Invest or Exit? Optimal Decisions in the Face of a Declining Profit Stream*

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Abstract

Even in the face of deteriorating and highly volatile demand, firms often invest in, rather than discard, aging technologies. In order to study this phenomenon, we model the firm's profit stream as a Brownian motion with negative drift. At each point in time, the firm can continue operations, or it can stop and exit the project. In addition, there is a one-time option to make an investment which boosts the project's profit rate. Using stochastic analysis, we show that the optimal policy always exists and that it is characterized by three thresholds. There are investment and exit thresholds before investment, and there is a threshold for exit after investment. We also effect a comparative statics analysis of the thresholds with respect to the drift and the volatility of the Brownian motion. When the profit boost upon investment is sufficiently large, we find a novel result: the investment threshold decreases in volatility.

1 Introduction

The computer disk drive industry underwent a series of disruptive architectural innovations (Christensen 1992). Until the mid-1970's, 14-inch hard disk drives dominated the mainframe computer disk drive market. Between 1978 and 1980, several new entrants introduced 8-inch disk drives which were initially sold to minicomputer manufacturers because their recording capacity was too small and the cost per megabyte was too high for mainframe computers. As the performance of 8-inch drives kept improving, the entrants quickly encroached upon the mainframe computer market. By the mid-1980's, 8-inch drives dominated the mainframe market and rendered 14-inch drives obsolete. Nevertheless, among the dozen or so established manufacturers of 14-inch drives, two thirds of them never introduced 8-inch drives. Instead, they

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continued to enhance the recording capacity of the extant 14-inch drives in order to appeal to the higher end mainframe market (Christensen 2000, p. 19). Eventually, all 14-inch drive manufacturers, except those that were vertically integrated, were forced out of the disk drive market. This pattern of industry-wide disruption emanating from the introduction of a successful new technology is a commonplace rather than an isolated incident; as such, it deserves serious attention. Even 8-inch drives were eventually superseded by 5.25-inch drives. Currently, the computer disk drive industry is in the process of yet another architectural transition, one from hard disk drives to flash solid state disks (used in USB stick drives).

This paper focuses upon the difficult investment and exit decisions of a firm facing a declining profit stream. With the onslaught of disruptive technological innovation, as in the example of the disk drive industry, a firm employing an extant technology faces a deteriorating profit stream due to declining demand and/or prices. Faced with a profit stream that has eroded, it might be optimal for the firm to cease operations and avoid recurring losses. On the other hand, if the erosion has not been too large, then it can be optimal for the firm to make an additional investment in the project. The pressing question is when, if ever, to invest and when to exit. Exit ought to occur when the current profit rate is sufficiently negative; a negative value of the profit rate, however, is not a sufficient condition to induce exit because the option to cease operations sometime in the future must be taken into account. Likewise, a firm must invest in its operations in a timely fashion before the desirable investment opportunity vanishes. In a highly volatile environment such as in the disk drive industry, however, it is difficult to calculate the optimal time to invest or exit because of the uncertainty in the future demand. After we obtain the optimal policy, we will examine how increases in uncertainty affect the optimal policy.

In light of the declining demand, it seems counter-intuitive to invest in the current operation. However, in the example of the computer hard disk drive industry, the manufacturers of 14-inch disk drives continued investment even though they faced a deteriorating profit stream and, as it turned out, eventual displacement from the industry. Christensen (2000) finds such examples in the mechanical excavator industry and the steel mill industry as well. Rosenberg (1976) also notes the major investments in obsolescent technologies during the transition from the wooden sailing ship to the iron-hull steamship and from the steam to the diesel engine, to name just two amongst many such examples.

The two salient features of our model are the possibility of exit and the declining stochastic profit stream. In particular, the firm can exit at any point in time, and we model the firm's uncertain profit stream as a Brownian motion X_t with drift μ and volatility σ where both μ and σ are time-independent constants known to the firm. Of course, the drift μ is the average rate of change in the profit rate, and the volatility σ measures the underlying uncertainty. In particular, we give special attention to the case in which μ is negative. With this representation, the firm's cumulative profit is the time-integral of the Brownian motion. The firm's investment and exit decisions are stopping times for the Brownian motion;

we utilize the well-known machinery of stochastic analysis (Alvarez 2001a, Borodin and Salminen 2002, Harrison 1985, Oksendal 2003, Peskir and Shiryaev 2006) to prove the existence of the optimal policy and find the optimal stopping times.

In Sec. 3, we present the basic model in which investment is not possible. At each point in time, the firm must decide whether to continue operations or irrevocably exit the project. The firm seeks to maximize its expected discounted cumulative profit by selecting the optimal time τ at which to exit, where τ is a stopping time for the Brownian motion. Utilizing results obtained by Alvarez (2001a), we show that the optimal policy is a threshold rule: it is optimal to continue operations until the profit rate X_t falls below a critical threshold ξ_0, at which time it is optimal to exit. The closed-form expression for ξ_0 is a decreasing function of μ and σ, and it reveals that ξ_0 is negative.

In Sec. 4, we extend the basic model to include a one-time opportunity to invest in improving the extant technology: at each point in time, the firm can (1) continue operations, (2) stop and irrevocably exit the project, or (3) invest in the operations. The investment increases both the current profit rate and the drift of the profit stream by known quantities. In view of the investment opportunity, the firm's policy is specified by three stopping times. The firm must specify when to exit and when to invest while the investment option is still available. If the firm already has made the investment, then the firm must decide when to exit. Each stopping time is characterized by a threshold. If investment has not been made, it is optimal to exit whenever the profit rate falls below a threshold ξ_E, and it is optimal to invest if the profit rate rises above a second threshold ξ_I. When the current profit rate is between ξ_E and ξ_I, it is optimal to maintain the *status quo*: continue operations but do not invest. Because there is only one opportunity to invest, after investment, the firm's decision problem reduces to that of the basic model, albeit with different drift: after investment, the firm exits when the current profit rate drops below a third threshold ξ_1.

After finding the optimal policy, we effect a comparative statics analysis of the thresholds ξ_I and ξ_E with respect to μ and σ. Although it is intuitively clear that the optimal policy is characterized by thresholds, the comparative statics analysis is neither intuitively obvious nor straightforward. In order to obtain ξ_I and ξ_E, we first need to solve an optimal stopping time problem with a reward which depends on the return from investment. The complication is that the return from investment in turn depends on both μ and σ because the firm will continue operations prior to eventual exit. Nevertheless, we have been able to effect a comparative statics analysis using a power-series expansion method without resorting to a numerical analysis.

We must proceed cautiously in applying intuition from real options theory to the comparative statics of the threshold for investment (ξ_I). For example, consider real options models in which the return from investment is independent of σ; real options theory has shown that, under certain mild conditions, it is optimal to wait longer before making an irrevocable investment if the volatility of the underlying asset increases (Dixit 1992, Alvarez 2003). Waiting and observing the evolution of the value of the asset enables the investor to avoid the downturn risk and take advantage of the upturn potential. In accord with this intuition, we anticipate that ξ_I increases in σ because the upturn potential of the profit stream increases in σ. Indeed, if the boost in the profit rate upon investment is small enough, then ξ_I increases in σ as expected. Surprisingly, if the boost is sufficiently large, then ξ_I decreases in σ; this novel comparative statics of ξ_I results because the return from investment increases in σ due to the embedded option to exit, and consequently, the investment option is more attractive with higher σ. The ever-present exit option is the salient feature which sets our model apart from the conventional real options models. In the operations context, our novel comparative statics result offers cautionary advice against blindly following the intuition inherited from real options theory. For example, Bollen (1999) shows that if the product life cycle (demand dynamics) is ignored, then the conventional real-option technique tends to undervalue capacity contraction and overvalue capacity expansion.

This paper is organized as follows. We review related literature in Sec. 2, and we present our basic model and some results about the basic model in Sec. 3. The basic model is extended to include one investment opportunity in Sec. 4, and we effect the comparative statics of the optimal policy in Sec. 4.3. Summary of the paper is given in Sec. 5.

2 Related Literature

There is a rich literature on technology and process adoption. (See, for example, Bridges et al. 1991 for a review.) In an early paper which formulates technology adoption as an investment problem, Barzel (1968) uses the net-present-value approach to obtain the optimal timing of a one-time investment in the adoption of technology when the future profit stream is deterministic. In the context of process improvement, Porteus (1985) uses the EOQ model to examine the economic trade-offs between the cost of investment which reduces the setup cost and the benefit from the reduced setup cost: the optimal policy is to invest if and only if the sales rate is above a threshold. Porteus (1986) extends this work by examining a model in which lower setup costs lead to improved quality control (lower defect rate).

An objective of the current paper is to obtain the optimal investment and exit policy under uncertainty. Many papers have modeled technology adoption as a stopping time problem. (See Hoppe 2002 for a survey of literature.) For example, Balcer and Lippman (1984) study the optimal time to adopt the best currently available technology when multiple adoptions are allowed. In their model, the *timing* and the *value* of future innovations is uncertain although the profitability of the currently available technology is known. They show that it is optimal to adopt the best currently available technology if the technological lag exceeds a threshold which depends upon the multi-dimensional state: the elapsed time since last innovation and the pace (rapidity) of technological progress. There is also a substantial literature on Bayesian models of investment and exit; see Jensen (1982), McCardle (1985), Ryan and Lippman (2003), and Ryan and Lippman (2005).

One important contribution of our paper is the results regarding the impact of uncertainty on the investment and exit decisions. Dixit (1992, p. 108) points out that, as uncertainty increases, it is optimal to wait longer before investment if (1) the investment is irreversible, (2) the uncertainty regarding the investment is being resolved gradually in time, and (3) the investment can be flexibly postponed. In this vein, McDonald and Siegel (1986) study investment in an asset whose value and price evolve as geometric Brownian motion. They find that the optimal policy is a threshold rule with respect to the ratio of the value to the price of the asset. Moreover, the investment threshold increases in the volatility: it is optimal to postpone investment longer as the uncertainty increases.

A number of papers address the effect of uncertainty on technology adoption using the real options approach. Essentially, they confirm the conventional intuition regarding the value of waiting. Farzin et al. (1998) study the optimal time to irreversibly switch to a new technology when the value and the arrival date of future improvements are uncertain. In their model, the improvement in the value of the currently available technology follows a compound Poisson process. They allow multiple investments in technology; again, the optimal policy is a threshold rule. In particular, they find that the pace of adoption is slower with the real-option method than with the suboptimal net-present-value method. Alvarez and Stenbacka (2001) also use the real options approach to study the optimal time to adopt a technology with an opportunity for improvement after adoption. Once the firm adopts the technology, it receives a revenue stream which evolves stochastically over time: at an exponential time, an improved technology becomes available to the firm. They show that increased market uncertainty (volatility) increases the real-option value of adopting the initial technology.

The real options method has also been applied to exit decisions in a duopoly game when the profit stream is stochastic. Fine and Li (1986) find a Nash equilibrium in stopping times in their discrete-time duopoly game of exit from a market with declining stochastic demand. Murto (2004) studies a similar duopoly exit game in an industry in which the declining demand follows a geometric Brownian motion; he obtains Markov-perfect equilibria. Although these two papers analyze a duopoly model, they also consider the exit problem of a monopolist which is similar to our basic model. Their focus, however, is on the strategic interaction rather than on the uncertainty.

Mathematical characteristics of general real options models have been extensively studied in the stochastic control theory literature. In particular, many real options models can be formulated as optimal stopping problems. The solution method of optimal stopping time problems consists of finding a candidate solution by solving a differential equation and applying a verification theorem which includes smooth-pasting conditions (that the optimal return function is continuously differentiable); for example, see Dayanik and Karatzas (2003), Chapter 10 of Oksendal (2003), and Chapter IV of Peskir and Shiryaev (2006). Alvarez (2001a) considers a class of optimal stopping problems with a profit rate and a salvage value which are functions of a linear diffusion process. He finds a representation of the optimal return function, a set of necessary conditions for the optimal solution, and also conditions under which the necessary conditions are sufficient. Alvarez (2003) studies a class of optimal stopping problems which often occur in economic decision problems and finds general comparative statics with respect to volatility. In particular, he characterizes a set of conditions under which the value function and the optimal continuation region increase in volatility.

The solution methods for stopping time problems have been applied to study mathematical properties of financial options (see, for example, Guo 2001 and Ekstrom 2004) as well as a wide variety of real options models. For example, Wang (2005) studies an optimal stopping time problem in which the decision-maker has a sequence of projects. For each project, there is a forced exit event at which time the decision-maker is forced to stop the project. The forced exits occur as a Poisson process, and the decision variable is the sequence of entry times. If the entry cost is large enough, then the presence of the forced exits enlarge the continuation region. The stopping time framework can be also applied to study real options problems under incomplete information. Decamps et al. (2005) study the optimal timing of investment in a noisy asset whose true underlying value (the drift of the Brownian motion) is unknown. The information about the value of the drift is Bayesian-updated based on the observed value of the Brownian motion. In their model, the return to investment is a function of the Brownian motion itself rather than its drift, so the optimal policy is path-dependent. Lastly, there are real options models in which the rate of production at each point in time is another control variable in addition to the stopping time. These problems are handled by the Hamilton-Jacobi-Bellman formulation of stochastic control theory; see, for example, Knudsen et al. (1998), Duckworth and Zervos (2000), Alvarez (2001b), and Kumar and Muthuraman (2004).

In addition to the uncertainty in the profit stream, there is another complicating but salient feature in our model: exit is possible after investment. Among the papers that include this feature, McDonald and Siegel (1985) study the valuation of a manufacturing firm facing a stochastic price for its output product using option pricing techniques. In their model, the product price is a geometric Brownian motion, and the firm can shutdown and re-open its plant without cost at any point in time. In contrast, Dixit (1989) considers fixed cost of entry and exit. In his model, the firm can enter and exit the industry as many times as the firm wishes, and the profit stream is a geometric Brownian motion. He shows that it is optimal to invest if the profit rate is above an upper threshold and exit if it is below a lower threshold. He performs a numerical comparative statics analysis and finds that the upper (lower) threshold increases (decreases) in the volatility. In his model, the investment (entry) decision can be exercised only by an inactive firm; of course, the exit decision can be exercised only by active firms. Our paper studies investment and exit decisions in a quite different model: the firm has one opportunity to invest in its operations while being active in the industry, and it can exit at any point in time. Moreover, our comparative statics results are analytical.

In the literatures on technology adoption and on exit, there is a paucity of work on investment when the firm faces a declining profit stream. To our knowledge, the current paper is the first to study the impact of uncertainty on investment in an on-going project with an exit option available both before and after an investment.

3 The Basic Model

Consider a manufacturing firm whose product is produced with an aging technology or process. Because of obsolescence, its profit stream is in decline (perhaps because a substitute product produced with a new technology is encroaching upon the market). At any point in time, the firm can stop the project by permanently closing its production plant.

The firm, seeking to maximize the expected discounted value of its profit stream over an infinite horizon, must determine the best time to cease operations and exit the market. The firm's profit rate at time t is a random variable X_t where $\{X_t : t \geq 0\}$ is a stochastic process with continuous sample paths whose law of motion we will specify shortly. We refer to $\{X_t : 0 \leq t \leq \tau\}$ as the firm's *profit stream* where the stopping time $\tau \leq \infty$ is the time of exit.

We model the firm's profit stream as a Brownian motion with constant drift μ and volatility σ. Specifically, let X_t denote the profit at time t with $X_t = X_0 + \nu t + \sigma B_t$ where $\{B_t : t \geq 0\}$ is a one-dimensional standard Brownian motion, so the profit stream has constant drift ν and constant volatility σ. We pay particular attention to the case $\nu < 0$ because our main focus is modeling a declining profit stream. If the firm begins operations at time 0 and exits at a stopping time τ, the discounted value of its profit stream is $\int_0^\tau e^{-\alpha t} X_t dt$, where α is the strictly positive discount rate. (To be more precise, $\{X_t : t \geq 0\}$ is a one-dimensional Brownian motion adapted to a filtration $\{\mathcal{F}_t\}$ of a probability space $(\Omega, \mathcal{F}, P)$. The random variable τ is an element of $\mathcal{T}$, the set of all non-negative stopping times with respect to the filtration $\{\mathcal{F}_t\}$.)

To illustrate, suppose that the demand D_t per unit time for the firm's product is a Brownian motion with drift ν/p, where p is the sales price per unit, and let c be the fixed cost of operation per unit time. Then the relationship between the demand and the profit stream is linear:

$$X_t = pD_t - c\,. \tag{1}$$

Next, we review the solution and a few characteristics of the optimal solution to the basic model. Our results below can be derived from the more general results of Alvarez (2001a) and Alvarez (2003). In our basic model, at each point in time, the firm must elect either to continue operations or to exit irrevocably. The firm seeks a stopping time τ which maximizes

$$E^x[\int_0^\tau X_t e^{-\alpha t} dt], \tag{2}$$

where $E^x[\cdot] \equiv E[\cdot|X_0 = x]$, the expectation conditioned on $X_0 = x$. The initial profit rate x can be any real number.

The firm receives cumulative discounted profit from continuation and, implicitly, a reward of zero after exit. Hence, if the firm were to exit at $\tau = 0$, then it receives zero profit. For convenience, we transform the problem to one with a reward from exit using the strong Markov property of X and the fact that $E^x \int_0^\infty |X_t| e^{-\alpha t} dt < \infty$ (Alvarez 2001a, p. 318):

$$\begin{aligned} E^x[\int_0^\tau X_t e^{-\alpha t} dt] =& E^x[\int_0^\infty X_t e^{-\alpha t} dt - \int_\tau^\infty X_t e^{-\alpha t} dt] \\ =& E^x[\int_0^\infty X_t e^{-\alpha t} dt] - E^x[E^{X_\tau}[\int_0^\infty X_t e^{-\alpha t} dt]] \\ =& \alpha^{-1}(x + \nu/\alpha) - \alpha^{-1} E^x[e^{-\alpha\tau}(X_\tau + \nu/\alpha)], \end{aligned} \tag{3}$$

which conforms to the standard stopping time problem with a reward from exit. Here we used the identity

$$E^x[\int_0^\infty X_t e^{-\alpha t} dt] = \alpha^{-1}(x + \nu/\alpha),$$

which can be derived using the Green function method (Alvarez 2001a, p. 319).

If the exit policy is stationary, then we can represent the stopping time as τ_D which denotes the *first exit time of the process X_t from a measurable set $D \subset \mathbb{R}$*:

$$\tau_D \equiv \inf\{t > 0 : X_t \notin D\}.$$

In other words, the policy represented by the stopping time τ_D is to continue operations as long as $X_t \in D$ and stop when $X_t \notin D$. The set D is called a *continuation region*. The objective function in Eq. (2) or Eq. (3) has no time-dependence other than through the process X_t and the discount factor $e^{-\alpha t}$; hence, we can show directly (or use the argument of Oksendal 2003, p. 220) that the optimal policy, *if it exists*, is stationary. Throughout this section, we set $V(x;\nu) \equiv \sup_\tau E^x[\int_0^\tau X_t e^{-\alpha t} dt]$. If $V(x;\nu) = E^x[\int_0^{\tau_D} X_t e^{-\alpha t} dt]$, then D is the *optimal continuation region*, and an optimal policy exists.

Proposition 1 *The optimal policy for the exit problem in Eq. (2) exists, and the optimal return is given by*

$$V(x;\nu) = \begin{cases} x/\alpha + \nu/\alpha^2 + \frac{\sigma^2/\alpha}{\nu+\sqrt{\nu^2+2\alpha\sigma^2}} \exp[\frac{-\nu-\sqrt{\nu^2+2\alpha\sigma^2}}{\sigma^2}(x-\xi(\nu))] & \text{if } x > \xi(\nu)\,, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

where $\xi(\nu)$ *is a negative number given by*

$$\xi(\nu) = -\frac{\nu}{\alpha} - \frac{\sigma^2}{\nu+\sqrt{\nu^2+2\alpha\sigma^2}} = \frac{\sigma^2}{\nu-\sqrt{\nu^2+2\alpha\sigma^2}}\,. \tag{5}$$

Moreover, the optimal continuation region is $(\xi(\nu),\infty)$.

Proof: The proof follows directly from Proposition 2 of Alvarez (2001a). In the terminology used by Alvarez (2001a), the decreasing fundamental solution is $e^{\phi x}$ where ϕ is defined in Eq. (6) below, the profit rate function is x, and the reward from exit is zero. From the fact that $E[\int_0^\infty X_t e^{-\alpha t} dt] = x/\alpha + \nu/\alpha^2$, it is straightforward to show that the optimal threshold is $\arg\max_x[-(x/\alpha+\nu/\alpha^2)/e^{\phi x}] = \xi(\nu)$ and to verify that the optimal return function given by Proposition 2 of Alvarez (2001a) reduces to Eq. (4). ■

The most conventional way to solve a stopping time problem such as the one in Eq. (3) is to propose a candidate function $V(\cdot;\nu)$ which is the return function from a candidate policy and verify that the candidate function $V(\cdot;\nu)$ satisfies a number of conditions as given by Theorem 10.4.1 of Oksendal (2003). One of the conditions is that $V(\cdot;\nu)$ should satisfy a second-order ordinary differential equation (ODE): $(-\alpha + \nu\partial_x + \frac{1}{2}\sigma^2\partial_x^2)V(\cdot;\nu) = -x$. Then $V(\cdot;\nu)$ is a sum of the term $(x/\alpha+\nu/\alpha^2)$ and a function $f(x)$ which satisfies the ODE $(-\alpha+\nu\partial_x+\frac{1}{2}\sigma^2\partial_x^2)f(x) = 0$. The general solution $f(x)$ is a linear combination of $e^{\psi x}$ and $e^{\phi x}$ where

$$\psi(\nu) = (-\nu+\sqrt{\nu^2+2\alpha\sigma^2})/\sigma^2 \qquad \text{and} \qquad \phi(\nu) = (-\nu-\sqrt{\nu^2+2\alpha\sigma^2})/\sigma^2\,. \tag{6}$$

(We thank an anonymous referee for pointing us to Proposition 2 of Alvarez (2001a): our exit problem in Eq. (3) is a special case of Proposition 2 of Alvarez (2001a) and admits a much shorter proof above.) Interestingly, the optimal policy exists, and the closed form solution $V(\cdot;\nu)$ is obtained.

Proposition 1 says that the optimal policy is to exit when the profit rate goes below the threshold $\xi(\nu)$. It is intuitively clear that the firm will exit if its profit rate has deteriorated below some threshold, but the fact that the threshold is negative is not obvious. The reason $\xi(\nu) < 0$ is that there is value in waiting before taking an irrevocable action: even if $\nu < 0$ and the current profit rate is slightly negative, it is possible for the profit rate to turn positive in the future. If the profit stream were deterministic and monotonically decreasing, then it would be optimal to exit when the profit rate hits zero. This intuition regarding the value of waiting is consistent with the fact that $\xi(\nu)$ increases to 0 as $\sigma \to 0$, which follows from Eq. (5)

when $\nu < 0$.

Now that we have a closed form solution for $V(\cdot;\cdot)$ and $\xi(\cdot)$ in terms of all of the model parameters, their comparative statics are straightforward as follows.

Proposition 2 *For any $\nu < 0$, (i) $\psi(\nu)$ decreases in σ and ν and increases in α; $\phi(\nu)$ increases in σ and decreases in ν and α. (ii) The exit threshold $\xi(\nu)$ decreases in σ and ν, and it increases in α. (iii) The optimal return $V(x;\nu)$ increases in σ and ν for $x > \xi(\nu)$.*

(The proof is in the e-companion to this paper.)

The fact that $\xi(\nu)$ decreases in σ is shown by Alvarez (2003) in a more general stopping time problem. The comparative statics of $V(\cdot;\nu)$ with respect to σ also follows from a more general characteristic of stopping time problems obtained by Theorem 5 of Alvarez (2003): as σ increases, there is more noise in the profit stream, so there is a larger upturn potential as well as a larger downturn risk. However, the firm can take advantage of the upturn potential while avoiding downturn risk by exit. Hence, the return function increases in σ. Similarly, because an increase in ν improves the profit stream X_t, the return function also increases for each continuation region D, so $V(\cdot;\nu)$ increases in ν.

In the next section, we consider a *one-time* opportunity to invest. The firm initially begins with drift $\mu < 0$, and the investment boosts the profit rate by b and the drift by $\delta \geq 0$. Because there is only one investment opportunity, the post-investment problem reduces to the basic model with drift $\nu = \mu + \delta$. For convenience, we define

$$V_0(\cdot) \equiv V(\cdot;\mu)$$

as the optimal return to the exit problem with drift μ,

$$\gamma_p \equiv \psi(\mu) = (-\mu + \sqrt{\mu^2 + 2\alpha\sigma^2})/\sigma^2 \quad \text{and} \quad \gamma_n \equiv \phi(\mu) = (-\mu - \sqrt{\mu^2 + 2\alpha\sigma^2})/\sigma^2 \,, \tag{7}$$

and

$$\xi_0 \equiv \xi(\mu)$$

as the optimal exit threshold with drift μ. Similarly, we define

$$V_0^+(\cdot) \equiv V(\cdot;\mu + \delta)$$

as the optimal return to the exit problem with the boosted drift $\mu + \delta$, and

$$\begin{aligned} \lambda \equiv \phi(\mu+\delta) &= [-(\mu+\delta) - \sqrt{(\mu+\delta) + 2\alpha\sigma^2}]/\sigma^2 \,, \\ \xi_1 \equiv \xi(\mu+\delta) &= -\frac{\mu+\delta}{\alpha} + \frac{1}{\lambda} \,, \end{aligned} \tag{8}$$

where ξ_1 is the post-investment optimal exit threshold. Since the comparative statics obtained in Proposition 2 applies for any negative drift, it applies to $V_0(\cdot)$, ξ_0, $V^+(\cdot)$, ξ_1, γ_p, γ_n, and λ as long as $\mu < 0$ and $\mu + \delta < 0$. In the next section, $\max\{V_0^+(x+b) - k, 0\}$ plays the role of the reward function from investment, where k is the cost of investment. By Proposition 2, the reward from investment increases in the volatility, a salient feature of our model.

4 The Model with One Investment Opportunity

Consider the possibility of a once-in-a-lifetime investment. For instance, manufacturers of 14-inch disk drives can, despite the writing on the wall, improve the performance (recording capacity) of 14-inch drives in order to immediately boost demand in the higher-end mainframe computer market (Christensen 2000, p. 19). Of course, eventual exit is inevitable when $\mu < 0$.

For analytical tractability, our model allows only one investment opportunity. As suggested by Fine and Porteus (1989), in practice, the firm might have multiple opportunities for gradual improvement in the technology/process. The impact of multiple investment opportunities is beyond the scope of this paper.

4.1 The Model

We now include a one-time opportunity to implement an innovation which improves the quality of the product or the process. The implementation cost is $k > 0$. If the quality of the product improves, then the demand for the product increases; moreover, the demand declines more slowly. Specifically, the investment boosts the current profit rate by b and increases the drift by δ. In terms of the example specified in Eq. (1) where the profit rate is linear in the demand D_t, investment induces an increase of b in pD_t (or, equivalently, a decrease of b in c) and an increase of δ in $p \cdot dD_t/dt$. If the firm invests at time τ, then the improved profit stream follows the process

$$Y_t \equiv X_t + \delta(t - \tau) + b\,, \quad \text{for } t > \tau$$

so that $dY_t = (\mu + \delta)dt + \sigma dB_t$.

Prior to investment, the firm needs to find the optimal stopping time τ at which to invest or to exit, whichever action results in a better payoff. If the firm invests at time τ, then its expected return starting at time τ is $V_0^+(X_\tau + b) - k$ because its expected cumulative profit stream after investment is $V_0^+(X_\tau + b)$ and the cost of investment is k. On the other hand, if the firm exits at time τ, then its return starting at time τ is 0. Hence, the firm receives the expected payoff of $\max\{V_0^+(X_\tau + b) - k, 0\}$ at time τ when it makes its investment or exit decision.

Let x^+ be the unique number which satisfies

$$V_0^+(x^+ + b) = k \quad . \tag{9}$$

(This definition uniquely determines x^+ because $V_0^+(x)$ is strictly increasing in x for all x such that $V_0^+(x) > 0$.) Then, at a stopping time τ, it is optimal to exit if $X_\tau < x^+$ and invest if $X_\tau > x^+$ because $V_0^+(x+b) - k > 0$ if $x > x^+$ and $V_0^+(x+b) - k < 0$ if $x < x^+$. If the current profit rate X_t is x^+, then immediate investment and immediate exit both yield zero expected return. Hence, our objective is to find

$$\bar{V}(x) \equiv \sup_{\tau \in \mathcal{T}} E^x[\int_0^\tau e^{-\alpha t} X_t dt + e^{-\alpha\tau} h(X_\tau)] \, , \tag{10}$$

where

$$h(x) = \max\{0, V_0^+(x+b) - k\} \tag{11}$$

is the lump sum payoff when x is the state when stopping occurs.

Next, we examine the conditions under which it is never optimal to invest. Define

$$g \equiv \alpha(\int_0^\infty (b + \delta t) e^{-\alpha t} dt - k) = b + \delta/\alpha - k\alpha \tag{12}$$

so that g/α is the net discounted gain from investment if exit never occurs.

Proposition 3 *Investment is never optimal if and only if $g \leq 0$.*

Proof: To prove this proposition, we study the difference between the return from immediate investment ($V_0^+(x+b) - k$) and the optimal return from waiting to exit ($V_0(x)$) for $x > x^+$:

$$[V_0^+(x+b) - k] - V_0(x) = \frac{g}{\alpha} - \frac{1}{\lambda\alpha} \exp[\lambda(x + b - \xi_1)] + \frac{1}{\gamma_n \alpha} \exp[\gamma_n(x - \xi_0)] \, . \tag{13}$$

Note that $-\frac{1}{\lambda\alpha}\exp[\lambda(x+b-\xi_1)] < -\frac{1}{\gamma_n\alpha}\exp[\gamma_n(x-\xi_0)]$ because $\lambda < \gamma_n < 0$ and $\xi_1 < \xi_0$ by Proposition 2 (i) and (ii). If $g \leq 0$, then the right-hand-side of Eq. (13) is negative for all $x > x^+$. Thus, at any given stopping time τ, the decision-maker is better off choosing $V_0(X_\tau)$ (return from waiting to exit) than $V_0^+(X_\tau + b) - k$ (return from immediate investment). We conclude that investment is never optimal if $g \leq 0$.

Now suppose that investment is also never optimal for $g > 0$. In this case, Eq. (13) is positive for sufficiently large x because the two exponential terms converge to zero as $x \to \infty$. Hence, at any τ such that X_τ is sufficiently large, immediate investment is a better option than waiting to exit. This contradicts the assumption that investment is never optimal. We conclude that the policy of waiting to exit without ever investing is not optimal when $g > 0$. ■

In light of Proposition 3, we assume $g > 0$ for the remainder of the paper unless otherwise noted.

4.2 Existence and Characterization of the Optimal Policy and the Optimal Return

In this subsection, we verify that an optimal policy always exists and that it is essentially unique. We show that, prior to investment, the optimal policy is completely characterized by a pair (ξ_E, ξ_I) of thresholds: exit if $X_t \leq \xi_E$ and invest if $X_t \geq \xi_I$; after investment, the problem reverts to the one analyzed at the end of Sec. 3 where we established that it is optimal to exit as soon as $X_t \leq \xi_1$. Partial recompense for the analytical difficulty implicit in our model is found in the closed form solution for the optimal return function as given in Eq. (16).

If the optimal policy exists, it is stationary because neither the payoff $\max\{V_0^+(X_t+b)-k, 0\}$ nor the profit stream has any time-dependence other than through X_t and $e^{-\alpha t}$. Thus, it suffices to consider the class of stopping times $\tau_D = \inf\{t > 0 : X_t \notin D\}$ expressed with respect to continuation regions D. Consequently, we can express the objective function as

$$R_D(x) = E^x[\int_0^{\tau_D} e^{-\alpha t} X_t dt + e^{-\alpha \tau_D} h(X_{\tau_D})] \ . \tag{14}$$

In this new representation, the firm's policy is to continue operations as long as $X_t \in D$ and to stop as soon as $X_t \notin D$, at which time the firm receives $h(X_t)$. Hence, if the optimal policy exists, our objective is to find the optimal continuation region D^* such that

$$\bar{V}(x) \equiv \sup_D R_D(x) = R_{D^*}(x) \ . \tag{15}$$

Proposition 4 *In the stopping time problem of Eq. (15), the optimal policy always exists; the optimal expected return is uniquely given by*

$$V_1(x) = \begin{cases} x/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p x} + a_2 e^{\gamma_n x} , & \text{for } x \in D^* = (\xi_E, \xi_I) \\ h(x) \quad , & \text{otherwise} \end{cases} \ , \tag{16}$$

and the optimal continuation region is $D^* = (\xi_E, \xi_I)$: it is optimal to exit when $x \leq \xi_E$, invest when $x \geq \xi_I$, and continue operations otherwise.

The proof of Proposition 4 is in the e-companion to this paper. The proof proceeds by considering $V_1(\cdot)$ as a candidate for being the optimal return function and then verifying that $V_1(\cdot)$ satisfies all the sufficient conditions for being the optimal return function specified in Theorem 10.4.1 of Oksendal (2003). (The general necessary conditions for optimality of a return function with a *bounded* continuation region are obtained by Alvarez 2001a; see also Guo 2001.) Surprisingly, the existence proof is rather difficult. It amounts to showing that the following boundary conditions

$$V_1(\xi_E) = \xi_E/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p \xi_E} + a_2 e^{\gamma_n \xi_E} = h(\xi_E) = 0\,, \tag{17}$$

$$\begin{aligned} V_1(\xi_I) &= \xi_I/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p \xi_I} + a_2 e^{\gamma_n \xi_I} \\ &= h(\xi_I) = (\xi_I + b)/\alpha + \mu^+/\alpha^2 - (\alpha\lambda)^{-1} e^{\lambda(\xi_I + b - \xi_1)} - k\,, \end{aligned} \tag{18}$$

along with the smooth-pasting conditions

$$\partial_x V_1(\xi_E) = \alpha^{-1} + \gamma_p a_1 e^{\gamma_p \xi_E} + \gamma_n a_2 e^{\gamma_n \xi_E} = \partial_x h(\xi_E) = 0\,, \tag{19}$$

$$\partial_x V_1(\xi_I) = \alpha^{-1} + \gamma_p a_1 e^{\gamma_p \xi_I} + \gamma_n a_2 e^{\gamma_n \xi_I} = \partial_x h(\xi_I) = \alpha^{-1}[1 - e^{\lambda(\xi_I + b - \xi_1)}]\,. \tag{20}$$

always have a solution with desirable properties as stipulated by Theorem 10.4.1 of Oksendal (2003).

The optimal return function $V_1(\cdot) = \bar{V}(\cdot)$ is, of course, unique. It follows that the optimal policy is unique: to stop when $X_t \in \{x : x < \xi_E \text{ or } x > \xi_I\}$ and continue otherwise. If the current profit rate is x^+, then there is positive probability that the profit rate will increase to a value bigger than x^+ in the immediate future. Hence, the expected return from waiting is positive, so $V_1(x^+) > 0$ and $x^+ \in D^* = (\xi_E, \xi_I)$. By Proposition 4, the firm's optimal policy is to stop the first time X_t hits ξ_E or ξ_I and receive the reward $h(X_t)$. Because $\xi_E < x^+ < \xi_I$, Eq. (9) reveals that $V_0^+(\xi_I + b) - k > 0$ and $V_0^+(\xi_E + b) - k < 0$. As anticipated, the firm's optimal action at the stopping time τ_{D^*} depends on which end of the interval (ξ_E, ξ_I) the profit rate X_t hits first. It is optimal to exit if X_t hits ξ_E at time τ_{D^*}, and it is optimal to invest if X_t hits ξ_I at time τ_{D^*}.

4.3 Comparative Statics

In this subsection, we effect a comparative statics analysis of $V_1(\cdot)$, ξ_E, and ξ_I. We first establish the convexity of $V_1(\cdot)$, which leads to the comparative statics with respect to σ.

Lemma 1 *The optimal return function $V_1(\cdot)$ is convex.*

(The proof is in the e-companion to this paper.)

Next, we examine the comparative statics of $V_1(\cdot)$ with respect to μ and σ.

Proposition 5 *For all $x \in \mathbb{R}$, $V_1(x)$ is non-decreasing in μ and σ. In particular, $V_1(x)$ is strictly increasing in μ for $x > \xi_E$.*

Proof: To begin, note that $h(\cdot)$ is convex and non-decreasing because $V_0^+(\cdot)$ is convex and non-decreasing. Also note that $h(\cdot)$ is non-decreasing in both μ and σ because $V_0(\cdot)$ is non-decreasing in μ and σ by Proposition 2.

To show that $V_1(\cdot)$ is non-decreasing in μ, let $V_1(x;\mu)$ and $h(x;\mu)$ denote the dependence of $V_1(x)$ and $h(x)$ on the initial (pre-investment) drift μ. Then for any $\beta > 0$ and $x > \xi_E$,

$$\begin{aligned} V_1(x;\mu) &= E^x[\int_0^{T_\mu} X_t e^{-\alpha t} dt + e^{-\alpha T_\mu} h(X_{T_\mu};\mu)] \\ &< E^x[\int_0^{T_\mu} (X_t + \beta t) e^{-\alpha t} dt + e^{-\alpha T_\mu} h(X_{T_\mu} + \beta T_\mu; \mu + \beta)] \le V_1(x;\mu+\beta) \end{aligned}$$

where T_μ is the optimal stopping time which maximizes $R_D(x)$ when the drift is μ. In establishing the strict inequality, we used the fact that $T_\mu > 0$ for $x > \xi_E$, $h(x;\mu)$ is non-decreasing in x and μ, and T_μ is suboptimal when the drift is $\mu + \beta$.

For comparative statics with respect to σ, we first assume that $h(\cdot)$ does *not* have functional dependence on σ. By Lemma 2 and Theorem 4 of Alvarez (2003) concerning the comparative statics of more general stopping problems, because $V_1(\cdot)$ is convex and it is obtained as the return from stopping at $\tau_{(\xi_E,\xi_I)}$, $V_1(\cdot)$ is non-decreasing in σ. (See also Ekstrom (2004)). Moreover, the reward function $h(\cdot)$ is non-decreasing in σ by Prop 2. Thus, $V_1(x)$ is non-decreasing in σ ■

By the same argument used in the proof of Theorem 5 of Alvarez (2003), the comparative statics of ξ_E follows from Proposition 5:

Corollary 1 *The exit threshold ξ_E satisfies $\partial_\mu \xi_E < 0$ and $\partial_{\sigma^2} \xi_E \le 0$.*

Proof: Noting that $\xi_E = \inf\{x : V_1(x) > 0\}$, this result follows from the fact that $V_1(\cdot)$ is strictly increasing in μ for $x > \xi_E$ and non-decreasing in σ (by Proposition 5). ■

In contrast, the comparative statics of ξ_I is considerably more complicated. Because $V_1(x) > V_0^+(x + b) - k$ if and only if $x < \xi_I$, we can write $\xi_I = \sup\{x : V_1(x) - [V_0^+(x+b) - k] > 0\}$. Hence, the dependence of both $V_1(\cdot)$ and $V_0^+(\cdot)$ on μ and σ determine the comparative statics of ξ_I. This is in stark contrast to models in which the reward functions have no σ or μ dependence. (See, for example, Theorem 5 of Alvarez, 2003.) In order to examine the comparative statics of ξ_I, we need to study the equations for both ξ_E and ξ_I as expressed by Eqs. (21) and (22) of Appendix B, where λ is given by (8).

Note that a closed-form expression for ξ_I and ξ_E can not be obtained from Eqs. (21) and (22). In contrast, using the closed-form expression for ξ_0, it was straightforward to effect a complete comparative statics analysis of ξ_0. Lack of a closed-form expression impairs our ability to effect a comparative statics analysis of ξ_I. However, we can obtain useful insights by examining the leading-order terms of ξ_I in power series expansions of g when b is close to $\alpha k - \delta/\alpha$ (g is small) and when b is large (g is large). We do not consider δ large because we restrict our discussions to the interesting case $\mu^+ < 0$, i.e., the profit stream is in decline even after investment.

Using the expansions given in Lemmas 4 and 5 of Appendix B, we obtain the limiting behavior of ξ_I and ξ_E. As $g \to 0$, we find $\xi_E \to \xi_0$ and $\xi_I \to \infty$; this echoes the intuition that it is almost never optimal to invest when g is close to zero. In the other limit where $b \to \infty$, we find $\xi_E \to -\infty$ and $\xi_I - \xi_E \to 0$; this occurs because it is optimal to invest whenever b is sufficiently large.

Lemma 2 *(i)* $\partial_{\sigma^2}\gamma_n > 0$ *and* $\partial_{\sigma^2}\lambda > 0$*; (ii)* $\partial_\mu \gamma_n < 0$ *and* $\partial_\mu \lambda < 0$*; (iii)* $\lambda/\gamma_n \geq 1$.

Proof: From Eqs. (7) and (8), (i) and (ii) can be shown after some algebra. Statement (iii) follows from statement (ii) because γ_n is equal to λ if μ is replaced by $\mu + \delta$ with $\delta \geq 0$. ■

Proposition 6 *For sufficiently small values of g, (i)* $\partial_{\sigma^2}\xi_I > 0$ *and* $\partial_{\sigma^2}\xi_E < 0$*, and also (ii)* $\partial_\mu \xi_I < 0$.

Proof: Take partial derivatives of leading order terms of Eqs. (23) and (24) with respect to μ and σ^2 and use statement (i) of Proposition 2. ■

Proposition 7 *For b sufficiently large, (i)* $\partial_{\sigma^2}\xi_I < 0$ *and* $\partial_{\sigma^2}\xi_E < 0$*, and also (ii)* $\partial_\mu \xi_I < 0$.

Proof: (i) From the definition of ξ_0 and Eq. (25), we have (a function $f(x)$ such that $f(x) \to 0$ as $x \to \infty$ is said to be $o(1)$)

$$\partial_{\sigma^2}\xi_E = -\gamma_n^{-2}\partial_{\sigma^2}\gamma_n + \partial_{\sigma^2}\theta + o(1) = -z(e^z - 1)^{-1}\lambda^{-2}\partial_{\sigma^2}\lambda + o(1)\,,$$

where θ is a positive number defined by Eq. (27), $z \equiv -\lambda(\theta + \alpha k + \gamma_n^{-1} - \lambda^{-1})$ is strictly positive by Lemma 6, and $\partial_{\sigma^2}\theta$ is given by Eq. (28). Note that z and θ are independent of b so that they are not affected when we take the limit as $b \to \infty$. Because $\partial_{\sigma^2}\lambda > 0$ from Proposition 2 (i), we have $\partial_{\sigma^2}\xi_E < 0$ for sufficiently large b. From Eq. (26), we have $\partial_{\sigma^2}(\xi_I - \xi_E) \to 0$ as $b \to \infty$ so that

$$\partial_{\sigma^2}\xi_I = \partial_{\sigma^2}\xi_E + \partial_{\sigma^2}(\xi_I - \xi_E) = -z(e^z - 1)^{-1}\lambda^{-2}\partial_{\sigma^2}\lambda + o(1)\,.$$

Thus, $\partial_{\sigma^2}\xi_I < 0$ for sufficiently large b.

(ii) By Corollary 1, $\partial_\mu \xi_E < 0$. From Eqs. (26) and (29),

$$\partial_\mu \xi_I = \partial_\mu \xi_0 + \partial_\mu(\xi_E - \xi_0) + \partial_\mu(\xi_I - \xi_E) = -\alpha^{-1} - z(e^z - 1)^{-1}\lambda^{-2}\partial_\mu \lambda_\mu + o(1)\,.$$

From the definition of λ in Eq. (8) and Proposition 2 (i), we have $-\alpha^{-1} - z(e^z - 1)^{-1}\lambda^{-2}\partial_\mu \lambda_\mu < 0$. Thus, $\partial_\mu \xi_I < 0$ for sufficiently large b. ■

In the conventional real options model, the reward function (corresponding to $h(\cdot)$ in our problem) has no σ-dependence. In this case, as shown by Theorems 6 and 7 of Alvarez (2003), the continuation region is enlarged as σ increases, so we anticipate that the entry (exit) threshold increases (decreases) in the volatility. In our model, however, $h(\cdot)$ has an explicit dependence on σ, so the σ-dependence of the thresholds does not necessarily follow the result by Alvarez (2003). When g is small, $\partial_{\sigma^2}\xi_E < 0$ and $\partial_{\sigma^2}\xi_I > 0$: as the volatility σ increases, it is optimal to wait longer to take advantage of the upturn potential before taking an irreversible action. This is similar to the result obtained numerically by Dixit (1989) and proved analytically by Alvarez (2003). However, when b is large, Proposition 7 (i) asserts that $\partial_{\sigma^2}\xi_E < 0$ and $\partial_{\sigma^2}\xi_I < 0$. Notice that the result $\partial_{\sigma^2}\xi_I < 0$ stands in contrast to the conventional intuition inherited from real options theory. This counterintuitive result obtains because the return from investment, $V_0^+(x+b)-k$, depends on σ. It is important to note that the return from investment has dependence on σ only because exit is possible after investment.

The thresholds ξ_I and ξ_E and their comparative statics with respect to σ^2 are illustrated by a numerical example in Figs. 1 and 2, in which we set $\alpha = 1$, $\sigma^2 = 0.5$, $\mu = -1$, $\delta = 0.1$, and $k = 0.5$. The graphs are shown as a function of $b+\delta/\alpha-\alpha k = b-0.4$. In Fig. 2, notice that $\partial_{\sigma^2}\xi_I$ is positive for $g < 0.96$ and negative for $g > 0.96$.

Another quantity of interest is the probability of investment prior to the eventual exit and its dependence on volatility. Let $p_I(x)$ denote the probability (conditional on $X_0 = x$ where $x \in (\xi_E, \xi_I)$) that the profit rate hits ξ_I before hitting ξ_E (investment is optimally made prior to exit). By II.4 and II.9 of Borodin and Salminen (2002),

$$p_I(x) = \frac{\exp(-\frac{2\mu}{\sigma^2}x) - \exp(-\frac{2\mu}{\sigma^2}\xi_E)}{\exp(-\frac{2\mu}{\sigma^2}\xi_I) - \exp(-\frac{2\mu}{\sigma^2}\xi_E)} = \frac{\exp[-\frac{2\mu}{\sigma^2}(x-\xi_E)]-1}{\exp[-\frac{2\mu}{\sigma^2}(\xi_I-\xi_E)]-1} .$$

Proposition 8 *For sufficiently small and large values of g, $p_I(\cdot)$ increases in σ.*

Proof: In the small-g limit, by Eqs. (23) and (24),

$$p_I(x) = \exp[\frac{-2\mu}{\mu+\sqrt{\mu^2+2\alpha\sigma^2}}\log(g)] \cdot \{\exp[-\frac{2\mu}{\sigma^2}(x-\xi_E)]-1\} \cdot (1+o(1)) .$$

Taking the derivative of the above with respect to σ^2, we obtain

$$\frac{d}{d\sigma^2}p_I(x) = p_I(x)\cdot\log(g)\frac{2\mu\alpha}{(\mu+\sqrt{\mu^2+2\alpha\sigma^2})^2\sqrt{\mu^2+2\alpha\sigma^2}} \cdot (1+o(1)) .$$

The leading-order term of $\frac{d}{d\sigma^2}p_I(x)$ is positive because $\log(g) < 0$ and $\mu < 0$.

Next, in the large-g limit, by Eq. (26), both $\xi_I - \xi_E$ and $x - \xi_E$ are bounded by Cg^{-1} for some positive constant C because $x \in (\xi_E, \xi_I)$. Hence,

$$p_I(x) = \frac{\exp[-\frac{2\mu}{\sigma^2}(x-\xi_E)]-1}{\exp[-\frac{2\mu}{\sigma^2}(\xi_I-\xi_E)]-1} = \frac{x-\xi_E}{\xi_I-\xi_E}(1+o(1)) .$$

By Proposition 7, both ξ_I and ξ_E decrease in σ, and so

$$\frac{d}{d\sigma^2}(\frac{x-\xi_E}{\xi_I-\xi_E}) = -\partial_{\sigma^2}\xi_I \frac{x-\xi_E}{(\xi_I-\xi_E)^2} - \partial_{\sigma^2}\xi_E \frac{\xi_I - x}{(\xi_I-\xi_E)^2} > 0 .$$

Hence, $\frac{d}{d\sigma^2} p_I(x)$ is positive for sufficiently large g. ■

With a declining profit rate ($\mu < 0$), the investment will only be made if the profit rate is boosted by random noise. Hence, the probability of making an investment before exit is expected to be increasing in volatility. Although we suspect that this is a general feature, the general comparative statics analysis is not available, and we only can confirm the comparative statics in the two limiting cases (small and large values of g) by Proposition 8.

4.4 Extension: Switching to New Technology

So far we have examined investment in the currently employed technology which is becoming obsolete. Another important decision concerns when to adopt the next-generation technology. In the context of the computer hard-drive industry, some of the 14-inch drive manufacturers were compelled to adopt 8-inch drives. In this subsection, we assume that the decision-maker can adopt a new technology upon exit from the current project and study how the exit value (expected profit from switching to the new technology) affects the optimal policy.

First, we examine the impact of adding a lump sum salvage value s receivable at the time of exit. If plant and equipment are sold upon exit, then we anticipate $s > 0$. However, if there is employee severance or liabilities associated with decommissioning of the business, then $s < 0$.

Lemma 3 *Let $V_0(\cdot; s)$ denote the optimal return function when s is the salvage value. Then*

$$V_0(x; s) = s + V_0(x - \alpha s) ,$$

and the exit threshold is $\xi(s) = \xi_0 + \alpha s$.

(The proof is in the e-companion to this paper.)

Next, we consider a decision of whether and when to switch to a new technology when there is a one-time investment opportunity to improve the current technology. Without loss of generality, we assume that there is no switching cost. Letting $s > 0$ denote the expected cumulative profit from switching to a new technology, we modify the investment model considered in Sec. 4 by adding a constant exit value s. Of course, even after investment in the old technology, the firm can still switch to the new technology and receive s. The objective is to find the optimal stopping time τ to maximize the following:

$$E^x\left[\int_0^\tau e^{-\alpha t} X_t dt + e^{-\alpha\tau}(h(X_\tau - \alpha s) + s)\right] .$$

Proposition 9 *With the switching value s, the optimal return from the investment decision problem is* $V_1(x - \alpha s) + s$*; the investment threshold is* $\xi_I + \alpha s$ *and the switching (exit) threshold is* $\xi_E + \alpha s$.

The proof of Proposition 9 is essentially the same as that of Lemma 3. With the opportunity to invest in a new technology, the firm has less incentive to invest in or hold on to the current technology. Hence, the thresholds for investment and switching are higher when there is a profitable alternative technology.

In this model of technology switching, we assumed that the cost of switching is constant (or zero without loss of generality) and that the expected profit from switching is independent of the current profit rate X_t. In general, the switching cost and the expected profit from switching may depend on the current profit rate if, for example, X_t represents the current demand (or market share) and the switched technology serves the same market. There is also the possibility of multiple and uncertain arrivals of improved technologies with different dynamics of the profit stream. These complications are beyond the scope of our paper. The problem of switching to future superior but uncertain technologies have been studied by Alvarez and Stenbacka (2001) and Balcer and Lippman (1984), and the difficulty with switching to disruptive technologies has been empirically studied by Christensen (1992 and 2000).

5 Summary

Our analysis of investment under deteriorating conditions is congruent with empirical reality as exemplified by the hard disk drive industry and the many examples of obsolescent technologies enumerated in Christensen (2000) and Rosenberg (1976): it can be optimal to invest even in the face of a declining profit stream and eventual displacement from the market. Moreover, it can be optimal to remain in the market even if the current profit rate is negative but above a threshold; it is optimal to exit only when the profit rate has deteriorated sufficiently.

In this paper, we studied a model of investment and exit decisions under deteriorating conditions, and we proved that there exists the optimal policy which is characterized by three thresholds: ξ_I, ξ_E, and ξ_1. Our comparative statics analysis with respect to the volatility provided a novel and counterintuitive result. As explained by Dixit (1992), illustrated by McDonald and Siegel (1986) and Dixit (1989), and generalized by Alvarez (2003), it is optimal to delay an irreversible action longer as the degree of uncertainty increases in conventional real options models. In the basic model of Sec. 3, for instance, the exit threshold ξ_0 always decreases in the volatility σ. Similarly, in the model of Sec. 4, the exit threshold ξ_E decreases in σ. The same intuition suggests that ξ_I increases in σ. Indeed, ξ_I increases in σ for sufficiently small g. However, we find that ξ_I decreases in σ for sufficiently large g: if the boost in the profit rate is sufficiently large, then it is optimal to invest earlier as the uncertainty about the future profit stream increases. (See Fig. 2.) This counterintuitive result is due to the firm's ability to control the time of its eventual exit, a salient feature of our model. Because post-investment exit is possible, the firm can take advantage of the volatility after investment, so an increase in volatility induces an increase in the expected return from investment and an increase in ξ_I for sufficiently large g.

Appendix: Equations for Thresholds

Consider a solution (ξ_I, ξ_E, a_1, a_2) to Eqs. (17) – (20). For notational convenience, we define $\Delta_{IE} \equiv \xi_I - \xi_E$ and $\Delta_{E0} \equiv \xi_E - \xi_0$. We eliminate a_1 and a_2 from Eqs. (17) – (20), and we obtain

$$\Delta_{E0} = -ge^{-\gamma_p \Delta_{IE}} + (\lambda^{-1} - \gamma_n^{-1}) e^{\lambda(\Delta_{IE} + \Delta_{E0} + b + \xi_0 - \xi_1)} e^{-\gamma_p \Delta_{IE}} \tag{21}$$

$$= -ge^{-\gamma_n \Delta_{IE}} + (\gamma_p^{-1} - \gamma_n^{-1}) + (\lambda^{-1} - \gamma_p^{-1}) e^{\lambda(\Delta_{IE} + \Delta_{E0} + b + \xi_0 - \xi_1)} e^{-\gamma_n \Delta_{IE}} , \tag{22}$$

where g is defined by Eq. (12).

In order to keep track of leading-order terms of power expansions of g, we introduce a notation to denote the subleading order terms: we say that $f(x) = o\,(j(x))$ if $f(x)/j(x) \to 0$ as $x \to 0$, where $f(x)$ and $j(x)$ are functions of x.

Lemma 4 *In the small-g limit,*

$$\Delta_{E0} = -g^{1-\gamma_p/\gamma_n} C(\delta)(1 + o(1)) , \tag{23}$$

$$\Delta_{IE} = -\gamma_n^{-1} \ln(g^{-1})(1 + o(1)) , \tag{24}$$

where $C(\delta) = [(\gamma_p^{-1} - \gamma_n^{-1})]^{\gamma_p/\gamma_n}$ *if* $\delta > 0$ *and* $C(\delta) = [(\gamma_p^{-1} - \gamma_n^{-1})(1 - e^{\gamma_n b})]^{\gamma_p/\gamma_n}$ *if* $\delta = 0$.

(The proof is in the e-companion to this paper.)

Similarly, we also say that $f(x) = o\,(j(x))$ if $f(x)/j(x) \to 0$ as $x \to \infty$.

Lemma 5 *In the large-b limit,*

$$\Delta_{E0} = -g + \theta + o(1) \tag{25}$$

$$\Delta_{IE} \quad = \quad -g^{-1}(\gamma_p\gamma_n)^{-1}(1-\lambda\theta-\lambda/\gamma_n)+o(g^{-1}) \tag{26}$$

where θ *is the unique positive solution to the equation*

$$\theta = -\gamma_n^{-1} + \lambda^{-1} e^{\lambda(\theta+\alpha k-\delta/\alpha+\xi_0-\xi_1)} \,. \tag{27}$$

(The proof is in the e-companion to this paper.)

We need to obtain the comparative statics of θ in order to examine the comparative statics of ξ_I and ξ_E in the large-b limit in Sec. 4.3. From Eq. (27) and the implicit function theorem, the partial derivatives of θ with respect to σ^2 and μ are given by

$$\partial_{\sigma^2}\theta = \gamma_n^{-2}\partial_{\sigma^2}\gamma_n + \frac{e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}{1-e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})\lambda^{-1}\partial_{\sigma^2}\lambda \,, \tag{28}$$

$$\partial_{\mu}\theta = \gamma_n^{-2}\partial_{\mu}\gamma_n + \frac{e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}{1-e^{\lambda(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})}}(\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1})\lambda^{-1}\partial_{\mu}\lambda \,. \tag{29}$$

Finally, the following is a useful property of θ:

Lemma 6 $\theta+\alpha k+\gamma_n^{-1}-\lambda^{-1} > 0$.

Proof: For any value of b, $\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1 > 0$ always holds. In the limit $b\to\infty$, by Lemma 5, $\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1 \longrightarrow \theta+\alpha k+\gamma_n^{-1}-\lambda^{-1}$, which also must be positive. ■

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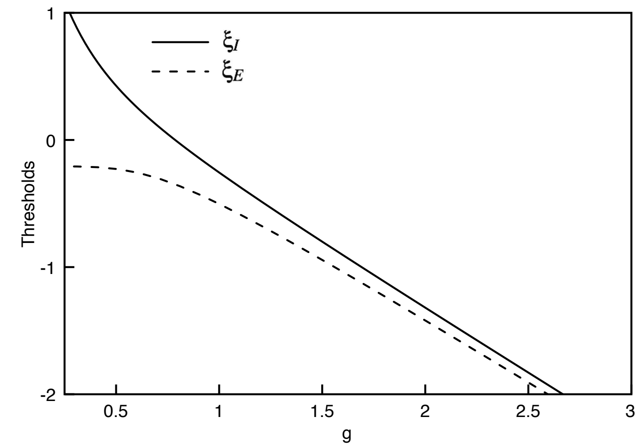

Figure 1: Thresholds as a function of g.

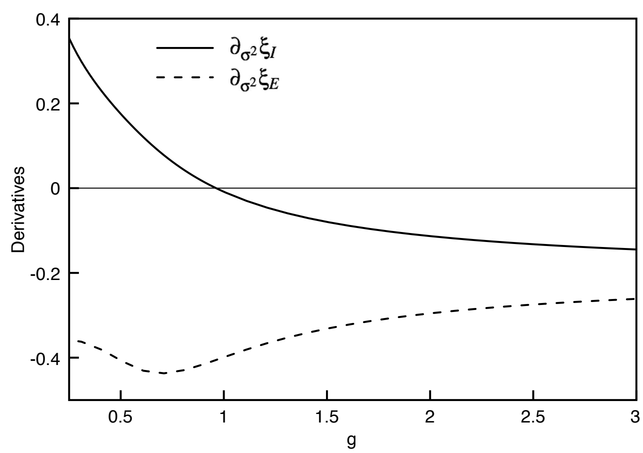

Figure 2: Derivatives of the thresholds with respect to σ^2 as a function of g.

On-Line Appendix

Appendix A: Proof of Proposition 4

We first suppose that there is a solution (ξ_E, ξ_I, a_1, and a_2) to Eqs. (17) – (20). (The existence of a solution is assured by Lemma 8.) Then Lemma 7 below establishes a condition under which the optimal policy exists.

Lemma 7 *Suppose that there exists a solution (ξ_E, ξ_I, a_1, and a_2) to Eqs. (17) – (20) that satisfies the constraints $\xi_E < x^+ < \xi_I$ and*

$$x/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p x} + a_2 e^{\gamma_n x} \geq h(x) \quad \textit{for} \quad x \in (\xi_E, \xi_I)\,, \tag{30}$$

$$\gamma_p^2 a_1 e^{\gamma_p x} + \gamma_n^2 a_2 e^{\gamma_n x} \geq h''(x) \quad \textit{for} \;\; x \in \{\xi_E, \xi_I\}\,. \tag{31}$$

Then the optimal policy exists, and its expected return is given by Eq. (16); moreover, the optimal continuation region is $D^* = (\xi_E, \xi_I)$.

Proof: Suppose there are ξ_E, ξ_I, a_1, and a_2 which satisfy Eqs. (17)–(20) and Eq. (30). The infinitesimal generator for the process (t, X_t) is given by $\partial_t + \mu\partial_x + \frac{1}{2}\sigma^2\partial_x^2$ (see Oksendal 2003, p.222); however, when the time-dependence of the return functions is only through the discount factor $e^{-\alpha t}$, the infinitesimal generator can be conveniently replaced by

$$\mathcal{A} \equiv -\alpha + \mu\partial_x + \frac{1}{2}\sigma^2\partial_x^2\,. \tag{32}$$

We consider $V_1(\cdot)$ defined in Eq. (16) as a candidate for the optimal return function. By Theorem 9.3.3, of Oksendal (2003), $V_1(x)$ is $R_{(\xi_E,\xi_I)}(x)$, the return function with a continuation region (ξ_E, ξ_I), because $V_1(x)$ satisfies $\mathcal{A}V_1(x) = -x$ and the boundary conditions $V_1(\xi_E) = h(\xi_E)$ and $V_1(\xi_I) = h(\xi_I)$.

Next, we show that $V_1(\cdot)$ indeed coincides with $\bar{V}(\cdot)$ defined in Eq. (10) and establish that (ξ_E, ξ_I) is the optimal continuation region. This is achieved by simply checking the conditions of the variational inequalities given by Theorem 10.4.1 of Oksendal (2003).

First, as a preliminary condition for the variational inequalities, $\{h^-(X_\tau) : \tau \in \mathcal{T}, \tau \leq \infty\}$ is uniformly integrable, which follows from the fact that $h^-(x) = 0$. For a second preliminary condition, we must show that $E^x[\int_0^\infty e^{-\alpha t}(X_t)^- dt] < \infty$. This follows from

$$E^x[\int_0^\infty e^{-\alpha t}(X_t)^- dt] \leq E^x[\int_0^\infty e^{-\alpha u}|X_u| du] \quad \leq \quad E^x[\int_0^\infty e^{-\alpha u}(|\mu u| + |x| + |\sigma B_u|) du] < \infty\,.$$

Now we apply Theorem 10.4.1 of Oksendal (2003) and find that $V_1(x) \geq \bar{V}(x)$ because the following conditions are satisfied: (a) $V_1(\cdot)$ is continuously differentiable [Eqs. (19) and (20)] in $\mathbb{R}$, (b) $V_1(x) \geq h(x)$ for all $x \in \mathbb{R}$ [Eq. (30)], (c) $V_1(x) = h(x)$ for $x \in \{\xi_E, \xi_I\}$ [Eqs. (17) and (18)], (d) $V_1(\cdot)$ is twice continuously differentiable except at $\{\xi_E, \xi_I\}$ (by the definition of $V_1(\cdot)$), (e) the magnitudes of the second-order derivatives of $V_1(\cdot)$ are finite near $x = \xi_E$ and ξ_I, and (f) $\mathcal{A}V_1(x) \leq -x$ for $x \in \mathbb{R}\backslash\{\xi_E, \xi_I\}$.

The last condition (f) has yet to be verified. By straightforward algebra, $\mathcal{A}V_1(x) = -x$ for $x \in (\xi_E, \xi_I)$ and

$$\mathcal{A}V_1(x) = \mathcal{A}h(x) = \begin{cases} -x - g + \alpha^{-1}\delta e^{\lambda(x+b-\xi_1)} & \text{if } x > \xi_I \,, \\ 0 & \text{if } x < \xi_E \end{cases} .$$

By Eqs. (31), (18) and (20), we have $\lim_{x \nearrow \xi_I} \mathcal{A}[V_1(x) - h(x)] \geq 0$, so $\lim_{x \nearrow \xi_I} \mathcal{A}V_1(x) = -\xi_I \geq \lim_{x \searrow \xi_I} \mathcal{A}h(x) = -\xi_I - g + \alpha^{-1}\delta e^{\lambda(\xi_I + b - \xi_1)}$. Since $e^{\lambda(x+b-\xi_1)}$ decreases in x, we conclude that $\mathcal{A}V_1(x) = \mathcal{A}h(x) < -x$ for $x > \xi_I$. Similarly, by Eqs. (31), (17) and (19), $\lim_{x \searrow \xi_E} \mathcal{A}[V_1(x) - h(x)] \geq 0$ so that $-\xi_E \geq 0$, so $\mathcal{A}V_1(x) = \mathcal{A}h(x) = 0 < -x$ for $x < \xi_E$.

Finally, because $V_1(x) = R_{(\xi_E, \xi_I)}(x) \leq \bar{V}(x)$, we conclude that $V_1(x) = \bar{V}(x)$ is the unique optimal return function and that $D^* = (\xi_E, \xi_I)$. ■

Next, we establish that there is a solution to Eqs. (17) – (20) with some desirable properties. Note that a solution to Eqs. (21) and (22) is also a solution to Eqs. (17) – (20).

Lemma 8 *There always exists a solution (ξ_E and ξ_I) to Eqs. (21) and (22) that satisfies the constraints $\xi_I - \xi_E > 0$ and $\xi_I + b - \xi_1 > 0$.*

Proof: For fixed Δ_{IE}, Eq. (21) has a unique solution for Δ_{E0} because the left-hand-side (LHS) is increasing while the right-hand-side (RHS) is decreasing in Δ_{E0}. Let $\Delta_{E0} = f(\Delta_{IE})$ be the solution to Eq. (21) as a function of Δ_{IE}.

At $\Delta_{IE} = 0$, we have $f(0) + g = (\lambda^{-1} - \gamma_n^{-1})e^{\lambda(f(0)+b+\xi_0-\xi_1)}$. Because $f(0) + b + \xi_0 - \xi_1 = f(0) + g + \alpha k - \lambda^{-1} + \gamma_n^{-1}$, if $f(0) + b + \xi_0 - \xi_1 \leq 0$, then $f(0) + g = (\lambda^{-1} - \gamma_n^{-1})e^{\lambda(f(0)+b+\xi_0-\xi_1)} \geq \lambda^{-1} - \gamma_n^{-1}$, which contradicts $f(0) + g + \alpha k - \lambda^{-1} + \gamma_n^{-1} \leq 0$. Hence, $f(0) + b + \xi_0 - \xi_1 > 0$. This also means that RHS of Eq. (22) is larger than RHS of Eq. (21) if we set $\Delta_{E0} = f(0)$ and $\Delta_{IE} = 0$. For large positive values of Δ_{IE}, on the other hand, RHS of Eq. (22) is less than RHS of Eq. (21) irrespective of the value of $f(\Delta_{IE})$. Hence, there is a value of $\Delta_{IE} > 0$ at which RHS of Eq. (21) equals RHS of Eq. (21). Thus, there is a solution to the simultaneous equations (21) and (22) with the constraint $\Delta_{IE} > 0$.

Suppose that the solution satisfies $\xi_I + b - \xi_1 \leq 0$. From $\xi_E < \xi_I$, it follows that $\xi_E + b - \xi_1 < 0$. Hence,

$$\Delta_{E0} = \xi_E - \xi_0 < -b + \xi_1 - \xi_0 = -g - \alpha k + (\lambda^{-1} - \gamma_n^{-1}) \,. \tag{33}$$

We also observe that $\Delta_{E0} < 0$ because $b > 0$ and $\xi_1 < \xi_0$. From Eq. (21) and by the assumption that

$e^{\lambda(\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1)} \geq 1$, we have

$$0 > \Delta_{E0} = -ge^{-\gamma_p\Delta_{IE}} + (\lambda^{-1}-\gamma_n^{-1})e^{\lambda(\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1)}e^{-\gamma_p\Delta_{IE}}$$
$$\geq e^{-\gamma_p\Delta_{IE}}[-g+(\lambda^{-1}-\gamma_n^{-1})] > -g+(\lambda^{-1}-\gamma_n^{-1}),$$

where the last inequality holds because $\lambda^{-1}-\gamma_n^{-1} \geq 0$ [by the fact that $\phi(\nu)$ decreases in ν from Proposition 2 (i)] and $e^{-\gamma_p\Delta_{IE}} < 1$. The inequality $\Delta_{E0} > -g+(\lambda^{-1}-\gamma_n^{-1})$ contradicts Eq. (33). ■

Lemma 8 ensures that there exists a solution to Eqs. (17) – (20) with $\xi_I > \xi_E$, and the solution satisfies the inequality $e^{\lambda(\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1)} < 1$. We have yet to prove that $\xi_I > x^+ > \xi_E$ and that Eqs. (30) and (31) hold. For the rest of this Appendix, we assume a solution (ξ_I, ξ_E, a_1, a_2) to Eqs. (17) – (20) that satisfies $\xi_I > \xi_E$ and $\xi_I + b - \xi_1 > 0$, and we consider a test function $\phi(\cdot)$ defined as follows:

$$\phi(x) = x/\alpha + \mu/\alpha^2 + a_1 e^{\gamma_p x} + a_2 e^{\gamma_n x}.$$

In the following Lemma, we establish a very convenient property of exponential functions which will be used in the proofs of forthcoming Lemmas. Recall that λ is defined in Eq. (8) and that $\lambda < \gamma_n$ by Proposition 2 (i).

Lemma 9 *Given any function of the form*

$$f(x) = C_1\exp(\gamma_p x) + C_2\exp(\gamma_n x) + C_3\exp(\lambda x)$$

where $C_1 > 0$, suppose (i) $C_2 > 0$ or (ii) $C_2 < 0$ and $C_3 < 0$. If $f(y) > 0$ for some y, then $f(x) > 0$ for all $x > y$.

Proof: (i) If $C_2 > 0$ and $C_3 > 0$, then the lemma is obvious. If $C_2 > 0$ and $C_3 < 0$, then $|C_3\exp(\lambda x)|$ is decreasing in x at a faster rate than $C_2\exp(\gamma_n x)$ while $C_1\exp(\gamma_p x)$ is increasing in x, so the lemma follows. (ii) If $C_2 < 0$ and $C_3 < 0$, then $|C_2\exp(\gamma_n x) + C_3\exp(\lambda x)|$ decreases in x while while $C_1\exp(\gamma_p x)$ increases in x, so the lemma follows again. ■

Lemma 10 *The coefficients a_1 and a_2 are positive.*

Proof: Given a solution to Eqs. (17) – (20) with the conditions $\xi_I > \xi_E$ and $\xi_I + b - \xi_1 > 0$, a_1 and a_2 can be obtained from Eqs. (18) and (20):

$$a_1 = \alpha^{-1}\gamma_p^{-1}\frac{e^{\gamma_n\xi_I}}{e^{\gamma_p\xi_I+\gamma_n\xi_E} - e^{\gamma_p\xi_E+\gamma_n\xi_I}}(1-e^{\lambda(\xi_I+b-\xi_1)}e^{-\gamma_n(\xi_I-\xi_E)}),$$

$$a_2 = -\alpha^{-1}\gamma_n^{-1}\frac{e^{\gamma_p\xi_I}}{e^{\gamma_p\xi_I+\gamma_n\xi_E} - e^{\gamma_p\xi_E+\gamma_n\xi_I}}\left(1 - e^{\lambda(\xi_I+b-\xi_1)}e^{-\gamma_p(\xi_I-\xi_E)}\right).$$

The denominator $e^{\gamma_p\xi_I+\gamma_n\xi_E} - e^{\gamma_p\xi_E+\gamma_n\xi_I}$ is always positive because $\xi_I > \xi_E$. Moreover, because $e^{\lambda(\xi_I+b-\xi_1)} < 1$ and $e^{-\gamma_p(\xi_I-\xi_E)} < 1$, a_2 is strictly positive.

Suppose $a_1 \le 0$. (i) If $\xi_E < \xi_I \le x^+$, then $\phi(\xi_E) = 0$ and $\phi(\xi_I) \le 0$, so the first-derivative $\phi'(y)$ takes a negative value at some point y in the interval (ξ_E, ξ_I). We also know that $\phi'(\xi_E) = 0$, so the second derivative $\phi''(x)$ takes a negative value somewhere in the interval (ξ_E, y); this is only possible if $a_1 < 0$. By Lemma 9, $\phi''(x)$ takes a negative value in the interval (y, ξ_I) so that $\phi'(\xi_I) < 0$. This contradicts the condition $\phi'(\xi_I) = \alpha^{-1}[1 - e^{\lambda(\xi_I+b-\xi_1)}] > 0$.

(ii) Suppose $x^+ \le \xi_E < \xi_I$, and consider the function $f(x) \equiv \phi(x) - h(x)$. By Eqs. (18) and (20), $f(\xi_I) = 0$ and $f'(\xi_I) = 0$. Because $\phi'(\xi_E) = 0$ and $h'(\xi_E) > 0$, we have $f'(\xi_E) < 0$. From the functional form $f'(x) = \gamma_p a_1 e^{\gamma_p x} + \gamma_n a_2 e^{\gamma_n x} + \alpha e^{\lambda(x+b-\xi_1)}$ and by Lemma 9, $f'(x) < 0$ for all $x > \xi_E$ because $\gamma_p a_1 \le 0$ and $\gamma_n a_2 < 0$, so $f(\cdot)$ strictly decreases for $x > \xi_E$. Hence $f(\xi_I) = 0$ is impossible.

(iii) The only remaining case is $\xi_E < x^+ < \xi_I$. As was argued in case (i), if $\phi(x^+) \le 0$, then $\phi(x)$ decreases in x for all $x > x^+$, in which case $f(\xi_I) = 0$ cannot be achieved. Suppose $\phi(x^+) > 0$. Then $f(x^+) > 0$ and $f(\xi_I) = 0$, so there is some $y \in (x^+, \xi_I)$ such that $f'(y) < 0$. By Lemma 9, $f'(x) < 0$ for all $x > y$, in which case $f'(\xi_I) = 0$ is impossible.

Because $a_1 \le 0$ is impossible in all possible cases (i)–(iii), we conclude that $a_1 > 0$. ■

The following Lemma ensures that the solution ξ_E and ξ_I satisfies $\xi_E < x^+ < \xi_I$, one of the constraints required by Lemma 7.

Lemma 11 *The inequality $\xi_E < x^+ < \xi_I$ is satisfied.*

Proof: Define $f(x) \equiv \phi(x) - h(x)$. By Lemma 10, $\phi(\cdot)$ is strictly convex and strictly increasing for $x > \xi_E$ because $\phi'(\xi_E) = 0$. Also, $\phi(x)$ is positive for $x > \xi_E$ because $\phi(\xi_E) = 0$.

(i) Suppose $\xi_E < \xi_I \le x^+$. Then $\phi(x) > 0$ for all $x > \xi_E$, but $(\xi_I + b)/\alpha + \mu^+/\alpha^2 - (\alpha\lambda)^{-1}e^{\lambda(\xi_I+b-\xi_1)} - k \le 0$ because $\xi_I \le x^+$. Hence, Eq. (18) cannot be satisfied.

(ii) Suppose $x^+ \le \xi_E < \xi_I$. Because $f(\xi_E) = 0$ and $f(\xi_I) > 0$, there is some $y \in (\xi_E, \xi_I)$ such that $f'(y) > 0$. By Lemmas 9 and 10, $f'(x) > 0$ for all $x > y$, which contradicts the condition $f'(\xi_I) = 0$. ■

Finally, it remains to show Eqs. (30) and (31).

Lemma 12 *The constraints Eqs. (30) and (31) are satisfied.*

Proof: Because $\phi(x)$ is positive for $x > \xi_E$, we have $\phi(x) > h(x) = 0$ for $x \in (\xi_E, x^+]$. Now consider the function $f(x) \equiv \phi(x) - h(x)$ for $x \in (x^+, \xi_I)$. Suppose that $f(y) < 0$ for some $y \in (x^+, \xi_I)$. Then $f'(z) > 0$

for some $z \in (y, \xi_I)$ because $f(\xi_I) = 0$. By Lemma 9, $f'(x) > 0$ for all $x > z$, which contradicts the condition $f'(\xi_I) = 0$. Therefore, $\phi(x) \geq h(x)$ for all $x \in (\xi_E, \xi_I)$.

Next, $\phi''(\xi_E) > 0 = h''(\xi_E)$ because $a_1 > 0$ and $a_2 > 0$. In the interval (x^+, ξ_I), $f(\cdot)$ decreased from $f(x^+) > 0$ to $f(\xi_I)$ so there is some $y \in (x^+, \xi_I)$ at which $f'(y) < 0$. Because $f'(\xi_I) = 0$, there is some $z \in (y, \xi_I)$ at which $f''(z) > 0$. By Lemma 9, $f''(\xi_I) > 0$. Hence, Eq. (31) is satisfied. ■

By Lemmas 7, 8, 11, and 12, Proposition 4 is proved.

Appendix B

Proof of Proposition 2: (i) From Eq. (6), it is straightforward to obtain the following inequalities:

$$\frac{\partial\psi(\nu)}{\partial(\sigma^2)} = \frac{-\nu^2 - \alpha\sigma^2 + \nu\sqrt{\nu^2 + 2\alpha\sigma^2}}{\sigma^4\sqrt{\nu^2 + 2\alpha\sigma^2}} < 0\,, \frac{\partial\phi(\nu)}{\partial(\sigma^2)} = \frac{\nu^2 + \alpha\sigma^2 + \nu\sqrt{\nu^2 + 2\alpha\sigma^2}}{\sigma^4\sqrt{\nu^2 + 2\alpha\sigma^2}} > 0\,,$$
$$\frac{\partial\psi(\nu)}{\partial\nu} = \frac{\nu - \sqrt{\nu^2 + 2\alpha\sigma^2}}{\sigma^2\sqrt{\nu^2 + 2\alpha\sigma^2}} < 0\,, \frac{\partial\phi(\nu)}{\partial\nu} = \frac{-\nu - \sqrt{\nu^2 + 2\alpha\sigma^2}}{\sigma^2\sqrt{\nu^2 + 2\alpha\sigma^2}} < 0$$
$$\frac{\partial\psi(\nu)}{\partial\alpha} = \frac{1}{\sqrt{\nu^2 + 2\alpha\sigma^2}} > 0\,, \frac{\partial\phi(\nu)}{\partial\alpha} = -\frac{1}{\sqrt{\nu^2 + 2\alpha\sigma^2}} < 0 \qquad .$$

(ii) We first recognize that $\xi(\nu) = -1/\psi(\nu)$. Using the chain rule of differentiation ($\frac{\partial\xi(\nu)}{\partial z} = \frac{1}{\psi^2(\nu)}\frac{\partial\psi(\nu)}{\partial z}$ for any model parameter z), we can obtain the comparative statics of $\xi(\nu)$ from the comparative statics of $\psi(\nu)$ above.

(iii) For $x > \xi(\nu)$, we can express

$$V(x;\nu) = \frac{x}{\alpha} + \frac{\nu}{\alpha^2} - \frac{1}{\alpha\phi(\nu)}\exp[\phi(\nu)(x - \xi(\nu))]\,.$$

Hence,

$$\begin{aligned}\frac{\partial V(x;\nu)}{\partial\sigma^2} =& \frac{1}{\alpha\phi^2(\nu)}\exp[\phi(\nu)(x - \xi(\nu))] \\ &\times [\frac{\partial\phi(\nu)}{\partial\sigma^2} - \phi(\nu)\frac{\partial\phi(\nu)}{\partial\sigma^2}(x - \xi(\nu)) - \frac{\phi^2(\nu)}{\psi^2(\nu)} \cdot \frac{\partial\psi(\nu)}{\partial\sigma^2}]\,.\end{aligned}$$

From the fact that $\phi(\nu) < 0$, $x > \xi(\nu)$, $\frac{\partial\psi(\nu)}{\partial\sigma^2} < 0$, and

$$\frac{\partial\phi(\nu)}{\partial\sigma^2} = \frac{\nu^2 + \alpha\sigma^2 + \nu\sqrt{\nu^2 + 2\alpha\sigma^2}}{\sigma^4\sqrt{\nu^2 + 2\alpha\sigma^2}} > 0\,,$$

we conclude that $\frac{\partial V(x;\nu)}{\partial\sigma^2} > 0$.

The comparative statics $\frac{\partial V(x;\nu)}{\partial\nu} > 0$ is easier to see from the objective function in Eq. (2) which increases in ν with a fixed τ. Alternatively, $\frac{\partial V(x;\nu)}{\partial \nu} > 0$ can be directly shown by algebra. ■

Proof of Lemma 1: Let $x_\theta = \theta x_1 + (1-\theta)x_2$ where $\theta \in (0,1)$ and $x_1 \neq x_2$, and let τ^* denote the optimal stopping time conditional on $X_0 = x_\theta$. From the fact that $h(\cdot)$ is convex and that

$$x_\theta + \mu t + \sigma B_t = \theta(x_1 + \mu t + \sigma B_t) + (1-\theta)(x_2 + \mu t + \sigma B_t)\,,$$

we obtain the following inequality:

$$\begin{aligned}V_1(x_\theta) =& E[\int_0^{\tau^*} e^{-\alpha t}(x_\theta + \mu t + \sigma B_t)dt + e^{-\alpha\tau^*}h(x_\theta + \mu\tau^* + \sigma B_{\tau^*})]\\ \leq& \theta E^{x_1}[\int_0^{\tau^*} e^{-\alpha t}X_t dt + e^{-\alpha\tau^*}h(X_{\tau^*})]\\ &+(1-\theta)E^{x_1}[\int_0^{\tau^*} e^{-\alpha t}X_t dt + e^{-\alpha\tau^*}h(X_{\tau^*})]\\ \leq& \theta V_1(x_1) + (1-\theta)V_1(x_2)\,.\end{aligned}$$

■

Proof of Lemma 3: Because $se^{-\alpha\tau} = s - \int_0^\tau \alpha s e^{-\alpha t}dt$,

$$\begin{aligned}V_0(x;s) =& E^x[\int_0^\tau X_t e^{-\alpha t}dt + se^{-\alpha\tau}]E^x = s + E^x[\int_0^\tau (X_t - \alpha s)e^{-\alpha t}dt]\\ =& s + E^{x-\alpha s}[\int_0^\tau X_t e^{-\alpha t}dt] = s + V_0(x - \alpha s)\,.\end{aligned}$$

Because $V_0(x;s)$ is increasing in s, the exit threshold is

$$\xi(s) = \inf\{x : V_0(x;s) > s\} = \xi_0 + \alpha s\,.$$

■

Proof of Lemma 4: First, we notice that if $g = 0$, then $\Delta_{E0} = \xi_E - \xi_0 = 0$ and $\Delta_{IE} = \xi_I - \xi_E = \infty$. Hence, $\Delta_{E0} \to 0$ and $\Delta_{IE} \to \infty$ as $g \to 0$.

Suppose that $\delta > 0$. The first term of the right-hand-side (RHS) of Eq. (21) strictly dominates the second term so that

$$g^{-1}\exp[\lambda(\Delta_{IE} + \Delta_{E0} + b + \xi_0 - \xi_1)] \to 0 \quad \text{as } g \to 0\,. \tag{34}$$

(Otherwise, if the second term of RHS of Eq. (21) is dominant, then $\Delta_{E0} > 0$ in the $g \to 0$ limit; if both terms converge to zero at the same rate as $g \to 0$, then the RHS of Eq. (22) converges to $(\gamma_p^{-1} - \gamma_n^{-1})$ as $g \to 0$.) From Eq. (34), the leading order terms in RHS of Eq. (22) are contained in the first two terms: $-ge^{-\gamma_n\Delta_{IE}} + (\gamma_p^{-1} - \gamma_n^{-1})$ in agreement with Eq. (24). From the fact that $\lim_{g\to 0}\Delta_{E0} = 0$, the only possible leading order term of Δ_{IE} is $\gamma_n^{-1}\ln[g(\gamma_p^{-1}-\gamma_n^{-1})^{-1}]$. The leading-order terms of $\Delta_{IE} = \gamma_n^{-1}\ln[g(\gamma_p^{-1}-\gamma_n^{-1})^{-1}]+o(1)$ is consistent with the condition in Eq. (34) because $\lambda/\gamma_n > 1$ by Proposition 2 (i). Finally, using the leading-order term of Δ_{IE} in Eq. (21), we obtain Eq. (23). We repeat the same procedure with $\delta = 0$ to arrive at the expression for $C(0)$. ■

Proof of Lemma 5: In the limit $b \to \infty$, we can show that $\Delta_{IE} = \xi_I - \xi_E \to 0$ and $\Delta_{E0} = \xi_E - \xi_0 \to -\infty$ are the only correct asymptotic behaviors. We notice that a necessary condition for the firm at time t to have non-negative return from investment is that the boosted profit rate $X_t + b$ exceeds ξ_1, so $\xi_I + b > \xi_1$ must be satisfied. Hence, in the limit $b \to \infty$, $e^{\lambda(\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1)}$ is bounded by 1 because $\Delta_{IE}+\Delta_{E0}+b+\xi_0-\xi_1 = \xi_I + b - \xi_1 > 0$ and $\lambda < 0$.

From RHS of Eq. (21), the leading-order term of Δ_{E0} is $-g$. We claim that the second-leading-order term of Δ_{E0} is a positive constant, independent of g. Suppose that the second-order term of Δ_{E0} grows in g, but does so more slowly than g. Then the first and second leading-order terms of Eqs. (21) and (22) are $-ge^{-\gamma_p\Delta_{IE}} = -g + g\gamma_p\Delta_{IE}(1+o(1))$ and $-ge^{-\gamma_n\Delta_{IE}} = -g + g\gamma_n\Delta_{IE}(1+o(1))$ respectively, which are inconsistent because $\gamma_p \neq \gamma_n$. Thus, the second leading order term of Δ_{E0} is a constant independent of g. Hence, we can express Δ_{E0} as in Eq. (25) where θ is a constant yet to be determined. Then Eqs. (21) and (22) can be re-expressed as

$$\Delta_{E0} = -g + g\gamma_p\Delta_{IE} + (\lambda^{-1}-\gamma_n^{-1})e^{\lambda(\theta-\delta/\alpha+k\alpha+\xi_0-\xi_1)} + o(1)\,, \tag{35}$$

$$\Delta_{E0} = -g + g\gamma_n\Delta_{IE} + (\gamma_p^{-1}-\gamma_n^{-1}) + (\lambda^{-1}-\gamma_p^{-1})e^{\lambda(\theta-\delta/\alpha+k\alpha+\xi_0-\xi_1)} + o(1)\,. \tag{36}$$

Thus, the leading-order term of Δ_{IE} converges to zero at least as fast as g^{-1} because otherwise Δ_{E0} has a second leading order term growing in g. Let us set $\Delta_{IE} = C/g + o(g^{-1})$ for some constant C. From Eqs. (25), (35) and (36), we arrive at $C = -(\gamma_p\gamma_n)^{-1}(1-\lambda\theta-\lambda/\gamma_n)$ where θ satisfies Eq. (27). ■