

# EVERY ZERO-DIMENSIONAL HOMOGENEOUS SPACE IS STRONGLY HOMOGENEOUS UNDER DETERMINACY

RAPHAËL CARROY, ANDREA MEDINI, AND SANDRA MÜLLER

**ABSTRACT.** All spaces are assumed to be separable and metrizable. We show that, assuming the Axiom of Determinacy, every zero-dimensional homogeneous space is strongly homogeneous (that is, all its non-empty clopen subspaces are homeomorphic), with the trivial exception of locally compact spaces. In fact, we obtain a more general result on the uniqueness of zero-dimensional homogeneous spaces which generate a given Wadge class. This extends work of van Engelen (who obtained the corresponding results for Borel spaces), complements a result of van Douwen, and gives partial answers to questions of Terada and Medvedev.

## 1. INTRODUCTION

Throughout this article, unless we specify otherwise, we will be working in the theory  $\text{ZF} + \text{DC}$ , that is, the usual axioms of Zermelo-Fraenkel (without the Axiom of Choice) plus the principle of Dependent Choices (see Section 2 for more details). By *space* we will always mean separable metrizable topological space, unless we specify otherwise. A space  $X$  is *homogeneous* if for every  $x, y \in X$  there exists a homeomorphism  $h : X \rightarrow X$  such that  $h(x) = y$ . For example, using translations, it is easy to see that every topological group is homogeneous (as [vE3, Corollary 3.6.6] shows, the converse is not true, not even for zero-dimensional Borel spaces). Homogeneity is a classical notion in topology, which has been studied in depth (see for example [AvM]). In particular, in his remarkable doctoral thesis [vE3] (see also [vE1] and [vE2]), Fons van Engelen gave a complete classification of the homogeneous zero-dimensional Borel spaces. In fact, as we will make more precise, this article is inspired by his work and relies heavily on some of his techniques.

A space  $X$  is *strongly homogeneous* (or *h-homogeneous*) if every non-empty clopen subspace of  $X$  is homeomorphic to  $X$ . This notion has been studied by several authors, both “instrumentally” and for its own sake (see the list of references in [Me1]). It is well-known that every zero-dimensional strongly homogeneous space is homogeneous (see for example [vE3, 1.9.1] or [Me2, Proposition 3.32]). Our main result shows that, under the Axiom of Determinacy (briefly, AD) the converse

---

*Date:* February 26, 2020.

2010 *Mathematics Subject Classification.* 54H05, 03E15, 03E60.

*Key words and phrases.* Homogeneous, strongly homogeneous, h-homogeneous, zero-dimensional, determinacy, Wadge theory, Hausdorff operation,  $\omega$ -ary Boolean operation.

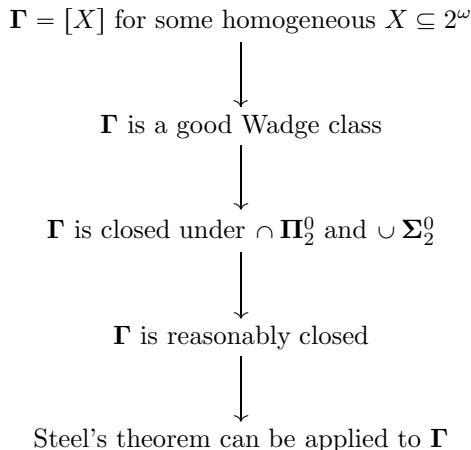
The first-listed author acknowledges the support of the FWF grant P 28153-N35. The second-listed author acknowledges the support of the FWF grant P 30823-N35. The third-listed author (formerly known as Sandra Uhlenbrock) acknowledges the support of the FWF grant P 28157-N35. The authors are grateful to Alessandro Andretta for valuable bibliographical help, and to Alain Louveau for allowing them to use his unpublished book [Lo2].

also holds (with the trivial exception of locally compact spaces, see Proposition 2.5). For the proof, see Corollary 15.3.

**Theorem 1.1.** *Assume AD. If  $X$  is a zero-dimensional homogeneous space that is not locally compact then  $X$  is strongly homogeneous.*

The above theorem follows from a uniqueness result about zero-dimensional homogeneous spaces, namely Theorem 15.2, which is of independent interest.<sup>1</sup> This theorem essentially states that, for every sufficiently high level of complexity  $\Gamma$ , there are at most two homogeneous zero-dimensional spaces of complexity exactly  $\Gamma$  (one meager and one Baire).

Our fundamental tool will be Wadge theory, which was founded by William Wadge in his doctoral thesis [Wa1] (see also [Wa2]), and has become a classical topic in descriptive set theory. We believe that [vEMS, Theorem 2.4] and our results are the only applications to topology of an analysis of the full (as opposed to just Borel) Wadge hierarchy.<sup>2</sup> In fact, most of this article (Sections 3 to 13) is purely Wadge-theoretic in character. The ultimate goal of the Wadge-theoretic portion of the paper is to show that good Wadge classes are closed under intersection with  $\Pi_2^0$  sets (see Section 12), hence they are reasonably closed (see Section 13). Homogeneity comes into play in Section 14, where we show that  $[X]$  is a good Wadge class whenever  $X$  is a homogeneous space of sufficiently high complexity. This will allow us to use a theorem of Steel from [St2], which will in turn yield the uniqueness result mentioned above (see Section 15). The following diagram summarizes the structure of the proof of our main result. In the preceding sections, the necessary tools are developed. More specifically, Section 4 is devoted to the analysis of the selfdual Wadge classes, Sections 5 to 8 develop the machinery of relativization through Hausdorff operations, and Sections 9 to 11 develop the notions of level and expansion. While none of the ideas contained in these preliminary sections are new (except, to the best of our knowledge, Theorem 12.4), satisfactory references are hard to come by, especially for the required level of generality. For this reason, we will always restate the needed results (and sometimes give full proofs).



<sup>1</sup> In fact, the only consequences of AD that Theorem 15.2 (hence Theorem 1.1) requires are BP (see Section 2) and Lemma 3.2.

<sup>2</sup> While all of these results ultimately rely on [St2, Theorem 2], the techniques used in the proof of [vEMS, Theorem 2.4] are very different from those used here.

The application of Wadge theory to the study of homogeneous spaces was pioneered by van Engelen in [vE3], where he obtained the classification mentioned above. As a corollary (namely, [vE3, Corollary 4.4.6]), he obtained the Borel version of Theorem 1.1. The reason why his results are limited to Borel spaces is that they are all based on the fine analysis of the Borel Wadge classes given by Louveau in [Lo1]. Fully extending this analysis beyond the Borel realm appears to be a very hard problem (although partial results have been obtained in [Fo]). Here, we will follow a different strategy, and we will “substitute” facts from [Lo1] about Borel Wadge classes with more general results about arbitrary Wadge classes (under AD). Furthermore, since most of the literature on Wadge theory only deals with  $\omega^\omega$  as the ambient space, while Steel’s theorem is stated for  $2^\omega$ , we decided to work in the context of arbitrary zero-dimensional uncountable Polish spaces. With regard to these issues, Louveau’s book [Lo2] and Van Wesep’s results on Hausdorff operations from [VW1] were crucial. For other applications of Wadge theory to topology, see the characterizations of Borel filters and semifilters given respectively in [vE4] and [Me3].

At this point, it is natural to ask whether assuming AD is really necessary in the above results. As the following theorem shows, the answer is “yes”. This result was essentially proved in [vD], but our exposition is based on [vM, Theorem 5.1]. Following [vM], we will say that  $X \subseteq \mathbb{R}$  is a *bi-Bernstein set*<sup>3</sup> if  $K \cap X \neq \emptyset$  and  $K \cap (\mathbb{R} \setminus X) \neq \emptyset$  for every  $K \subseteq \mathbb{R}$  that is homeomorphic to  $2^\omega$ .

**Theorem 1.2** (van Douwen). *There exists a ZFC example  $X$  of a homogeneous zero-dimensional space that is not locally compact and not strongly homogeneous.*

*Proof.* Let  $X$  be the space given by [vM, proof of Theorem 5.1]. Notice that  $X$  is homogeneous because  $X$  is a subgroup of  $\mathbb{R}$ . Furthermore,  $X$  is a bi-Bernstein set by [vM, Proposition 4.5]. It follows that both  $X$  and  $\mathbb{R} \setminus X$  are dense in  $\mathbb{R}$ . In particular,  $X$  is zero-dimensional and not locally compact.

Given any Borel subset  $B$  of  $X$ , pick a Borel subset  $A$  of  $\mathbb{R}$  such that  $A \cap X = B$ , then define  $\bar{\mu}(B) = \mu(A)$ , where  $\mu$  denotes the Lebesgue measure on  $\mathbb{R}$ . Using the fact that  $X$  is bi-Bernstein, it is easy to check that  $\bar{\mu}$  is a well-defined measure on the Borel subsets of  $X$ . The crucial property of  $\bar{\mu}$ , as given by the statement of [vM, Theorem 5.1], is that if  $B$  and  $C$  are homeomorphic Borel subspaces of  $X$ , then  $\bar{\mu}(B) = \bar{\mu}(C)$ .

Now pick  $a, b, c \in \mathbb{R} \setminus X$  such that  $a < b < c$ . Observe that  $U = (a, b) \cap X$  and  $V = (a, c) \cap X$  are non-empty clopen subsets of  $X$ . Furthermore, it is clear from the definition of  $\bar{\mu}$  that  $\bar{\mu}(U) = b - a \neq c - a = \bar{\mu}(V)$ . Therefore  $U$  and  $V$  are not homeomorphic, which concludes the proof.  $\square$

However, we do not know the answer to the following question. Recall that, when  $\Gamma = \Sigma_n^1$  or  $\Gamma = \Pi_n^1$  for some  $n \geq 1$ , a space is  $\Gamma$  if it is homeomorphic to a  $\Gamma$  subspace of some Polish space (see [MZ, Section 4] for a more detailed treatment).

**Question 1.3.** Assuming  $V = L$ , is it possible to construct a zero-dimensional  $\Pi_1^1$  or  $\Sigma_1^1$  space that is homogeneous, not locally compact, and not strongly homogeneous?

---

<sup>3</sup>These sets are commonly referred to simply as *Bernstein sets*.

The above question is natural because there are many examples of properties (such as the perfect set property<sup>4</sup>) that are known to hold for all spaces under AD, for which definable counterexamples can be constructed under  $\mathbf{V} = \mathbf{L}$ . Notice that  $\mathbf{\Pi}_1^1$  and  $\mathbf{\Sigma}_1^1$  are optimal by [vE3, Corollary 4.4.6]. For other relevant examples, see [vEMS, Theorem 2.6], [Mi], and [Vi].

Finally, we mention three applications of Theorem 1.1. The first is that Theorem 1.2 cannot be proved without using the Axiom of Choice (more precisely, it cannot be proved in  $\mathbf{ZF} + \mathbf{DC}$  alone). The second concerns the following question from [Te, Section 3] (see [Me1, Section 3] and [MvMZ, Section 5] for more on this topic).

**Question 1.4** (Terada). Is  $X^\omega$  strongly homogeneous for every zero-dimensional first-countable space  $X$ ?

Since Lawrence showed that  $X^\omega$  is homogeneous for every zero-dimensional space  $X$  (see [La], or [DP] for a more general result), it follows from Theorem 1.1 that the answer to Question 1.4 in the separable metrizable realm is “yes” under AD.<sup>5</sup> As the third application, we obtain that the answer to the following question (see [Mv, Question 1]) is also “yes” under AD.<sup>6</sup>

**Question 1.5** (Medvedev). Is every zero-dimensional meager homogeneous space strongly homogeneous?

## 2. PRELIMINARIES AND NOTATION

Let  $Z$  be a set, and let  $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$ . Define  $\check{\mathbf{\Gamma}} = \{Z \setminus A : A \in \mathbf{\Gamma}\}$ . We will say that  $\mathbf{\Gamma}$  is *selfdual* if  $\mathbf{\Gamma} = \check{\mathbf{\Gamma}}$ . Also define  $\Delta(\mathbf{\Gamma}) = \mathbf{\Gamma} \cap \check{\mathbf{\Gamma}}$ . Given a function  $f : Z \rightarrow W$ ,  $A \subseteq Z$ , and  $B \subseteq W$ , we will use the notation  $f[A] = \{f(x) : x \in A\}$  and  $f^{-1}[B] = \{x \in Z : f(x) \in B\}$ .

**Definition 2.1** (Wadge). Let  $Z$  be a space, and let  $A, B \subseteq Z$ . We will write  $A \leq B$  if there exists a continuous function  $f : Z \rightarrow Z$  such that  $A = f^{-1}[B]$ .<sup>7</sup> In this case, we will say that  $A$  is *Wadge-reducible* to  $B$ , and that  $f$  *witnesses* the reduction. We will write  $A < B$  if  $A \leq B$  and  $B \not\leq A$ . We will write  $A \equiv B$  if  $A \leq B$  and  $B \leq A$ .

**Definition 2.2** (Wadge). Let  $Z$  be a space. Given  $A \subseteq Z$ , define

$$[A] = \{B \subseteq Z : B \leq A\}.$$
<sup>8</sup>

We will say that  $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$  is a *Wadge class* if there exists  $A \subseteq Z$  such that  $\mathbf{\Gamma} = [A]$ . We will say that  $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$  is *continuously closed* if  $[A] \subseteq \mathbf{\Gamma}$  for every  $A \in \mathbf{\Gamma}$ .

<sup>4</sup>To see that every space has the perfect set property under AD, proceed as in [Ke, Section 21.A]. For the counterexample under  $\mathbf{V} = \mathbf{L}$ , see [Ka, Theorem 13.12].

<sup>5</sup>Notice that if a zero-dimensional infinite power is locally compact then it is compact, hence either it has size 1, or it is homeomorphic to  $2^\omega$  by [Ke, Theorem 7.4].

<sup>6</sup>Recall that locally compact spaces are Baire.

<sup>7</sup>Wadge-reduction is usually denoted by  $\leq_W$ , which allows to distinguish it from other types of reduction (such as Lipschitz-reduction). Since we will not consider any other type of reduction in this article, we decided to simplify the notation.

<sup>8</sup>Usually, this denotes the *Wadge degree* of  $A$ , that is  $\{B \subseteq Z : B \equiv A\}$ . Our notation follows [vE1], [vE3], [vE4], and [Lo1], since these were among our main inspirations and sources. Even the book [Lo2] uses the similar (but slightly more cumbersome) notation  $\mathbf{\Gamma}[A]$ .

Both of the above definitions depend of course on the space  $Z$ . Often, for the sake of clarity, we will specify what the ambient space is by saying, for example, that “ $A \leq B$  in  $Z$ ” or “ $\Gamma$  is a Wadge class in  $Z$ ”. We will say that  $A \subseteq Z$  is *selfdual* if  $A \equiv Z \setminus A$  in  $Z$ . It is easy to check that  $A$  is selfdual iff  $[A]$  is selfdual. Given a space  $Z$ , we will also use the following shorthand notation:

- $\text{SD}(Z) = \{\Gamma : \Gamma \text{ is a selfdual Wadge class in } Z\}$ ,
- $\text{NSD}(Z) = \{\Gamma : \Gamma \text{ is a non-selfdual Wadge class in } Z\}$ .

Our reference for descriptive set theory is [Ke]. In particular, we assume familiarity with the basic theory of Borel sets and Polish spaces, and use the same notation as in [Ke, Section 11.B]. For example, given a space  $Z$ , we use  $\Sigma_1^0(Z)$ ,  $\Pi_1^0(Z)$ , and  $\Delta_1^0(Z)$  to denote the collection of all open, closed, and clopen subsets of  $Z$  respectively. Our reference for other set-theoretic notions is [Je].

The classes defined below constitute the so-called *difference hierarchy* (or *small Borel sets*). For a detailed treatment, see [Ke, Section 22.E] or [vE3, Chapter 3]. Here, we will only mention that the  $\text{D}_\eta(\Sigma_\xi^0(Z))$  are among the simplest concrete examples of Wadge classes (see Proposition 9.3 and Corollary 9.5).

**Definition 2.3** (Kuratowski). Let  $Z$  be a space, let  $1 \leq \eta < \omega_1$  and  $1 \leq \xi < \omega_1$ . Given a sequence of sets  $(A_\mu : \mu < \eta)$ , define

$$\text{D}_\eta(A_\mu : \mu < \eta) = \begin{cases} \bigcup \{A_\mu \setminus \bigcup_{\zeta < \mu} A_\zeta : \mu < \eta \text{ and } \mu \text{ is odd}\} & \text{if } \eta \text{ is even,} \\ \bigcup \{A_\mu \setminus \bigcup_{\zeta < \mu} A_\zeta : \mu < \eta \text{ and } \mu \text{ is even}\} & \text{if } \eta \text{ is odd.} \end{cases}$$

Define  $A \in \text{D}_\eta(\Sigma_\xi^0(Z))$  if there exist  $A_\mu \in \Sigma_\xi^0(Z)$  for  $\mu < \eta$  such that  $A = \text{D}_\eta(A_\mu : \mu < \eta)$ .<sup>9</sup>

For an introduction to the topic of games, we refer the reader to [Ke, Section 20]. Here, we only want to give the precise definition of determinacy. A *play* of the game  $G(\omega, X)$  is described by the diagram

$$\begin{array}{cccc} \text{I} & a_0 & a_2 & \cdots \\ \text{II} & & a_1 & a_3 \cdots, \end{array}$$

in which  $a_n \in \omega$  for every  $n \in \omega$  and  $X \subseteq \omega^\omega$  is called the *payoff set*. We will say that Player I *wins* this play of the game  $G(\omega, X)$  if  $(a_0, a_1, \dots) \in X$ . Player II *wins* if Player I does not win.

A *strategy* for a player is a function  $\sigma : \omega^{<\omega} \rightarrow \omega$ . We will say that  $\sigma$  is a *winning strategy* for Player I if setting  $a_{2n} = \sigma(a_1, a_3, \dots, a_{2n-1})$  for each  $n$  makes Player I win for every  $(a_1, a_3, \dots) \in \omega^\omega$ . A winning strategy for Player II is defined similarly. We will say that the game  $G(\omega, X)$  (or simply the set  $X$ ) is *determined* if (exactly) one of the players has a winning strategy. The *Axiom of Determinacy* (briefly, AD) states that every  $X \subseteq \omega^\omega$  is determined.<sup>10</sup> We will denote by BP the axiom stating that for every Polish space  $Z$ , every subset of  $Z$  has the Baire property. Using the arguments in [Ke, Section 21.C], it can be shown that AD implies BP.

It is well-known that AD is incompatible with the Axiom of Choice (see [Je, Lemma 33.1]). This is the reason why, throughout this article, we will be working

<sup>9</sup>Notice that the definition of  $\text{D}_\eta(\Sigma_\xi^0(Z))$  does not change if one adds the requirement that  $(A_\mu : \mu < \eta)$  is  $\subseteq$ -increasing.

<sup>10</sup>Quite amusingly, Van Wesep referred to AD as a “frankly heretical postulate” (see [VW1, page 64]), and Steel deemed it “probably false” (see [St1, page 63]).

in  $\mathbf{ZF} + \mathbf{DC}$ .<sup>11</sup> Recall that the *principle of Dependent Choices* (briefly, DC) states that if  $R$  is a binary relation on a non-empty set  $A$  such that for every  $a \in A$  there exists  $b \in A$  such that  $(b, a) \in R$ , then there exists a sequence  $(a_0, a_1, \dots) \in A^\omega$  such that  $(a_{n+1}, a_n) \in R$  for every  $n \in \omega$ . An equivalent formulation of DC is that a relation  $R$  on a set  $A$  is well-founded iff there exists no sequence  $(a_0, a_1, \dots) \in A^\omega$  such that  $(a_{n+1}, a_n) \in R$  for every  $n \in \omega$  (see [Je, Lemma 5.5.ii]). Furthermore, DC implies the Countable Axiom of Choice (see [Je, Exercise 5.7]). To the reader who is unsettled by the lack of the full Axiom of Choice, we recommend [HR].

We conclude this section with some miscellaneous topological definitions and results. We will write  $X \approx Y$  to mean that the spaces  $X$  and  $Y$  are homeomorphic. A subset of a space is *clopen* if it is closed and open. A space is *zero-dimensional* if it is non-empty and it has a base consisting of clopen sets.<sup>12</sup> Given a function  $s : F \rightarrow 2$ , where  $F \subseteq \omega$  is finite, we will use the notation  $[s] = \{z \in 2^\omega : s \subseteq z\}$ . A space is *crowded* if it is non-empty and it has no isolated points. A space  $X$  is *Baire* if every non-empty open subset of  $X$  is non-meager in  $X$ . A space  $X$  is *meager* if  $X$  is a meager subset of  $X$ . Proposition 2.4 is a particular case of [FZ, Lemma 3.1] (see also [vE3, 1.12.1]). Proposition 2.5 is the reason why we refer to locally compact spaces as the “trivial exceptions”. Theorem 2.8 is a special case of [Te, Theorem 2.4] (see also [Me1, Theorem 2 and Appendix A] or [Me2, Theorem 3.2 and Appendix B]).

**Proposition 2.4** (Fitzpatrick, Zhou). *Let  $X$  be a homogeneous space. Then  $X$  is either a meager space or a Baire space.*

**Proposition 2.5.** *Let  $X$  be a zero-dimensional locally compact space. Then  $X$  is homogeneous iff  $X$  is discrete,  $X \approx 2^\omega$ , or  $X \approx \omega \times 2^\omega$ .*

*Proof.* The right-to-left implication is trivial. For the left-to-right implication, use the well-known characterization of  $2^\omega$  as the unique zero-dimensional crowded compact space (see [Ke, Theorem 7.4]).  $\square$

**Proposition 2.6.** *Let  $X$  be a zero-dimensional homogeneous space. If there exists a non-empty Polish  $U \in \Sigma_1^0(X)$  then  $X$  is Polish.*

*Proof.* Let  $U \in \Sigma_1^0(X)$  be non-empty and Polish. Since  $X$  is zero-dimensional, we can assume without loss of generality that  $U \in \Delta_1^0(X)$ . Let  $\mathcal{U} = \{h[U] : h \text{ is a homeomorphism of } X\}$ . Notice that  $\mathcal{U}$  is a cover of  $X$  because  $X$  is homogeneous and  $U$  is non-empty. Let  $\{U_n : n \in \omega\}$  be a countable subcover of  $\mathcal{U}$ . Define  $V_n = U_n \setminus \bigcup_{k < n} U_k$  for  $n \in \omega$ , and observe that each  $V_n$  is Polish. Since  $V_n \cap V_m = \emptyset$  whenever  $m \neq n$ , it follows from [Ke, Proposition 3.3.iii] that  $X = \bigcup_{n \in \omega} V_n$  is Polish.  $\square$

**Proposition 2.7.** *Assume BP. Let  $Z$  be a Polish space, and let  $X$  be a dense Baire subspace of  $Z$ . Then  $X$  is comeager in  $Z$ .*

*Proof.* Since  $X$  has the Baire property, we can write  $X = G \cup M$  by [Ke, Proposition 8.23.ii], where  $G \in \Pi_2^0(Z)$  and  $M$  is meager in  $Z$ . It will be enough to show that  $G$  is dense in  $Z$ . Assume, in order to get a contradiction, that there exists a non-empty open subset  $U$  of  $Z$  such that  $U \cap G = \emptyset$ . Observe that  $U \cap X$  is a non-empty

<sup>11</sup> The consistency of  $\mathbf{ZF} + \mathbf{DC} + \mathbf{AD}$  can be obtained under suitable large cardinal assumptions (see [Ka, Proposition 11.13] and [Ne]).

<sup>12</sup> The empty space has dimension  $-1$ .

open subset of  $X$  because  $X$  is dense in  $Z$ . Furthermore, using the density of  $X$  again, it is easy to see that  $M = M \cap X$  is meager in  $X$ . Since  $U \cap X \subseteq M$ , this contradicts the fact that  $X$  is a Baire space.  $\square$

**Theorem 2.8** (Terada). *Let  $X$  be a non-compact space. Assume that  $X$  has a base  $\mathcal{B} \subseteq \Delta_1^0(X)$  such that  $U \approx X$  for every  $U \in \mathcal{B}$ . Then  $X$  is strongly homogeneous.*

### 3. THE BASICS OF WADGE THEORY

The following simple lemma will allow us to generalize many Wadge-theoretic results from  $\omega^\omega$  to an arbitrary zero-dimensional Polish space. This approach has already appeared in [An, Section 5], where it is credited to Marcone. Recall that, given a space  $Z$  and  $W \subseteq Z$ , a *retraction* is a continuous function  $\rho : Z \rightarrow W$  such that  $\rho \upharpoonright W = \text{id}_W$ . By [Ke, Theorem 7.8], every zero-dimensional Polish space is homeomorphic to a closed subspace  $Z$  of  $\omega^\omega$ , and by [Ke, Proposition 2.8] there exists a retraction  $\rho : \omega^\omega \rightarrow Z$ .

**Lemma 3.1.** *Let  $Z \subseteq \omega^\omega$ , and let  $\rho : \omega^\omega \rightarrow Z$  be a retraction. Fix  $A, B \subseteq Z$ . Then  $A \leq B$  in  $Z$  iff  $\rho^{-1}[A] \leq \rho^{-1}[B]$  in  $\omega^\omega$ .*

*Proof.* If  $f : Z \rightarrow Z$  witnesses that  $A \leq B$  in  $Z$ , then  $f \circ \rho : \omega^\omega \rightarrow \omega^\omega$  will witness that  $\rho^{-1}[A] \leq \rho^{-1}[B]$  in  $\omega^\omega$ . On the other hand, if  $f : \omega^\omega \rightarrow \omega^\omega$  witnesses that  $\rho^{-1}[A] \leq \rho^{-1}[B]$  in  $\omega^\omega$ , then  $\rho \circ (f \upharpoonright Z) : Z \rightarrow Z$  will witness that  $A \leq B$  in  $Z$ .  $\square$

The following result (commonly known as ‘‘Wadge’s Lemma’’) shows that antichains with respect to  $\leq$  have size at most 2.

**Lemma 3.2** (Wadge). *Assume AD. Let  $Z$  be a zero-dimensional Polish space, and let  $A, B \subseteq Z$ . Then either  $A \leq B$  or  $Z \setminus B \leq A$ .*

*Proof.* For the case  $Z = \omega^\omega$ , see [Ke, proof of Theorem 21.14]. To obtain the full result from this particular case, use Lemma 3.1 and the remarks preceding it.  $\square$

**Theorem 3.3** (Martin, Monk). *Assume AD. Let  $Z$  be a zero-dimensional Polish space. Then the relation  $\leq$  on  $\mathcal{P}(Z)$  is well-founded.*

*Proof.* For the case  $Z = \omega^\omega$ , see [Ke, proof of Theorem 21.15]. To obtain the full result from this particular case, use Lemma 3.1 and the remarks preceding it.  $\square$

Given a zero-dimensional Polish space  $Z$ , define

$$W(Z) = \{\{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\} : \mathbf{\Gamma} \text{ is a Wadge class in } Z\}.$$

Given  $p, q \in W(Z)$ , define  $p < q$  if  $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$  for every  $\mathbf{\Gamma} \in p$  and  $\mathbf{\Lambda} \in q$ . Using the two previous results, one sees that the ordering  $<$  on  $W(Z)$  is a well-order. Therefore, there exists an order-isomorphism  $\phi : W(Z) \rightarrow \Theta$  for some ordinal  $\Theta$ .<sup>13</sup> The reason for the ‘‘1+’’ in the definition below is simply a matter of technical convenience (see [AHN, page 45]).

**Definition 3.4.** Let  $Z$  be a zero-dimensional Polish space, and let  $\mathbf{\Gamma}$  be a Wadge class in  $Z$ . Define

$$\|\mathbf{\Gamma}\| = 1 + \phi(\{\mathbf{\Gamma}, \check{\mathbf{\Gamma}}\}).$$

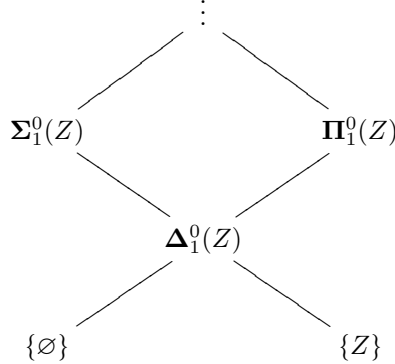
We will say that  $\|\mathbf{\Gamma}\|$  is the *Wadge-rank* of  $\mathbf{\Gamma}$ .

<sup>13</sup>For a characterization of  $\Theta$ , see [So, Definition 0.1 and Lemma 0.2].

It is easy to check that  $\{\{\emptyset\}, \{Z\}\}$  is the minimal element of  $W(Z)$ . Furthermore, elements of the form  $\{\Gamma, \check{\Gamma}\}$  for  $\Gamma \in \text{NSD}(Z)$  are always followed by  $\{\Delta\}$  for some  $\Delta \in \text{SD}(Z)$ , while elements of the form  $\{\Delta\}$  for  $\Delta \in \text{SD}(Z)$  are always followed by  $\{\Gamma, \check{\Gamma}\}$  for some  $\Gamma \in \text{NSD}(Z)$ . This was proved by Van Wesep for  $Z = \omega^\omega$  (see [VW1, Corollary to Theorem 2.1]), and it can be generalized to arbitrary uncountable zero-dimensional Polish spaces using Corollary 4.2 and the machinery of relativization that we will develop in Sections 6 to 8. Since these facts will not be needed in the remainder of the paper, we omit the proof.

In fact, as Proposition 6.5 (together with Theorem 8.3) will show, the ordering of the non-selfdual classes is independent of the space  $Z$ . However, the situation is more delicate for selfdual classes. For example, it follows easily from Corollary 4.2 that if  $\Gamma$  is a Wadge class in  $2^\omega$  such that  $\|\Gamma\|$  is a limit ordinal of countable cofinality, then  $\Gamma$  is non-selfdual. On the other hand, if  $\Gamma$  is a Wadge class in  $\omega^\omega$  such that  $\|\Gamma\|$  is a limit ordinal of countable cofinality, then  $\Gamma$  is selfdual (see [VW1, Corollary to Theorem 2.1] again).

The collection of all Wadge classes on a given space  $Z$ , ordered by  $\subseteq$ , is known as the *Wadge hierarchy*. The following diagram shows how this hierarchy looks like when  $Z$  is a zero-dimensional Polish space.<sup>14</sup>



We conclude this section with an elementary result, which shows that clopen sets are “neutral sets” for Wadge-reduction (the simple proof is left to the reader). By this we mean that, apart from trivial exceptions, intersections or unions with these sets do not change the Wadge class. In Section 12, we will prove more sophisticated closure properties.

**Proposition 3.5.** *Let  $Z$  be a space, let  $\Gamma$  be a Wadge class in  $Z$ , and let  $A \in \Gamma$ .*

- *Assume that  $\Gamma \neq \{Z\}$ . Then  $A \cap V \in \Gamma$  for every  $V \in \Delta_1^0(Z)$ .*
- *Assume that  $\Gamma \neq \{\emptyset\}$ . Then  $A \cup V \in \Gamma$  for every  $V \in \Delta_1^0(Z)$ .*

#### 4. THE ANALYSIS OF SELFDUAL SETS

In this section we will simply collect well-known results which show that every selfdual set can be built using non-selfdual sets of lower complexity (apply Corollary 4.2 with  $V = Z$ ). We will refer to the proof of [MR, Theorem 5.3], which in turn generalizes [AM, Theorem 16] (see also [Lo2, Lemma 7.3.4]).

<sup>14</sup>Notice that this hierarchy can collapse rather soon when  $Z$  is countable. For example, when  $Z$  is the discrete space  $\omega$ , the Wadge hierarchy consists only of the three bottom classes.



**Theorem 4.1.** *Assume BP. Let  $Z$  be a zero-dimensional Polish space, let  $V \in \Delta_1^0(Z)$ , and let  $A$  be a selfdual subset of  $Z$ . Assume that  $A \notin \Delta_1^0(Z)$ . Then there exist pairwise disjoint  $V_n \in \Delta_1^0(V)$  for  $n \in \omega$  such that  $\bigcup_{n \in \omega} V_n = V$  and  $A \cap V_n < A$  in  $Z$  for each  $n$ .*

*Proof.* This is proved like [MR, Theorem 5.3], with  $Z$  instead of  $\omega^\omega$  (which is denoted by  $\mathbb{R}$  there) and  $D_0 = V$ , where  $\mathcal{F}$  is the collection of all continuous  $f : Z \rightarrow Z$  and  $\Delta_{\mathcal{F}} = \Delta_1^0(Z)$ .  $\square$

**Corollary 4.2.** *Assume AD. Let  $Z$  be a zero-dimensional Polish space, let  $V \in \Delta_1^0(Z)$ , and let  $A$  be a selfdual subset of  $Z$ . Then there exist pairwise disjoint  $V_n \in \Delta_1^0(V)$  and non-selfdual  $A_n < A$  in  $Z$  for  $n \in \omega$  such that  $\bigcup_{n \in \omega} V_n = V$  and  $\bigcup_{n \in \omega} (A_n \cap V_n) = A \cap V$ .*

*Proof.* As one can easily check, it will be enough to show that there exist pairwise disjoint  $V_n \in \Delta_1^0(V)$  for  $n \in \omega$  such that  $\bigcup_{n \in \omega} V_n = V$  and for every  $n \in \omega$  either  $A \cap V_n \in \Delta_1^0(Z)$  or  $A \cap V_n$  is non-selfdual in  $Z$ . If this were not the case, then, using Theorem 4.1, one could recursively construct a strictly  $\leq$ -decreasing sequence of subsets of  $Z$ , which would contradict Theorem 3.3.  $\square$

## 5. BASIC FACTS ON HAUSDORFF OPERATIONS

For a history of the following important notion, see [Ha, page 583]. For a modern survey, we recommend [Za]. Most of the proofs in this section are straightforward, hence we leave them to the reader.

**Definition 5.1.** Given a set  $Z$  and  $D \subseteq \mathcal{P}(\omega)$ , define

$$\mathcal{H}_D(A_0, A_1, \dots) = \{x \in Z : \{n \in \omega : x \in A_n\} \in D\}$$

whenever  $A_0, A_1, \dots \subseteq Z$ . Functions of this form are called *Hausdorff operations* (or  $\omega$ -ary *Boolean operations*).

Of course, the function  $\mathcal{H}_D$  depends on the set  $Z$ , but what  $Z$  is will usually be clear from the context. In case there might be uncertainty about the ambient space, we will use the notation  $\mathcal{H}_D^Z$ . Notice that, once  $D$  is specified, the corresponding Hausdorff operation simultaneously defines functions  $\mathcal{P}(Z)^\omega \rightarrow \mathcal{P}(Z)$  for every  $Z$ .

The following proposition lists the most basic properties of Hausdorff operations. Given  $n \in \omega$ , define  $S_n = \{A \subseteq \omega : n \in A\}$ .

**Proposition 5.2.** *Let  $I$  be a non-empty set, and let  $D_i \subseteq \mathcal{P}(\omega)$  for every  $i \in I$ . Fix an ambient set  $Z$  and  $A_0, A_1, \dots \subseteq Z$ .*

- $\mathcal{H}_{S_n}(A_0, A_1, \dots) = A_n$  for all  $n \in \omega$ .
- $\bigcap_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \dots) = \mathcal{H}_D(A_0, A_1, \dots)$ , where  $D = \bigcap_{i \in I} D_i$ .
- $\bigcup_{i \in I} \mathcal{H}_{D_i}(A_0, A_1, \dots) = \mathcal{H}_D(A_0, A_1, \dots)$ , where  $D = \bigcup_{i \in I} D_i$ .
- $Z \setminus \mathcal{H}_D(A_0, A_1, \dots) = \mathcal{H}_{\mathcal{P}(\omega) \setminus D}(A_0, A_1, \dots)$  for all  $D \subseteq \mathcal{P}(\omega)$ .

The point of the above proposition is that any operation obtained by combining unions, intersections and complements can be expressed as a Hausdorff operation. For example, if  $D = \bigcup_{n \in \omega} (S_{2n+1} \setminus S_{2n})$ , then  $\mathcal{H}_D(A_0, A_1, \dots) = \bigcup_{n \in \omega} (A_{2n+1} \setminus A_{2n})$ .

The following proposition shows that the composition of Hausdorff operations is again a Hausdorff operation. We will assume that a bijection  $\pi : \omega \times \omega \rightarrow \omega$  has been fixed, and use the notation  $\langle m, n \rangle = \pi(m, n)$ .

**Proposition 5.3.** *Let  $Z$  be a set, let  $D \subseteq \mathcal{P}(\omega)$  and  $E_m \subseteq \mathcal{P}(\omega)$  for  $m \in \omega$ . Then there exists  $F \subseteq \mathcal{P}(\omega)$  such that*

$$\mathcal{H}_D(B_0, B_1, \dots) = \mathcal{H}_F(A_0, A_1, \dots)$$

for all  $A_0, A_1, \dots \subseteq Z$ , where  $B_m = \mathcal{H}_{E_m}(A_{\langle m,0 \rangle}, A_{\langle m,1 \rangle}, \dots)$ .

*Proof.* Define  $z \in F$  if  $\{m \in \omega : \{n \in \omega : \langle m, n \rangle \in z\} \in E_m\} \in D$ . The rest of the proof is a straightforward verification.  $\square$

We conclude this section with a result that will easily imply the fundamental Lemma 6.4.

**Proposition 5.4.** *Let  $Z$  and  $W$  be sets, let  $D \subseteq \mathcal{P}(\omega)$ , let  $A_0, A_1, \dots \subseteq Z$  and  $B_0, B_1, \dots \subseteq W$ .*

- (1)  $W \cap \mathcal{H}_D^Z(A_0, A_1, \dots) = \mathcal{H}_D^W(A_0 \cap W, A_1 \cap W, \dots)$  whenever  $W \subseteq Z$ .
- (2)  $f^{-1}[\mathcal{H}_D(B_0, B_1, \dots)] = \mathcal{H}_D(f^{-1}[B_0], f^{-1}[B_1], \dots)$  for all  $f : Z \rightarrow W$ .
- (3)  $f[\mathcal{H}_D(A_0, A_1, \dots)] = \mathcal{H}_D(f[A_0], f[A_1], \dots)$  for all bijections  $f : Z \rightarrow W$ .

## 6. WADGE CLASSES AND HAUSDORFF OPERATIONS

When one tries to give a systematic exposition of Wadge theory, it soon becomes apparent that it would be very useful to be able to talk about “abstract” Wadge classes, as opposed to Wadge classes in a particular space. More precisely, given a Wadge class  $\Gamma$  in some space  $Z$ , one would like to find a way to define what a “ $\Gamma$  subset of  $W$ ” is, for every other space  $W$ , while of course preserving suitable coherence properties. It turns out that Hausdorff operations allow us to do exactly that in a rather elegant way, provided that  $\Gamma$  is a non-selfdual Wadge class,  $Z$  and  $W$  are uncountable zero-dimensional Polish spaces, and AD holds (see also the discussion in Section 3). For an early instance of this idea, see [LSR2, Theorem 4.2].<sup>15</sup> The following is the crucial definition. In fact, the aim of this section and the next two is to show that the classes  $\Gamma_D(Z)$  have nice properties (see Lemma 6.4 and Proposition 6.5), and that, under AD, they are exactly the non-selfdual Wadge classes on  $Z$  (see Theorem 8.3).

**Definition 6.1.** Given a space  $Z$  and  $D \subseteq \mathcal{P}(\omega)$ , define

$$\Gamma_D(Z) = \{\mathcal{H}_D(A_0, A_1, \dots) : A_n \in \Sigma_1^0(Z) \text{ for every } n \in \omega\}.$$

As examples (that will be useful later), consider the following two simple propositions.

**Proposition 6.2.** *Let  $1 \leq \eta < \omega_1$ . Then there exists  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_D(Z) = D_\eta(\Sigma_1^0(Z))$  for every space  $Z$ .*

*Proof.* This follows from Propositions 5.2 and 5.3 (in case  $\eta > \omega$ , use a bijection  $\pi : \eta \rightarrow \omega$ ).  $\square$

**Proposition 6.3.** *Let  $1 \leq \xi < \omega_1$ . Then there exists  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_D(Z) = \Sigma_\xi^0(Z)$  for every space  $Z$ .*

*Proof.* This can be proved by induction on  $\xi$ , using Propositions 5.2 and 5.3.  $\square$

<sup>15</sup>This result is limited to the Borel context. On the other hand, the ambient space is allowed to be analytic, as opposed to Polish.

Next, we obtain a very useful lemma, which shows that this notion behaves well with respect to subspaces and continuous functions. This lemma is essentially what we refer to when we speak about the “machinery of relativization”. It extends (and is inspired by) [vE4, Lemma 2.3].

**Lemma 6.4.** *Let  $Z$  and  $W$  be spaces, and let  $D \subseteq \mathcal{P}(\omega)$ .*

- (1) *Assume that  $W \subseteq Z$ . Then  $B \in \Gamma_D(W)$  iff there exists  $A \in \Gamma_D(Z)$  such that  $B = A \cap W$ .*
- (2) *If  $f : Z \rightarrow W$  is continuous and  $B \in \Gamma_D(W)$  then  $f^{-1}[B] \in \Gamma_D(Z)$ .*
- (3) *If  $h : Z \rightarrow W$  is a homeomorphism then  $A \in \Gamma_D(Z)$  iff  $h[A] \in \Gamma_D(W)$ .*

*Proof.* This is a straightforward consequence of Proposition 5.4.  $\square$

The following simple result, together with Theorem 8.3, shows that the ordering of the non-selfdual Wadge classes is independent of the ambient space  $Z$  (provided that AD holds).

**Proposition 6.5.** *Let  $Z$  and  $W$  be zero-dimensional spaces that contain a copy of  $2^\omega$ , and let  $D, E \subseteq \mathcal{P}(\omega)$ . Then  $\Gamma_D(Z) \subseteq \Gamma_E(Z)$  iff  $\Gamma_D(W) \subseteq \Gamma_E(W)$ .*

*Proof.* Assume that  $\Gamma_D(Z) \subseteq \Gamma_E(Z)$ . Since  $Z$  contains a copy of  $2^\omega$  and  $W$  is zero-dimensional, we see that  $Z$  contains a copy of  $W$ . Using Lemma 6.4.3, we can assume without loss of generality that  $W \subseteq Z$ . Then

$$\Gamma_D(W) = \{A \cap W : A \in \Gamma_D(Z)\} \subseteq \{A \cap W : A \in \Gamma_E(Z)\} = \Gamma_E(W),$$

where the first and last equalities hold by Lemma 6.4.1. The proof of the other implication is similar.  $\square$

## 7. UNIVERSAL SETS

The aim of this section is to prove the easier half of Theorem 8.3 (namely, Theorem 7.5). The ideas presented here are well-known, but since we could not find a satisfactory reference, we will give all the details. Our approach is inspired by [Ke, Section 22.A].

**Definition 7.1.** Let  $Z$  and  $W$  be spaces, and let  $D \subseteq \mathcal{P}(\omega)$ . Given  $U \subseteq W \times Z$  and  $x \in W$ , let  $U_x = \{y \in Z : (x, y) \in U\}$  denote the vertical section of  $U$  above  $x$ . We will say that  $U \subseteq W \times Z$  is a  $W$ -universal set for  $\Gamma_D(Z)$  if the following two conditions hold:

- $U \in \Gamma_D(W \times Z)$ ,
- $\{U_x : x \in W\} = \Gamma_D(Z)$ .

Notice that, by Proposition 6.3, the above yields the definition of a  $W$ -universal set for  $\Sigma_\xi^0(Z)$  whenever  $1 \leq \xi < \omega_1$ . Furthermore, this definition agrees with [Ke, Definition 22.2].

**Proposition 7.2.** *Let  $Z$  be a space, and let  $D \subseteq \mathcal{P}(\omega)$ . Then there exists a  $2^\omega$ -universal set for  $\Gamma_D(Z)$ .*

*Proof.* By [Ke, Theorem 22.3], we can fix a  $2^\omega$ -universal set  $U$  for  $\Sigma_1^0(Z)$ . Let  $h : 2^\omega \rightarrow (2^\omega)^\omega$  be a homeomorphism, and let  $\pi_n : (2^\omega)^\omega \rightarrow 2^\omega$  be the projection on the  $n$ -th coordinate for  $n \in \omega$ . Notice that, given any  $n \in \omega$ , the function  $f_n : 2^\omega \times Z \rightarrow 2^\omega \times Z$  defined by  $f_n(x, y) = (\pi_n(h(x)), y)$  is continuous. Let  $V_n = f_n^{-1}[U]$  for each  $n$ , and observe that each  $V_n \in \Sigma_1^0(2^\omega \times Z)$ . Set  $V = \mathcal{H}_D(V_0, V_1, \dots)$ .

We claim that  $V$  is a  $2^\omega$ -universal set for  $\mathbf{\Gamma}_D(Z)$ . It is clear that  $V \in \mathbf{\Gamma}_D(2^\omega \times Z)$ . Furthermore, using Lemma 6.4, one can easily check that  $V_x \in \mathbf{\Gamma}_D(Z)$  for every  $x \in 2^\omega$ . To complete the proof, fix  $A \in \mathbf{\Gamma}_D(Z)$ . Let  $A_0, A_1, \dots \in \Sigma_1^0(Z)$  be such that  $A = \mathcal{H}_D(A_0, A_1, \dots)$ . Since  $U$  is  $2^\omega$ -universal, we can fix  $z_n \in 2^\omega$  such that  $U_{z_n} = A_n$  for every  $n \in \omega$ . Set  $z = h^{-1}(z_0, z_1, \dots)$ . It is straightforward to verify that  $V_z = A$ .  $\square$

**Corollary 7.3.** *Let  $Z$  be a space that contains a copy of  $2^\omega$ , and let  $D \subseteq \mathcal{P}(\omega)$ . Then there exists a  $Z$ -universal set for  $\mathbf{\Gamma}_D(Z)$ .*

*Proof.* By Proposition 7.2, we can fix a  $2^\omega$ -universal set  $U$  for  $\mathbf{\Gamma}_D(Z)$ . Let  $W \subseteq Z$  be such that  $W \approx 2^\omega$ , and fix a homeomorphism  $h : 2^\omega \rightarrow W$ . Notice that  $(h \times \text{id}_Z)[U] \in \mathbf{\Gamma}_D(W \times Z)$  by Lemma 6.4.3. Therefore, by Lemma 6.4.1, there exists  $V \in \mathbf{\Gamma}_D(Z \times Z)$  such that  $V \cap (W \times Z) = (h \times \text{id}_Z)[U]$ . Using Lemma 6.4 again, one can easily check that  $V$  is a  $Z$ -universal set for  $\mathbf{\Gamma}_D(Z)$ .  $\square$

**Lemma 7.4.** *Let  $Z$  be a space, and let  $D \subseteq \mathcal{P}(\omega)$ . Assume that there exists a  $Z$ -universal set for  $\mathbf{\Gamma}_D(Z)$ . Then  $\mathbf{\Gamma}_D(Z)$  is non-selfdual.*

*Proof.* Fix a  $Z$ -universal set  $U \subseteq Z \times Z$  for  $\mathbf{\Gamma}_D(Z)$ . Assume, in order to get a contradiction, that  $\mathbf{\Gamma}_D(Z)$  is selfdual. Let  $f : Z \rightarrow Z \times Z$  be the function defined by  $f(x) = (x, x)$ , and observe that  $f$  is continuous. Since  $f^{-1}[U] \in \mathbf{\Gamma}_D(Z) = \check{\mathbf{\Gamma}}_D(Z)$ , we see that  $Z \setminus f^{-1}[U] \in \mathbf{\Gamma}_D(Z)$ . Therefore, since  $U$  is  $Z$ -universal, we can fix  $z \in Z$  such that  $U_z = Z \setminus f^{-1}[U]$ . If  $z \in U_z$  then  $f(z) = (z, z) \in U$  by the definition of  $U_z$ , contradicting the fact that  $U_z = Z \setminus f^{-1}[U]$ . On the other hand, if  $z \notin U_z$  then  $f(z) = (z, z) \notin U$  by the definition of  $U_z$ , contradicting the fact that  $Z \setminus U_z = f^{-1}[U]$ .  $\square$

The case  $Z = \omega^\omega$  of the following result is [VW1, Proposition 5.0.3], and it is credited to Addison by Van Wesep.

**Theorem 7.5.** *Let  $Z$  be a zero-dimensional space that contains a copy of  $2^\omega$ , and let  $D \subseteq \mathcal{P}(\omega)$ . Then  $\mathbf{\Gamma}_D(Z) \in \text{NSD}(Z)$ .*

*Proof.* The fact that  $\mathbf{\Gamma}_D(Z)$  is non-selfdual follows from Corollary 7.3 and Lemma 7.4. Therefore, it will be enough to show that  $\mathbf{\Gamma}_D(Z)$  is a Wadge class. By Proposition 7.2, we can fix a  $2^\omega$ -universal set  $U \subseteq 2^\omega \times Z$  for  $\mathbf{\Gamma}_D(Z)$ . Let  $W \subseteq Z$  be such that  $W \approx 2^\omega \times Z$ , and fix a homeomorphism  $h : 2^\omega \times Z \rightarrow W$ . By Lemma 6.4, we can fix  $A \in \mathbf{\Gamma}_D(Z)$  such that  $A \cap W = h[U]$ . We claim that  $\mathbf{\Gamma}_D(Z) = [A]$ . The inclusion  $\supseteq$  follows from Lemma 6.4.2. In order to prove the other inclusion, pick  $B \in \mathbf{\Gamma}_D(Z)$ . Since  $U$  is  $2^\omega$ -universal, we can fix  $z \in 2^\omega$  such that  $B = U_z$ . Consider the function  $f : Z \rightarrow 2^\omega \times Z$  defined by  $f(x) = (z, x)$ , and observe that  $f$  is continuous. It is straightforward to check that  $h \circ f : Z \rightarrow Z$  witnesses that  $B \leq A$  in  $Z$ .  $\square$

## 8. VAN WESEP'S THEOREM

The following is one of the main results of Van Wesep's doctoral thesis (see [VW1, Theorem 5.3.1], whose proof also employs results of Steel from [St1] and results of Radin), and it will allow us to obtain the harder half of Theorem 8.3. Notice how Corollary 8.2 guarantees that every non-selfdual Wadge class is amenable to the machinery of relativization.

**Theorem 8.1** (Van Wesep). *Assume AD. For every  $\Gamma \in \text{NSD}(\omega^\omega)$  there exists  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma = \Gamma_D(\omega^\omega)$ .*

**Corollary 8.2.** *Assume AD. Let  $Z$  be a zero-dimensional Polish space, and let  $\Gamma \in \text{NSD}(Z)$ . Then there exists  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma = \Gamma_D(Z)$ .*

*Proof.* By [Ke, Theorem 7.8], there exists a closed  $W \subseteq \omega^\omega$  such that  $Z \approx W$ . Therefore, using Lemma 6.4.3, we can assume without loss of generality that  $Z$  is a closed subspace of  $\omega^\omega$ . Hence, by [Ke, Proposition 2.8], we can fix a retraction  $\rho : \omega^\omega \rightarrow Z$ . Let  $A \subseteq Z$  be such that  $\Gamma = [A]$ . Set  $B = \rho^{-1}[A]$ , and let  $\Lambda = [B]$  be the Wadge class generated by  $B$  in  $\omega^\omega$ .

Using Lemma 3.1, it is easy to see that  $\Lambda \in \text{NSD}(\omega^\omega)$ . Therefore, by Theorem 8.1, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\Lambda = \Gamma_D(\omega^\omega)$ . We claim that  $\Gamma = \Gamma_D(Z)$ . Notice that  $A = B \cap Z \in \Gamma_D(Z)$  by Lemma 6.4.1, hence  $\Gamma \subseteq \Gamma_D(Z)$  by Lemma 6.4.2. Finally, to see that  $\Gamma_D(Z) \subseteq \Gamma$ , pick  $C \in \Gamma_D(Z)$ . Observe that  $\rho^{-1}[C] \in \Gamma_D(\omega^\omega) = \Lambda$  by Lemma 6.4.2. This means that  $\rho^{-1}[C] \leq B = \rho^{-1}[A]$  in  $\omega^\omega$ , hence  $C \leq A$  in  $Z$  by Lemma 3.1. So  $C \in [A] = \Gamma$ , which concludes the proof.  $\square$

Finally, we can “put everything together” and state the full result promised in the introduction to Section 6.

**Theorem 8.3.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space. Then*

$$\text{NSD}(Z) = \{\Gamma_D(Z) : D \subseteq \mathcal{P}(\omega)\}$$

*Proof.* This follows immediately from Theorem 7.5 and Corollary 8.2.  $\square$

## 9. BASIC FACTS ON EXPANSIONS

The following notion is essentially due to Wadge (see [Wa1, Chapter IV]), and it is inspired by work of Kuratowski. Recall that, given  $1 \leq \xi < \omega_1$  and spaces  $Z$  and  $W$ , a function  $f : Z \rightarrow W$  is  $\Sigma_\xi^0$ -measurable if  $f^{-1}[U] \in \Sigma_\xi^0(Z)$  for every  $U \in \Sigma_1^0(W)$ .

**Definition 9.1.** Let  $Z$  be a space, and let  $\xi < \omega_1$ . Given  $\Gamma \subseteq \mathcal{P}(Z)$ , define

$$\Gamma^{(\xi)} = \{f^{-1}[A] : A \in \Gamma \text{ and } f : Z \rightarrow Z \text{ is } \Sigma_{1+\xi}^0\text{-measurable}\}.$$

We will refer to  $\Gamma^{(\xi)}$  as an *expansion* of  $\Gamma$ .

The following is the corresponding definition in the context of Hausdorff operations. Corollary 10.4 below shows that this is in fact the “right” definition.

**Definition 9.2.** Let  $Z$  be a space, let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ . Define

$$\Gamma_D^{(\xi)}(Z) = \{\mathcal{H}_D(A_0, A_1, \dots) : A_n \in \Sigma_{1+\xi}^0(Z) \text{ for every } n \in \omega\}.$$

As an example (that will be useful later), consider the following simple observation.

**Proposition 9.3.** *Let  $1 \leq \eta < \omega_1$ . Then there exists  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_D^{(\xi)}(Z) = \text{D}_\eta(\Sigma_{1+\xi}^0(Z))$  for every space  $Z$  and every  $\xi < \omega_1$ .*

*Proof.* This is proved like Proposition 6.2 (in fact, the same  $D$  will work).  $\square$

The following proposition shows that Definition 9.2 actually fits in the context provided by Section 6.

**Proposition 9.4.** *Let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ . Then there exists  $E \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_D^{(\xi)}(Z) = \Gamma_E(Z)$  for every space  $Z$ .*

*Proof.* This is proved by combining Propositions 6.3 and 5.3.  $\square$

**Corollary 9.5.** *Let  $Z$  be an uncountable zero-dimensional Polish space, let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ . Then  $\Gamma_D^{(\xi)}(Z) \in \text{NSD}(Z)$ .*

*Proof.* This is proved by combining Proposition 9.4 and Theorem 7.5.  $\square$

The following useful result is the analogue of Lemma 6.4 in the present context.

**Lemma 9.6.** *Let  $Z$  and  $W$  be spaces, let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ .*

- (1) *Assume that  $W \subseteq Z$ . Then  $B \in \Gamma_D^{(\xi)}(W)$  iff there exists  $A \in \Gamma_D^{(\xi)}(Z)$  such that  $B = A \cap W$ .*
- (2) *If  $f : Z \rightarrow W$  is continuous and  $B \in \Gamma_D^{(\xi)}(W)$  then  $f^{-1}[B] \in \Gamma_D^{(\xi)}(Z)$ .*
- (3) *If  $f : Z \rightarrow W$  is  $\Sigma_{1+\xi}^0$ -measurable and  $B \in \Gamma_D(W)$  then  $f^{-1}[B] \in \Gamma_D^{(\xi)}(Z)$ .*
- (4) *If  $h : Z \rightarrow W$  is a homeomorphism then  $A \in \Gamma_D^{(\xi)}(Z)$  iff  $h[A] \in \Gamma_D^{(\xi)}(W)$ .*

*Proof.* This is a straightforward consequence of Proposition 5.4.  $\square$

## 10. KURATOWSKI'S TRANSFER THEOREM

The aim of this section is to collect the tools needed to successfully employ the notion of expansion. For example, Corollary 10.3 will be a crucial ingredient in the proof of Theorem 11.3. A stronger form of Theorem 10.1 appears as [Lo2, Theorem 7.1.6], where it is called ‘‘Kuratowski’s transfer theorem’’.

**Theorem 10.1** (Kuratowski). *Let  $(Z, \tau)$  be a Polish space, let  $1 < \xi < \omega_1$ , and let  $\mathcal{A} \subseteq \Sigma_\xi^0(Z, \tau)$  be countable. Then there exists a zero-dimensional Polish topology  $\sigma$  on the set  $Z$  such that  $\tau \subseteq \sigma \subseteq \Sigma_\xi^0(Z, \tau)$  and  $\mathcal{A} \subseteq \sigma$ .*

*Proof.* This follows from [Ke, Exercise 22.20], using the fact that every element of  $\Sigma_\xi^0(Z, \tau)$  can be written as a countable union of elements of  $\Delta_\xi^0(Z, \tau)$ .  $\square$

**Corollary 10.2.** *Let  $Z$  be a zero-dimensional Polish space, let  $1 \leq \xi < \omega_1$ , and let  $\mathcal{A} \subseteq \Sigma_\xi^0(Z)$  be countable. Then there exists a zero-dimensional Polish space  $W$  and a  $\Sigma_\xi^0$ -measurable bijection  $f : Z \rightarrow W$  such that  $f[A] \in \Sigma_1^0(W)$  for every  $A \in \mathcal{A}$ .*

*Proof.* The case  $\xi = 1$  is trivial, so assume that  $\xi > 1$ . The space  $W$  is simply the set  $Z$  with the finer topology given by Theorem 10.1, while  $f = \text{id}_Z$ .  $\square$

**Corollary 10.3.** *Let  $Z$  be a zero-dimensional Polish space, let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ . Assume that  $\mathcal{A} \subseteq \Gamma_D^{(\xi)}(Z)$  and  $\mathcal{B} \subseteq \Sigma_{1+\xi}^0(Z)$  are countable. Then there exists a zero-dimensional Polish space  $W$  and a  $\Sigma_{1+\xi}^0$ -measurable bijection  $f : Z \rightarrow W$  such that  $f[A] \in \Gamma_D(W)$  for every  $A \in \mathcal{A}$  and  $f[B] \in \Sigma_1^0(W)$  for every  $B \in \mathcal{B}$ .*

*Proof.* Let  $\mathcal{A} = \{A_m : m \in \omega\}$ . Given  $m \in \omega$ , fix  $A_{m,n} \in \Sigma_{1+\xi}^0(Z)$  for  $n \in \omega$  such that  $A_m = \mathcal{H}_D(A_{m,0}, A_{m,1}, \dots)$ . Define  $\mathcal{C} = \{A_{m,n} : m, n \in \omega\} \cup \mathcal{B}$ . By Corollary 10.2, we can fix a Polish space  $W$  and a  $\Sigma_{1+\xi}^0$ -measurable bijection  $f : Z \rightarrow W$  such that  $f[C] \in \Sigma_1^0(W)$  for every  $C \in \mathcal{C}$ . It remains to observe that

$$f[A_m] = f[\mathcal{H}_D(B_{m,0}, B_{m,1}, \dots)] = \mathcal{H}_D(f[B_{m,0}], f[B_{m,1}], \dots) \in \Gamma_D(W)$$

for every  $m \in \omega$ , where the second equality follows from Proposition 5.4.3.  $\square$

**Corollary 10.4.** *Let  $Z$  be an uncountable zero-dimensional Polish space, let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ . Then  $\mathbf{\Gamma}_D(Z)^{(\xi)} = \mathbf{\Gamma}_D^{(\xi)}(Z)$ .*

*Proof.* The inclusion  $\mathbf{\Gamma}_D(Z)^{(\xi)} \subseteq \mathbf{\Gamma}_D^{(\xi)}(Z)$  follows from Lemma 9.6.3. In order to prove the other inclusion, pick  $A \in \mathbf{\Gamma}_D^{(\xi)}(Z)$ . By Corollary 10.3, we can fix a zero-dimensional Polish space  $W$  and a  $\Sigma_{1+\xi}^0$ -measurable bijection  $f : Z \rightarrow W$  such that  $f[A] \in \mathbf{\Gamma}_D(W)$ . Since  $Z$  contains a copy of  $2^\omega$  and  $W$  is zero-dimensional, using Lemma 6.4.3 we can assume without loss of generality that  $W$  is a subspace of  $Z$ , so that  $f : Z \rightarrow Z$ . By Lemma 6.4.1, we can fix  $B \in \mathbf{\Gamma}_D(Z)$  such that  $B \cap W = f[A]$ . It is easy to check that  $A = f^{-1}[B]$ , which concludes the proof.  $\square$

**Corollary 10.5.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, and let  $\xi < \omega_1$ . Then  $\mathbf{\Gamma}^{(\xi)} \in \text{NSD}(Z)$  for every  $\mathbf{\Gamma} \in \text{NSD}(Z)$ .*

*Proof.* This follows from Corollary 8.2, Corollary 10.4, and Proposition 9.5.  $\square$

**Corollary 10.6.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, and let  $\xi < \omega_1$ . Then  $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$  iff  $\mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}$  for every  $\mathbf{\Gamma}, \mathbf{\Lambda} \in \text{NSD}(Z)$ .*

*Proof.* The fact that  $\mathbf{\Gamma} \subseteq \mathbf{\Lambda}$  implies  $\mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}$  is a trivial consequence of the definition of expansion. Now fix  $\mathbf{\Gamma}, \mathbf{\Lambda} \in \text{NSD}(Z)$  such that  $\mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}$ . Assume, in order to get a contradiction, that  $\mathbf{\Gamma} \not\subseteq \mathbf{\Lambda}$ . Then  $\check{\mathbf{\Lambda}} \subseteq \mathbf{\Gamma}$  by Lemma 3.2, hence

$$\widetilde{\check{\mathbf{\Lambda}}}^{(\xi)} = \check{\mathbf{\Lambda}}^{(\xi)} \subseteq \mathbf{\Gamma}^{(\xi)} \subseteq \mathbf{\Lambda}^{(\xi)}.$$

Since  $\mathbf{\Lambda}^{(\xi)}$  is non-selfdual by Corollary 10.5, this is a contradiction.  $\square$

## 11. THE EXPANSION THEOREM

The main result of this section is Theorem 11.3, which will be a crucial tool in obtaining the closure properties in the next section, and will be referred to as the expansion theorem. The proof given here is essentially the same as [Lo2, proof of Theorem 7.3.9.ii]. This result can be traced back to [LSR1, Théorème 8], which is however limited to the Borel context. We need to introduce the following two notions. The first is [Wa1, Definition D1], while the second is taken from [LSR1] (see also [Lo2, Section 7.3.4]).<sup>16</sup>

**Definition 11.1** (Wadge). Let  $Z$  be a space, let  $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$ , and let  $\xi < \omega_1$ . Define  $\text{PU}_\xi(\mathbf{\Gamma})$  to be the collection of all sets of the form

$$\bigcup_{n \in \omega} (A_n \cap V_n),$$

where each  $A_n \in \mathbf{\Gamma}$ , each  $V_n \in \mathbf{\Delta}_{1+\xi}^0(Z)$ , the  $V_n$  are pairwise disjoint, and  $\bigcup_{n \in \omega} V_n = Z$ . A set in this form is called a *partitioned union* of sets in  $\mathbf{\Gamma}$ .

Notice that the sets  $V_n$  in the above definition are not required to be non-empty. It is easy to check that  $\text{PU}_\xi(\mathbf{\Gamma})$  is continuously closed whenever  $\mathbf{\Gamma}$  is. Furthermore, it is clear that

$$\mathbf{\Gamma} \subseteq \text{PU}_\xi(\mathbf{\Gamma}) \subseteq \text{PU}_\eta(\mathbf{\Gamma})$$

whenever  $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$  and  $\xi \leq \eta < \omega_1$ .

**Definition 11.2** (Louveau, Saint-Raymond). Let  $Z$  be a space, let  $\mathbf{\Gamma} \subseteq \mathcal{P}(Z)$  be continuously closed, and let  $\xi < \omega_1$ . Define

<sup>16</sup>In [LSR1], the notation  $\mathbf{\Delta}_{1+\xi}^0$ -PU is used instead of  $\text{PU}_\xi$ , and  $\lambda_C$  is used instead of  $\ell$ .

- $\ell(\Gamma) \geq \xi$  if  $\text{PU}_\xi(\Gamma) = \Gamma$ ,
- $\ell(\Gamma) = \xi$  if  $\ell(\Gamma) \geq \xi$  and  $\ell(\Gamma) \not\geq \xi + 1$ ,
- $\ell(\Gamma) = \omega_1$  if  $\ell(\Gamma) \geq \eta$  for every  $\eta < \omega_1$ .

We refer to  $\ell(\Gamma)$  as the *level* of  $\Gamma$ .

As a trivial example, observe that  $\ell(\{\emptyset\}) = \ell(\{Z\}) = \omega_1$ . Using the definition of Wadge-reduction, it is a simple exercise to see that  $\ell(\Gamma) \geq 0$  for every Wadge class  $\Gamma$ . We remark that it is not clear at this point whether for every non-selfdual Wadge class  $\Gamma$  there exists  $\xi \leq \omega_1$  such that  $\ell(\Gamma) = \xi$ .<sup>17</sup> This happens to be true under AD, and it can be proved using techniques from [Lo2] (see [CMM, Corollary 18.3]). However, this fact is never used in this article.

**Theorem 11.3.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, let  $\Gamma \in \text{NSD}(Z)$ , and let  $\xi < \omega_1$ . Then the following conditions are equivalent:*

- (1)  $\ell(\Gamma) \geq \xi$ ,
- (2)  $\Gamma = \Lambda^{(\xi)}$  for some  $\Lambda \in \text{NSD}(Z)$ .

*Proof.* In order to show that (1)  $\rightarrow$  (2), assume that  $\ell(\Gamma) \geq \xi$ . Let  $\Lambda \in \text{NSD}(Z)$  be minimal with respect to the property that  $\Gamma \subseteq \Lambda^{(\xi)}$ . Assume, in order to get a contradiction, that  $\Lambda^{(\xi)} \not\subseteq \Gamma$ . It follows from Lemma 3.2 that  $\Gamma \subseteq \check{\Lambda}^{(\xi)}$ , hence  $\Gamma \subseteq \Delta(\Lambda^{(\xi)})$ . Fix  $A \subseteq Z$  such that  $\Gamma = [A]$ . Also fix  $D, E \subseteq \mathcal{P}(\omega)$  such that  $\Gamma = \Gamma_D(Z)$  and  $\Lambda = \Gamma_E(Z)$ . Then  $\{A, Z \setminus A\} \subseteq \Gamma_E(Z)^{(\xi)} = \Gamma_E^{(\xi)}(Z)$ , where the equality holds by Corollary 10.4. Then, by Corollary 10.3, we can fix a zero-dimensional Polish space  $W$  and a  $\Sigma_{1+\xi}^0$ -measurable bijection  $f : Z \rightarrow W$  such that  $\{f[A], f[Z \setminus A]\} \subseteq \Gamma_E(W)$ .

Next, we will show that  $[f[A]] \in \text{SD}(W)$ . Assume, in order to get a contradiction, that this is not the case. By Corollary 8.2, we can fix  $F \subseteq \mathcal{P}(\omega)$  such that  $[f[A]] = \Gamma_F(W)$ . Notice that  $\Gamma_F(W) \subseteq \Gamma_E(W)$ . Furthermore  $W \setminus f[A] = f[Z \setminus A] \in \Gamma_E(W)$ , hence  $\check{\Gamma}_F(W) \subseteq \Gamma_E(W)$ . Since  $\Gamma_E(W)$  is non-selfdual by Theorem 7.5, it follows that  $\Gamma_F(W) \subsetneq \Gamma_E(W)$ . Therefore,  $\Gamma_F(Z) \subsetneq \Gamma_E(Z) = \Lambda$  by Proposition 6.5. On the other hand, Lemma 9.6.3 and Corollary 10.4 show that  $A = f^{-1}[f[A]] \in \Gamma_F^{(\xi)}(Z) = \Gamma_F(Z)^{(\xi)}$ . Hence  $\Gamma \subseteq \Gamma_F(Z)^{(\xi)}$ , which contradicts the minimality of  $\Lambda$ .

Since  $[f[A]] \in \text{SD}(W)$ , by Corollaries 4.2 and 8.2, we can fix  $A_n \subseteq W$ ,  $G_n \subseteq \mathcal{P}(\omega)$  and pairwise disjoint  $V_n \in \Delta_1^0(W)$  for  $n \in \omega$  such that  $f[A] = \bigcup_{n \in \omega} (A_n \cap V_n)$  and  $A_n \in \Gamma_{G_n}(W) \subsetneq \Gamma_E(W)$  for each  $n$ . Notice that  $\Gamma_{G_n}(Z) \subsetneq \Gamma_E(Z)$  for each  $n$  by Proposition 6.5, hence  $\Gamma_D(Z) \not\subseteq \Gamma_{G_n}(Z)^{(\xi)}$  for each  $n$  by the minimality of  $\Lambda$ . It follows from Corollary 10.4 and Lemma 3.2 that  $\check{\Gamma}_{G_n}^{(\xi)}(Z) \subseteq \Gamma_D(Z)$ . Then, using Propositions 9.4 and 6.5, one sees that  $\check{\Gamma}_{G_n}^{(\xi)}(W) \subseteq \Gamma_D(W)$ .

Set  $B_n = W \setminus A_n \in \check{\Gamma}_{G_n}(W)$  for  $n \in \omega$ . Observe that  $f^{-1}[B_n] \in \check{\Gamma}_{G_n}^{(\xi)}(Z) \subseteq \Gamma_D(Z) = \Gamma$  for each  $n$  by Lemma 9.6.3. Furthermore, it is clear that  $f^{-1}[V_n] \in \Delta_{1+\xi}^0(Z)$  for each  $n$ . In conclusion, since  $W \setminus f[A] = \bigcup_{n \in \omega} (B_n \cap V_n)$ , we see that

$$Z \setminus A = \bigcup_{n \in \omega} (f^{-1}[B_n] \cap f^{-1}[V_n]) \in \text{PU}_\xi(\Gamma) = \Gamma,$$

where the last equality uses the assumption that  $\ell(\Gamma) \geq \xi$ . This contradicts the fact that  $\Gamma$  is non-selfdual.

<sup>17</sup>For example, it is conceivable that  $\text{PU}_\eta(\Gamma) = \Gamma$  for all  $\eta < \xi$ , where  $\xi$  is a limit ordinal, while  $\text{PU}_\xi(\Gamma) \neq \Gamma$ .



In order to show that (2)  $\rightarrow$  (1), let  $\mathbf{\Lambda} \in \text{NSD}(Z)$  be such that  $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}$ . Pick  $A_n \in \mathbf{\Gamma}$  and pairwise disjoint  $V_n \in \mathbf{\Delta}_{1+\xi}^0(Z)$  for  $n \in \omega$  such that  $\bigcup_{n \in \omega} V_n = Z$ . We need to show that  $\bigcup_{n \in \omega} (A_n \cap V_n) \in \mathbf{\Gamma}$ . By Corollary 8.2, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\mathbf{\Lambda} = \mathbf{\Gamma}_D(Z)$ . By Corollary 10.3, we can fix a Polish space  $W$  and a  $\Sigma_{1+\xi}^0$ -measurable bijection  $f : Z \rightarrow W$  such that each  $f[A_n] \in \mathbf{\Gamma}_D(W)$  and each  $f[V_n] \in \mathbf{\Delta}_1^0(W)$ . Let  $B = \bigcup_{n \in \omega} (f[A_n] \cap f[V_n])$ . Since  $\mathbf{\Gamma}_D(W)$  is a Wadge class in  $W$  by Theorem 7.5, one sees that  $B \in \text{PU}_0(\mathbf{\Gamma}_D(W)) = \mathbf{\Gamma}_D(W)$ . It follows from Lemma 9.6.3 that

$$\bigcup_{n \in \omega} (A_n \cap V_n) = f^{-1}[B] \in \mathbf{\Gamma}_D^{(\xi)}(Z) = \mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma},$$

where the second equality holds by Corollary 10.4.  $\square$

**Corollary 11.4.** *Assume AD. Let  $Z$  and  $W$  be uncountable zero-dimensional Polish spaces, let  $D \subseteq \mathcal{P}(\omega)$ , and let  $\xi < \omega_1$ . Then  $\ell(\mathbf{\Gamma}_D(Z)) \geq \xi$  iff  $\ell(\mathbf{\Gamma}_D(W)) \geq \xi$ .*

*Proof.* We will only prove the left-to-right implication, as the other one can be proved similarly. Assume that  $\ell(\mathbf{\Gamma}_D(Z)) \geq \xi$ . Then, by Theorems 7.5 and 11.3, there exists  $\mathbf{\Lambda} \in \text{NSD}(Z)$  such that  $\mathbf{\Lambda}^{(\xi)} = \mathbf{\Gamma}_D(Z)$ . By Corollary 8.2, we can fix  $E \subseteq \mathcal{P}(\omega)$  such that  $\mathbf{\Lambda} = \mathbf{\Gamma}_E(Z)$ . By Proposition 9.4, we can fix  $F \subseteq \mathcal{P}(\omega)$  such that  $\mathbf{\Gamma}_F(Z) = \mathbf{\Gamma}_E^{(\xi)}(Z)$  and  $\mathbf{\Gamma}_F(W) = \mathbf{\Gamma}_E^{(\xi)}(W)$ . Notice that  $\mathbf{\Gamma}_F(Z) = \mathbf{\Gamma}_D(Z)$  by Corollary 10.4, hence  $\mathbf{\Gamma}_F(W) = \mathbf{\Gamma}_D(W)$  by Proposition 6.5. By applying Corollary 10.4 again, we see that  $\mathbf{\Gamma}_D(W) = \mathbf{\Gamma}_E(W)^{(\xi)}$ , hence  $\ell(\mathbf{\Gamma}_D(W)) \geq \xi$  by Theorems 7.5 and 11.3.  $\square$

## 12. GOOD WADGE CLASSES

The following key notion is essentially due to van Engelen, although he did not give it a name. One important difference is that van Engelen's treatment of this notion is fundamentally tied to Louveau's classification of the Borel Wadge classes from [Lo1], hence it is limited to the Borel context. The notion of level and the expansion theorem allow us to completely bypass [Lo1], and extend this concept to arbitrary Wadge classes.

**Definition 12.1.** Let  $Z$  be a space, and let  $\mathbf{\Gamma}$  be a Wadge class in  $Z$ . We will say that  $\mathbf{\Gamma}$  is *good* if the following conditions are satisfied:

- $\mathbf{\Gamma}$  is non-selfdual,
- $\Delta(\text{D}_\omega(\Sigma_2^0(Z))) \subseteq \mathbf{\Gamma}$ ,
- $\ell(\mathbf{\Gamma}) \geq 1$ .

The following proposition gives some concrete examples of good Wadge classes.

**Proposition 12.2.** *Let  $Z$  be an uncountable zero-dimensional Polish space, let  $\omega \leq \eta < \omega_1$ , and let  $2 \leq \xi < \omega_1$ . Then  $\text{D}_\eta(\Sigma_\xi^0(Z))$  is a good Wadge class in  $Z$ .*

*Proof.* Set  $\mathbf{\Gamma} = \text{D}_\eta(\Sigma_\xi^0(Z))$ . The fact that  $\mathbf{\Gamma} \in \text{NSD}(Z)$  follows from Propositions 9.3 and 9.5. The inclusion  $\Delta(\text{D}_\omega(\Sigma_2^0(Z))) \subseteq \mathbf{\Gamma}$  holds trivially. Finally, using Corollary 10.4 and [LSR1, Théorème 8],<sup>18</sup> one sees that  $\ell(\mathbf{\Gamma}) \geq 1$ .  $\square$

<sup>18</sup>Here, we apply [LSR1, Théorème 8] instead of Theorem 11.3 simply because the former does not require AD.

The main result of this section is Theorem 12.4, which will be crucial in showing that good Wadge classes are reasonably closed (see Lemma 13.2). The case  $Z = \omega^\omega$  of the following lemma is due to Andretta, Hjorth, and Neeman (see [AHN, Lemma 3.6.a]), and the general case follows easily from this particular case (thanks to the machinery of relativization).

**Lemma 12.3.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, and let  $\Gamma \in \text{NSD}(Z)$ . Assume that  $D_n(\Sigma_1^0(Z)) \subseteq \Gamma$  for every  $n \in \omega$ .*

- If  $A \in \Gamma$  and  $C \in \Pi_1^0(Z)$  then  $A \cap C \in \Gamma$ .
- If  $A \in \Gamma$  and  $U \in \Sigma_1^0(Z)$  then  $A \cup U \in \Gamma$ .

*Proof.* Observe that, since  $\tilde{\Gamma}$  also satisfies the assumptions of the lemma, it will be enough to prove the first statement. So pick  $A \in \Gamma$  and  $C \in \Pi_1^0(Z)$ . By Corollary 8.2, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma = \Gamma_D(Z)$ . Set  $\Lambda = \Gamma_D(\omega^\omega)$ . Using Lemma 6.4.3, we can assume without loss of generality that  $Z$  is a closed subspace of  $\omega^\omega$ . By [Ke, Proposition 2.8], we can fix a retraction  $\rho : \omega^\omega \rightarrow Z$ . Notice that  $\rho^{-1}[A] \in \Lambda$  by Lemma 6.4.2. Furthermore, it is clear that  $C \in \Pi_1^0(\omega^\omega)$ .

Next, we claim that  $\|\Lambda\| \geq \omega$  (see Definition 3.4). Since  $D_n(\Sigma_1^0(Z)) \subseteq \Gamma$  for every  $n \in \omega$ , using Propositions 6.2 and 6.5 one sees that  $D_n(\Sigma_1^0(\omega^\omega)) \subseteq \Lambda$  for every  $n \in \omega$ . Since these are Wadge classes by Theorem 7.5, and they form a strictly increasing sequence by [Ke, Exercise 22.26.iv], our claim is proved. Therefore, we can apply [AHN, Lemma 3.6.a], which shows that  $\rho^{-1}[A] \cap C \in \Lambda$ . Finally, Lemma 6.4.1 shows that  $A \cap C = (\rho^{-1}[A] \cap C) \cap Z \in \Gamma_D(Z) = \Gamma$ .  $\square$

**Theorem 12.4.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, and let  $\Gamma \in \text{NSD}(Z)$ . Assume that  $D_n(\Sigma_2^0(Z)) \subseteq \Gamma$  for every  $n \in \omega$  and  $\ell(\Gamma) \geq 1$ .*

- If  $A \in \Gamma$  and  $G \in \Pi_2^0(Z)$  then  $A \cap G \in \Gamma$ .
- If  $A \in \Gamma$  and  $F \in \Sigma_2^0(Z)$  then  $A \cup F \in \Gamma$ .

*In particular, the above two statements hold for every good Wadge class  $\Gamma$  in  $Z$ .*

*Proof.* Observe that, since  $\tilde{\Gamma}$  also satisfies the assumptions of the theorem, it will be enough to prove the first statement. So pick  $A \in \Gamma$  and  $G \in \Pi_2^0(Z)$ . By Theorem 11.3, we can pick  $\Lambda \in \text{NSD}(Z)$  such that  $\Lambda^{(1)} = \Gamma$ . By Corollary 8.2, we can fix  $E \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_E(Z) = \Lambda$ .

Since  $\Lambda^{(1)} = \Gamma$ , there exists a  $\Sigma_2^0$ -measurable function  $f : Z \rightarrow Z$  and  $B \in \Lambda$  such that  $A = f^{-1}[B]$ . Furthermore, using Corollary 10.4 for a suitable choice of  $D$ , it is easy to check that  $\Pi_1^0(Z)^{(1)} = \Pi_2^0(Z)$ . Therefore, there exists a  $\Sigma_2^0$ -measurable function  $g : Z \rightarrow Z$  and  $C \in \Pi_1^0(Z)$  such that  $G = g^{-1}[C]$ . By applying Lemma 6.4.2 to the projection on the first coordinate  $\pi : Z \times Z \rightarrow Z$ , one sees that  $B \times Z \in \Gamma_E(Z \times Z)$ . Furthermore, it is clear that  $Z \times C \in \Pi_1^0(Z \times Z)$ .

We claim that  $D_n(\Sigma_1^0(Z \times Z)) \subseteq \Gamma_E(Z \times Z)$  for every  $n \in \omega$ . So fix  $n \in \omega$ , and let  $D \subseteq \mathcal{P}(\omega)$  be the set given by Proposition 9.3 when  $\eta = n$ . Notice that

$$\Gamma_D(Z)^{(1)} = \Gamma_D^{(1)}(Z) = D_n(\Sigma_2^0(Z)) \subseteq \Gamma = \Gamma_E(Z)^{(1)},$$

where the first equality holds by Corollary 10.4. Therefore  $\Gamma_D(Z) \subseteq \Gamma_E(Z)$  by Corollary 10.6. An application of Proposition 6.5 with  $W = Z \times Z$  concludes the proof of our claim.

Therefore, we can apply Lemma 12.3, which shows that  $B \times C = (B \times Z) \cap (Z \times C) \in \Gamma_E(Z \times Z)$ . Consider the function  $(f, g) : Z \rightarrow Z \times Z$  defined by

$(f, g)(x) = (f(x), g(x))$ , and observe that  $(f, g)$  is  $\Sigma_2^0$ -measurable. By Lemma 9.6.3, it follows that

$$A \cap G = (f, g)^{-1}[B \times C] \in \Gamma_E^{(1)}(Z) = \mathbf{\Lambda}^{(1)} = \mathbf{\Gamma},$$

where the second equality holds by Corollary 10.4.  $\square$

### 13. REASONABLY CLOSED WADGE CLASSES

In this section we will define reasonably closed Wadge classes and prove that every good Wadge class is reasonably closed. This notion is an ad hoc definition, and it is the key idea of an ingenious lemma due to Harrington (see [St2, Lemma 3]). This lemma is a crucial ingredient in the proof of Theorem 15.1. Here, we will follow the approach of [vE3, Section 4.1].

Given  $i \in 2$ , set

$$Q_i = \{x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega\}.$$

Notice that every element of  $2^\omega \setminus (Q_0 \cup Q_1)$  is obtained by alternating finite blocks of zeros and finite blocks of ones. Define the function  $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \rightarrow 2^\omega$  by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length,} \\ 1 & \text{otherwise,} \end{cases}$$

where we start counting with the  $0^{\text{th}}$  block of zeros. It is easy to check that  $\phi$  is continuous.

**Definition 13.1.** Let  $\mathbf{\Gamma}$  be a Wadge class in  $2^\omega$ . We will say that  $\mathbf{\Gamma}$  is *reasonably closed* if  $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$  for every  $A \in \mathbf{\Gamma}$ .

The following result is essentially the same as [vE3, Lemma 4.2.17], except that it is not limited to the Borel context.

**Lemma 13.2.** *Assume AD. Let  $\mathbf{\Gamma}$  be a good Wadge class in  $2^\omega$ . Then  $\mathbf{\Gamma}$  is reasonably closed.*

*Proof.* By Corollary 8.2, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\mathbf{\Gamma} = \mathbf{\Gamma}_D(2^\omega)$ . Set  $Z = 2^\omega \setminus (Q_0 \cup Q_1)$ . Pick  $A \in \mathbf{\Gamma}$ . Notice that  $\phi^{-1}[A] \in \mathbf{\Gamma}_D(Z)$  by Lemma 6.4.2. Therefore, by Lemma 6.4.1, there exists  $B \in \mathbf{\Gamma}$  such that  $B \cap Z = \phi^{-1}[A]$ . Since  $\mathbf{\Gamma}$  is a good Wadge class and  $Z \in \mathbf{\Pi}_2^0(2^\omega)$ , it follows from Theorem 12.4 that  $\phi^{-1}[A] \in \mathbf{\Gamma}$ . Finally, again by Theorem 12.4, we see that  $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$ , which concludes the proof.  $\square$

### 14. WADGE CLASSES OF HOMOGENEOUS SPACES ARE GOOD

The main result of this section is that  $[X]$  is a good Wadge class whenever  $X$  is a homogeneous space of sufficiently high complexity (see Theorem 14.4 for the precise statement). Together with Lemma 13.2, this will allow us to apply Theorem 15.1 in the next section.

We will need three preliminary results. Lemmas 14.1, 14.2, and 14.3 correspond to [vE3, Lemma 4.2.16], [vE3, Lemma 4.4.2], and [vE3, Lemma 4.4.1] respectively, while Theorem 14.4 corresponds to [vE3, Lemma 4.4.3]. Once again, the difference is that we work with arbitrary sets instead of just Borel sets. In the case of Lemma 14.3, this yields at the same time a substantially simpler proof, inspired by [Lo2, proof of Theorem 7.3.10.ii].

**Lemma 14.1.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, and let  $\Gamma$  be a good Wadge class in  $Z$ . Assume that  $A$  and  $B$  are subspaces of  $Z$  such that  $B \in \Gamma$  and  $A \approx B$ . Then  $A \in \Gamma$ .*

*Proof.* Let  $h : A \rightarrow B$  be a homeomorphism. By [Ke, Theorem 3.9], we can fix  $G, H \in \Pi_2^0(Z)$  and a homeomorphism  $f : G \rightarrow H$  such that  $h \subseteq f$ . By Corollary 8.2, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma = \Gamma_D(Z)$ . Notice that  $B \in \Gamma_D(H)$  by Lemma 6.4.1. It follows from Lemma 6.4.2 that  $A \in \Gamma_D(G)$ . Therefore, according to Lemma 6.4.1, there exists  $C \in \Gamma_D(Z)$  such that  $C \cap G = A$ . Since  $G \in \Pi_2^0(Z)$ , an application of Theorem 12.4 concludes the proof.  $\square$

**Lemma 14.2.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, let  $\Gamma$  be a good Wadge class in  $Z$ , and let  $X$  be a homogeneous subspace of  $Z$ . Assume that  $A \in \Sigma_1^0(X)$  is non-empty and  $A \in \Gamma$ . Then  $X \in \Gamma$ .*

*Proof.* Define  $\mathcal{U} = \{h[A] : h \text{ is a homeomorphism of } X\}$ . Notice that  $\mathcal{U}$  is a cover of  $X$  because  $X$  is homogeneous and  $A$  is non-empty. Let  $\{A_n : n \in \omega\}$  be a countable subcover of  $\mathcal{U}$ . Observe that each  $A_n \in \Gamma$  by Lemma 14.1. Fix  $U_n \in \Sigma_1^0(Z)$  for  $n \in \omega$  such that  $U_n \cap X = A_n$  for each  $n$ . Set  $V_n = U_n \setminus \bigcup_{k < n} U_k$  for  $n \in \omega$ , and observe that  $V_n \in \Delta_2^0(Z)$  for each  $n$ . Furthermore, it is easy to check that

$$X = \bigcup_{-1 \leq n < \omega} (V_n \cap A_n),$$

where  $V_{-1} = Z \setminus \bigcup_{n < \omega} V_n = Z \setminus \bigcup_{n < \omega} U_n$  and  $A_{-1} = \emptyset$ . In conclusion, we see that  $X \in \text{PU}_1(\Gamma)$ . Since  $\ell(\Gamma) \geq 1$ , it follows that  $X \in \Gamma$ .  $\square$

**Lemma 14.3.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, let  $\Gamma \in \text{NSD}(Z)$  be such that  $\ell(\Gamma) = 0$ , and let  $X \in \Gamma$  be codense in  $Z$ . Then there exists a non-empty  $U \in \Delta_1^0(Z)$  and  $\Lambda \in \text{NSD}(Z)$  such that  $\Lambda \subsetneq \Gamma$  and  $X \cap U \in \Lambda$ .*

*Proof.* Since  $\ell(\Gamma) = 0$ , using Lemma 3.2 and the fact that  $\text{PU}_1(\Gamma)$  is continuously closed, it is easy to see that  $\check{\Gamma} \subseteq \text{PU}_1(\Gamma)$ . Therefore, we can fix  $A_n \in \Gamma$  and pairwise disjoint  $V_n \in \Delta_2^0(Z)$  for  $n \in \omega$  such that  $\bigcup_{n \in \omega} (A_n \cap V_n) = Z \setminus X$ . Since  $Z$  is a Baire space, we can fix  $n \in \omega$  and a non-empty  $U \in \Delta_1^0(Z)$  such that  $U \subseteq V_n$ .

Notice that  $\Gamma \neq \{\emptyset\}$  and  $\check{\Gamma} \neq \{\emptyset\}$  because  $\ell(\Gamma) = 0$ , hence it is possible to apply Proposition 3.5. In particular, one sees that  $U \setminus X = A_n \cap U \in \Gamma$ , hence  $Z \setminus (X \cap U) = (Z \setminus U) \cup (U \setminus X) \in \Gamma$ . So, we have  $X \cap U \in \Gamma$  (again by Proposition 3.5) and  $Z \setminus (X \cap U) \in \Gamma$ . This easily yields the desired result if  $X \cap U$  is non-selfdual, so assume that  $X \cap U$  is selfdual. By Corollary 4.2, we can fix pairwise disjoint  $U_n \in \Delta_1^0(U)$  and non-selfdual  $B_n < X \cap U$  in  $Z$  for  $n \in \omega$  such that  $\bigcup_{n \in \omega} U_n = U$  and  $\bigcup_{n \in \omega} (B_n \cap U_n) = X \cap U$ . If we had  $B_n = Z$  for each  $n$  such that  $U_n \neq \emptyset$  then the assumption that  $X$  is codense in  $Z$  would be contradicted, so assume that  $n \in \omega$  is such that  $B_n \neq Z$  and  $U_n \neq \emptyset$ . To conclude the proof, set  $\Lambda = [B_n]$  and observe that  $X \cap U_n = B_n \cap U_n \leq B_n$  by Proposition 3.5.  $\square$

**Theorem 14.4.** *Assume AD. Let  $Z$  be an uncountable zero-dimensional Polish space, and let  $X$  be a homogeneous dense subspace of  $Z$  such that  $X \notin \Delta(\text{D}_\omega(\Sigma_2^0(Z)))$ . Then  $[X]$  is a good Wadge class in  $Z$ .*

*Proof.* Fix  $\Gamma \in \text{NSD}(Z)$  minimal with respect to the property that  $X \cap U \in \Gamma \cup \check{\Gamma}$  for some non-empty  $U \in \Delta_1^0(Z)$ . Fix a non-empty  $U \in \Delta_1^0(Z)$  such that  $X \cap U \in$

$\Gamma \cup \check{\Gamma}$ . Assume without loss of generality that  $X \cap U \in \Gamma$  (the case  $X \cap U \in \check{\Gamma}$  is similar). First we will prove that  $\Gamma$  is a good Wadge class, then we will show that  $[X] = \Gamma$ . Observe that  $D_\omega(\Sigma_2^0(Z))$  and  $\check{D}_\omega(\Sigma_2^0(Z))$  are good Wadge classes in  $Z$  by Proposition 12.2. We claim that  $X \cap U \notin \Delta(D_\omega(\Sigma_2^0(Z)))$ . Assume, in order to get a contradiction, that  $X \cap U \in \Delta(D_\omega(\Sigma_2^0(Z)))$ . Then, by the density of  $X$ , it is possible to apply Lemma 14.2 (twice), obtaining that  $X \in \Delta(D_\omega(\Sigma_2^0(Z)))$ . Since this contradicts our assumptions, our claim is proved. By Lemma 3.2, it follows that  $\Delta(D_\omega(\Sigma_2^0(Z))) \subseteq \Gamma$ .

Next, we claim that  $\ell(\Gamma) \geq 1$ . Assume, in order to get a contradiction, that  $\ell(\Gamma) = 0$ . By Corollary 8.2, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_D(Z) = \Gamma$ . Since  $X$  is dense in  $Z$  and homogeneous, if  $U$  were countable then  $X$  would be countable, by the same argument as in the proof of Proposition 2.6. So  $U$  is an uncountable zero-dimensional Polish space, and  $\ell(\Gamma_D(U)) = 0$  by Corollary 11.4. Furthermore,  $X$  must be codense in  $Z$ , otherwise it would follow that  $X$  is Polish by Proposition 2.6, hence  $X \in \Pi_2^0(Z)$  by [Ke, Theorem 3.11]. Therefore, by Lemma 14.3, there exists a non-empty  $V \in \Delta_1^0(U)$  and  $\Lambda \in \text{NSD}(U)$  such that  $\Lambda \subseteq \Gamma_D(U)$  and  $X \cap V \in \Lambda$ . By Corollary 8.2, we can fix  $E \subseteq \mathcal{P}(\omega)$  such that  $\Gamma_E(U) = \Lambda$ . Observe that  $\Gamma_E(Z) \not\subseteq \Gamma_D(Z)$  by Proposition 6.5. Therefore, in order to contradict the minimality of  $\Gamma$ , it remains to show that  $X \cap V \in \Gamma_E(Z)$ . By Lemma 6.4.1, there exists  $A \in \Gamma_E(Z)$  such that  $A \cap U = X \cap V$ . Notice that  $\Gamma_E(Z) \neq \{Z\}$ , otherwise it would follow that  $X = Z$ , which contradicts our assumptions. Therefore  $X \cap V = A \cap U \in \Gamma_E(Z)$  by Proposition 3.5.

At this point, we know that  $\Gamma$  is a good Wadge class, so we can apply Lemma 14.2, obtaining that  $X \in \Gamma$ . To conclude the proof, it will be enough to show that  $X$  is non-selfdual, as it will follow from the minimality of  $\Gamma$  and Proposition 3.5 that  $[X] = \Gamma$ . Assume, in order to get a contradiction, that  $X$  is selfdual. Then, by Corollary 4.2, there exist a non-empty  $V \in \Delta_1^0(Z)$  and a non-selfdual  $A < X$  in  $Z$  such that  $A \cap V = X \cap V$ . Set  $\Lambda = [A]$ , and observe that  $\Lambda \subseteq \Gamma$ . Notice that  $\Lambda \neq \{Z\}$ , otherwise it would follow that  $V \subseteq X$ , hence  $X$  would not be codense in  $Z$ . Therefore  $X \cap V = A \cap V \in \Lambda$  by Proposition 3.5. This contradicts the minimality of  $\Gamma$ .  $\square$

## 15. THE MAIN RESULTS

This sections contains our main results. Theorem 15.2 extends (and is inspired by) [vE4, Lemma 2.7]. All the work done so far was aimed at applying the following result, which is a particular case of [St2, Theorem 2]. Given a Wadge class  $\Gamma$  in  $2^\omega$  and  $X \subseteq 2^\omega$ , we will say that  $X$  is *everywhere properly*  $\Gamma$  if  $X \cap [s] \in \Gamma \setminus \check{\Gamma}$  for every  $s \in 2^{<\omega}$ .

**Theorem 15.1** (Steel). *Assume AD. Let  $\Gamma$  be a reasonably closed Wadge class in  $2^\omega$ . Assume that  $X$  and  $Y$  are subsets of  $2^\omega$  that satisfy the following conditions:*

- $X$  and  $Y$  are everywhere properly  $\Gamma$ ,
- $X$  and  $Y$  are either both meager in  $2^\omega$  or both comeager in  $2^\omega$ .

*Then there exists a homeomorphism  $h : 2^\omega \rightarrow 2^\omega$  such that  $h[X] = Y$ .*

**Theorem 15.2.** *Assume AD. Let  $X$  and  $Y$  be homogeneous dense subspaces of  $2^\omega$ . Assume that  $X \notin \Delta(D_\omega(\Sigma_2^0(2^\omega)))$ , and that the following conditions are satisfied:*

- $[X] = [Y]$ ,
- $X$  and  $Y$  are either both meager spaces or both Baire spaces.

Then there exists a homeomorphism  $h : 2^\omega \longrightarrow 2^\omega$  such that  $h[X] = Y$ .

*Proof.* Let  $\Gamma = [X]$ . Notice that  $\Gamma$  is a good Wadge class by Theorem 14.4, hence it is reasonably closed by Lemma 13.2. It is clear that if  $X$  and  $Y$  are both meager spaces, then they are both meager in  $2^\omega$ . On the other hand, if  $X$  and  $Y$  are both Baire spaces, then they are comeager in  $2^\omega$  by Proposition 2.7. Hence, by Theorem 15.1, it will be enough to show that  $X$  and  $Y$  are everywhere properly  $\Gamma$ . We will only prove this for  $X$ , since the proof for  $Y$  is perfectly analogous. Pick  $s \in 2^{<\omega}$ . Using Proposition 3.5, one sees that  $X \cap [s] \in \Gamma$ . In order to get a contradiction, assume that  $X \cap [s] \in \check{\Gamma}$ . Since  $\check{\Gamma}$  is also a good Wadge class, it follows from Lemma 14.2 that  $X \in \check{\Gamma}$ , which contradicts the fact that  $\Gamma$  is non-selfdual.  $\square$

**Corollary 15.3.** *Assume AD. Let  $X$  be a zero-dimensional homogeneous space that is not locally compact. Then  $X$  is strongly homogeneous.*

*Proof.* Notice that  $X$  is crowded, otherwise it would be discrete by homogeneity. Therefore, we can assume without loss of generality that  $X$  is a dense subspace of  $2^\omega$ . If  $X \in \Delta(\mathbb{D}_\omega(\Sigma_2^0(2^\omega)))$ , then the desired result follows from [vE3, Corollary 4.4.6]. So assume that  $X \notin \Delta(\mathbb{D}_\omega(\Sigma_2^0(2^\omega)))$ .

By Theorem 2.8, it will be enough to show that  $X \cap [s] \approx X$  for every  $s \in 2^{<\omega}$ . Pick  $s \in 2^{<\omega}$ . Let  $h : [s] \longrightarrow 2^\omega$  be a homeomorphism, and let  $Y = h[X \cap [s]]$ . It is easy to check that  $Y$  is a homogeneous dense subspace of  $2^\omega$ . Furthermore, it is clear that  $X$  and  $Y$  are either both meager spaces or both Baire spaces. We claim that  $[X] = [Y]$ . By Theorem 15.2, this will conclude the proof.

Set  $\Gamma = [X]$ , and observe that  $\Gamma$  is a good Wadge class by Theorem 14.4. In particular,  $\Gamma$  is non-selfdual. Hence, by Corollary 8.2, we can fix  $D \subseteq \mathcal{P}(\omega)$  such that  $\Gamma = \Gamma_D(2^\omega)$ . Notice that  $X \cap [s] \in \Gamma_D([s])$  by Lemma 6.4.1, hence  $Y \in \Gamma_D(2^\omega)$  by Lemma 6.4.3. This shows that  $[Y] \subseteq [X]$ . In order to prove the other inclusion, by Lemma 3.2, it will be enough to show that  $Y \notin \check{\Gamma}_D(2^\omega)$ . Assume, in order to get a contradiction, that  $Y \in \check{\Gamma}_D(2^\omega)$ . Then  $X \cap [s] \in \check{\Gamma}_D([s])$  by Lemma 6.4.3. It follows easily from Lemma 6.4.1 and Proposition 3.5 that  $X \cap [s] \in \check{\Gamma}_D(2^\omega) = \check{\Gamma}$ . This implies that  $X \in \check{\Gamma}$  by Lemma 14.2, which contradicts the fact that  $\Gamma$  is non-selfdual.  $\square$

## REFERENCES

- [An] A. ANDRETTA. The SLO principle and the Wadge hierarchy. In: *Foundations of the Formal Sciences V. Infinite Games*, 1–38, Stud. Log. **11**, Coll. Publ., London, 2007.
- [AHN] A. ANDRETTA, G. HJORTH, I. NEEMAN. Effective cardinals of boldface pointclasses. *J. Math. Log.* **7:1** (2007), 35–82.
- [AM] A. ANDRETTA, D. A. MARTIN. Borel-Wadge degrees. *Fund. Math.* **177:2** (2003), 173–190.
- [AvM] A. V. ARKHANGEL'SKIĬ, J. VAN MILL. Topological homogeneity. In: *Recent Progress in General Topology III*. Atlantis Press, 2014. 1–68.
- [CMM] R. CARROY, A. MEDINI, S. MÜLLER. Constructing Wadge classes. Preprint (2019). Available at <http://arxiv.org/abs/1907.07612>.
- [vD] E. K. VAN DOUWEN. A compact space with a measure that knows which sets are homeomorphic. *Adv. in Math.* **52:1** (1984), 1–33.
- [DP] A. DOW, E. PEARL. Homogeneity in powers of zero-dimensional first-countable spaces. *Proc. Amer. Math. Soc.* **125** (1997), 2503–2510.
- [vE1] F. VAN ENGELEN. Homogeneous Borel sets of ambiguous class two. *Trans. Am. Math. Soc.* **290** (1985), 1–39.
- [vE2] F. VAN ENGELEN. Homogeneous Borel sets. *Proc. Am. Math. Soc.* **96** (1986), 673–682.

- [vE3] A. J. M. VAN ENGELEN. *Homogeneous zero-dimensional absolute Borel sets*. CWI Tract, 27. Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1986. iv+133 pp. Available at <http://repository.cwi.nl/>.
- [vE4] F. VAN ENGELEN. On Borel ideals. *Ann. Pure Appl. Logic* **70:2** (1994), 177–203.
- [vEMS] F. VAN ENGELEN, J. STEEL, A. W. MILLER. Rigid Borel sets and better quasiorder theory. *Logic and combinatorics, Proc. AMS-IMS-SIAM Conf., Arcata/Calif. 1985, Contemp. Math.* **65** (1987), 199–222.
- [FZ] B. FITZPATRICK JR., H. X. ZHOU. Countable dense homogeneity and the Baire property. *Topology Appl.* **43:1** (1992), 1–14.
- [Fo] K. FOURNIER. Wadge hierarchy of differences of co-analytic sets. *J. Symb. Log.* **81:1** (2016), 201–215.
- [Ha] F. HAUSDORFF. *Gesammelte Werke. Band III: Mengenlehre (1927, 1935). Deskriptive Mengenlehre und Topologie*. Edited by U. Felgner, H. Herrlich, M. Hušek, V. Kanovei, P. Koepke, G. Preuß, W. Purkert und E. Scholz. Springer, Berlin, 2010. xx+878 pp.
- [HR] P. HOWARD, J. E. RUBIN. *Consequences of the Axiom of Choice*. Mathematical Surveys and Monographs, 59. American Mathematical Society, Providence, Rhode Island, 1998. 432 pp.
- [Je] T. JECH. *Set theory*. The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. xiv+769 pp.
- [Ka] A. KANAMORI. *The higher infinite*. Second edition. Paperback reprint of the 2003 edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2009. xxii+536 pp.
- [Ke] A. S. KECHRIS. *Classical descriptive set theory*. Graduate Texts in Mathematics, 156. Springer-Verlag, New York, 1995. xviii+402 pp.
- [La] L. B. LAWRENCE. A rigid subspace of the real line whose square is a homogeneous subspace of the plane. *Trans. Amer. Math. Soc.* **357:7** (2005), 2535–2556.
- [Lo1] A. LOUVEAU. Some results in the Wadge hierarchy of Borel sets. In: *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, 3–23, Lect. Notes Log. **37**, Assoc. Symbol. Logic, La Jolla, CA, 2012. Originally published in: *Cabal seminar 79-81*, 28–55, Lecture Notes in Math. **1019**, Springer, Berlin, 1983.
- [Lo2] A. LOUVEAU. *Effective Descriptive Set Theory*. Unpublished manuscript, 1997.
- [LSR1] A. LOUVEAU, J. SAINT-RAYMOND. Les propriétés de réduction et de norme pour les classes de Boréliens. *Fund. Math.* **131:3** (1988), 223–243.
- [LSR2] A. LOUVEAU, J. SAINT-RAYMOND. The strength of Borel Wadge determinacy. In: *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, 74–101, Lect. Notes Log. **37**, Assoc. Symbol. Logic, La Jolla, CA, 2012. Originally published in: *Cabal seminar 81-85*, 1–30, Lecture Notes in Math. **1333**, Springer, Berlin, 1988.
- [Me1] A. MEDINI. Products and h-homogeneity. *Topology Appl.* **158:18** (2011), 2520–2527.
- [Me2] A. MEDINI. *The topology of ultrafilters as subspaces of the Cantor set and other topics*. Ph.D. Thesis. University of Wisconsin - Madison. ProQuest LLC, Ann Arbor, MI, 2013. 110 pp. Available at <http://www.math.wisc.edu/~lempp/theses/medini.pdf>.
- [Me3] A. MEDINI. On Borel semifilters. *Topology Proc.* **53** (2019), 97–122.
- [Mv] S. V. MEDVEDEV. About closed subsets of spaces of first category. *Topology Appl.* **159:8** (2012), 2187–2192.
- [MvMZ] A. MEDINI, J. VAN MILL, L. ZDOMSKYY. Infinite powers and Cohen reals. *Canad. Math. Bull.* **61:4** (2018), 812–821.
- [MZ] A. MEDINI, L. ZDOMSKYY. Between Polish and completely Baire. *Arch. Math. Logic.* **54:1-2** (2015), 231–245.
- [vM] J. VAN MILL. Sierpiński’s technique and subsets of  $\mathbb{R}$ . *Topology Appl.* **44:1-3** (1992), 241–261.
- [Mi] A. W. MILLER. Infinite combinatorics and definability. *Ann. Pure Appl. Logic* **41:2** (1989), 179–203.
- [MR] L. MOTTO ROS. Borel-amenable reducibilities for sets of reals. *J. Symb. Log.* **74:1** (2009), 27–49.
- [Ne] I. NEEMAN. Determinacy in  $L(\mathbb{R})$ . In: *Handbook of Set Theory*. Springer, Dordrecht, 2010, 1877–1950.
- [So] R. M. SOLOVAY. The independence of DC from AD. In: *Cabal seminar 76-77*, 171–183, Lecture Notes in Math. **689**, Springer, Berlin, Heidelberg, 1978.
- [St1] J. R. STEEL. *Determinateness and Subsystems of Analysis*. Ph.D. Thesis. University of California, Berkeley, 1977.
- [St2] J. R. STEEL. Analytic sets and Borel isomorphisms. *Fund. Math.* **108:2** (1980), 83–88.

- [Te] T. TERADA. Spaces whose all nonempty clopen subsets are homeomorphic. *Yokohama Math. Jour.* **40** (1993), 87–93.
- [VW1] R. A. VAN WESEP. *Subsystems of second-order arithmetic, and descriptive set theory under the axiom of determinateness*. Ph.D. Thesis. University of California, Berkeley, 1977.
- [VW2] R. A. VAN WESEP. Wadge degrees and descriptive set theory. In: *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, 24–42, Lect. Notes Log. **37**, Assoc. Symbol. Logic, La Jolla, CA, 2012. Originally published in: *Cabal seminar 76-77*, 151–170, Lecture Notes in Math. **689**, Springer, Berlin, 1978.
- [Vi] Z. VIDNYÁNSZKY. Transfinite inductions producing coanalytic sets. *Fund. Math.* **224:2** (2014), 155–174.
- [Wa1] W. W. WADGE. *Reducibility and Determinateness on the Baire Space*. Ph.D. Thesis. University of California, Berkeley, 1984.
- [Wa2] W. W. WADGE. Early investigations of the degrees of Borel sets. In: *Wadge degrees and projective ordinals. The Cabal Seminar. Volume II*, 3–23, Lect. Notes Log. **37**, Assoc. Symbol. Logic, La Jolla, CA, 2012.
- [Za] S. ZAFRANY. *On the Topological Strength of Boolean Set Operations*. Unpublished manuscript, 1989. Available at <http://aspbak.netanya.ac.il/~samy/papers/paper6.ps>.

DEPARTMENT OF MATHEMATICS “GIUSEPPE PEANO”  
 PALAZZO CAMPANA, UNIVERSITY OF TURIN  
 VIA CARLO ALBERTO 10  
 10123 TURIN, ITALY  
*E-mail address:* [raphael.carroy@unito.it](mailto:raphael.carroy@unito.it)  
*URL:* <http://www.logique.jussieu.fr/~carroy/indexeng.html>

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC  
 INSTITUTE OF MATHEMATICS, UNIVERSITY OF VIENNA  
 AUGASSE 2-6, UZA 1 - BUILDING 2  
 1090 VIENNA, AUSTRIA  
*URL:* <http://www.logic.univie.ac.at/~medinia2/>

KURT GÖDEL RESEARCH CENTER FOR MATHEMATICAL LOGIC  
 INSTITUTE OF MATHEMATICS, UNIVERSITY OF VIENNA  
 AUGASSE 2-6, UZA 1 - BUILDING 2  
 1090 VIENNA, AUSTRIA  
*E-mail address:* [mueller.sandra@univie.ac.at](mailto:mueller.sandra@univie.ac.at)  
*URL:* <https://muellersandra.github.io>