$  is an  $a$ -automatic sequence and  $g$  is a  $b$ -automatic sequence, we obtain our bounds on the length of the common prefix by following the proof of Krebs and making explicit several of the bounds that appear in this proof. Our result answers a question posed by Zamboni (personal communication), who asked how long a sequence generated by a  $b$ -uniform morphism and one generated by an  $a$ -uniform morphism can agree before the two sequences are forced to be equal.

This problem of bounding the length of the common prefix of  $f$  and  $g$  is related to the concept of *b-automaticity* of infinite sequences [9], which measures the minimum number of states of a base- $b$  automaton that computes the length- $n$  prefix of the sequence. In particular, we are able to get a lower bound on the  $b$ -automaticity of an  $a$ -automatic sequence.

Regarding the property of having “arbitrarily large factors in common”, it is not difficult to see that even distinct aperiodic  $a$ -automatic and  $b$ -automatic sequences can have arbitrarily large factors in common. For example, the characteristic sequences of powers of 2 and 3 respectively are 2-automatic and 3-automatic respectively, and clearly have arbitrarily large runs of 0’s in common. The problem in this case is to show that in general such large factors necessarily have some simple structure; however, we do not address this question in this paper.

If we now change the family  $\mathcal{G}$  of sequences from  $a$ -automatic to Sturmian, then it is somewhat easier to answer these kinds of questions. *Sturmian sequences* are those given by the first differences of sequences of the form

$$(\lfloor n\alpha + \beta \rfloor)_{n \geq 1},$$

where  $0 \leq \alpha, \beta < 1$  and  $\alpha$  is irrational [3]. The number  $\alpha$  is called the *slope* of the Sturmian sequence and the number  $\beta$  is called the *intercept*. It is well-known that a Sturmian sequence cannot be  $b$ -automatic. This follows from the fact that the limiting frequency of 1’s in a Sturmian sequence is  $\alpha$ , whereas if a letter in a  $b$ -automatic sequence has a limiting frequency, that frequency must be rational [6, Thm. 6, p. 180].

The problem of determining the maximum length of a common prefix of a  $b$ -automatic sequence and a Sturmian sequence was examined by Shallit [9]. Upper bounds on the length

of the common prefix can be deduced from the automaticity results given by Shallit. In the present paper we answer, in the negative, the question, “Can a Sturmian sequence and a  $b$ -automatic sequence have arbitrarily large finite factors in common?”

Byszewski and Konieczny [4] examine these questions for the family of *generalized polynomial functions* (these are sequences defined by expressions involving algebraic operations along with the floor function). This family contains the family of Sturmian sequences as a subset. In recent work [5], they have extended some of the results of this paper to this more general class.

We also mention the work of Tapsoba [11]. Recall that the complexity of a word  $s$  is the function counting the number of distinct factors of length  $n$  in  $s$ . It is also well-known that Sturmian words have the minimum possible complexity  $n + 1$  achievable by an aperiodic infinite word. Tapsoba shows another distinction between automatic sequences and Sturmian words by giving a formula for the minimal complexity function of the fixed point of an injective  $k$ -uniform binary morphism and comparing this to the complexity function of Sturmian words.

## 2 Common prefix of $a$ -automatic and $b$ -automatic sequences

This section is largely based on the work of Krebs [8] and so we will mostly stick to the notation used in his paper. The reader should read this section in conjunction with Krebs’ paper; we occasionally omit details that can be found there.

### 2.1 Definitions and notation

Let  $b \geq 2$  and let  $w \in \{0, 1, 2, \dots\}^*$ . Write  $w = w_{n-1}w_{n-2}\cdots w_0$ , where each  $w_i \in \{0, 1, 2, \dots\}$ . We define the number  $[w]_b$  by

$$[w]_b = w_{n-1}b^{n-1} + w_{n-2}b^{n-2} + \cdots + w_1b + w_0.$$

Typically, one restricts  $w$  to be over the *canonical digit set*  $\{0, 1, \dots, b - 1\}$ , in which case every natural number  $x$  has a unique representation  $w$  such that  $x = [w]_b$  and  $w$  does not begin with a 0 (the number 0 is represented by the empty string). In this case, we use  $\langle x \rangle_b$  to denote this representation  $w$ .

However, Krebs’ proof requires the use of a larger digit set. Let  $D_b$  denote the digit set  $\{0, \dots, 2b\}$ . Over this digit set, numbers may no longer have unique representations, even with the restriction that the representation must begin with a non-zero digit. We use the notation  $(x)_{D_b}$  to refer to some particular representation of  $x$  over the digit set  $D_b$  that does not begin with the digit zero, without necessarily specifying which representation it is. Note also that if some representation  $(x)_{D_b}$  has length  $n$ , then

$$x \leq 2b \sum_{i=0}^{n-1} b^i = \frac{2b(b^n - 1)}{b - 1} \leq 2b^{n+1}.$$

A *deterministic finite automaton with output* (DFAO) is a 6-tuple  $(S, D, \delta, s_0, \Delta, F)$ , where  $S$  is a finite set of *states*,  $D$  is a finite *input alphabet*,  $\delta : S \times D \rightarrow S$  is the *transition function*,  $s_0 \in S$  is the *initial state*,  $\Delta$  is a finite *output alphabet*, and  $F : S \rightarrow \Delta$  is the *output function*. See [2] for more details.

Let  $D$  be a set of non-negative digits containing  $\{0, 1, \dots, b-1\}$ . A sequence  $(f_x)_{x \in \mathbb{N}}$  is  $(b, D)$ -*automatic* if there is a DFAO  $M = (S, D, \delta, s_0, \Delta, F)$  such that  $f_{[w]_b} = F(\delta(s_0, w))$  for all  $w \in D^*$ . Note that for each  $x$ , the DFAO  $M$  must produce the same output for all  $w \in D^*$  satisfying  $x = [w]_b$ . The DFAO  $M$  is called a  $(b, D)$ -*DFAO*. A sequence is  $b$ -*automatic* if it is  $(b, \{0, 1, \dots, b-1\})$ -automatic, and the automaton  $M$  in this case is called a  $b$ -*DFAO*.

## 2.2 Normalization

Krebs begins his proof by showing that a sequence  $f$  is automatic with respect to representations over the canonical base- $b$  digit set if and only if it is automatic with respect to representations over the digit set  $D_b$ . The reverse direction can be seen by noting that given a  $(b, D_b)$ -DFAO generating  $f$ , one obtains a  $b$ -DFAO generating  $f$  simply by deleting the transitions on all digits other than  $\{0, 1, \dots, b-1\}$ . The forward direction is proved using two results: the first is a modification of [2, Theorem 6.8.6] and the second can be found in [7, Proposition 7.1.4]. The first result [2, Theorem 6.8.6] states that if a sequence  $f$  is generated by a  $b$ -DFAO  $M$ , then so is the sequence obtained by first applying a transducer  $T$  to the input and then feeding the output of  $T$  to  $M$ . As presented in [2], this result requires  $T$  to map words over the digits set  $\{0, 1, \dots, b-1\}$  to words over the same digit set; however, the proof is easily modified to allow  $T$  to map words over any digit set to words over  $\{0, 1, \dots, b-1\}$ . Krebs therefore applies this modified version of [2, Theorem 6.8.6] where  $T$  is the transducer of [7, Proposition 7.1.4], which converts input over a non-canonical digit set (in our case  $D_b$ ) to the canonical digit set for a given base  $b$  (this is called *normalization*). The result of this operation is therefore a  $(b, D_b)$ -DFAO computing  $f$ . We now discuss the details of this construction with the aim of obtaining a reasonably small  $(b, D_b)$ -DFAO computing  $f$ .

Let  $N$  be the transducer of [7, Lemma 7.1.1], which converts from the digit set  $D_b$  to the digit set  $\{0, 1, \dots, b-1\}$  and reads its input from least significant digit to most significant digit. The number of states of  $N$  is determined by the quantity

$$m = \max\{|e - d| : e \in D_b, d \in \{0, 1, \dots, b-1\}\};$$

in particular, the state set of  $N$  is defined to be  $Q = \{s \in \mathbb{N} : s < m/(b-1)\}$ . In our case, we have  $m = 2b$ , and furthermore, for  $b = 2$  we have  $2b/(b-1) = 4$  and for  $b > 2$  we have  $2 < 2b/(b-1) \leq 3$ . We therefore set  $\gamma = 4$  if  $b = 2$  and  $\gamma = 3$  if  $b > 2$ , so that  $Q = \{s \in \mathbb{N} : s < \gamma\}$ .

The set of transitions of  $N$  is

$$E = \{s \xrightarrow{e|d} s' : s + e = bs' + d\}.$$

The initial state is 0 and the output function  $\omega$  maps each state  $s \in Q$  to  $\langle s \rangle_b$ . Note that  $N$  is *subsequential*, or “input-deterministic”. To see this, suppose we have two transitions

$$s \xrightarrow{e|d'} s' \text{ and } s \xrightarrow{e|d''} s''.$$

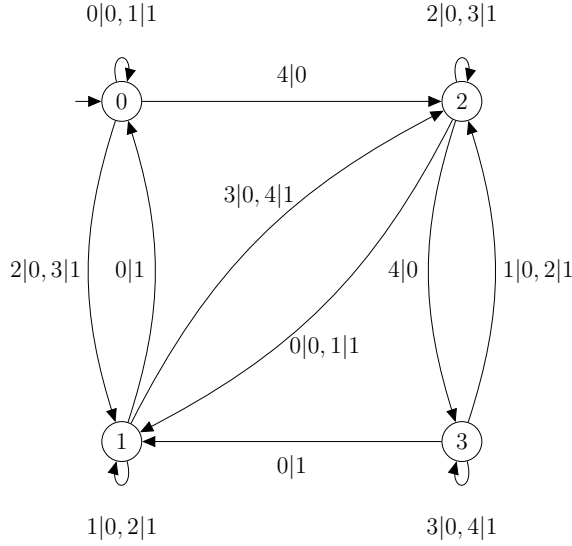


Figure 1: The transducer  $N$  in base 2 converting  $D_2$  into  $\{0, 1\}$ .

Then  $bs' + d' = bs'' + d''$ , which we can rewrite as  $(s' - s'')b = d'' - d'$ . However, we have  $|d'' - d'| < b$ , so  $|s' - s''| < 1$ , which implies  $s' = s''$  and  $d' = d''$ .

On input  $u = e_n e_{n-1} \cdots e_0$  over  $D_b$ , the transducer  $N$  produces output  $v = \omega(s)d_n d_{n-1} \cdots d_0$  over  $\{0, 1, \dots, b-1\}$ , where  $s$  is the state reached by  $N$  after reading  $u$ , and  $[u^R]_b = [v^R]_b$ .

**Example 1.** Throughout this section, we illustrate the proof with the case  $b = 2$ . In this case, the transducer  $N$  is the one given in Figure 1. For instance, on input  $u = 4032$  over  $D_2$ , the transitions of  $N$  are

$$0 \xrightarrow{2|0} 1 \xrightarrow{3|0} 2 \xrightarrow{0|0} 1 \xrightarrow{4|1} 2,$$

so  $N$  outputs  $v = \langle 2 \rangle_2 1000 = 101000$ , which is the canonical base-2 expansion of  $u$ .

Let  $M = (S, \{0, 1, \dots, b-1\}, \delta, I, \Delta, F)$  be a  $b$ -DFAO generating a  $b$ -automatic sequence  $f$ . Recall that our convention is that a  $b$ -DFAO reads its input from most significant digit to least significant digit.

**Example 1 (Continued).** We now consider the Thue–Morse sequence  $t = 01101001 \cdots$  which is the fixed point of the morphism  $\tau : 0 \mapsto 01, 1 \mapsto 10$ . It is well known that the Thue–Morse sequence  $t$  is 2-automatic and can be generated by the 2-DFAO  $M = (S, \{0, 1\}, \delta, I, \Delta, F)$  with  $S = \{0, 1\} = F$  and  $I = 0$  drawn in Figure 2.

Let  $M' = (S', D_b, \delta', I', \Delta, F')$ , be the  $(b, D_b)$ -DFAO defined as follows (again, it reads its input from most significant digit to least significant digit). We define

$$S' = \{ \{ (s_0, 0), (s_1, 1), \dots, (s_{\gamma-1}, \gamma-1) \} : s_0, s_1, \dots, s_{\gamma-1} \in S \}, \text{ and}$$

$$I' = \{ (\delta(I, \langle q \rangle_b), q) : 0 \leq q < \gamma \}.$$

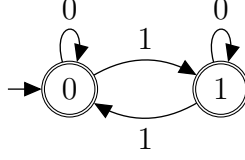


Figure 2: The 2-DFAO  $M$  generating the Thue–Morse sequence.

Clearly we have  $I' \in S'$ . For any  $t \in S'$  and  $e \in D_b$ , we define

$$\delta'(t, e) = \bigcup_{(s, q) \in t} \left\{ (\delta(s, d), q') : q' \xrightarrow{e|d} q \text{ in } N \right\}.$$

Finally, for  $t \in S'$ , define  $F'(t) = F(s)$ , where  $(s, 0) \in t$  (by the definition of  $S'$ , there is a unique such  $s \in S$ ).

We first show that  $\delta'$  is well-defined. Let  $t \in S'$  and  $e \in D_b$ , and we will show that  $\delta'(t, e) \in S'$ . We need to show that for every state  $p$  of  $N$  (i.e., every  $p \in Q$ ) the set  $\delta'(t, e)$  contains a unique element of the form  $(s, p)$ , where  $s \in S$ . Let  $p \in Q$  be a state of  $N$ . Since  $N$  is input-deterministic, there is exactly one outgoing transition from  $p$  in  $N$  with input symbol  $e$ , say  $p \xrightarrow{e|d} q$  in  $N$ . Since  $(s, q) \in t$  for exactly one  $s \in S$  (by definition of  $S'$ ), we conclude that  $(\delta(s, d), p) \in \delta'(t, e)$ , and it is the unique element in  $\delta'(t, e)$  with second coordinate  $p$ .

Now we show that  $M'$  computes the same automatic sequence as  $M$ . For any  $u = u_m \cdots u_0 \in D_b^*$  that doesn't begin with 0, there exists exactly one  $v = v_n \cdots v_0 \in \{0, 1, \dots, b-1\}^*$  that doesn't begin with 0 such that  $[u]_b = [v]_b$ . Namely,  $v = \langle [u]_b \rangle_b$ . Note that  $m \leq n \leq m+2$ . We need to show that if  $(s, 0) \in \delta'(I', u)$ , then  $\delta(I, v) = s$ . Suppose that  $(s, 0) \in \delta'(I', u)$ . Then in  $N$ , we have

$$0 \xrightarrow{u_0|v_0} q_0 \xrightarrow{u_1|v_1} q_1 \xrightarrow{u_2|v_2} \cdots \xrightarrow{u_m|v_m} q_m,$$

and  $\langle q_m \rangle_b = v_n \cdots v_{m+1}$ . Therefore, we have  $(\delta(I, v_n \cdots v_{m+1}), q_m) \in I'$ , and retracing the steps of  $M'$ , we conclude that

$$\delta(I, v) = s.$$

Informally,  $M'$  works through the transducer  $N$  in the reverse direction, while computing the transitions of  $M$  on the output. Since we are working through the transducer backwards, there are  $\gamma$  possible places to start, each corresponding to a different backwards path through the transducer. Further, if we start working backwards from state  $q$  in the transducer, then the output function of the transducer will be  $\langle q \rangle_b$ . The output function of the transducer is read first by  $M'$ , which explains the definition of  $I'$ . Only when we reach the end of the input string do we know which backwards path through the transducer was correct (the one that started at state 0), so  $M'$  computes the transitions of  $M$  for all  $\gamma$  paths along the way.

We have therefore shown how, given a  $b$ -DFAO  $M$  for  $f$ , to produce a  $(b, D_b)$ -DFAO  $M'$  that also generates  $f$ . Furthermore, the  $(b, D_b)$ -DFAO  $M'$  has at most

$$|S|^\gamma \leq |S|^4 \tag{1}$$

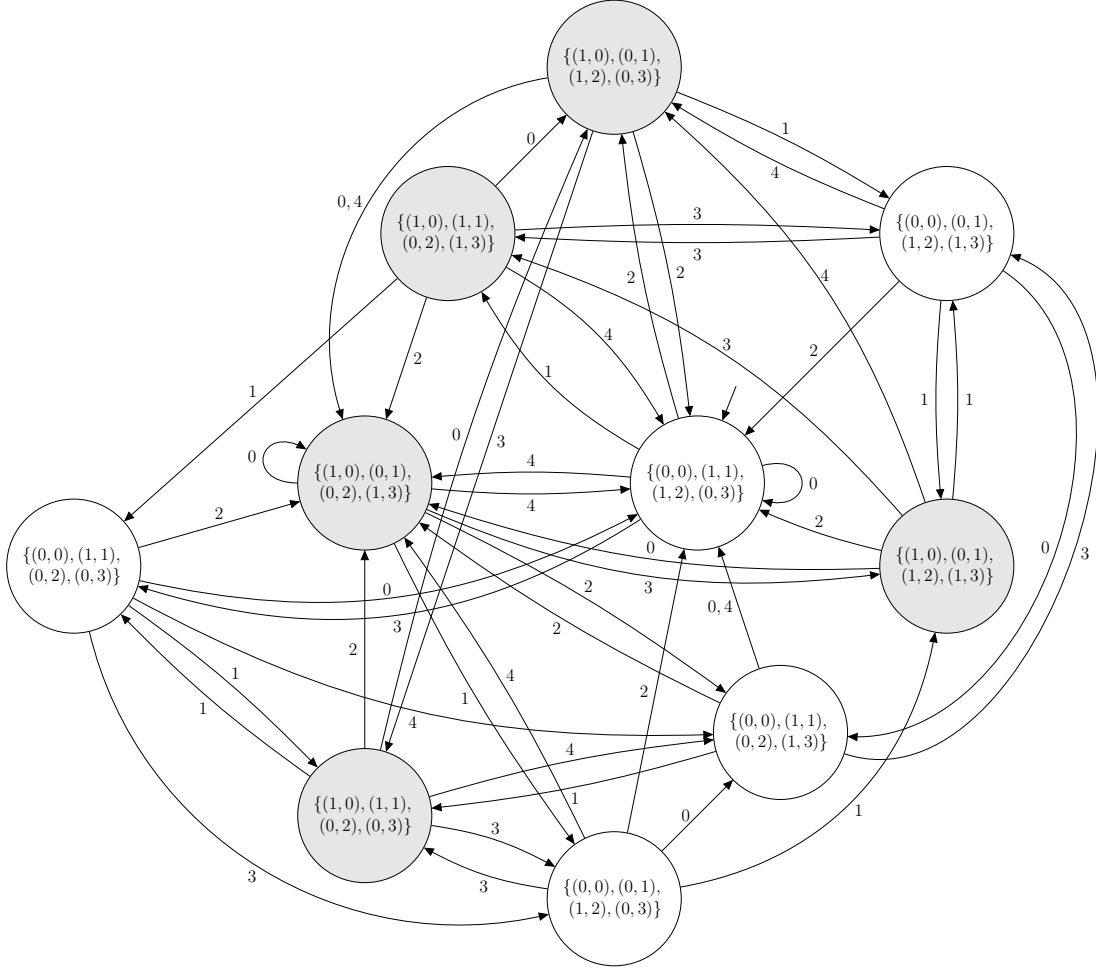


Figure 3: The  $(2, D_2)$ -DFAO  $M'$  computing the Thue–Morse sequence (“white” states output 0; “grey” states output 1).

states.

**Example 1** (Continued). In Figure 3, we give the  $(2, D_2)$ -DFAO  $M'$  (omitting all unreachable states) that computes the Thue–Morse sequence. We also give its transition table in Table 1. To that aim, recall that  $\gamma = 4$ . From Figure 2, we also get

$$\begin{aligned}
 I' &= \{ (\delta(I, \langle q \rangle_b), q) : 0 \leq q < \gamma \} \\
 &= \{ (\delta(I, \epsilon), 0), (\delta(I, 1), 1), (\delta(I, 10), 2), (\delta(I, 11), 3) \} \\
 &= \{ (0, 0), (1, 1), (1, 2), (0, 3) \}.
 \end{aligned}$$

We also compute  $M'$  on two different words  $u \in D_2^*$ . Take  $u = 4032 \in D_2^*$  whose canonical

$\delta'(t, e)$	$e \in \{0, 1, 2\}$		
$t \in S'$	0	1	2
$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$
$\{(1, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$
$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$
$\{(0, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$
$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$
$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$
$\{(1, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$
$\{(1, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$
$\{(0, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$
$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$

∞

$\delta'(t, e)$	$e \in \{3, 4\}$	
$t \in S'$	3	4
$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$
$\{(1, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$
$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$
$\{(0, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$
$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$
$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$
$\{(1, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(1, 0), (0, 1), (1, 2), (0, 3)\}$
$\{(1, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$
$\{(0, 0), (0, 1), (1, 2), (0, 3)\}$	$\{(1, 0), (1, 1), (0, 2), (0, 3)\}$	$\{(1, 0), (0, 1), (0, 2), (1, 3)\}$
$\{(0, 0), (1, 1), (0, 2), (1, 3)\}$	$\{(0, 0), (0, 1), (1, 2), (1, 3)\}$	$\{(0, 0), (1, 1), (1, 2), (0, 3)\}$

Table 1: The transition function  $\delta'$  of  $M'$  as a function of  $t \in S'$  and  $e \in \{0, 1, 2, 3, 4\}$ .



base-2 expansion is  $v = 101000$ . The transitions are

$$\begin{aligned}
I' = \{(0, 0), (1, 1), (\mathbf{1}, \mathbf{2}), (0, 3)\} &\xrightarrow{4} \{(1, 0), (\mathbf{0}, \mathbf{1}), (0, 2), (1, 3)\} \\
&\xrightarrow{0} \{(1, 0), (0, 1), (\mathbf{0}, \mathbf{2}), (1, 3)\} \\
&\xrightarrow{3} \{(1, 0), (\mathbf{0}, \mathbf{1}), (1, 2), (1, 3)\} \\
&\xrightarrow{2} \{(\mathbf{0}, \mathbf{0}), (1, 1), (1, 2), (0, 3)\}.
\end{aligned}$$

By definition of  $F'$ , we have  $F'(\{(0, 0), (1, 1), (1, 2), (0, 3)\}) = F(0) = 0$ . Thus the automaton  $M'$  outputs 0 after reading  $u$ , just as the automaton  $M$  does when reading  $v$ . The second coordinates of the ordered pairs in bold are the states of the “correct path” through the transducer  $N$ , in reverse:

$$0 \xrightarrow{2|0} 1 \xrightarrow{3|0} 2 \xrightarrow{0|0} 1 \xrightarrow{4|1} 2.$$

The first coordinate of the bolded pair in  $I'$  is  $\delta(I, \langle 2 \rangle_2) = \delta(I, 10) = 1$ , and the first coordinates of the remaining bolded pairs are determined by starting from state  $\delta(I, 10) = 1$  in  $M$  and following the transitions of  $M$  given by the output labels of the above path through  $N$  (again, working backwards through  $N$ ):

$$\delta(I, 10) = 1 \xrightarrow{1} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 = \delta(I, v).$$

This illustrates how, on input  $u$ ,  $M'$  computes  $F(\delta(I, v))$ , which is exactly the output of  $M$  on input  $v$ .

As a second illustration, take  $u' = 2014 \in D_2^*$  whose canonical base-2 expansion is  $v' = 10110$ . On the input  $u'$ , the transitions of  $M'$  are

$$\begin{aligned}
I' = \{(0, 0), (\mathbf{1}, \mathbf{1}), (1, 2), (0, 3)\} &\xrightarrow{2} \{(\mathbf{1}, \mathbf{0}), (0, 1), (1, 2), (0, 3)\} \\
&\xrightarrow{0} \{(1, 0), (\mathbf{0}, \mathbf{1}), (0, 2), (1, 3)\} \\
&\xrightarrow{1} \{(0, 0), (0, 1), (\mathbf{1}, \mathbf{2}), (0, 3)\} \\
&\xrightarrow{4} \{(\mathbf{1}, \mathbf{0}), (0, 1), (0, 2), (1, 3)\}.
\end{aligned}$$

Similarly,  $F'(\{(1, 0), (0, 1), (0, 2), (1, 3)\}) = F(1) = 1$ , so the automaton  $M'$  outputs 1 after reading  $u'$ , agreeing with the output of  $M$  on input  $v'$ . Again, we have bolded the ordered pairs corresponding to the “correct path” through the transducer  $N$ .

We end this section with some remarks on the construction. We hope that the reader is convinced that the construction we have described works for any digit set containing  $\{0, 1, \dots, b-1\}$  and not just the digit set  $D_b$ . Furthermore, Krebs has pointed out (private communication) that the number of states needed for the construction can be improved by changing the digit set from  $D_b$  to  $\{0, 1, \dots, 2b-2\}$ . Recall that our construction results in a DFAO with  $|S|^\gamma$  states. If  $b = 2$ , then we have  $\gamma = 4$ , while if  $b > 2$ , then we have  $\gamma = 3$ . However, if we change the digit set as suggested by Krebs, we improve this to  $|S|^2$  states. Krebs’ proof of Cobham’s Theorem works just as well with this new choice of digit set; however, a number of bounds and constants in his proof would have to be modified. We do not present these modifications here; we just note that it is possible to do it.

## 2.3 Upper bound on longest common prefix

Having dealt with the conversion to the larger digit set required by Krebs, we now proceed with the Diophantine approximation result used by Krebs.

**Lemma 2.** *Let  $a, b \geq 2$  be integers and let  $\epsilon$  be a positive real number. Define*

$$\eta := \max\{\lceil \log_a b \rceil, \lceil \log_b a \rceil\}.$$

*There are non-negative integers  $m, n < \eta((b-1)/\epsilon + 1)$  such that  $|a^m - b^n| \leq \epsilon b^n$ .*

*Proof.* First suppose that  $a \geq b$ . Let  $(f_x)_{x \in \mathbb{N}}$  be the sequence such that  $a^x b^{-f_x} \in [1, b)$  for all  $x \in \mathbb{N}$ . Then  $0 \leq (\log_b a)x - f_x$ , so  $f_x \leq (\log_b a)x$ . Now by the pigeonhole principle there exist  $x < y \leq (b-1)/\epsilon + 1$  such that  $|a^y b^{-f_y} - a^x b^{-f_x}| \leq \epsilon$ ; i.e.,

$$|a^{y-x} - b^{f_y-f_x}| \leq \epsilon b^{f_y} a^{-x} \leq \epsilon b^{f_y-f_x}.$$

Thus, we have  $m = y - x \leq y \leq (b-1)/\epsilon + 1$  and

$$n = f_y - f_x \leq f_y \leq (\log_b a)y \leq (\log_b a)((b-1)/\epsilon + 1) \leq \eta((b-1)/\epsilon + 1),$$

as required.

Now suppose that  $a < b$ . Applying the previous argument with  $a^{\lceil \log_a b \rceil}$  in place of  $a$  (where  $\lceil \rho \rceil$  denotes the least integer greater than or equal to  $\rho$ ), we find that

$$m = \lceil \log_a b \rceil (y - x) \leq \lceil \log_a b \rceil y \leq \lceil \log_a b \rceil ((b-1)/\epsilon + 1) \leq \eta((b-1)/\epsilon + 1),$$

and

$$n = f_y - f_x \leq f_y \leq \lceil \log_a b \rceil (\log_b a)y \leq \lceil \log_a b \rceil (\log_b a)((b-1)/\epsilon + 1) \leq \eta((b-1)/\epsilon + 1),$$

as required (the final inequality above follows from the fact that  $\log_b a < 1$  in this case).  $\square$

As in Lemma 2, define  $\eta := \max\{\lceil \log_a b \rceil, \lceil \log_b a \rceil\}$  and also define  $\theta := \max\{a, b\}$ . We now define

$$\begin{aligned} E(a, b, R, S) &:= \eta \left[ 6 \left( 2\theta^{(S+1)(R+1)} + 1 \right) (\theta - 1) + 1 \right], \\ A(a, b, R, S) &:= \left( 2\theta^{(S+1)(R+1)} + 2 \right) \theta^{E(a, b, R, S)}, \end{aligned}$$

and note that both these functions are symmetric under exchange of their first two arguments and also under exchange of their last two arguments.

**Theorem 3.** *Let  $a, b \geq 2$  be multiplicatively independent integers. Let  $g = (g_x)_{x \in \mathbb{N}}$  be computed by a DFAO  $M_a = (S_a, D_a, \delta_a, s_{0,a}, \Delta_a, F_a)$  in base  $a$  and let  $f = (f_x)_{x \in \mathbb{N}}$  be computed by a DFAO  $M_b = (S_b, D_b, \delta_b, s_{0,b}, \Delta_b, F_b)$  in base  $b$ . Suppose that  $f$  and  $g$  agree on a prefix of length  $A(a, b, |S_a|, |S_b|)$ . Then  $f$  and  $g$  are equal and ultimately periodic.*

*Proof.* Let  $S_\infty$  be the subset of states of  $M_b$  consisting of all states  $s$  with the property that there are infinitely many numbers  $x$  such that some representation  $(x)_{D_b}$  reaches state  $s$  in  $M_b$ . For each  $s \in S_\infty$ , we claim that there must exist a state  $t \in S_a$  and positive integers  $x_{st}$  and  $y_{st}$  such that some base- $b$  representations  $(x_{st})_{D_b}$  and  $(y_{st})_{D_b}$  both lead to state  $s$  in  $M_b$  and some base- $a$  representations  $(x_{st})_{D_a}$  and  $(y_{st})_{D_a}$  both lead to state  $t$  in  $M_a$ . We show this by giving an explicit upper bound on  $x_{st}$  and  $y_{st}$ .

If a string  $W$  has length at least  $|S_b|$ , then any computation of  $M_b$  on  $W$  repeats a state. Since for each  $s \in S_\infty$  there are infinitely many  $(x)_{D_b}$  that reach state  $s$ , there must exist some number  $x_0$ , some representation  $(x_0)_{D_b}$ , and some factorization  $(x_0)_{D_b} = uv$  with the following properties:

- $|(x_0)_{D_b}| \leq |S_b|$ .
- There exists a non-empty  $v$  such that  $|v| \leq |S_b|$  and  $uv^i w$  reaches  $s$  for all  $i \geq 0$ .

For  $1 \leq i \leq |S_a|$ , let  $x_i$  be the integer such that  $(x_i)_{D_b} = uv^i w$ . Then the numbers  $x_i$  are all distinct. Now consider the states reached in  $M_a$  by some choice of representations  $(x_i)_{D_a}$ , for  $0 \leq i \leq |S_a|$ . There must be two such numbers  $x_i$  and  $x_j$  such that  $(x_i)_{D_a}$  and  $(x_j)_{D_a}$  reach the same state  $t$  in  $M_a$ . We choose these as our  $x_{st}$  and  $y_{st}$ . Finally, we note that for  $0 \leq i \leq |S_a|$ , we have  $|(x_i)_{D_b}| \leq |S_b|(|S_a| + 1)$ , which gives the bound  $x_{st}, y_{st} \leq 2b^{|S_b|(|S_a|+1)+1}$ .

Let  $\xi := \max\{x_{st}, y_{st} \mid s \in S_\infty\} + 1 \leq 2b^{|S_b|(|S_a|+1)+1} + 1$ . By Lemma 2, there exist

$$m, n \leq \eta[6\xi(b-1) + 1] \leq E(a, b, |S_a|, |S_b|)$$

such that  $\xi|a^m - b^n| \leq \frac{1}{6}b^n$ . As defined in [8], let  $p_{st} := (x_{st} - y_{st})(a^m - b^n)$  (swapping  $x_{st}$  and  $y_{st}$  if necessary, so that  $p_{st} > 0$ ), and note that from [8] we have  $p_{st} \leq \frac{1}{6}b^n$ . Let  $z$  be any integer such that  $z, z + p_{st} \in [\frac{1}{3}b^n, \frac{5}{3}b^n]$ . In particular, there exist representations  $(z)_{D_b}$  and  $(z + p_{st})_{D_b}$  such that  $|(z)_{D_b}|, |(z + p_{st})_{D_b}| \leq n$ . In what follows, we specifically use the representations of  $z$  and  $z + p_{st}$  that satisfy this condition on their lengths. We also note that by the calculation in [8], we have  $z - y_{st}(a^m - b^n) \leq 2a^m$ , so there is also a representation  $(z - y_{st}(a^m - b^n))_{D_a}$  of length at most  $m$ .

Let  $x$  be any integer such that some representation  $(x)_{D_b}$  goes to state  $s$  in  $M_b$ . Recall that  $(x_{st})_{D_b}$  and  $(y_{st})_{D_b}$  go to state  $s$  in  $M_b$  and  $(x_{st})_{D_a}$  and  $(y_{st})_{D_a}$  go to state  $t$  in  $M_a$ . If  $f$  and  $g$  agree on a sufficiently long prefix (to be specified later), then we have

$$\begin{aligned} f_{xb^n+z} &= f_{y_{st}b^n+z} && \text{(since } (x)_{D_b} \text{ and } (y_{st})_{D_b} \text{ go to state } s \text{ in } M_b) \\ &= f_{y_{st}a^m+z-y_{st}(a^m-b^n)} && \text{(rewriting the index)} \\ &= g_{y_{st}a^m+z-y_{st}(a^m-b^n)} && \text{(since } f \text{ and } g \text{ agree)} \\ &= g_{x_{st}a^m+z-y_{st}(a^m-b^n)} && \text{(since } (y_{st})_{D_a} \text{ and } (x_{st})_{D_a} \text{ go to state } t \text{ in } M_a) \\ &= g_{x_{st}b^n+z+p_{st}} && \text{(rewriting the index)} \\ &= f_{x_{st}b^n+z+p_{st}} && \text{(since } f \text{ and } g \text{ agree)} \\ &= f_{xb^n+z+p_{st}} && \text{(since } (x_{st})_{D_b} \text{ and } (x)_{D_b} \text{ go to state } s \text{ in } M_b). \end{aligned}$$

For this calculation to be correct, the two sequences  $f$  and  $g$  should agree on a prefix of

length

$$\begin{aligned} \max\{y_{st}, x_{st} \mid s \in S_\infty\}a^m + z - y_{st}(a^m - b^n) &\leq (\xi - 1)a^m + z - y_{st}(a^m - b^n) \\ &\leq (\xi - 1)a^m + 2a^m \\ &\leq (\xi + 1)a^m. \end{aligned}$$

Now  $\xi \leq 2b^{|S_b|(|S_a|+1)+1} + 1$ , so we have

$$\begin{aligned} (\xi + 1)a^m &\leq (2b^{|S_b|(|S_a|+1)+1} + 2) a^m \\ &\leq (2b^{|S_b|(|S_a|+1)+1} + 2) a^{E(a,b,|S_a|,|S_b|)} \\ &\leq A(a, b, |S_a|, |S_b|). \end{aligned}$$

Thus, if  $f$  and  $g$  agree on a prefix of length  $A(a, b, |S_a|, |S_b|)$ , then  $f$  has a local period

$$p_{st} \leq \frac{1}{6}b^n \leq \frac{1}{6}b^{E(a,b,|S_a|,|S_b|)}$$

on the interval  $[(x + 1/3)b^n, (x + 5/3)b^n]$ . By the same argument as in [8], the sequence  $f$  is ultimately periodic. We will show further that the periodicity begins after a prefix of length at most

$$\left(2b^{|S_b|+1} + \frac{1}{3}\right) b^n \leq \left(2b^{|S_b|+1} + \frac{1}{3}\right) b^{E(a,b,|S_a|,|S_b|)}.$$

Any representation  $(x)_{D_b}$  of length  $|S_b|$  must reach a state in  $S_\infty$ . Therefore if  $x = 2b^{|S_b|+1}$ , then for every  $y \geq x$ , every representation  $(y)_{D_b}$  reaches a state in  $S_\infty$ . Now by the argument of [8], the sequence  $f$  has period  $p_f := p_{st}$  starting from index

$$i_f := \left(2b^{|S_b|+1} + \frac{1}{3}\right) b^n.$$

By a similar argument (with the roles of  $f$  and  $g$  reversed) we find that if  $f$  and  $g$  agree on a prefix of length  $A(a, b, |S_a|, |S_b|)$ , then  $g$  has period  $p_g$  starting from some index  $i_g$ , where  $p_g$  and  $i_g$  are defined analogously to  $p_f$  and  $i_f$ . Now, we have

$$\max\{p_f, p_g\} \leq \frac{1}{6}\theta^{E(a,b,|S_a|,|S_b|)},$$

and

$$\begin{aligned} \max\{i_f, i_g\} &\leq \max\left\{\left(2b^{|S_b|+1} + \frac{1}{3}\right) b^{E(a,b,|S_a|,|S_b|)}, \left(2a^{|S_a|+1} + \frac{1}{3}\right) a^{E(b,a,|S_b|,|S_a|)}\right\} \\ &\leq \max\left\{\left(2b^{|S_b|+1} + \frac{1}{3}\right), \left(2a^{|S_a|+1} + \frac{1}{3}\right)\right\} \theta^{E(a,b,|S_a|,|S_b|)}, \end{aligned}$$

so

$$\begin{aligned} \max\{i_f, i_g\} + p_f + p_g &\leq \max\left\{\left(2b^{|S_b|+1} + \frac{2}{3}\right), \left(2a^{|S_a|+1} + \frac{2}{3}\right)\right\} \theta^{E(a,b,|S_a|,|S_b|)} \\ &\leq A(a, b, |S_a|, |S_b|). \end{aligned}$$

Therefore, by the Fine–Wilf theorem [10, Theorem 2.3.5], the sequences  $f$  and  $g$  are equal.  $\square$

In the next corollary, let  $\exp_r(x)$  denote the function  $r^x$ .

**Corollary 4.** *Let  $a, b \geq 2$  be multiplicatively independent integers. Let  $g = (g_x)_{x \in \mathbb{N}}$  and  $f = (f_x)_{x \in \mathbb{N}}$  be sequences over a set  $\Delta$  of size  $d$ . Suppose that  $g$  is computed by an  $a$ -DFAO  $M'_a$  with  $R$  states and  $f$  is computed by a  $b$ -DFAO  $M'_b$  with  $S$  states. There is a positive constant  $C$  (depending only on  $a$  and  $b$ ) such that if  $f$  and  $g$  agree on a prefix of length*

$$\exp_\theta(\exp_\theta(CR^4S^4)) \tag{2}$$

then  $f$  and  $g$  are equal and ultimately periodic.

*Proof.* We have previously observed that conversion from a  $b$ -DFAO to a  $(b, D_b)$ -DFAO increases the number of states to at most the quantity (1). We apply the bound of Theorem 3 with

$$|S_a| = R^4 \text{ and } |S_b| = S^4.$$

Simplifying the resulting expression, we find that there is a positive constant  $C$  such that the bound of Theorem 3 is at most the quantity (2).  $\square$

Note that the bound on the length of the common prefix that we obtain seems absurdly large compared to what seems likely to be the optimal bound. It is not too difficult to give an example where the common prefix has length that is (singly) exponential in the size of the defining automata. For instance, let  $g$  be the constant (and hence  $a$ -automatic) sequence  $(0, 0, 0, 0, \dots)$ . Fix some positive integer  $N$  and let  $f$  be the characteristic sequence of the set  $\{b^n - 1 : n \geq N\}$ . Then  $f$  is an aperiodic  $b$ -automatic sequence. Indeed, a  $b$ -DFAO  $M$  generating  $f$  can be obtained from the  $N + 2$  state DFA accepting the regular language  $0^*(b - 1)^N(b - 1)^*$  by making the accepting state output 1 and all other states output 0. Then  $M$  has  $N + 2$  states and the sequences  $f$  and  $g$  agree on a prefix of length  $b^N - 1$ .

Now we examine the connection to *automaticity*. The  $b$ -automaticity of a sequence  $g$  is the function  $A_g^b(n)$  whose value is the least number of states required in a  $b$ -DFAO that computes a prefix of  $g$  of length  $n$ . Shallit [9, Proposition 1(c)] showed that if  $g$  is not  $b$ -automatic, then there is a constant  $c$  such that  $A_g^b(n) \geq c \log_b n$  for infinitely many  $n$ .

**Corollary 5.** *Let  $a, b$  be multiplicatively independent integers with  $a, b \geq 2$ . There is a positive constant  $D$  such that the  $b$ -automaticity  $A_g^b(n)$  of an aperiodic  $a$ -automatic sequence  $g$  satisfies*

$$A_g^b(n) > D(\log \log n)^{1/4},$$

for all  $n \geq 0$ .

*Proof.* Let  $M_a$  be an  $a$ -DFAO computing  $g$  and let  $M_{b,n}$  be a  $b$ -DFAO computing a sequence  $f$  that agrees with  $g$  on a prefix of length  $n$ . Suppose that  $M_a$  has  $E$  states and that  $M_{b,n}$  has  $S_n$  states. Since  $g$  is aperiodic, by (2) we have

$$n < \exp_\theta(\exp_\theta(CE^4(S_n)^4))$$

Treating  $E$  as a constant, we get

$$S_n > \left( \frac{1}{C^{1/4}E} \right) (\log_\theta \log_\theta n)^{1/4} = D(\log \log n)^{1/4},$$

for some positive constant  $D$ .  $\square$

Note that while this may seem weaker than the  $c \log_b n$  lower bound mentioned previously, the former only holds for infinitely many  $n$ , whereas our lower bound holds for all  $n$ . Without the assumption that  $g$  is  $a$ -automatic, the  $b$ -automaticity of  $g$  could potentially be constant for long stretches, and only for very sparsely distributed values of  $n$  satisfy  $A_g^b(n) \geq c \log_b n$ . Our result shows that under the assumption that  $g$  is  $a$ -automatic, the function  $A_g^b(n)$  cannot be constant for too long.

On the other hand, our lower bound on the  $b$ -automaticity does seem to be rather weak compared to what can be proved for specific sequences. Shallit [9] showed that if  $p$  is the fixed point of  $0 \rightarrow 01, 1 \rightarrow 00$ , then for  $k$  odd, we have  $A_p^k(n) = \Omega(n^{1/2}/k)$ , and if  $t$  is the fixed point of  $0 \rightarrow 01, 1 \rightarrow 10$  (the Thue–Morse word), then for  $k$  odd, we have  $A_t^k(n) = \Omega(n^{1/4}/k^{1/2})$ .

### 3 Common factors of $b$ -automatic and Sturmian sequences

As mentioned in the introduction, the problem of bounding the length of the longest common prefix of a  $b$ -automatic sequence and a Sturmian sequence was addressed by Shallit [9]. In this section, we show that two such sequences cannot have arbitrarily large factors in common. Our main result is the following:

**Theorem 6.** *Let  $f$  be a  $b$ -automatic sequence and let  $g$  be a Sturmian sequence. There exists a constant  $C$  (depending on  $f$  and  $g$ ) such that if  $f$  and  $g$  have a factor in common of length  $n$ , then  $n \leq C$ .*

Note that this result would follow fairly easily from the frequency results mentioned previously, if  $f$  is *uniformly recurrent* (meaning that every factor  $z$  of  $f$  occurs infinitely often, and with bounded gap size between two consecutive occurrences). However, unlike Sturmian sequences, automatic sequences need not be uniformly recurrent: consider, for example, the 2-automatic sequence that is the characteristic sequence of the powers of 2. Our proof is therefore based on the finiteness of the  $b$ -kernel of  $f$ , along with the *uniform distribution* property of Sturmian sequences (this is similar to the techniques used in [9]).

*Proof.* Let  $f = f_0 f_1 \cdots$  and  $g = g_0 g_1 \cdots$ , where  $g$  has slope  $\alpha$  and intercept  $\beta$ . Since the factors of a Sturmian word do not depend on  $\beta$ , without loss of generality, we may suppose that  $\beta = 0$  (or, in other words, that  $g$  is a *characteristic word*). Then  $g$  can be defined by the following rule:

$$g_n = \begin{cases} 1, & \text{if } \{(n+1)\alpha\} < \alpha; \\ 0, & \text{otherwise.} \end{cases}$$

Here  $\{\cdot\}$  denotes the fractional part of a real number.

Suppose that for some integer  $L$ , the words  $f$  and  $g$  have a factor of length  $L$  in common: i.e., for some  $i \leq j$ , we have

$$f_i \cdots f_{i+L-1} = g_j \cdots g_{j+L-1}.$$

(We may assume that  $i \leq j$  since  $g$  is recurrent, but this is not important for what follows.) Suppose that the  $b$ -kernel of  $f$ ,

$$\{(f_{nb^r+s})_{n \geq 0} : r \geq 0 \text{ and } 0 \leq s < b^r\},$$

has  $Q$  distinct elements. Let  $r$  satisfy  $b^r > Q$ . There there exist integers  $s_1, s_2$  with  $0 \leq s_1 < s_2 < b^r$  such that

$$(f_{nb^r+s_1})_{n \geq 0} = (f_{nb^r+s_2})_{n \geq 0}.$$

Define

$$\begin{aligned} d_1 &:= s_1 + j - i + 1, \\ d_2 &:= s_2 + j - i + 1. \end{aligned}$$

For all  $n$  satisfying  $i \leq nb^r + s_1$  and  $nb^r + s_2 \leq i + L - 1$  we have  $f_{nb^r+s_1} = g_{nb^r+d_1-1}$  and  $f_{nb^r+s_2} = g_{nb^r+d_2-1}$ . Since  $f_{nb^r+s_1} = f_{nb^r+s_2}$ , we have  $g_{nb^r+d_1-1} = g_{nb^r+d_2-1}$ . This means that either the inequalities

$$\{(nb^r + d_1)\alpha\} < \alpha \text{ and } \{(nb^r + d_2)\alpha\} < \alpha \tag{3}$$

both hold, or the inequalities

$$\{(nb^r + d_1)\alpha\} \geq \alpha \text{ and } \{(nb^r + d_2)\alpha\} \geq \alpha \tag{4}$$

both hold.

If  $L$  is arbitrarily large, then there exist arbitrarily large sets  $I$  of consecutive positive integers such that every  $n \in I$  satisfies either (3) or (4). Without loss of generality, suppose that  $\{d_2\alpha\} > \{d_1\alpha\}$ . Choose  $\epsilon > 0$  such that  $\epsilon < \{d_2\alpha\} - \{d_1\alpha\}$ . Note that  $d_2 - d_1 = s_2 - s_1$ , so  $\epsilon$  does not depend on  $L$  (or  $I$ ). Since  $b^r\alpha$  is irrational, if  $I$  is sufficiently large, then by Kronecker's theorem (which asserts that the set of points  $\{n\alpha\}$  is dense in  $(0, 1)$ ) there exists  $N \in I$  such that

$$\{N(b^r\alpha) + d_2\alpha\} \in [\alpha, \alpha + \epsilon].$$

By the choice of  $\epsilon$ , this implies that

$$\{N(b^r\alpha) + d_2\alpha\} \geq \alpha \text{ and } \{N(b^r\alpha) + d_1\alpha\} < \alpha,$$

contradicting the assumption that  $N$  satisfies one of (3) or (4). The contradiction means that  $L$  must be bounded by some constant  $C$ , which proves the theorem.  $\square$

**Example 7.** Consider the Thue-Morse word  $t = 01101001\dots$ , and the Fibonacci word  $f = 01001010\dots$  given by the fixed point of  $0 \rightarrow 01$  and  $1 \rightarrow 0$ . The latter is Sturmian. The set of common factors is

$$\begin{aligned} &\{\epsilon, 0, 1, 00, 01, 10, 001, 010, 100, 101, 0010, 0100, 0101, 1001, 1010, \\ &\quad 00101, 01001, 10010, 10100, 010010, 100101, 101001, 0100101, 1010010, 10100101\}, \end{aligned}$$

so  $C = 8$ .

## 4 Final thoughts

As noted at the end of Section 2, the  $\Omega((\log \log n)^{1/4})$  lower bound we obtain on the  $b$ -automaticity of an aperiodic  $a$ -automatic sequence is surely not optimal. Sequences with  $O(\log n)$  (i.e., “low”)  $b$ -automaticity are called *b-quasiautomatic* in [9]. It seems unlikely that an aperiodic  $a$ -automatic sequence can even be  $b$ -quasiautomatic. Known examples of  $b$ -quasiautomatic sequences strongly resemble  $b$ -automatic sequences. For example, the fixed point of the morphism  $c \rightarrow cba$ ,  $a \rightarrow aa$ ,  $b \rightarrow b$ , starting with  $c$ , is 2-quasiautomatic, but not 2-automatic [9]. Similarly, the fixed point of  $1 \rightarrow 121$ ,  $2 \rightarrow 12221$  is not 2-automatic [1] but is conjectured to be 2-quasiautomatic [9]. Curiously, this latter sequence is automatic with respect to the positional numeration system (and a certain choice of canonical representations) whose place values are given by the sequence  $((2^n - (-1)^n)/3)_{n \geq 1}$  [1].

We conclude by again mentioning the problem stated in the introduction of characterizing the common factors of a  $b$ -automatic sequence and an  $a$ -automatic sequence. Can the method of Krebs be applied to this problem?

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