NON-DEFINABILITY OF RINGS OF INTEGERS IN MOST ALGEBRAIC FIELDS

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Abstract. We show that the set of algebraic extensions F of $\mathbb{Q}$ in which $\mathbb{Z}$ or the ring of integers $\mathcal{O}_F$ are definable is meager in the set of all algebraic extensions.

It is proven in [EMSW20, Theorem 1.1, Corollary 5.7] that the set of subfields F of $\overline{\mathbb{Q}}$ in which one of $\mathbb{Z}$, $\mathbb{Q}\setminus\mathbb{Z}$, $\mathcal{O}_F$, $F\setminus\mathcal{O}_F$ is existentially definable is a meager subset of the space $\mathcal{E}$ of all subfields E of $\overline{\mathbb{Q}}$, in the topology induced from $2^{\overline{\mathbb{Q}}}$. In this short note we explain how a stronger statement can be deduced from known results from field arithmetic (which in particular studies certain properties of algebraic extensions of $\mathbb{Q}$) and model theory (which studies definable subsets in structures with certain properties).

Recall that a field F is *PAC* if every geometrically irreducible F-variety has an F-rational point, *ω-free* if every finite embedding problem for the absolute Galois group G_F is solvable, and *Hilbertian* if $\mathbb{A}^1(F)$ is not thin, i.e. for every finitely many absolutely irreducible $f_1,\dots,f_n \in F[X,Y]$ monic of degree at least 2 in Y, and $0 \neq g \in F[X]$ there exists $x \in F$ such that $g(x) \neq 0$ and $f_1\cdots f_n(x,Y)$ has no zero in F, see chapters 11, 27, 12 and section 13.5 of [FJ08].

Proposition 1. *The set of subfields F of $\overline{\mathbb{Q}}$ which are ω-free and PAC is comeager in $\mathcal{E}$.*

Proof. We claim that both the set $\mathcal{P}$ of PAC fields in $\mathcal{E}$ and the set $\mathcal{H}$ of Hilbertian fields in $\mathcal{E}$ are dense G_δ-sets and therefore comeager. Since the union of two meager sets is meager, and Hilbertian PAC fields are ω-free [Jar11, Theorem 5.10.3], this then implies the claim.

The set $\mathcal{P}$ is dense in $\mathcal{E}$, since for any finite extensions $\mathbb{Q} \subseteq K \subsetneq L$, Jarden's PAC Nullstellensatz [FJ08, Theorem 18.6.1] gives a PAC field $K \subseteq F \subseteq \overline{\mathbb{Q}}$ with $F \cap L = K$. Moreover, $\mathcal{P}$ is the intersection of the countably many open sets

$$U_f = \{F \in \mathcal{E} : f \notin F[X,Y]\} \cup \bigcup_{x,y\in\overline{\mathbb{Q}}, f(x,y)=0} \big\{F \in \mathcal{E} : x, y \in F\big\}$$

for $f \in \overline{\mathbb{Q}}[X,Y]$ irreducible, and hence a G_δ-set.

The set $\mathcal{H}$ is dense in $\mathcal{E}$ since every number field is Hilbertian (this is Hilbert's irreducibility theorem, see [Ser92, Theorem 3.4.1] or [FJ08, Theorem 13.3.5]). Moreover, $\mathcal{H}$ is

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the intersection of the countably many open sets

$$V_{f_1,\dots,f_n,g} = \mathcal{E} \setminus \{F \in \mathcal{E} : f_1, \dots, f_n, g \in F[X,Y]\} \cup \bigcup_{x \in \overline{\mathbb{Q}}, g(x) \neq 0} \bigcap_{i=1}^{n} \bigcap_{y \in \overline{\mathbb{Q}}, f_i(x,y)=0} \{F \in \mathcal{E} : x \in F, y \notin F\}$$

where $n > 0$, $f_1, \dots, f_n \in \overline{\mathbb{Q}}[X,Y]$ monic of degree at least 2 in Y and irreducible, and $0 \neq g \in \overline{\mathbb{Q}}[X]$. □

Remark 2. By [FJ08, Theorem 11.2.3], it would suffice to take U_f with $f \in \mathbb{Q}[X,Y]$. The fact that the set of Hilbertian PAC fields $F \subseteq \overline{\mathbb{Q}}$ is dense in $\mathcal{E}$ could also be deduced directly by applying [Jar97, Theorem 2.7] instead of the PAC Nullstellensatz.

Proposition 3. *In an ω-free PAC field F, every definable subring $R \subseteq F$ is a field.*

Proof. An integral domain R is partially ordered by the relation

$$a \preceq b \iff a = b \vee (a \mid b \wedge b \nmid a).$$

If R is not a field, the powers of a non-zero non-unit form an infinite chain with respect to $\preceq$, which shows that R has the strict order property [She96, Definition 2.1], cf. the argument in [Poi01, Chapter 1.2 Lemma 1]. The strict order property implies the strong order property SOP [She96, Definition 2.2, Claim 2.3(1)], which in turn implies the 3-strong order property SOP_3 [She96, Definition 2.5, Claim 2.6]. However, ω-free PAC fields do not have SOP_3 by Chatzidakis's result [Cha19, Theorem 3.10], hence so has any structure definable in them. □

Remark 4.
(1) The same conclusion holds if the PAC field F is 'bounded' (rather than ω-free), e.g. G_F is finitely generated, since then its theory is even *simple* [CP98, Corollary 4.8], in particular it does not have SOP_3 [She96, Claim 2.7].
(2) Moreover, a PAC field of characteristic zero also has no definable proper sub*fields* [JK10, Lemma 6.1 and Proposition 4.1].
(3) It is known that ω-free PAC fields satisfy not even the weaker property SOP_1 (rather than SOP_3), see [CR16, Corollary 6.8] and [KR20, Section 9.3].

Corollary 5. *The set of subfields F of $\overline{\mathbb{Q}}$ in which $\mathbb{Z}$ or $\mathcal{O}_F$ are definable is meager in $\mathcal{E}$.*

Remark 6. The same arguments go through for separable algebraic extensions of $\mathbb{F}_p(t)$ instead of $\mathbb{Q}$. If one is interested only in $\mathbb{Z}$ not being *existentially* definable, one could apply the much more elementary [Feh10, Theorem 2] and [Ans19, Theorem 1], which work more generally for *large* fields, instead of Proposition 3.

Remark 7. By combining Proposition 1 and Remark 4(2) we also obtain a strengthening of [EMSW20, Corollary 5.8]: For every number field K, the set of fields $F \subseteq \overline{\mathbb{Q}}$ containing K in which K is definable is meager in $\mathcal{E}$.

Remark 8. Similarly, we obtain a strengthening of [EMSW20, Corollary 5.14]: If $\bar{\mathcal{E}}$ denotes the space $\mathcal{E}$ modulo isomorphism of fields, the set of isomorphism classes of fields $F \subseteq \overline{\mathbb{Q}}$

in which $\mathbb{Z}$, $\mathcal{O}_F$, or some some fixed number field K are definable is meager in $\bar{\mathcal{E}}$. Indeed, as the sets $\mathcal{P}$ and $\mathcal{H}$ (notation from the proof of Proposition 1) are dense G_δ-sets invariant under isomorphism, and the quotient map $\mathcal{E} \to \bar{\mathcal{E}}$ is continuous and closed, also the images of $\mathcal{P}$ and $\mathcal{H}$ are dense G_δ-sets, and therefore comeager in $\bar{\mathcal{E}}$.

Remark 9. We sketch how a strengthening of [EMSW20, Theorem 5.11] can also be obtained: The set of computable and decidable fields $F \subseteq \overline{\mathbb{Q}}$ in which neither $\mathbb{Z}$ nor $\mathcal{O}_F$ are definable is dense in $\mathcal{E}$. Indeed, given finite extensions $\mathbb{Q} \subseteq K \subseteq L$, let e be the minimal number of generators of the Galois group of the Galois closure $\hat{L}$ of L/K. By slightly adapting the proof of [JS17, Proposition 2.5] one finds a computable and decidable PAC field $K \subseteq F \subseteq \overline{\mathbb{Q}}$ with absolute Galois group free profinite on e generators and $F \cap \hat{L} = K$, and Remark 4(1) applies to F.

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