Notre Dame Journal of Formal Logic
Volume 38, Number 1, Winter 1997

A New Semantics for Positive Modal Logic

S. CELANI and R. JANSANA

Abstract The paper provides a new semantics for *positive modal logic* using Kripke frames having a quasi ordering $\leq$ on the set of possible worlds and an accessibility relation R connected to the quasi ordering by the conditions (1) that the composition of $\leq$ with R is included in the composition of R with $\leq$ and (2) the analogous for the inverse of $\leq$ and R. This semantics has an advantage over the one used by Dunn in "Positive modal logic," *Studia Logica* (1995) and works fine for extensions of the minimal system of normal positive modal logic.

1 Introduction In [4] Dunn begins the study of *positive modal logic*, modal logic without negation and without implication—that is, modal logic with the connectives $\wedge, \vee, \Box, \Diamond$, and also modal logic with the mentioned connectives plus the propositional constants $\top$ and $\bot$. The question addressed in the paper is which set of postulates characterizes the definition of these connectives (and propositional constants) in the usual Kripke semantics: that is, the semantics where (1) frames are pairs consisting of a set of possible worlds and a binary relation on that set, (2) valuations are any function from the propositional functions to sets of possible worlds, and (3) the semantical clauses in the definition of truth in a world are the usual classical ones for the connectives involved. [4] answers the question by introducing the systems K_+ (with the connectives $\wedge, \vee, \Box, \Diamond$) and $K_+^{\top\bot}$ (with the connectives $\wedge, \vee, \Box, \Diamond$ and the propositional constants $\top, \bot$) of *positive minimal normal modal logic* defined by means of calculi on consequence pairs, that is, pairs of formulas (φ, ψ), written $\varphi \vdash \psi$, that can be identified with sequents. Dunn's systems have the following property for the formulas in the languages involved:

$$\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \varphi \text{ is deducible} \quad \text{iff} \quad \varphi_1, \ldots, \varphi_n \models_K \varphi,$$

where $\models_K$ is the local consequence associated to the minimal normal modal logic K, that is, the one defined by

$$\varphi_1, \ldots, \varphi_n \models_K \varphi \quad \text{iff} \quad \varphi_1 \wedge \cdots \wedge \varphi_n \to \varphi \text{ is a theorem of K.}$$

Received December 3, 1996; revised April 24, 1997

Hence, Dunn's systems are essentially, and respectively, the $\wedge$, $\vee$, $\Box$, $\Diamond$-fragment and the $\wedge$, $\vee$, $\Box$, $\Diamond$, $\top$, $\bot$-fragment of the deductive system (on the sets of formulas with all the connective) K^{MP} whose consequence relation is the relation $\models_{\mathrm{K}}$ just mentioned.

The semantics for postive modal logic considered by Dunn is, as we said, the usual Kripke semantics. With this semantics, a consequence pair $\varphi \vdash \psi$ is valid in a model if the set of worlds where φ is true is included in the set of worlds where ψ is true. Dunn not only studies the *positive minimal normal modal logic*, but also several of its extensions by means of consequence pairs like $\Box\varphi \vdash \Box\Box\varphi$, $\Box\varphi \vdash \varphi$ and their duals $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ and $\varphi \vdash \Diamond\varphi$ as well. His semantics has a shortcoming that, for example, if one adds $\Box\varphi \vdash \Box\Box\varphi$ to his basic system, one obtains a system that is frame incomplete: $\Diamond\Diamond p \vdash \Diamond p$ is valid in all the frames where $\Box p \vdash \Box\Box p$ is valid but is not deducible in the system. This is not a good feature. It seems that the semantics must reflect the fact that without negation the consequence pair schemes $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ and $\Box\varphi \vdash \Box\Box\varphi$ are no longer dependent on each other.

Two years ago, unaware of Dunn's work, we began to study what turned out to be the $(\wedge, \vee, \Box, \Diamond, \top, \bot)$-fragment of the modal deductive system K^{MP}. We started our study by using a different semantics than Dunn's in order to develop a duality theory for bounded distributive lattices with modal operators by extending the well-known Priestley duality between bounded distributive lattices and Priestley spaces. This duality theory is mainly the subject of the Ph.D. dissertation [2] of the first author and will be the subject of another paper. In order to find the semantics, we looked at the mentioned fragment as a fragment of a possible intuitionistic modal logic with the axioms

$$\Box(\varphi \vee \psi) \to \Box\varphi \vee \Diamond\psi \quad \text{and} \quad \Box\varphi \wedge \Diamond\psi \to \Diamond(\varphi \wedge \psi).$$

In Kripke semantics for intuitionistic modal logic, frames are triples with a set of possible worlds, a quasi ordering $\leq$ on it, and an accessibility relation R, and the valuations used are the increasing ones. So we considered structures of this kind as our frames and the increasing valuations as our valuations. Moreover, we used the classical semantic conditions for $\Diamond$ and $\Box$ in the definition of truth in a world, as in [1]. It turns out that (1) both axioms are intuitionistically valid in any frame and (2) any increasing valuation extends to all formulas in such a way that the set of worlds where a formula is true is an increasing set (relative to $\leq$) if and only if the two conditions on frames

$$(\leq \circ R) \subseteq (R \circ \leq)$$

and

$$(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$$

hold. The first condition is the one used by Božić and Došen in [1] (see also [3]) to define the frames for their system of intuitionistic modal logic $\mathrm{HK}\Box$ and the second one is the condition used by these authors to define the frames for their system of intuitionistic modal logic $\mathrm{HK}\Diamond$. We imposed these two conditions on our structures, so our frames are the structures mentioned that satisfy both conditions. We introduced a deductive system called $\mathcal{S}_m$ by means of a Gentzen calculus sound and complete for this class of frames. This system is equivalent to Dunn's system $\mathrm{K}_{+}^{\top\bot}$ but our semantics has the advantage that it works well for extensions. For example, with our semantics the extensions by the sequents $\Diamond\Diamond\varphi \vdash \Diamond\varphi$ and by $\Box\varphi \vdash \Box\Box\varphi$ are both frame complete. The purpose of this paper is to present the deductive system $\mathcal{S}_m$, the semantics just mentioned, and the completeness proof, as well as the study of some canonical extensions.

One specific point should be stressed. Since a (finitary) deductive system can be seen (as pointed out in Section 2) as a set of sequents, we will extend the usual notions of truth in a world, validity in a model, and so on, for formulas to sequents. In this way, even if we do not have a conditional we will be able to express properties of the accessibility relation. We will not do this by means of formulas but by means of sequents.

The paper is divided into seven sections, apart from this introduction. In the next section, the *Preliminaries*, the basic notions of modal deductive system and Gentzen system as well as related notions are introduced. In Section 3 the basic deductive system $\mathcal{S}_m$ is introduced by means of a Gentzen calculus. Section 4 deals with the Kripke semantics for $\mathcal{S}_m$ and extensions. In Section 5 several sequents are considered and the properties on frames that correspond to them are studied. Section 6 is devoted to the proof of the completeness theorems for $\mathcal{S}_m$ by using the canonical model built by means of the prime theories. To conclude the paper, Section 7 is devoted to the study of several canonical extensions of $\mathcal{S}_m$.

2 Preliminaries We will deal with the modal propositional language L with a denumerable set of propositional variables whose connectives are the elements of the set $\{\wedge, \vee, \Box, \Diamond\}$ and that, in addition, has two propositional constants $\top, \bot$. Fm will denote indistinctly the set of formulas and the algebra of formulas.

A deductive system is a pair $\mathcal{S} = \langle Fm, \vdash_{\mathcal{S}}\rangle$ where $\vdash_{\mathcal{S}}$ is a finitary and structural consequence relation on Fm: that is, a relation that satisfies the following conditions.

1. If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{S}} \varphi$.
2. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and for every $\psi \in \Gamma$ $\Delta \vdash_{\mathcal{S}} \psi$, then $\Delta \vdash_{\mathcal{S}} \varphi$.
3. For any homomorphism σ from Fm into itself (i.e., a substitution), if $\Gamma \vdash_{\mathcal{S}} \varphi$, then $\sigma[\Gamma] \vdash_{\mathcal{S}} \sigma(\varphi)$.
4. If $\Gamma \vdash_{\mathcal{S}} \varphi$ then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{S}} \varphi$.

From (1) and (2) it follows that

5. If $\Gamma \vdash_{\mathcal{S}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathcal{S}} \varphi$.

Condition 3 is called *the structurality condition*.

A deductive system $\mathcal{S}'$ is an *extension* of a deductive system $\mathcal{S}$ if the relation $\vdash_{\mathcal{S}}$ is a subrelation of the relation $\vdash_{\mathcal{S}'}$.

A *sequent* of L is a pair $\langle \Gamma, \varphi\rangle$ where Γ is a finite set of formulas and φ is a formula. As usual we will write $\Gamma \vdash \varphi$ for a sequent. The set of all the sequents of L is denoted by *Seq*. A *Gentzen system* is a pair $\mathcal{G} = \langle Seq, \mid\!\sim_{\mathcal{G}}\rangle$ where $\mid\!\sim_{\mathcal{G}}$ is a finitary consequence relation on *Seq*, that is, a relation that satisfies the conditions analogous to conditions 1, 2, and 4 but for sequents and sets of sequents instead of formulas and sets of formulas, and such that it satisfies the following structurality condition: for any family $\{\Gamma_i \vdash \varphi_i : i \in I\} \cup \{\Gamma \vdash \varphi\}$ of sequents and any substitution σ,

$$\{\sigma[\Gamma_i] \vdash \sigma(\varphi_i) : i \in I\} \mid\!\sim_{\mathcal{G}} \sigma[\Gamma] \vdash \sigma(\varphi)$$

whenever,

$$\{\Gamma_i \vdash \varphi_i : i \in I\} \mathrel{|\!\sim}_{\mathcal{G}} \Gamma \vdash \varphi.$$

A sequent $\Gamma \vdash \varphi$ is *derivable in* $\mathcal{G}$ if $\varnothing \mathrel{|\!\sim}_{\mathcal{G}} \Gamma \vdash \varphi$. Gentzen calculi with all the structural rules can be used to define Gentzen systems.

A *substitution instance of a sequent* $\Gamma \vdash \varphi$ is any sequent $\sigma[\Gamma] \vdash \sigma(\varphi)$ where σ is a substitution, and a *substitution instance of a formula* φ is any formula $\sigma(\varphi)$ where σ is a substitution.

Given a deductive system $\mathcal{S}$, a sequent $\Gamma \vdash \varphi$ is a *sequent of* $\mathcal{S}$ or an $\mathcal{S}$*-sequent* if $\Gamma \vdash_{\mathcal{S}} \varphi$. The set of sequents of a deductive system is closed under substitution instances and under the Gentzen rules of Reflexivity, Weakening, and Cut:

$$\text{(Ref.)}\ \frac{}{\varphi \vdash \varphi} \qquad \text{(Weak.)}\ \frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi} \qquad \text{(Cut)}\ \frac{\Gamma \vdash \varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

Any set of sequents Σ closed under substitution instances and Gentzen rules of Reflexivity, Weakening, and Cut can be used to define a deductive system $\mathcal{S}$ as follows.

$$\Gamma \vdash_{\mathcal{S}} \varphi \quad \text{iff} \quad \text{there is a finite } \Delta \subseteq \Gamma \text{ such that the sequent } \langle \Delta, \varphi \rangle \in \Sigma.$$

Because of these two facts we will identify deductive systems with sets of sequents closed under substitution instances and the Gentzen rules of Reflexivity, Weakening, and Cut. Therefore, a deductive system will be identified with its set of sequents.

To any Gentzen system $\mathcal{G}$ we can associate the deductive system

$$\mathcal{S}_{\mathcal{G}} = \langle Fm, \vdash_{\mathcal{S}_{\mathcal{G}}} \rangle$$

defined by

$$\Gamma \vdash_{\mathcal{S}_{\mathcal{G}}} \varphi \text{ iff there is a finite set } \Delta \subseteq \Gamma \text{ such that } \varnothing \mathrel{|\!\sim}_{\mathcal{G}} \Gamma \vdash \varphi.$$

According to the identification proposed above, this deductive system is the set of derivable sequents of $\mathcal{G}$.

Given a deductive system $\mathcal{S}$ and a set of sequents $\{\Gamma_i \vdash \varphi_i : i \in I\}$,

$$\mathcal{S} + \{\Gamma_i \vdash \varphi_i : i \in I\}$$

will denote the least deductive system $\mathcal{S}'$ that extends $\mathcal{S}$ and is such that for each $i \in I$ any substitution instance of $\Gamma_i \vdash \varphi_i$ is a sequent of $\mathcal{S}'$.

3 The basic deductive system We will introduce the basic deductive system of the paper as the deductive system associated to the Gentzen system $\mathcal{G}_m$ defined by means of the Gentzen calculus whose rules are the following:

$$\frac{}{\varphi \vdash \varphi} \qquad \frac{}{\vdash \top} \qquad \frac{}{\Diamond\bot \vdash \bot}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi} \qquad \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \qquad \frac{\Gamma \vdash \varphi \quad \Gamma, \varphi \vdash \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma, \varphi, \psi \vdash \alpha}{\Gamma, \varphi \wedge \psi \vdash \alpha} \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\Gamma, \varphi \vdash \alpha \quad \Gamma, \psi \vdash \alpha}{\Gamma, \varphi \vee \psi \vdash \alpha} \qquad \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

$$[\Box\Diamond]\ \frac{\Gamma, \varphi \vdash \psi \vee \alpha}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\psi \vee \Diamond\alpha} \qquad [\Diamond\Box]\ \frac{\Gamma \vdash \varphi \vee \psi}{\Box\Gamma \vdash \Box\varphi \vee \Diamond\psi}$$

where for any set Γ of formulas $\Box\Gamma = \{\Box\varphi : \varphi \in \Gamma\}$. The following rules are derived rules.

$$\frac{\Gamma \vdash \varphi}{\Box\Gamma \vdash \Box\varphi} \qquad \frac{\Gamma, \varphi \vdash \psi}{\Box\Gamma, \Diamond\varphi \vdash \Diamond\psi}$$

The following sequents are derivable sequents (the proofs are left to the reader):

1. $\Box(\varphi \wedge \psi) \vdash \Box\varphi \wedge \Box\psi$
2. $\Box\varphi \wedge \Box\psi \vdash \Box(\varphi \wedge \psi)$
3. $\Diamond(\varphi \vee \psi) \vdash \Diamond\varphi \vee \Diamond\psi$
4. $\Diamond\varphi \vee \Diamond\psi \vdash \Diamond(\varphi \vee \psi)$
5. $\Box\top \vdash \top$
6. $\vdash \Box\top$
7. $\bot \vdash \Diamond\bot$
8. $\Box(\varphi \vee \psi) \vdash \Box\varphi \vee \Diamond\psi$
9. $\Box\varphi \wedge \Diamond\psi \vdash \Diamond(\varphi \wedge \psi)$

The last two sequents are the sequents used by Dunn for his axiomatization of positive modal logic. The main difference between his presentation and ours is that he only deals with sequents of the form $\varphi \vdash \psi$ (his consequence pairs) and his calculus is, properly speaking, not a Gentzen calculus but an axiomatic calculus to deal with sequents of that type.

A property of our calculus is the following: for any formulas $\varphi_1, \ldots, \varphi_n, \varphi$,

$\{\varphi_1, \ldots, \varphi_n\} \vdash \varphi$ is derivable iff $\varphi_1 \wedge \cdots \wedge \varphi_n \vdash \varphi$ is derivable as well.

Using this property it can easily be seen that Dunn's positive logic is essentially the same as ours. Let us denote by $\mathcal{S}_m$ the deductive system just defined. We study this system and some of its extensions.

4 Kripke semantics for $\mathcal{S}_m$ The main difference between our Kripke style semantics and the one used by Dunn lies in the fact that he uses classical Kripke frames, a set of worlds plus a binary accessibility relation, and we use structures that in addition have a quasi ordering relation with some special connections with the accessibility relation. Moreover, for Dunn any valuation is admissible but for us only the increasing valuations relative to the quasi ordering will be admissible. These differences allow us to have completeness theorems for systems that are incomplete with Dunn's semantics: for instance, the deductive system $\mathcal{S}_m + \{\Box p \vdash \Box\Box p\}$. This is a strong reason in favor of our semantics.

Definition 4.1 A *frame* is a triple $\mathcal{F} = \langle M, \leq, R\rangle$ where $\leq$ is a quasi ordering on M, that is, a binary reflexive and transitive relation on M, R is a binary relation on M, and the following two conditions hold:

1. $(\leq \circ R) \subseteq (R \circ \leq)$, and
2. $(\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$,

where $\circ$ denotes the composition between binary relations.

Let $\mathcal{F} = \langle M, \leq, R \rangle$ be a frame. A subset X of M is *increasing* if for every $x \in X$ and every $y \in M$ such that $x \leq y$, it holds that $y \in X$. An *increasing valuation* on the frame $\mathcal{F}$ (a valuation from now on) is a function V from the set of variables into the set of all increasing subsets of M. Note that we do not consider arbitrary valuations, only the increasing ones as in intuitionistic logic. A valuation V can be extended recursively to the set of all formulas by means of the following clauses:

1. $V(\top) = M$,
2. $V(\bot) = \varnothing$,
3. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$,
4. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,
5. $V(\Box\varphi) = \{x \in M : \forall y \in M(xRy \Longrightarrow y \in V(\varphi))\}$,
6. $V(\Diamond\varphi) = \{x \in M : \exists y \in M(xRy \text{ and } y \in V(\varphi))\}$.

First of all we see that any valuation has the property that assigns an increasing set to each formula.

Lemma 4.2 *Let $\mathcal{F}$ be a frame and V a valuation on it. Then for any formula φ the set $V(\varphi)$ is increasing.*

Proof: By induction. We deal only with the modal connectives. Assume that $V(\varphi)$ is increasing and that $x \in V(\Box\varphi)$ is such that $x \leq y$. In order to see that $y \in V(\Box\varphi)$ assume that yRz. Since $x \leq y$ and yRz, $\langle x, z \rangle \in \ \leq \circ R$. Therefore, by condition 1 of 4.1, $\langle x, z \rangle \in R \circ \leq$. Let w be such that xRw and $w \leq z$. Since $x \in V(\Box\varphi)$, $w \in V(\varphi)$. Therefore, since $V(\varphi)$ is increasing, $z \in V(\varphi)$. Analogously one proves that $V(\Diamond\varphi)$ is increasing using condition 2 of 4.1. $\Box$

Now we introduce a notation that will be useful in the paper. Given a frame $\mathcal{F} = \langle M, \leq, R \rangle$, a valuation V on it and a set of formulas Γ,

$$V(\Gamma) = \bigcap_{\psi \in \Gamma} V(\psi).$$

If Γ is empty, $V(\Gamma) = M$.

A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$ where $\mathcal{F}$ is a frame and V is a valuation on it. We define the semantical notions of truth and validity in a model and validity in a frame for formulas and extend them to sequents. Given a model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and a point $x \in M$ we say that a formula φ is *true* at x in $\mathcal{M}$, in symbols $\mathcal{M}, x \Vdash \varphi$, if $x \in V(\varphi)$. A formula φ is *valid in a model* $\mathcal{M}$, in symbols $\mathcal{M} \models \varphi$, if it is true at every point in M. A formula φ is *valid in a frame* $\mathcal{F}$, in symbols $\mathcal{F} \models \varphi$, if for any valuation V on $\mathcal{F}$, φ is valid in the model $\langle \mathcal{F}, V \rangle$.

The previous notions extend to sequents as follows. Let $\mathcal{M} = \langle \mathcal{F}, V \rangle$ be a model and $x \in M$. A sequent $\Gamma \vdash \varphi$ is true at x in $\mathcal{M}$, in symbols $\mathcal{M}, x \Vdash \Gamma \vdash \varphi$, if $x \notin V(\Gamma)$ or $x \in V(\varphi)$, that is, when $V(\Gamma) \subseteq V(\varphi)$. A sequent $\Gamma \vdash \varphi$ is *valid in a model* $\mathcal{M}$, in symbols $\mathcal{M} \models \Gamma \vdash \varphi$, if it is true at every point in M, and it is *valid in a frame* $\mathcal{F}$, in symbols $\mathcal{F} \models \Gamma \vdash \varphi$, if it is valid in $\langle \mathcal{F}, V \rangle$ for any valuation V on $\mathcal{F}$.

It is standard to show that if a formula is valid in a frame so are all its substitution instances. In the same way as one shows this, one shows that if a sequent is valid in a frame so are all its substitution instances.

With these notions we can define the notion of a Gentzen rule being sound for model validity and for frame validity. Let $\{\Gamma_i \vdash \varphi_i : i \in I\} \cup \{\Gamma \vdash \varphi\}$ be a set of sequents. We say that the Gentzen rule

$$\frac{\{\Gamma_i \vdash \varphi_i : i \in I\}}{\Gamma \vdash \varphi}$$

is *sound for model validity* if for any model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ such that for all $i \in I$, $\mathcal{M} \models \Gamma_i \vdash \varphi_i$, it holds also that $\mathcal{M} \models \Gamma \vdash \varphi$, that is, when $V(\Gamma) \subseteq V(\varphi)$ whenever $V(\Gamma_i) \subseteq V(\varphi_i)$, for all $i \in I$. And we say that the Gentzen rule is *sound for frame validity* when for any frame $\mathcal{F}$, if $\mathcal{F} \models \Gamma_i \vdash \varphi_i$ for all $i \in I$ then $\mathcal{F} \models \Gamma \vdash \varphi$.

Clearly, if the rules of a Gentzen calculus are sound for model validity then all the derived rules and the derived sequents are valid in any model. Moreover, soundness for model validity implies soundness for frame validity.

Theorem 4.3 *All the rules of the Gentzen calculus $\mathcal{G}_m$ are sound for model validity and for frame validity. Therefore any derivable sequent is valid in any model and any derived rule is also sound for model validity and frame validity.*

Proof: It is straightforward to check that all the rules are sound. □

Let $\mathcal{S}$ be any deductive system that is an extension of the deductive system $\mathcal{S}_m$. We will denote by $\mathrm{Fr}(\mathcal{S})$ the class of all frames where every sequent of $\mathcal{S}$ is valid. Now let F be a class of frames. $Sq(\mathsf{F})$ denotes the class of all sequents that are valid in every frame in F: that is,

$$\Gamma \vdash \varphi \in Sq(\mathsf{F}) \quad \text{iff} \quad \text{for all } \mathcal{F} \in \mathsf{F},\ \mathcal{F} \models \Gamma \vdash \varphi.$$

$Sq(\mathsf{F})$ is a deductive system that extends $\mathcal{S}_m$ because it is closed under the Gentzen rules of our Gentzen calculus and under substitution instances. It is called the *deductive system* of F.

If M is a class of models, $Th_{sq}(\mathsf{M})$ is the class of all sequents that are valid in every model in the class M and is called the *sequential theory* of M. There are classes of models whose sequential theory is not a deductive system. The sequential theory of a class of models is closed under the rules of our Gentzen calculus but it is not necessarily closed under substitution instances.

A deductive system $\mathcal{S}$ is *characterized* by a class F of frames or is *complete* relative to a class F of frames, F-*complete* for short, if it is the deductive system of the class of frames F. Moreover, it is *frame complete* if the set of $\mathcal{S}$-sequents is $Sq(\mathrm{Fr}(\mathcal{S}))$. The next lemma has an obvious proof.

Lemma 4.4 *A deductive system $\mathcal{S}$ is frame complete if and only if it is characterized by some class of frames.*

Given a frame $\mathcal{F} = \langle M, \leq, R \rangle$ we can define the relations

$$R_\Box = R \circ \leq \quad \text{and} \quad R_\Diamond = R \circ \leq^{-1} .$$

Then we have the following lemma.

Lemma 4.5 *Let $\mathcal{F} = \langle \mathcal{M}, \leq, R\rangle$ be a frame and V a valuation on it. Then for any formula φ,*

1. $x \in V(\Box\varphi)$ *if and only if* $\forall y \in M(xR_{\Box}y \Longrightarrow y \in V(\varphi))$,
2. $x \in V(\Diamond\varphi)$ *if and only if* $\exists y \in M(xR_{\Diamond}y$ *and* $y \in V(\varphi))$.

Proof: We prove (1). Assume that $x \in V(\Box\varphi)$ and that $xR_{\Box}y$. Let $z \in M$ be such that xRz and $z \leq y$. Then $z \in V(\varphi)$ and, since $V(\varphi)$ is increasing, $y \in V(\varphi)$. Now, assume that for all $y \in M$ such that $xR_{\Box}y$, $y \in V(\varphi)$, and assume that xRz. Since $\leq$ is reflexive, it holds that $xR_{\Box}z$. Therefore, we conclude that $z \in V(\varphi)$. Hence, we obtain that $x \in V(\Box\varphi)$. The proof of (2) is analogous. $\Box$

If we use a semantics with Kripke frames with two relations, one to deal with $\Box$ and the other one to deal with $\Diamond$ but no quasi ordering, and we admit any valuation, then in the frames where the rules of our Gentzen calculus are sound the two relations are equal. In our situation we cannot conclude that in an arbitrary frame the relations $R_{\Box}$ and $R_{\Diamond}$ are equal because we only consider increasing valuations. It is precisely this that allows us to distinguish semantically between a sequent and its dual.

We will now prove a lemma that will be useful in the next section.

Lemma 4.6 *Given a frame $\mathcal{F} = \langle M, \leq, R\rangle$ and a point $x \in M$ the functions $V^x_{\Box}$ and $V^x_{\Diamond}$ defined by*

1. $V^x_{\Box}(p) = \{y \in M : xR_{\Box}y\}$ *and*
2. $V^x_{\Diamond}(p) = \{y \in M :$ *not* $xR_{\Diamond}y\}$

for every propositional variable p, are valuations (i.e., are increasing).

Proof: We prove (2). Assume that $y \leq z$, $y \in V^x_{\Diamond}(p)$, and $z \notin V^x_{\Diamond}(p)$. So, $xR_{\Diamond}z$. Therefore, let $w \in M$ be such that xRw and $z \leq w$. Hence, $y \leq w$. Therefore, $xR_{\Diamond}y$ which is absurd because $y \in V^x_{\Diamond}(p)$. The proof for (1) is even easier. $\Box$

5 Correspondence results In this section we introduce several sequents that will be used to define sequential extensions of the deductive system $\mathcal{S}_m$ and we prove correspondence results for them.

$T_{\Box}$	$\Box\varphi \vdash \varphi$
$T_{\Diamond}$	$\varphi \vdash \Diamond\varphi$
$4_{\Box}$	$\Box\varphi \vdash \Box\Box\varphi$
$4_{\Diamond}$	$\Diamond\Diamond\varphi \vdash \Diamond\varphi$
B_1	$\varphi \vdash \Box\Diamond\varphi$
B_2	$\Diamond\Box\varphi \vdash \varphi$
S	$\Box\varphi \vdash \Diamond\varphi$
E_1	$\Diamond\varphi \vdash \Box\Diamond\varphi$
E_2	$\Diamond\Box\varphi \vdash \Box\varphi$
D	$\Diamond\Box\varphi \vdash \Box\Diamond\varphi$

These sequents correspond to usual axioms considered in modal logic. $T_{\Box}$ corresponds to axiom T and $T_{\Diamond}$ to its dual, and so forth. Since in our language there is no negation we need to consider a sequent and its dual independently. (Note that S and D are their own duals.) When they are different, each one is independent of the other.

The previous sequents correspond to properties of frames. We will state these properties in terms of the relations $R_\Box$ and $R_\Diamond$.

Theorem 5.1 *Let $\mathcal{F} = \langle M, \leq, R\rangle$ be a frame. Then*

1. $\Box p \vdash p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Box$ *is reflexive;*
2. $p \vdash \Diamond p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Diamond$ *is reflexive;*
3. $\Box p \vdash \Box\Box p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Box$ *is transitive;*
4. $\Diamond\Diamond p \vdash \Diamond p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Diamond$ *is transitive;*
5. $p \vdash \Box\Diamond p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Box \subseteq R_\Diamond^{-1}$*;*
6. $\Diamond\Box p \vdash p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Diamond^{-1} \subseteq R_\Box$*;*
7. $\Box p \vdash \Diamond p$ *is valid in* $\mathcal{F}$ *if and only if* $R_\Box \cap R_\Diamond$ *is serial.*

Proof: The proofs of the implications from right to left are straightforward. The proofs of the other implications are similar to the ones for the parallel classical cases. For (1), (3), (6), and (7) one uses the valuations of the form $V_\Box^x$; for (2), (4), and (5) the valuations of the form $V_\Diamond^x$. □

Theorem 5.2 *Let $\mathcal{F} = \langle M, \leq, R\rangle$ be a frame. Then*

1. $\Diamond p \vdash \Box\Diamond p$ *is valid in* $\mathcal{F}$ *if and only if the following condition holds:*

   if $xR_\Box y$ *and* $xR_\Diamond z$ *then* $yR_\Diamond z$*;*

2. $\Diamond\Box p \vdash \Box p$ *is valid in* $\mathcal{F}$ *if and only if the following condition holds:*

   if $xR_\Diamond y$ *and* $xR_\Box z$ *then* $yR_\Box z$;

3. $\Diamond\Box p \vdash \Box\Diamond p$ *is valid in* $\mathcal{F}$ *if and only if the following condition holds:*

   if $xR_\Box y$ *and* $xR_\Diamond z$ *then there is* $u \in M$ *such that* $yR_\Diamond u$ *and* $zR_\Box u$.

Proof: As in the previous theorem the proofs of the implications from right to left are straightforward, and in order to prove only the implications from left to right one needs to consider for (1) a valuation of the form $V_\Diamond^x$, for (2) and (3) valuations of the form $V_\Box^x$. □

6 Canonical frames and models In this section we introduce the canonical models and canonical frames for extensions of the deductive system $\mathcal{S}_m$ and prove completeness theorems for $\mathcal{S}_m$. Let us fix a deductive system $\mathcal{S}$ that is an extension of the deductive system $\mathcal{S}_m$. A set of formulas is a *theory* of $\mathcal{S}$, or an $\mathcal{S}$-theory, if it is closed under the consequence relation $\vdash_\mathcal{S}$. A theory is *consistent* if it is not the set of all formulas, equivalently, if the formula $\perp$ does not belong to it. A *prime theory* of $\mathcal{S}$, or a prime $\mathcal{S}$-theory, is a consistent $\mathcal{S}$-theory Γ with the following property:

$$\text{if} \quad (\varphi \vee \psi) \in \Gamma \quad \text{then} \quad \varphi \in \Gamma \text{ or } \psi \in \Gamma.$$

We will use the letters P, Q, D, and K with possible subscripts and superscripts to refer to prime theories and $Th(\mathcal{S})$ to denote the set of all $\mathcal{S}$-theories.

Let us denote by $M_{\mathcal{S}}$ the set of all prime $\mathcal{S}$-theories. We define in this set the following relation $R_{\mathcal{S}}$ by

$$\langle P, Q\rangle \in R_{\mathcal{S}} \quad \text{iff} \quad \Box^{-1}(P) \subseteq Q \subseteq \Diamond^{-1}(P),$$

where $\Box^{-1}(P) = \{\varphi : \Box\varphi \in P\}$ and $\Diamond^{-1}(P) = \{\varphi : \Diamond\varphi \in P\}$. We will see that the structure

$$\mathcal{F}_{\mathcal{S}} = \langle M_{\mathcal{S}}, \subseteq, R_{\mathcal{S}}\rangle$$

is indeed a frame. It will be called the *canonical frame* for the deductive system $\mathcal{S}$. We need to establish some facts on prime theories. First of all we need the following observation.

Observation 6.1 For any prime theory P,

1. $\Box^{-1}(P)$ is an $\mathcal{S}$-theory and therefore is closed under conjunctions;
2. the complement of $\Diamond^{-1}(P)$ is closed under disjunctions.

Proof: (1) follows from the fact that if $\Gamma \vdash_{\mathcal{S}} \varphi$ then $\Box\Gamma \vdash_{\mathcal{S}} \Box\varphi$. To prove (2) let $\varphi, \psi \notin \Diamond^{-1}(P)$ and assume that $(\varphi \vee \psi) \in \Diamond^{-1}(P)$. Then $\Diamond(\varphi \vee \psi) \in P$ and since $\Diamond(\varphi \vee \psi) \vdash_{\mathcal{S}} (\Diamond\varphi \vee \Diamond\psi)$, we have $(\Diamond\varphi \vee \Diamond\psi) \in P$. Since P is prime, $\Diamond\varphi \in P$ or $\Diamond\psi \in P$. Hence, $\varphi \in \Diamond^{-1}(P)$ or $\psi \in \Diamond^{-1}(P)$, which is absurd. $\Box$

The following proposition is the logical analogy of the prime filter theorem for bounded distributive lattices.

Proposition 6.2 *Let Γ be a consistent $\mathcal{S}$-theory and let Δ be a set of formulas closed under disjunctions (i.e., if $\varphi, \psi \in \Delta$ then $\varphi \vee \psi \in \Delta$) and such that $\Gamma \cap \Delta = \varnothing$. Then there is a prime theory P such that $\Gamma \subseteq P$ and $P \cap \Delta = \varnothing$.*

Proof: Let us consider the set

$$W = \{T \in Th(\mathcal{S}) : T \text{ is consistent, } \Gamma \subseteq T \text{ and } T \cap \Delta = \varnothing\}.$$

W is nonempty because $\Gamma \in T$. It is easy to see that W, ordered by inclusion, is closed under unions of nonempty chains. Therefore by Zorn's lemma there is a maximal element. Let P be such a maximal element. We prove that P is a prime theory. Assume that $\varphi \vee \psi \in P$ and $\varphi \notin P$ and $\psi \notin P$. Let us consider the $\mathcal{S}$-theories T and T' generated, respectively, by $P \cup \{\varphi\}$ and $P \cup \{\psi\}$. These theories are consistent. We prove only that T is consistent since the proof that T' is consistent is analogous. If T is not consistent $P, \varphi \vdash_{\mathcal{S}} \psi$. Therefore $P, \varphi \vee \psi \vdash_{\mathcal{S}} \psi$. Hence, $P \vdash_{\mathcal{S}} \psi$ because $\varphi \vee \psi \in P$. Therefore $\psi \in P$, which is absurd. Now T and T' being consistent, since P is a maximal element in W we must have $T \cap \Delta \neq \varnothing$ and $T' \cap \Delta \neq \varnothing$. Let $\alpha, \beta \in \Delta$ be such that $P, \varphi \vdash_{\mathcal{S}} \alpha$ and $P, \psi \vdash_{\mathcal{S}} \beta$. Then $P, \varphi \vee \psi \vdash_{\mathcal{S}} \alpha \vee \beta$. Hence, $P \vdash_{\mathcal{S}} \alpha \vee \beta$, which is absurd because $\alpha \vee \beta \in \Delta$ and $\Delta \cap P = \varnothing$. $\Box$

Lemma 6.3 *If P and Q are prime theories such that $\Box^{-1}(P) \subseteq Q$, then there is a prime theory D such that $\langle P, D\rangle \in R_{\mathcal{S}}$ and $D \subseteq Q$.*

Proof: Assume that $\Box^{-1}(P) \subseteq Q$. It is not difficult to see that the set of formulas $\Box^{-1}(P)$ is an $\mathcal{S}$-theory. Let Σ be the closure under disjunctions of the set

$$\{\varphi : \varphi \notin Q \text{ or } \varphi \notin \Diamond^{-1}(P)\}.$$

Since Q is consistent, Σ is nonempty. We will prove that

$$\Box^{-1}(P) \cap \Sigma = \varnothing. \tag{1}$$

From this follows that $\Box^{-1}(P)$ is consistent. Thus we can apply Proposition 6.2 to obtain a prime theory D such that

$$\Box^{-1}(P) \subseteq D \text{ and } D \cap \Sigma = \varnothing.$$

Therefore for this prime theory it holds that

$$\Box^{-1}(P) \subseteq D \subseteq \Diamond^{-1}(P) \quad \text{and} \quad D \subseteq Q,$$

and thus $\langle P, D\rangle \in R_{\mathcal{S}}$ and $D \subseteq Q$.

In order to prove (1) we assume the opposite. So let $\varphi \in \Box^{-1}(P) \cap \Sigma$. Since $\perp \notin Q$, $\perp \notin \Diamond^{-1}(P)$ and the complements of Q and of $\Diamond^{-1}(P)$ are closed under disjunctions we can assume without loss of generality that there are $\alpha \notin Q$ and $\beta \notin \Diamond^{-1}(P)$ such that φ is (equivalent to) $\alpha \vee \beta$. Then, since $\Box(\alpha \vee \beta) \vdash_{\mathcal{S}} \Box\alpha \vee \Diamond\beta$ and $\Box(\alpha \vee \beta) \in P$,

$$\Box\alpha \vee \Diamond\beta \in P.$$

If $\Box\alpha \in P$, $\alpha \in Q$, which is absurd. So $\Diamond\beta \in P$ because P is prime. But this is absurd too because $\beta \notin \Diamond^{-1}(P)$. This concludes the proof. $\Box$

Lemma 6.4 *If P and Q are prime theories such that $Q \subseteq \Diamond^{-1}(P)$, then there is a prime theory D such that $\langle P, D\rangle \in R_{\mathcal{S}}$ and $Q \subseteq D$.*

Proof: Assume that $Q \subseteq \Diamond^{-1}(P)$. Let us consider the theory T generated by the set

$$\{\varphi : \varphi \in \Box^{-1}(P) \cap Q\}.$$

We prove that

$$T \subseteq \Diamond^{-1}(P). \tag{2}$$

Assume that $\alpha \in T$. Since $\Box^{-1}(P)$ and Q are closed under conjunctions, there are $\varphi \in \Box^{-1}(P)$, $\psi \in Q$ such that

$$\varphi \wedge \psi \vdash_{\mathcal{S}} \alpha.$$

By the rule $[\Box\Diamond]$ we obtain,

$$\Box\varphi \wedge \Diamond\psi \vdash_{\mathcal{S}} \Diamond\alpha.$$

Since $\psi \in Q$, $\Diamond\psi \in P$, and since $\Box\varphi \in P$ we obtain that $\Box\varphi \wedge \Diamond\psi \in P$. Thus it follows that $\Diamond\alpha \in P$ and $\alpha \in \Diamond^{-1}(P)$. Now from (2) it follows that T is consistent, because otherwise $\perp \in \Diamond^{-1}(P)$ and this implies that $\perp \in P$ which is not the case. To conclude the proof, since the complement of $\Diamond^{-1}(P)$ is closed under disjunctions, we can use Proposition 6.2 to obtain a prime theory D such that $T \subseteq D$ and $D \subseteq \Diamond^{-1}(P)$. Then

$$\Box^{-1}(P) \subseteq D \subseteq \Diamond^{-1}(P) \text{ and } Q \subseteq D.$$

Therefore $\langle P, D\rangle \in R_{\mathcal{S}}$ and $Q \subseteq D$. $\square$

Proposition 6.5 *The relation $R_{\mathcal{S}}$ has the following two properties,*

1. $(\subseteq \circ R_{\mathcal{S}}) \subseteq (R_{\mathcal{S}} \circ \subseteq^{-1})$, *and*
2. $(\subseteq^{-1} \circ R_{\mathcal{S}}) \subseteq (R_{\mathcal{S}} \circ \subseteq^{-1})$.

Therefore the structure $\mathcal{F}_{\mathcal{S}} = \langle M_{\mathcal{S}}, \subseteq, R_{\mathcal{S}}\rangle$ is a frame.

Proof: (6.5.1): Assume that $\langle P, Q\rangle \in \subseteq \circ R_{\mathcal{S}}$. Then let D be a prime theory such that $P \subseteq D$ and $\langle D, Q\rangle \in R_{\mathcal{S}}$. That is,

$$P \subseteq D \quad \text{and} \quad \Box^{-1}(D) \subseteq Q \subseteq \Diamond^{-1}(D). \tag{3}$$

Therefore $\Box^{-1}(P) \subseteq Q$. Hence by Lemma 6.3 there is a prime theory D' such that $\langle P, D'\rangle \in R_{\mathcal{S}}$ and $D' \subseteq Q$. Thus, $\langle P, Q\rangle \in R_{\mathcal{S}} \circ \subseteq$. (6.5.2) is proved analogously using Lemma 6.4. $\square$

We now prove two propositions that will give us the proof of the canonical model lemma.

Proposition 6.6 *Let P be a prime theory. Then $\Diamond\varphi \in P$ if and only if there is a prime theory Q such that $\langle P, Q\rangle \in R_{\mathcal{S}}$ and $\varphi \in Q$.*

Proof: The implication from right to left is immediate. To prove the other implication suppose that $\Diamond\varphi \in P$. Consider the theory T generated by the set $\Box^{-1}(P) \cup \{\varphi\}$. By a similar argument to the one used to prove (2) in the proof of Lemma 6.4 we obtain that

$$T \subseteq \Diamond^{-1}(P). \tag{4}$$

And from this follows that T is consistent because $\top \notin \Diamond^{-1}(P)$. Applying Proposition 6.2 to obtain a prime theory Q such that

$$T \subseteq Q \quad \text{and} \quad Q \subseteq \Diamond^{-1}(P)$$

we obtain a prime theory Q such that $\langle P, Q\rangle \in R_{\mathcal{S}}$ and $\varphi \in Q$. $\square$

Proposition 6.7 *Let P be a prime theory. Then $\Box\varphi \in P$ if and only if $\varphi \in Q$ for every prime theory Q such that $\langle P, Q\rangle \in R_{\mathcal{S}}$.*

Proof: The implication from left to right is immediate. To prove the other implication suppose $\Box\varphi \notin P$. Consider the set

$$\Sigma = \{\varphi \vee \alpha : \alpha \notin \Diamond^{-1}(P)\}$$

and the closure Σ' of Σ under disjunctions. Since the complement of $\Diamond^{-1}(P)$ is closed under disjunctions, any formula in Σ' is equivalent to formula in Σ. Let us show that

$$\Box^{-1}(P) \cap \Sigma' = \varnothing \tag{5}$$

If $\psi \in \Box^{-1}(P) \cap \Sigma'$ then there is $\alpha \notin \Diamond^{-1}(P)$ such that

$$\psi \vdash_{\mathcal{S}} \varphi \vee \alpha.$$

By the rule [$\Diamond\Box$] we obtain,

$$\Box\psi \vdash_{\mathcal{S}} \Box\varphi \vee \Diamond\alpha,$$

and since $\Box\psi \in P$ and $\Box\varphi \notin P$ we have $\Diamond\alpha \in P$ which is absurd. Therefore we obtain (5). Now we can apply Proposition 6.2 in order to obtain a prime theory Q such that

$$\Box^{-1}(P) \subseteq Q \quad \text{and} \quad Q \cap \Sigma' = \varnothing.$$

Thus

$$\Box^{-1}(P) \subseteq Q \subseteq \Diamond^{-1}(P) \quad \text{and} \quad \varphi \notin Q.$$

Therefore $\langle P, Q\rangle \in R_{\mathcal{S}}$ and $\varphi \notin Q$. □

We can define the *canonical model* for $\mathcal{S}$ as the model $\langle \mathcal{F}_{\mathcal{S}}, V_{\mathcal{S}}\rangle$ on the canonical frame where $V_{\mathcal{S}}$ is the valuation defined by

$$V_{\mathcal{S}}(p) = \{P \in M_{\mathcal{S}} : p \in P\},$$

for any variable p. It is clear that $V_{\mathcal{S}}$ is a valuation since the sets $\{P \in M_{\mathcal{S}} : p \in P\}$ are clearly increasing.

Lemma 6.8 (Canonical Model Lemma) *In the canonical model it holds that for any prime theory P, any formula φ, and any sequent $\Gamma \vdash \varphi$,*

1. $\langle \mathcal{F}_{\mathcal{S}}, V_{\mathcal{S}}\rangle, P \Vdash \varphi$ *if and only if* $\varphi \in P$;
2. $\langle \mathcal{F}_{\mathcal{S}}, V_{\mathcal{S}}\rangle, P \Vdash \Gamma \vdash \varphi$ *if and only if* $\Gamma \nsubseteq P$ *or* $\varphi \in P$;
3. $\langle \mathcal{F}_{\mathcal{S}}, V_{\mathcal{S}}\rangle \models \Gamma \vdash \varphi$ *if and only if* $\Gamma \vdash_{\mathcal{S}} \varphi$.

Proof: (2) follows from (1). (1) is proved by induction using Proposition 6.6 and Proposition 6.7. (3) is proved as follows. If $\Gamma \vdash_{\mathcal{S}} \varphi$, because of soundness it is clear that $V_{\mathcal{S}}(\Gamma) \subseteq V_{\mathcal{S}}(\varphi)$, and therefore $\langle \mathcal{F}_{\mathcal{S}}, V_{\mathcal{S}}\rangle \models \Gamma \vdash \varphi$. Now, if $\Gamma \nvdash_{\mathcal{S}} \varphi$, by Proposition 6.2 there is a prime theory P such that $\Gamma \subseteq P$ and $\varphi \notin P$. Hence, by (2), $\langle \mathcal{F}_{\mathcal{S}}, V_{\mathcal{S}}\rangle \not\models \Gamma \vdash \varphi$. □

Theorem 6.9 *Any deductive system $\mathcal{S}$ that is an extension of $\mathcal{S}_m$ is complete relative to its models: that is, any sequent valid in all of its models is a sequent of $\mathcal{S}$.*

Proof: Assume that a sequent $\Gamma \vdash \varphi$ is valid in every model of $\mathcal{S}$. So since by the canonical model lemma, the canonical model is a model of $\mathcal{S}$, $\Gamma \vdash \varphi$ is valid in the canonical model of $\mathcal{S}$. Therefore by (3) of the canonical model lemma, $\Gamma \vdash_{\mathcal{S}} \varphi$. □

A deductive system $\mathcal{S}$ that extends $\mathcal{S}_m$ is *canonical* if its canonical frame is a frame of $\mathcal{S}$: that is, if every $\mathcal{S}$-sequent is valid on it.

Observation 6.10 A deductive system $\mathcal{S}$ that extends $\mathcal{S}_m$ is canonical if and only if the deductive system of its canonical frame is $\mathcal{S}$. Therefore, any canonical system is frame complete.

Proof: If $\mathcal{S}$ is canonical any $\mathcal{S}$-sequent is valid in the canonical frame. Moreover, if a sequent is valid in the canonical frame it is valid in the canonical model and then using (3) of the canonical model lemma it must be an $\mathcal{S}$-sequent. The other implication is clear. Now, if $\mathcal{S}$ is canonical it is characterized by the class of frames $\{\mathcal{F}_S\}$. Therefore, it is frame complete. □

Theorem 6.11 *The deductive system $\mathcal{S}_m$ is canonical and hence frame complete.*

Proof: The $\mathcal{S}_m$-sequents are valid in any frame and if a sequent is valid in the canonical frame it is valid in the canonical model. Therefore by (3) of the canonical model lemma we obtain that it is an $\mathcal{S}_m$-sequent. □

To conclude this section we will prove that $\mathcal{S}_m$ is indeed the $\vee, \wedge, \top, \bot, \Box, \Diamond$-fragment of the modal deductive system K^{MP}: that is, the deductive system obtained from the minimum classical normal modal logic K by considering the local consequence relation that can be defined as in the introduction.

Theorem 6.12 *The deductive system $\mathcal{S}_m$ is the $\vee, \wedge, \top, \bot, \Box, \Diamond$-fragment of the modal deductive system* K^{MP}.

Proof: On the one hand, it is easy to see that if $\Gamma \vdash_{\mathcal{S}_m} \varphi$, for a set of formulas $\Gamma \cup \{\varphi\}$ of the language $\{\vee, \wedge, \top, \bot, \Box, \Diamond\}$, then $\Gamma \vdash_{\mathrm{K}^{\mathrm{MP}}} \varphi$ because any rule and axiom of the Gentzen calculus is a sound axiom or rule of K. On the other hand, if $\Gamma \vdash_{\mathrm{K}^{\mathrm{MP}}} \varphi$, assume without loss of generality that Γ is finite. Then if $\Gamma \nvdash_{\mathcal{S}_m} \varphi$, in the canonical model $\langle M_{\mathcal{S}_m}, \subseteq, R_{\mathcal{S}_m}, V_{\mathcal{S}_m}\rangle$ there is a prime theory P such that every formula in Γ is true at P but φ is false at P. Clearly $\langle M_{\mathcal{S}_m}, R_{\mathcal{S}_m}, V_{\mathcal{S}_m}\rangle$ is a model for K on which every formula in Γ is true at P but φ is false at P. Therefore $\Gamma \nvdash_{\mathrm{K}^{\mathrm{MP}}} \varphi$, against the assumption. □

7 Some canonical deductive systems In this section we will prove that any extension of $\mathcal{S}_m$ by some subset of the the set of sequents

$$\{\mathrm{T}_\Box, \mathrm{T}_\Diamond, 4_\Box, 4_\Diamond, \mathrm{B}_1, \mathrm{B}_2, \mathrm{S}, \mathrm{E}_1, \mathrm{E}_2, \mathrm{D}\}$$

is canonical and therefore frame complete. This solves a problem which arises in Dunn's paper, as we said in the introduction. With his semantics the deductive system $\mathcal{S}_m + 4_\Box$ is not frame complete because $4_\Diamond$ is valid in all frames of the system but is not a sequent of it. With our semantics we obtain frame completeness for both deductive systems $\mathcal{S}_m + 4_\Box$ and $\mathcal{S}_m + 4_\Diamond$. This is a desirable situation because in the absence of negation the sequents $4_\Box$ and $4_\Diamond$ are no longer dependent and this fact must be reflected in the semantics.

To prove that the mentioned deductive systems are canonical we will use the correspondence results by seeing that for each one of these sequents the relations $R_\Box$ and $R_\Diamond$ of the canonical frame have the properties that characterize the frames where they are valid.

Theorem 7.1 *Let $\mathcal{S}$ be a deductive system that extends $\mathcal{S}_m$ and let $R_\Box$ and $R_\Diamond$ be the corresponding relations of its canonical frame. Then*

1. *if* $\mathrm{T}_\Box$ *is an $\mathcal{S}$-sequent then $R_\Box$ is reflexive;*

2. *if* $\mathrm{T}_\Diamond$ *is an* $\mathcal{S}$*-sequent then* $R_\Diamond$ *is reflexive;*
3. *if* $4_\Box$ *is an* $\mathcal{S}$*-sequent then* $R_\Box$ *is transitive;*
4. *if* $4_\Diamond$ *is an* $\mathcal{S}$*-sequent then* $R_\Diamond$ *is transitive;*
5. *if* B_1 *is an* $\mathcal{S}$*-sequent then* $R_\Box \subseteq R_\Diamond^{-1}$*;*
6. *if* B_2 *is an* $\mathcal{S}$*-sequent then* $R_\Diamond^{-1} \subseteq R_\Box$*;*
7. *if* S *is an* $\mathcal{S}$*-sequent then* $R_\mathcal{S}$ *is serial and so* $R_\Box \cap R_\Diamond$ *is also serial.*

Proof: (1) Suppose that $\mathrm{T}_\Box$ is an $\mathcal{S}$-sequent. Let P be a prime $\mathcal{S}$-theory. It happens that $\Box^{-1}(P) \subseteq P$ because if $\Box\varphi \in P$, since $\Box\varphi \vdash_\mathcal{S} \varphi$, $\varphi \in P$. Therefore by Lemma 6.3 there is a prime theory D such that $\langle P, D\rangle \in R_\mathcal{S}$ and $D \subseteq P$. Therefore $\langle P, P\rangle \in R_\Box$. The proof of (2) is similar by using Lemma 6.4 instead of Lemma 6.3.

(3) Assume that $4_\Box$ is an $\mathcal{S}$-sequent and that P, Q, D are prime theories such that $\langle P, Q\rangle \in R_\Box$ and $\langle Q, D\rangle \in R_\Box$. So there are prime theories P' and Q' such that

$$\langle P, P'\rangle \in R_\mathcal{S},\ P' \subseteq Q,\ \langle Q, Q'\rangle \in R_\mathcal{S},\ \text{ and } Q' \subseteq D.$$

Let us see that $\Box^{-1}(P) \subseteq D$. If $\Box\varphi \in P$ then $\Box\Box\varphi \in P$ because $4_\Box$ is an $\mathcal{S}$-sequent. So, $\Box\varphi \in \Box^{-1}(P)$ and therefore $\Box\varphi \in P' \subseteq Q$. Thus, $\varphi \in \Box^{-1}(Q) \subseteq Q' \subseteq D$, as desired. Now we can apply Lemma 6.3 to obtain a prime theory D' such that $\langle P, D'\rangle \in R_\mathcal{S}$ and $D' \subseteq D$. Therefore $\langle P, D\rangle \in R_\Box$, and this concludes the proof. The proof of (4) can be dealt with similarly using Lemma 6.4.

(5) Suppose that B_1 is an $\mathcal{S}$-sequent and that $\langle P, Q\rangle \in R_\Box$. Then let K be a prime theory such that $\langle P, K\rangle \in R_\mathcal{S}$ and $K \subseteq Q$. We prove that $P \subseteq \Diamond^{-1}(Q)$. If $\varphi \in P$ then $\Box\Diamond\varphi \in P$ because B_1 is an $\mathcal{S}$-sequent. Therefore, $\Box\varphi \in K \subseteq Q$. Hence $\varphi \in \Diamond^{-1}(Q)$. Now we use Lemma 6.4 to obtain a prime theory D such that $\langle Q, D\rangle \in R_\mathcal{S}$ and $P \subseteq D$. Thus $\langle Q, P\rangle \in R_\Diamond$. (6) can be proved analogously using Lemma 6.4.

(7) Suppose that S is an $\mathcal{S}$-sequent. Let us see that $R_\mathcal{S}$ is serial. Let P be a prime theory. Then $\Box^{-1}(P) \subseteq \Diamond^{-1}(P)$ because if $\Box\varphi \in P$ then, since $\Box\varphi \vdash_\mathcal{S} \varphi$, $\Diamond\varphi \in P$. Moreover $\Box^{-1}(P)$ is a theory and it is consistent because otherwise $\Diamond\bot \in P$, which is impossible. In addition, the complement of $\Diamond^{-1}(P)$ is closed under disjunctions. Therefore we can apply Proposition 6.2 to obtain a prime theory D such that

$$\Box^{-1}(P) \subseteq D \quad \text{and} \quad D \subseteq \Diamond^{-1}(P),$$

which implies that $\langle P, D\rangle \in R_\mathcal{S}$. □

Theorem 7.2 *Let* $\mathcal{S}$ *be a deductive system that extends* $\mathcal{S}_m$ *and let* $R_\Box$ *and* $R_\Diamond$ *be the corresponding relations of its canonical frame. Then*

1. *if* E_1 *is an* $\mathcal{S}$*-sequent and* $\langle P, Q\rangle \in R_\Box$ *and* $\langle P, D\rangle \in R_\Diamond$*, then* $\langle Q, D\rangle \in R_\Diamond$*.*
2. *if* E_2 *is an* $\mathcal{S}$*-sequent and* $\langle P, Q\rangle \in R_\Diamond$ *and* $\langle P, D\rangle \in R_\Box$*, then* $\langle Q, D\rangle \in R_\Box$*.*
3. *if* D *is an* $\mathcal{S}$*-sequent and* $\langle P, Q\rangle \in R_\Box$ *and* $\langle P, D\rangle \in R_\Diamond$*, then there is a prime theory* K *such that* $\langle Q, K\rangle \in R_\Diamond$ *and* $\langle D, K\rangle \in R_\Box$*.*

Proof: (1) Suppose that E_1 is an $\mathcal{S}$-sequent and that $\langle P, Q\rangle \in R_\Box$ and $\langle P, D\rangle \in R_\Diamond$. Then let P' and P'' be two prime theories such that

$$\langle P, P'\rangle \in R_\mathcal{S},\ P' \subseteq Q,\ \langle P, P''\rangle \in R_\mathcal{S},\ \text{ and } D \subseteq P''.$$

We prove that $D \subseteq \Diamond^{-1}(Q)$. Assume that $\varphi \in D$. Then $\varphi \in P''$ and since $P'' \subseteq \Diamond^{-1}(P)$, $\Diamond\varphi \in P$. Therefore, using E_1, $\Box\Diamond\varphi \in P$ and hence, since $\Box^{-1}(P) \subseteq P' \subseteq Q$, $\Diamond\varphi \in Q$ and $\varphi \in \Diamond^{-1}(Q)$. Now by Lemma 6.4 there is a prime theory D' such that $\langle Q, D'\rangle \in R_{\mathcal{S}}$ and $D \subseteq D'$. Therefore, $\langle Q, D\rangle \in R_{\Diamond}$. Similarly one can prove (2) using Lemma 6.3.

(3) Suppose that D is an $\mathcal{S}$-sequent, $\langle P, Q\rangle \in R_{\Box}$ and $\langle P, D\rangle \in R_{\Diamond}$. Then let P' and P'' be two prime theories such that

$$\langle P, P'\rangle \in R_{\mathcal{S}}, P' \subseteq Q, \langle P, P''\rangle \in R_{\mathcal{S}} \text{ and } D \subseteq P''.$$

We prove that $\Box^{-1}(D) \subseteq \Diamond^{-1}(Q)$. Assume that $\Box\varphi \in D$. Then $\Box\varphi \in P''$ and since $P'' \subseteq \Diamond^{-1}(P)$, $\Diamond\Box\varphi \in P$. Therefore, using D, $\Box\Diamond\varphi \in P$ and hence, since $\Box^{-1}(P) \subseteq P' \subseteq Q$, $\Diamond\varphi \in Q$ and $\varphi \in \Diamond^{-1}(Q)$. Now, $\Box^{-1}(D)$ is a consistent theory since otherwise $\Diamond\bot \in P$, which is impossible. Moreover, the complement of $\Diamond^{-1}(Q)$ is closed under disjunctions. Thus, by Proposition 6.2, there is a prime theory K such that

$$\Box^{-1}(D) \subseteq K \quad \text{and} \quad K \subseteq \Diamond^{-1}(Q).$$

Then by Lemma 6.3 there is a prime theory D' such that

$$\langle D, D'\rangle \in R_{\mathcal{S}} \quad \text{and} \quad D' \subseteq K,$$

and by Lemma 6.4 there is a prime theory D'' such that

$$\langle Q, D''\rangle \in R_{\mathcal{S}} \quad \text{and} \quad K \subseteq D''.$$

Therefore, we have a prime theory, that is, K, such that $\langle Q, K\rangle \in R_{\Diamond}$ and $\langle D, K\rangle \in R_{\Box}$. $\Box$

The last two theorems allow us to prove the following frame completeness theorem.

Theorem 7.3 *Any extension of $\mathcal{S}_m$ obtained by adding to the deductive system any subset of the following set of sequents*

$$\{\mathrm{T}_{\Box}, \mathrm{T}_{\Diamond}, 4_{\Box}, 4_{\Diamond}, \mathrm{B}_1, \mathrm{B}_2, \mathrm{S}, \mathrm{E}_1, \mathrm{E}_2, \mathrm{D}\}$$

is canonical and therefore frame complete.

Proof: Let X be one of these subsets. Consider the properties that characterize its frames stated in Theorems 5.1 and 5.2. Then Theorems 7.1 and 7.2 establish that the canonical frame has these properties. Therefore it is a frame of the deductive system, that is, the deductive system is canonical. $\Box$

To conclude this section we will see that several deductive systems obtained by extending $\mathcal{S}_m$ by pairs of dual sequents in the set $\{T_{\Box}, T_{\Diamond}, 4_{\Box}, 4_{\Diamond}, B_1, B_2, S, E_1, E_2, D\}$ are characterized by a class of frames that can be described by a property of the accessibility relation. First of all we will state an interesting fact concerning the canonical frames.

Lemma 7.4 *Let $\mathcal{S}$ be an extension of $\mathcal{S}_m$. Then in the canonical frame $\mathcal{F}_{\mathcal{S}}$ it holds that*

$$R_{\mathcal{S}} = R_{\Box} \cap R_{\Diamond}.$$

Proof: The inclusion $R_{\mathcal{S}} \subseteq R_\Box \cap R_\Diamond$ holds because in any frame the corresponding inclusion holds due to the fact that the relation $\leq$ is reflexive. To prove the other inclusion suppose that P and Q are prime theories such that $\langle P, Q\rangle \in R_\Box \cap R_\Diamond$. Then there are prime theories D and D' such that $\Box^{-1}(P) \subseteq D \subseteq \Diamond^{-1}(P)$, $D \subseteq Q$, $\Box^{-1}(P) \subseteq D' \subseteq \Diamond^{-1}(P)$, and $Q \subseteq D'$. Hence $\Box^{-1}(P) \subseteq Q \subseteq \Diamond^{-1}(P)$. Therefore $\langle P, Q\rangle \in R_{\mathcal{S}}$. $\Box$

In general the following proposition holds.

Proposition 7.5 *Let $\langle M, \leq, R\rangle$ be a frame. Then*

1. *if R is reflexive then $R_\Box$ and $R_\Diamond$ are reflexive;*
2. *if R is transitive then $R_\Box$ and $R_\Diamond$ are transitive;*
3. *if R is symmetric then $R_\Box = R_\Diamond^{-1}$;*
4. *if R is euclidean then the following conditions hold:*
   (a) *if $\langle x, y\rangle \in R_\Box$ and $\langle x, z\rangle \in R_\Diamond$ then $\langle y, z\rangle \in R_\Diamond$,*
   (b) *if $\langle x, y\rangle \in R_\Diamond$ and $\langle x, z\rangle \in R_\Box$ then $\langle y, z\rangle \in R_\Box$.*

Proof: (1) is immediate because $\leq$ is reflexive. (2) Assume that R is transitive and that $xR_\Box y$ and $yR_\Box z$. Then there are $u, w \in M$ such that xRu, $u \leq y$, yRw, and $w \leq z$. Therefore, since $u \leq y$ and yRw, $\langle u, w\rangle \in (\leq \circ R) \subseteq (R \circ \leq)$. So there is $v \in M$ such that uRv and $v \leq w$. Since R is transitive, xRv, and since $\leq$ is transitive too, $v \leq z$. Therefore, $xR_\Box z$. Thus we conclude that $R_\Box$ is transitive. In an analogous way it can be proved that $R_\Diamond$ is transitive.

(3) Assume that R is symmetric and that $xR_\Box y$. Then let $z \in M$ such that xRz and $z \leq y$. Therefore $\langle y, x\rangle \in (\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$. But $R \circ \leq^{-1}$ is $R_\Diamond$. Thus $R_\Box \subseteq R_\Diamond^{-1}$. The other inclusion is proved in a similar way.

(4) Assume that R is euclidean. We prove the first condition, the other one is proved analogously. Suppose that $xR_\Box y$ and $xR_\Diamond z$. Then there are $u, v \in M$ such that xRu, $u \leq y$, xRv, and $z \leq v$. Since R is euclidean, uRv. Therefore, since $u \leq y$ and uRv, $\langle y, v\rangle \in (\leq^{-1} \circ R) \subseteq (R \circ \leq^{-1})$. Let $w \in M$ be such that yRw and $v \leq w$. Then, since $z \leq v$, yRw, and $z \leq w$, and therefore $yR_\Diamond z$. $\Box$

Proposition 7.6 *Let $\mathcal{S}$ be an extension of $\mathcal{S}_m$ and consider the canonical frame $\mathcal{F}_{\mathcal{S}} = \langle M_{\mathcal{S}}, \subseteq, R_{\mathcal{S}}\rangle$. Then*

1. *$R_{\mathcal{S}}$ is reflexive if and only if $R_\Box$ and $R_\Diamond$ are reflexive;*
2. *$R_{\mathcal{S}}$ is transitive if and only if $R_\Box$ and $R_\Diamond$ are transitive;*
3. *$R_{\mathcal{S}}$ is symmetric if and only if $R_\Box = R_\Diamond^{-1}$;*
4. *$R_{\mathcal{S}}$ is euclidean if and only if the following conditions hold:*
   (a) *if $\langle P, Q\rangle \in R_\Box$ and $\langle P, D\rangle \in R_\Diamond$ then $\langle Q, D\rangle \in R_\Diamond$,*
   (b) *if $\langle P, Q\rangle \in R_\Diamond$ and $\langle P, D\rangle \in R_\Box$ then $\langle Q, D\rangle \in R_\Box$.*

Proof: The implications from right to left follow easily from Lemma 7.4 and the implications from left to right from the previous proposition. $\Box$

From the previous propositions follows the next theorem.

Theorem 7.7

1. *$\mathcal{S}_m + \{\mathrm{T}_\Box, \mathrm{T}_\Diamond\}$ is characterized by the class of frames with a reflexive accessibility relation.*
2. *$\mathcal{S}_m + \{4_\Box, 4_\Diamond\}$ is characterized by the class of frames with a transitive accessibility relation.*
3. *$\mathcal{S}_m + \{\mathrm{B}_1, \mathrm{B}_2\}$ is characterized by the class of frames with a symmetric accessibility relation.*
4. *$\mathcal{S}_m + \{\mathrm{E}_1, \mathrm{E}_2\}$ is characterized by the class of frames with a euclidean accessibility relation.*
5. *$\mathcal{S}_m + \{\mathrm{S}\}$ is characterized by the class of frames with a serial accessibility relation.*

By using this theorem the reader can obtain similar characterization theorems by extending $\mathcal{S}_m$ with pairs of dual sequents among the ones just considered.[1]

Acknowledgments The work of the second author was partially supported by Spanish DGICYT grant PB94-0920.

NOTE

1. After the paper was accepted we were informed by J. Jaspars of the close connections between postive modal logic and *partial modal logic* as presented in [6] and [5]. The reader can compare the systems **M** of Partial Modal Logic in [6] with the system $\mathcal{S}_m$ of the present paper as well as the respective completeness proofs. Moreover, as pointed out to us by J. Jaspars, the proof of Theorem 5.5 in [6] given in [5] can be easily adapted to obtain the analogous result for Positive Modal Logic.

REFERENCES

[1] Božić, M., and K. Došen, "Models for normal intuitionistic modal logics," *Studia Logica*, vol. 43 (1984), pp. 217–45. Zbl 0634.03014 MR 86h:03020 1, 1

[2] Celani, S., *Lógicas Modales distributivas y de De Morgan*, Ph.D. dissertation, Universitat de Barcelona, Barcelona, 1995. 1

[3] Došen, K., "Models for stronger normal intuitionistic modal logics," *Studia Logica*, vol. 44 (1985), pp. 39–70. Zbl 0634.03015 MR 87d:03042 1

[4] Dunn, J. M., "Positive modal logic," *Studia Logica*, vol. 55 (1995), pp. 301–17. Zbl 0831.03007 96f:03013 1, 1

[5] Jaspars, J., *Calculi for Constructive Communication: A Study of the Dynamics of Partial States.* Doctoral dissertation. ILLC Dissertation Series 1994-4 and ITK Dissertation Series 1994-1, University of Amsterdam, Amsterdam, 1994. 7, 7

[6] Jaspars, J., and E. Thijsee, "Fundamentals of partial modal logic" pp. 111–42 in *Partiality, Modality and Nonmonotonicity*, edited by P. Doherty, CSLI, Stanford, 1995. Zbl 0904.03013 MR 1427286 7, 7, 7

Facultad de Ciencias Exactas
Universidad Nacional del Centro
de la Provincia de Buenos Aires
ARGENTINA
email: *scelani@tandil.edu.ar*

Departament de Lògica,
Història i Filosofia de la Ciència
Universitat de Barcelona
08028 Barcelona
SPAIN
email: *jansana@cerber.mat.ub.es*