# arXiv:2112.05055v3 [math.NA] 27 Apr 2023 arXiv:2112.05055v3 [math.NA] 27 Apr 2023

# MULTIVARIATE ANALYSIS-SUITABLE T-SPLINES OF ARBITRARY DEGREE

ROBIN GÖRMER<sup>\*</sup> AND PHILIPP MORGENSTERN<sup>†</sup>

Abstract. This paper defines analysis-suitable T-splines for arbitrary degree (including even and mixed degrees) and arbitrary dimension. We generalize the concept of anchor elements known from the two-dimensional setting, extend existing concepts of analysis-suitability and show their sufficiency for linearly independent T-Splines.

Key words. multivariate T-splines, Analysis-Suitability, Dual-Compatibility

AMS subject classifications. 65D07, 65D99, 65K99

1. Introduction. T-splines were introduced in 2003 in computer-aided design as a new realization for B-splines on non-uniform meshes [\[1\]](#page-23-0) with local mesh refinement [\[2\]](#page-23-1). Shortly after, Isogeometric Analysis was introduced, and T-splines were applied as ansatz functions for Galerkin schemes with promising results [\[3,](#page-23-2) [4\]](#page-23-3), but were proven to lack linear independence in certain cases [\[5\]](#page-23-4), which actually excludes them from the application in a Galerkin method. The issue was solved in 2012 [\[6\]](#page-24-0), proving that linear independence is guaranteed if meshline extensions at the hanging nodes, called T-junction extensions, do not intersect. This criterion is referred to as analysis-suitability in the literature, however we denote it as geometric analysissuitability in this paper for distinction against abstract analysis-suitability below. Still in 2012, the introduction of dual-compatibility and its equivalence to analysissuitability [\[7\]](#page-24-1) provided new insight on the linear independence of T-splines, and in 2013, analysis-suitability was generalized to T-splines of arbitrary polynomial degree [\[8\]](#page-24-2), still restricted to the two-dimensional case, while dual-compatibility could easily be generalized to higher dimensions [\[9,](#page-24-3) Definition 7.2]. At that time, techniques for the construction of 3D T-spline meshes from boundary representations were introduced [\[10,](#page-24-4) [11\]](#page-24-5), motivating the theoretical research on T-splines in three space dimensions, but in particular the linear independence of higher-dimensional T-splines was only characterized through the dual-compatibility criterion, until in 2016, an abstract version of analysis-suitability in three dimensions [\[12\]](#page-24-6) was introduced and, in 2017, generalized to arbitrary dimension [\[13\]](#page-24-7), but only for odd polynomial degrees. Throughout this paper, we refer to this version as abstract analysis-suitability (AAS), and to its equivalent strong version of dual-compatibility as SDC, while we abbreviate the weaker version from [\[9\]](#page-24-3) with WDC.

This paper generalizes abstract analysis-suitability from [\[13\]](#page-24-7) to arbitrary degrees and geometric analysis-suitability from  $[6]$  to arbitrary dimensions. We investigate the sufficiency for linearly independent spline bases as well as the relations and implications between all above-mentioned versions of analysis-suitability and dualcompatibility (see [Figure 1](#page-1-0) for a visualization of the results).

The paper is organized as follows. In [section 2,](#page-1-1) we investigate T-junctions in the high-dimensional setting, i.e. hanging  $(d-2)$ -dimensional interfaces in d-dimensional box meshes. In [section 3,](#page-6-0) we generalize the concept of anchor elements from [\[8\]](#page-24-2) to arbitrary dimension, as outlined in [\[14\]](#page-24-8). This allows a straight-forward generalization of [\[13\]](#page-24-7) to arbitrary degrees in [section 4.](#page-8-0) The generalization of T-junction extensions

<sup>∗</sup>Leibniz University Hannover, Institue of Applied Mathematics, Welfengarten 1, 30167 Hannover, Germany. Email: [goermer@ifam.uni-hannover.de,](mailto:goermer@ifam.uni-hannover.de) [morgenstern@ifam.uni-hannover.de](mailto:morgenstern@ifam.uni-hannover.de)

<span id="page-1-0"></span>

Fig. 1. Nesting behavior of the mesh classes considered in this paper.

from [\[8\]](#page-24-2) is more technical, but yields geometric criteria for linearly independent splines that can easily be visualized and checked. We define a weak and a strong version of geometric analysis-suitability (WGAS and SGAS, respectively). For the strong version, we prove sufficiency for linearly independence of the T-splines, for the weak version we conjecture it, see [Conjecture 6.3,](#page-20-0) providing two incomplete proofs. [Section 5](#page-16-0) recalls the concept of dual-compatibility, which is already available for arbitrary degree and dimension  $[9, 13]$  $[9, 13]$  $[9, 13]$  and does not need further generalization. In [section 6,](#page-20-1) we show that the equivalence of AAS and SDC is valid analogously to the odd-degree case from [\[13\]](#page-24-7). We further show that SGAS implies AAS and argument, however with incomplete proof, that WGAS implies WDC. To apply results of dual-compatible splines such as linear independence or projection properties, it is hence sufficient that the considered mesh is analysis-suitable in the geometric or abstract sense. Conclusions and outlook to future work are given in [section 7.](#page-23-5)

<span id="page-1-1"></span>2. T-junctions in high-dimensional box meshes. We consider a box-shaped open index domain  $\Omega = \mathsf{X}_{k=1}^d(0, N_k)$ , with  $N_k \in \mathbb{N}$  for  $k = 1, ..., d$  and an associated parametric domain  $\hat{\Omega} = \mathsf{X}_{k=1}^d(\xi_0^{(k)}, \xi_{N_k}^{(k)})$ , with global  $p_k$ -open knot vectors  $\Xi^{(k)} = \{\xi_0^{(k)}, \ldots, \xi_{N_k}^{(k)}\},\$  for polynomial degrees  $p_k \in \mathbb{N}$ . Let  $\mathcal{T}$  be a mesh of  $\Omega$ , consisting of open axis-parallel boxes with integer vertices, and constructed via symmetric bisections of boxes from an initial tensor-product mesh, which is described in detail in [Algorithm 2.1.](#page-3-0) We assume to obtain integer vertices from [Algorithm 2.1,](#page-3-0) i.e. that for the bisection of a cell  $\mathbf{Q}$  in direction j we get  $m = \frac{1}{2}(\inf \mathbf{Q}_j + \sup \mathbf{Q}_j) \in \mathbb{N}$ . This excludes for example mesh configurations as shown in [Figure 2.](#page-1-2) Further, T contains all lower-dimensional entities such as hyperfaces, faces, edges and vertices of these boxes. For  $k = 1, \ldots, d$ , we denote by  $\mathcal{H}^{(k)}$  the set of open k-dimensional mesh entities of T, e.g. by  $\mathcal{H}^{(0)}$  the set of nodes, by  $\mathcal{H}^{(1)}$  the set of one-dimensional edges without start and end point, by  $\mathcal{H}^{(2)}$  the set of two-dimensional faces without the boundary edges, and so on, such that the union  $\overline{\Omega} = \bigcup \mathcal{T}$ , with  $\mathcal{T} = \bigcup_{j=0}^d \mathcal{H}^{(j)}$ , is disjoint. The union of all element boundaries  $\text{Sk} = \bigcup_{\mathbf{Q} \in \mathcal{H}^{(d)}} \partial \mathbf{Q} = \bigcup_{j=0}^{d-1} \mathcal{H}^{(j)} = \overline{\Omega} \setminus \mathcal{H}^{(d)}$  is called the *skeleton* 

<span id="page-1-2"></span>

Fig. 2. Two examples of excluded mesh configurations.  $\overline{2}$ 

<span id="page-2-0"></span>

Fig. 3. A 3-dimensional mesh, refined in the front corner (top left), and the corresponding 1-orthogonal, 2-orthogonal and 3-orthogonal skeleton (top right, bottom left, bottom right, respectively).

of T. Note that this includes not only the 1-dimensional edges, but also the faces and hyperfaces up to dimension  $d-1$ . For an index set  $\kappa = {\kappa_1, \ldots, \kappa_k} \subset {1, \ldots, d}$ and a *d*-dimensional (volumetric) element  $\mathbf{Q} = \mathbf{Q}_1 \times \cdots \times \mathbf{Q}_d \in \mathcal{H}^{(d)}$  composed from open intervals  $\mathbf{Q}_1, \ldots, \mathbf{Q}_d$ , we denote the  $(d - \iota)$ -dimensional,  $\kappa$ -orthogonal interfaces by  $H^{(\kappa)}(\mathbb{Q}),$  i.e.

$$
(2.1) \qquad \mathbf{H}^{(\kappa)}(\mathbf{Q}) := \{ \widetilde{\mathbf{Q}} = \widetilde{\mathbf{Q}}_1 \times \cdots \times \widetilde{\mathbf{Q}}_d \mid \widetilde{\mathbf{Q}}_j \subsetneq \partial \mathbf{Q}_j \text{ for } j \in \kappa, \widetilde{\mathbf{Q}}_j = \mathbf{Q}_j \text{ for } j \notin \kappa \},
$$

where the components  $\tilde{\mathbf{Q}}_i$  are either singleton sets or open intervals with start and end points in  $\{0, \ldots, N_j\}.$ 

The global set of  $\kappa$ -orthogonal mesh entities is denoted by  $H^{(\kappa)} = \bigcup_{\mathbf{Q} \in \mathcal{H}^{(d)}} H^{(\kappa)}(\mathbf{Q}),$ with  $H^{(\emptyset)}(\mathbf{Q}) = \{\mathbf{Q}\}\$ and  $H^{(\emptyset)} = \mathcal{H}^{(d)}$ . For singleton index sets, we write  $H^{(j)} := H^{(\{j\})}$ , and we call  $\text{Sk}_j \coloneqq \bigcup_{E \in \text{H}(j)} \overline{E}$  the j-orthogonal skeleton of T. Note that it is composed of  $(d-1)$ -dimensional hyperfaces, see [Figure 3](#page-2-0) for an example.

For polynomial degrees  $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d$ , we split the index domain  $\Omega$  into an *active region*  $AR_p$  and a *frame region*  $FR_p$ , with

(2.2) 
$$
AR_{\mathbf{p}} := \bigtimes_{k=1}^{d} \left[ \left\lfloor \frac{p_k+1}{2} \right\rfloor, N_k - \left\lfloor \frac{p_k+1}{2} \right\rfloor \right] \text{ and } FR_{\mathbf{p}} := \overline{\Omega \setminus AR_{\mathbf{p}}}.
$$

Consider two cells  $\mathbf{Q}^{(1)}$ ,  $\mathbf{Q}^{(2)} \in \mathcal{H}^{(d)}$  that share a common face  $\mathbf{P} \in \mathcal{H}^{(d-1)}$ ,  $\partial \mathbf{Q}^{(1)} \cap \partial \mathbf{Q}^{(2)} = \overline{\mathbf{P}}$ . The *j*-orthogonal subdivision of  $\mathbf{Q}^{(1)}$ , i.e. the bisection of  $\mathbf{Q}_j^{(1)}$ , for some direction j which is not orthogonal to P, removes all mesh entities  $E = E_1 \times \cdots \times E_d$ with  $\mathbf{E}_j = \mathbf{Q}_j^{(1)}$  and inserts child entities  $\mathbf{E}^{(1)}$ ,  $\mathbf{E}^{(2)}$ ,  $\mathbf{E}^{(3)}$  including the children  $\mathbf{Q}^{(1,1)}$  and  $\mathbf{Q}^{(1,2)}$  of  $\mathbf{Q}^{(1)}$ , with mid  $\mathbf{Q}_j^{(1)} = \frac{1}{2}(\inf \mathbf{Q}_j^{(1)} + \sup \mathbf{Q}_j^{(1)})$ . This procedure is summarized 3

Algorithm 2.1 Subdivision of a cell.

<span id="page-3-0"></span>procedure  $SUBDIV(\mathcal{T}, \mathbb{Q}, i)$ assert that  $Q \subset AR_{p}$  $D \leftarrow \overline{\mathsf{Q}}$ for all  $\ell = 1, \ldots, d, \ell \neq j$  do if  $\min D_\ell = \lfloor \frac{p_k+1}{2}$  $D_{\ell} \leftarrow D_{\ell} \cup \left[0, \left\lfloor \frac{p_k+1}{2} \right\rfloor \right]$  $\triangleright$  If D touches the frame region, then  $\triangleright$  extend it to the end of the domain. end if  $\triangleright$  See [Remark 3.2](#page-6-1) for an explanation.  $\mathbf{if} \, \max D_\ell = N_\ell - \lfloor \frac{p_k+1}{2} \rfloor \ \mathbf{then}$  $D_{\ell} \leftarrow D_{\ell} \cup \left[ N_{\ell} - \left[ \frac{p_{k}+1}{2} \right], N_{\ell} \right]$ end if end for for all  $E \in \mathcal{T}, E \subset D, E_j = Q_j$  do  $m \leftarrow \frac{1}{2}(\inf \mathsf{Q}_j + \sup \mathsf{Q}_j)$  $\texttt{E}^{(1)} \leftarrow \texttt{E}_1 \times \cdots \times \texttt{E}_{j-1} \times (\inf \texttt{Q}_j, m) \times \texttt{E}_{j+1} \times \cdots \times \texttt{E}_d$  $\texttt{E}^{(2)} \leftarrow \texttt{E}_1 \times \cdots \times \texttt{E}_{j-1} \times \{m\} \times \texttt{E}_{j+1} \times \cdots \times \texttt{E}_{d}$  $\texttt{E}^{(3)} \leftarrow \texttt{E}_1 \times \cdots \times \texttt{E}_{j-1} \times (m, \sup \texttt{Q}_j) \times \texttt{E}_{j+1} \times \cdots \times \texttt{E}_d$  $\mathfrak{I} \leftarrow \mathfrak{I} \setminus \lbrace \mathtt{E} \rbrace \cup \lbrace \mathtt{E}^{(1)}, \mathtt{E}^{(2)}, \mathtt{E}$  $\triangleright$  Since D is a superset of Q, end for  $\rho$  at least  $\varphi$  is subdivided. return T end procedure

in [Algorithm 2.1,](#page-3-0) where additional subdivisions are done, whenever the cell to be subdivided touches the frame region, see [Remark 3.2](#page-6-1) for an explanation. Since the children inherit all but the j-th component of  $\mathbf{Q}^{(1)}$ , they satisfy  $\partial \mathbf{Q}^{(1,1)} \cap \partial \mathbf{Q}^{(2)} \neq \emptyset$ and  $\partial \mathbf{Q}^{(1,2)} \cap \partial \mathbf{Q}^{(2)} \neq \emptyset$ . Furthermore, we see that  $\mathbf{Q}^{(1,1)}$  and  $\mathbf{Q}^{(1,2)}$  share a face  $\mathbf{F} = \mathbf{Q}_1^{(1)} \times \cdots \times \mathbf{Q}_{j-1}^{(1)} \times \{\text{mid } \mathbf{Q}_j^{(1)}\} \times \mathbf{Q}_{j+1}^{(1)} \times \cdots \times \mathbf{Q}_d^{(1)} \in \mathcal{H}^{(d-1)}$ . By subdividing  $\mathbf{Q}^{(1)}$  we have thus generated an interface  $\overline{\mathbf{I}} = \overline{\mathbf{F}} \cap \partial \mathbf{Q}^{(2)}$ ,  $\mathbf{I} \subset \mathcal{H}^{(d-2)}$ , that is in the boundary of exactly three cells  $\mathbf{Q}^{(2)}$ ,  $\mathbf{Q}^{(1,1)}$ , and  $\mathbf{Q}^{(1,2)}$ . We classify this type of entities in the following definition.

DEFINITION 2.1 (T-junctions). We call an interface  $T \in \mathcal{H}^{(d-2)}$  with  $T \nsubseteq \partial \Omega$ a hanging interface or T-junction if it has valence  $|\{F \in \mathcal{H}^{(d-1)} | T \subset \partial F\}| < 4$ , or equivalently, if it is in the boundary of a cell  $Q = Q_1 \times \cdots \times Q_d \in \mathcal{T}$  without being connected to any of its vertices,  $T \subset \partial \mathbb{Q}$ ,  $\overline{T} \cap \partial \mathbb{Q}_1 \times \cdots \times \partial \mathbb{Q}_d = \emptyset$ . We then call  $\mathbb{Q}$  the associated cell of T and write  $Q = \text{ascell}(T)$ . Since  $T = T_1 \times \cdots \times T_d \in \mathcal{H}^{(d-2)}$ , there are two unique and distinct directions  $i, j \in \{1, ..., d\}$  such that  $T_i, T_j$  are singletons,  $T \in H^{(\{i,j\})}$ ,  $T_i \subsetneq Q_i$  and  $T_j \subsetneq \partial Q_j$ . We call i the orthogonal direction and j the pointing direction of T, and write odir(T) = i, pdir(T) = j.

PROPOSITION 2.2. For any T-junction T, the above-defined ascell(T), odir(T) and pdir(T) are unique.

*Proof.* Consider any  $(d-2)$ -dimensional mesh entity  $T \in \mathcal{H}^{(d-2)}$  that is not contained in the boundary of  $\Omega$ . Then T is of the form  $T = T_1 \times \cdots \times T_d$  and there exist exactly two indices  $i, j \in \{1, ..., d\}$  such that  $T_i$  and  $T_j$  are singletons and all other components  $\mathsf{T}_k$ ,  $i \neq k \neq j$ , are open intervals. Since T is a mesh entity of a d-dimensional box mesh constructed via refinement of a tensor-product mesh as assumed above, there is by construction a (possibly non-unique) cell  $\mathbf{Q} \in \mathcal{H}^{(d)}$  with  $\mathbf{Q} = \mathbf{Q}_1 \times \cdots \times \mathbf{Q}_d$  and

# (2.3)  $\mathsf{T}_i \subset \partial \mathsf{Q}_i$ ,  $\mathsf{T}_j \subset \partial \mathsf{Q}_j$ , and  $\mathsf{T}_k \subseteq \mathsf{Q}_k$  for  $i \neq k \neq j$ .

 $\mathsf Q$  is bounded by 2.d (or more, in case of T-junctions in its boundary) hyperfaces, and for each  $k \in \{1, \ldots, d\}$  and  $n_k \in \partial \mathbb{Q}_k = \{\inf \mathbb{Q}_k, \sup \mathbb{Q}_k\}$ , there is a hyperface  $\mathbf{F} \in \mathcal{H}^{(d-1)}$  with  $\mathbf{F} = \mathbf{F}_1 \times \cdots \times \mathbf{F}_d$ ,  $\mathbf{F}_k = \{n_k\}$  and  $\mathbf{F}_\ell \subseteq \mathbf{Q}_\ell$  for  $\ell \neq k$ . In particular, there are two such hyperfaces  $F^{(i)}$ ,  $F^{(j)}$  with  $F_i^{(i)} = T_i$  and  $F_j^{(j)} = T_j$ .  $F^{(i)}$  neighbors T in positive (resp. negative) j-th direction if  $T_j = \inf F_j^{(i)}$  (resp.  $T_j = \sup F_j^{(i)}$ ), and  $F^{(j)}$  neighbors T in positive (resp. negative) *i*-th direction if  $T_i = \inf F_i^{(j)}$  (resp.  $\mathbf{T}_i = \sup \mathbf{F}_i^{(j)}$ ). Together, T is neighbored by at least two  $(d-1)$ -dimensional interfaces in different directions. We assume without loss of generality that T has neighbor interfaces in positive *i*-th and *j*-th direction, i.e. that  $T_j = \inf F_j^{(i)}$  and  $T_i = \inf F_i^{(j)}$ .

Let  $h_{\mathcal{T}} = \min \{ \sup \mathbb{Q}_k - \inf \mathbb{Q}_k \mid \mathbb{Q} = \mathbb{Q}_1 \times \cdots \times \mathbb{Q}_d \in \mathcal{H}^{(d)}, k \in \{1, ..., d\} \}$  be the minimal mesh size. If there is no neighbor interface in negative  $i$ -th direction, then for any point  $x \in \mathcal{T}$  and  $0 < \varepsilon < h_{\mathcal{T}}$ , the point  $x - \varepsilon e_i$  (with  $e_i$  being the *i*-th unit vector) is in the interior of some cell  $\tilde{\mathfrak{q}} \in \mathfrak{H}^{(d)}$ , as well as the points  $x - \varepsilon e_i + \varepsilon e_j$  and  $x - \varepsilon e_i - \varepsilon e_j$ , since there is no j-orthogonal hyperface separating them.

If similarly  $T$  has no neighbor interface in negative j-th direction, then the points  $x - \varepsilon e_j + \varepsilon e_i$  and  $x - \varepsilon e_j - \varepsilon e_i$  are in the interior of Q.

If T does not have neighbor interfaces neither in negative i-th nor in negative j-th direction, then the three points  $x^{(1)} = x - \varepsilon e_i - \varepsilon e_j$ ,  $x^{(2)} = x - \varepsilon e_i + \varepsilon e_j$ ,  $x^{(3)} = x - \varepsilon e_j + \varepsilon e_i$  are in  $\tilde{\mathbb{Q}}$ , but the midpoint  $\frac{1}{2}(x^{(2)} + x^{(3)}) = x \notin \tilde{\mathbb{Q}}$  since  $x \in \mathbb{T} \subset \partial \tilde{\mathbb{Q}}$ and  $\tilde{\varphi}$  is open. This means that  $\tilde{\varphi}$  is not convex in contradiction to the assumption that  $\mathfrak{H}^{(d)}$  consists of open axis-aligned (and hence convex) boxes.

Together, any  $T \in H^{(\{i,j\})}$  is neighbored by at least three and at most four  $(d-1)$ dimensional faces. Thus, all  $T$ -junctions have valence 3. Let  $j$  be the unique direction in which there is no neighbor interface, and let  $s \in \{-1, 1\}$  indicate whether there is no neighbor face in negative  $(s=-1)$  or positive  $(s=1)$  j-th direction. Then odir(T) = i, pdir(T) = j, and ascell(T) is the unique neighbor cell containing the point  $x + \varepsilon \varepsilon_i$  for any  $x \in \mathcal{T}$ .  $\Box$ 

We give brief examples for odir( $T$ ) and  $\text{pdir}(T)$  for a hanging interface  $T$  in 2D and 3D, see also [Figure 4](#page-5-0) for related sketches. For 2D, let  $T = \{n\} \times \{m\}$  be a hanging node, and assume that it is of type  $\perp$  or  $\top$ . Then there is an associated cell ascell(T) =  $\mathbf{Q} = \mathbf{Q}_1 \times \mathbf{Q}_2$  such that the integer *n* is in the interior of  $\mathbf{Q}_1$  and *m* is the upper or lower bound of  $\mathbb{Q}_2$ , i.e.

 $n \in \mathbb{Q}_1$  and  $m \in \{\inf \mathbb{Q}_2, \sup \mathbb{Q}_2\},\$  or equivalently  $\{n\} \subseteq \mathbb{Q}_1$  and  $m \subseteq \partial \mathbb{Q}_2$ .

We hence have odir(T) = 1 and pdir(T) = 2. Similarly, for T-junctions of type  $\vdash$  or  $\neg$ we have  $odir(T) = 2$  and  $pdir(T) = 1$ .

As a 3D example, consider a hanging edge of the type  $T = \{n\} \times (m, \overline{m}) \times \{\ell\}$ with an associated cell ascell(T) =  $\mathbf{Q} = \mathbf{Q}_1 \times \mathbf{Q}_2 \times \mathbf{Q}_3$  such that

 $\ell \in \mathbb{Q}_3$  and  $n \in \{\inf \mathbb{Q}_1, \sup \mathbb{Q}_1\},$  or equivalently  $\{\ell\} \subsetneq \mathbb{Q}_3$  and  $\{n\} \subsetneq \partial \mathbb{Q}_1$ ,

which yields odir(T) = 3 and pdir(T) = 1.

The above-defined properties of T-junctions are essential for the analysis-suitability described in [section 4.](#page-8-0) Each T-junction is extended in its pointing direction and, for  $d > 2$ , by a larger amount in all other directions except the orthogonal direction,

<span id="page-5-0"></span>

Fig. 4. Examples for T-junctions and associated cells in 2D (left) and 3D (right).

and T-junction extensions with different pointing/orthogonal direction are required to be disjoint. Details are given in [section 4](#page-8-0) below. We end this section with a Lemma used for the proofs in [section 6,](#page-20-1) using the notation conv  $Z$  for the convex hull of a set Z.

<span id="page-5-1"></span>LEMMA 2.3. If two points x, y are aligned in i-direction, and  $x \in Sk_i \not\ni y$ , then there is an i-orthogonal T-junction and its associated cell between these points, i.e.

 $(2.4)$   $\forall x, y \in \Omega, i \in \{1, \ldots, d\}, x_i = y_i, x \in \mathrm{Sk}_i \not\ni y \exists \mathrm{T}, \mathrm{odir}(\mathrm{T}) = i, \mathbf{Q} = \mathrm{ascell}(\mathrm{T})$ :  $\overline{T} \cap \text{conv}\{x, y\} \neq \emptyset$ ,  $x_{\text{pdir}(T)} \neq y_{\text{pdir}(T)}$ ,  $\mathbb{Q}_{\text{pdir}(T)} \cap \text{conv}\{x_{\text{pdir}(T)}, y_{\text{pdir}(T)}\} \neq \emptyset$ .

Note that this implies  $\mathbf{T}_i = \{x_i\} = \{y_i\}.$ 

*Proof.* Define the function  $f : [0,1] \rightarrow \{0,1\}$  with

(2.5) 
$$
f(t) = \begin{cases} 1 & \text{if } (1-t)x + ty \in Sk_i \\ 0 & \text{otherwise.} \end{cases}
$$

Since  $Sk_i$  is a finite union of closed sets,  $Sk_i$  is closed as well. Consequently, the value of f at jump locations is always 1. Since  $f(0) = 1$  and  $f(1) = 0$ , there is at least one jump location  $t^* \in (0,1)$  with  $f(t^*) = 1$  and  $f(t^* + \varepsilon) = 0$  for arbitrarily small  $\varepsilon > 0$ . This means that  $x^{(t^*)} = (1 - t^*)x + t^*y \in \overline{F}$  for some *i*-orthogonal face  $F \in H^{(i)}$ , while  $x^{(t^*+\varepsilon)}$  is not in  $\overline{F'}$  for any  $F' \in H^{(i)}$ .

Moreover, since  $x^{(t^*+\varepsilon)} \in \text{conv}\{x,y\} \subset \overline{\Omega}$ , we have  $x^{(t^*+\varepsilon)} \in \overline{\mathfrak{Q}}$  for some cell  $\mathfrak{Q}$ such that any vertex v of Q satisfies  $v_i \neq x_i$ , since otherwise Q has an *i*-orthogonal hyperface in  $Sk_i$  and  $x^{(t^*+\varepsilon)}$  lies in  $Sk_i$  in contradiction to  $f(t^*+\varepsilon)=0$ . Since  $\overline{\mathbb{Q}}$  is closed and  $x^{(t^*+\varepsilon)} \in \overline{\mathbb{Q}}$  holds for arbitrarily small  $\varepsilon$ , we also have  $x^{(t^*)} \in \overline{\mathbb{Q}}$ . However,  $f(t^*) = 1$  tells us that also  $x^{(t^*)} \in \overline{\mathbb{Q}}$  holds for a different cell  $\mathbb{Q}'$  that has the *i*orthogonal hyperface F in its boundary. Hence  $x^{(t^*)} \in \partial \mathbf{Q}$ . The fact that  $x^{(t^*)} \in \overline{\mathbf{F}}$  but  $x^{(t^*+\varepsilon)} \notin \overline{\mathbf{F}}$  means that  $x^{(t^*)} \in \partial \mathbf{F}$  and hence that  $x^{(t^*)} \in \overline{\mathbf{T}} \subset \partial \mathbf{F}$  for some *i*-orthogonal entity **T**  $\in \mathcal{H}^{(d-2)}$ .

If  $x^{(t^*)} \in \mathcal{T} \in \mathcal{H}^{(i,j)} \subset \mathcal{H}^{(d-2)}$ , then  $\mathcal{T}, \mathcal{F}, \mathsf{Q}$  are unique,  $\mathcal{T} \subset \partial \mathsf{Q}$  and  $\mathcal{T}$  is a T-junction with  $Q = \text{ascell}(T)$  since it is a  $(d-2)$ -dimensional entity in the boundary of Q without being connected to any of its vertices. From  $x^{(t^*+\varepsilon)} \notin \mathcal{T}$  we conclude that  $x^{(t^*)}$  and  $x^{(t^*+\varepsilon)}$  differ in the *i*-th or *j*-th component. From  $x_i = y_i$  we get  $x_i^{(t^*)} = x_i^{(t^*+\varepsilon)}$  and hence  $x_j^{(t^*)} \neq x_j^{(t^*+\varepsilon)}$  with pdir(T) = j, which yields  $x_j \neq y_j$ . Moreover,  $x^{(t^*+\varepsilon)} \in \mathbb{Q}$ yields  $x_j^{(t^*+\varepsilon)} \in \mathbb{Q}_j$ , and  $x^{(t^*+\varepsilon)} \in \text{conv}\{x, y\}$  yields  $x_j^{(t^*+\varepsilon)} \in \text{conv}\{x_j, y_j\}$  where conv $\{x_j, y_j\}$  is  $[x_j, y_j]$  or  $[y_j, x_j]$ . We thus have  $\mathbb{Q}_j \cap \text{conv}\{x_j, y_j\} \neq \emptyset$ .

If otherwise  $x^{(t^*)} \in \partial T$ , then we consider a perturbation  $u \in \mathbb{R}^d$  such that the same construction with  $\tilde{x} = x + \tilde{\varepsilon}u$  and  $\tilde{y} = y + \tilde{\varepsilon}u$ , for any sufficiently small  $\tilde{\varepsilon} > 0$ , yields  $\tilde{x}^{(t^*)} \in \mathcal{T}$  for some  $\mathcal{T} \in \mathcal{H}^{(d-2)}$  with  $x^{(t^*)} \in \partial \mathcal{T}$ . The claim follows for any  $\tilde{\varepsilon} > 0$ and remains true for  $\tilde{\varepsilon} \to 0$ .  $\Box$ 

<span id="page-6-2"></span><span id="page-6-0"></span>3. Multivariate T-splines. This section explains the construction of multivariate T-splines, following the construction in [\[9\]](#page-24-3).

DEFINITION 3.1 (admissible meshes). We define for  $k = 1, \ldots, d$  and  $n =$  $0, \ldots, N_k$  the slice

(3.1) 
$$
S_k(n) := \bigtimes_{j=1}^{k-1} [0, N_j] \times \{n\} \times \bigtimes_{j=k+1}^d [0, N_j] = \{(x_1, \dots, x_d) \in \overline{\Omega} \mid x_k = n\},
$$

and the k-th frame region

(3.2) 
$$
\text{FR}_{\mathbf{p}}^{(k)} \coloneqq \{x \in \overline{\Omega} \mid x_k \in \left[0, \left\lfloor \frac{p_k+1}{2} \right\rfloor \right] \cup \left[N_k - \left\lfloor \frac{p_k+1}{2} \right\rfloor, N_k\right]\}.
$$

A T-mesh  $\Im$  is called admissible if for  $k = 1, \ldots, d$ , there is no T-junction  $\Im$  with  $odir(T) = k$  or  $pdf(T) = k$  in the k-th frame region, and

(3.3) 
$$
S_k(n) \subseteq Sk_k \quad \text{for } n = 0, \ldots, \left\lfloor \frac{p_k+1}{2} \right\rfloor \text{ and } n = N_k - \left\lfloor \frac{p_k+1}{2} \right\rfloor, \ldots, N_k.
$$

<span id="page-6-1"></span>Remark 3.2. [Algorithm 2.1](#page-3-0) preserves admissibility in the above sense. When subdividing a cell that touches the k-th frame region, T-junctions with pointing direction  $k$  are avoided by extending the refinement to the domain boundary. Further, since only cells in the active region can be subdivided, no k-orthogonal T-junction can be created in the k-th frame region.

For the definition of anchors and knot vectors, we follow the ideas of [\[9\]](#page-24-3). Anchors are defined as a certain type of mesh entities, e.g. edges or faces in a certain direction, and the knot vectors and sets are constructed by ray tracing these entities along the mesh. Using the above introduced sets  $H^{(\kappa)}$ , the anchors can be generalized to arbitrary dimensions.

DEFINITION 3.3 (anchors). Let  $\mathbf{p} = (p_1, \ldots, p_d)$  be the vector of polynomial degrees of the T-splines. The set of anchors is then defined by

<span id="page-6-4"></span>
$$
(3.4) \qquad \mathcal{A}_{\mathbf{p}} := \{ \mathbf{A} \in \mathbf{H}^{(\kappa)} \mid \mathbf{A} \subset \mathbf{AR}_{\mathbf{p}} \} \quad with \ \kappa = \{ \ell \in \{1, \ldots, d\} \mid p_{\ell} \ \text{odd } \}.
$$

<span id="page-6-3"></span>Similar to the literature [\[6,](#page-24-0) [8,](#page-24-2) [9\]](#page-24-3), we assign to each anchor a knot vector in each axis direction. This is achieved by fixing the anchor's  $j$ -th component to an index  $n$ and checking for which indices  $n$  the result is part of the skeleton.

DEFINITION 3.4 (global and local knot vectors). For any mesh entity  $E = E_1 \times$  $\cdots \times E_d$  and  $j \in \{1, ..., d\}$ , we define the projection  $P_{j,n}(\mathbf{E}) = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times \{n\} \times$  $E_{j+1} \times \cdots \times E_d$  of E on the slice  $S_j(n)$ , and the global knot vector

(3.5) 
$$
\mathcal{I}_j(\mathbf{E}) \coloneqq \left( n \in \mathbb{N} \mid P_{j,n}(\mathbf{E}) \subset \text{Sk}_j \right)
$$

with entries in non-decreasing order. The local knot vector  $v_j(A)$  for an anchor  $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_d$  is given by the  $p_j + 2$  consecutive indices  $\ell_0, \ldots, \ell_{p_j+1} \in \mathcal{I}_j(\mathbf{A}),$ such that  $\ell_k = \inf \mathbf{A}_j$  for  $k = \lfloor \frac{p_j+1}{2} \rfloor$ . This is, if  $p_j$  is odd, the singleton  $\mathbf{A}_j$  contains the middle entry of  $v_j(A)$ , and if  $p_j$  is even, the two middle entries of  $v_j(A)$  are the boundary values of  $A_i$ .

Note that we treat global and local knot vectors as ordered sets in the sense that  $n \in v_i(A)$  means that  $v_i(A)$  has a component equal to n. As a consequence of [Definition 3.1,](#page-6-2) any global knot vector  $\mathcal{I}_j(\mathbf{E})$  in an admissible mesh contains the values  $n=0,\ldots,\lfloor \frac{p_j+1}{2} \rfloor$  and  $n=N_j-\lfloor \frac{p_j+1}{2} \rfloor,\ldots,N_j$ .

<span id="page-7-1"></span><span id="page-7-0"></span>

<span id="page-7-4"></span><span id="page-7-3"></span><span id="page-7-2"></span>(a) **p** mod  $2 \equiv (1, 1, 1)$ (b) **p** mod  $2 \equiv (0, 1, 1)$ (c) **p** mod  $2 \equiv (0, 1, 0)$ (d) **p** mod  $2 \equiv (0, 0, 1)$ 

FIG. 5. Different anchor types on a cell in  $\mathbb{R}^3$  for various degrees p. Since the specific degree of p is not of interest for the anchor elements, we just consider different parities of p.

An example of different anchor elements for 3D is given in [Figure 5.](#page-7-0) Each example illustrates the anchor entities of a cell in the active region of the mesh. Note that  $H^{(\kappa)}$  determines the anchor type, where each direction in  $\kappa$  is fixed to singletons. In [Figure 5a,](#page-7-1) the polynomial degree is odd in every direction, hence, we get  $\kappa = \{1, 2, 3\}$ and  $H^{(\kappa)}$  corresponds to the vertices of the mesh inside the active region. In Figure [5b](#page-7-2) the polynomial degrees in the second and third coordinate are odd. It follows,  $\kappa = \{2, 3\}$ , from which we infer  $H^{(\kappa)}$  as the entities with singletons in its second and third direction, i.e. lines along the x-axis. In [Figure 5c,](#page-7-3) resp. [5d,](#page-7-4) we have  $\kappa = \{3\}$ , resp.  $\kappa = \{2\}$ , hence the set  $H^{(\kappa)}$  are faces with singletons in direction 3, resp. 2.

[Figure 6](#page-8-1) shows two examples for the construction of local knot vectors in 3D. In each example, we show for two anchors the construction of one local knot vector. The anchors are faces, and the local knot vector in direction 1 is constructed for the anchors highlighted in light blue. By tracing the anchor along the first direction, we highlight the projections that lie in the skeleton.

[Figure 6a](#page-8-2) considers the case **p** mod  $2 = (1, 0, 0)$ , i.e. anchors consist of singletons in their first coordinate,  $\mathbf{A} = {\{\bar{m}\}\times(n_1,n_2)\times(l_1,l_2)}$ . We collect the global knot vector of each anchor by tracing it along direction 1 and including the indices  $m$ for which  $P_{1,m}(\mathbf{A})$  is in the skeleton of the mesh, i.e. we check for each m if  $\{m\}\times$  $(n_1, n_2) \times (l_1, l_2) \subset Sk_1$  and include m in  $\mathfrak{I}_1(\mathbf{A})$  if this is the case. We then pick the consecutive  $p_1 + 2$  indices from  $\mathcal{I}_1(\mathbf{A})$  as the local knot vector  $v_1(\mathbf{A})$ . For the anchor  ${\bf A}^{(1)}$  at the top of [Figure 6a,](#page-8-2) we get  $v_1({\bf A}^{(1)}) = (\bar{m}-2, \bar{m}-1, \bar{m}, \bar{m}+1, \bar{m}+2)$ , and for the anchor  $\mathbf{A}^{(2)}$  at the bottom, we get  $v_1(\mathbf{A}^{(2)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 2, \bar{m} + 3)$ .

In [Figure 6b](#page-8-3) we consider anchors with singletons in their second coordinate, i.e.  $\mathbf{A} = (m_1, m_2) \times {\overline{n}} \times (\ell_1, \ell_2)$ . Fixing the first coordinate to some index m, we test  $\{m\} \times \{\bar{n}\} \times (\ell_1, \ell_2) \subset Sk_1$ . For the anchor at the top, we then get  $v_1(A^{(1)}) =$  $(m_1 - 2, m_1 - 1, m_1, m_2, m_2 + 1, m_2 + 2)$  and for the anchor at the bottom  $v_1(A^{(2)}) =$  $(m_1 - 2, m_1 - 1, m_1, m_2 + 1, m_2 + 2, m_2 + 3).$ 

DEFINITION 3.5 (T-spline). For  $p_j \in \mathbb{N}$ , we denote by  $B_{\mathbf{v}_j(\mathbf{A})}$ :  $\widehat{\Omega} \to \mathbb{R}$  the univariate B-spline function of degree  $p_j$  that is returned by the Cox-deBoor recursion with knot vector  $\xi_{v_j(A)} = (\xi_{\ell_0}^{(j)})$  $(\xi^{(j)}_{\ell_0}, \ldots, \xi^{(j)}_{\ell_{p_j+1}})$ . We assume that  $\xi^{(j)}_{\ell_0}$  $\zeta^{(j)}_{\ell_0} < \xi^{(j)}_{\ell_{p_j+1}}$  is always fulfilled. The T-spline function associated with the anchor  $A$  is defined as

(3.6) 
$$
B_{\mathbf{A}}(\zeta_1,\ldots,\zeta_d) := \prod_{j=1}^d B_{\mathbf{v}_j(\mathbf{A})}(\zeta_j), \quad \text{for } (\zeta_1,\ldots,\zeta_d) \in \widehat{\Omega},
$$

and the corresponding T-spline space is given by  $S_{\mathcal{T},p}(\widehat{\Omega}) = \text{span}\{B_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}_{\mathbf{p}}\}.$ 8

<span id="page-8-2"></span><span id="page-8-1"></span>

(a) Example for  $\mathbf{p} = (3, 2, 2)$ . The illustrated local knot vectors are  $\nu_1(\mathbf{A}^{(1)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 1, \bar{m} + 2)$  and  $\nu_1(\mathbf{A}^{(2)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 2, \bar{m} + 3).$ 

<span id="page-8-3"></span>

 $\nu_1(\mathbf{A}^{(2)}) = (m_1 - 2, m_1 - 1, m_1, m_2 + 1, m_2 + 2, m_2 + 3).$ 

FIG. 6. Construction of  $v_1(A)$  for the given anchors marked in light blue for various degrees p.

The index support of  $B_{\mathbf{A}}$  will be denoted by  $\text{supp}_{\Omega} B_{\mathbf{A}} = \chi_{k=1}^d \text{conv } \mathbf{v}_k(\mathbf{A}),$  where conv  $v_k(A) = \text{conv}(\ell_0, \ldots, \ell_{p_k+1}) = [\ell_0, \ell_{p_k+1}]$  is the closed interval from the first to the last entry of  $v_k(A)$ .

<span id="page-8-0"></span>4. Analysis-Suitability. We introduce below two versions of analysis-suitability. As shown in [section 6,](#page-20-1) both are sufficient criteria for the linear independence of the T-splines associated with the considered mesh, and we conjecture that the geometric version can be weakened, see [Conjecture 6.3.](#page-20-0)

<span id="page-8-5"></span>DEFINITION 4.1 (Abstract T-junction extensions and analysis-suitability). We define for all  $j = 1, ..., d$  and  $n = 0, ..., N_j$  the abstract T-junction extension

<span id="page-8-4"></span>(4.1) 
$$
ATJ_j(n) = S_j(n) \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \in \mathcal{I}_j(\mathbf{A})}} \text{supp}_{\Omega} B_{\mathbf{A}} \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \notin \mathcal{I}_j(\mathbf{A})}} \text{supp}_{\Omega} B_{\mathbf{A}}
$$

We call the mesh  $\mathcal T$  abstractly analysis-suitable (AAS) if the abstract T-junction extensions do not intersect in different directions, i.e. if  $\text{ATJ}_i(n) \cap \text{ATJ}_i(m) = \emptyset$  for any  $i \neq j$  and  $n \in \{0, \ldots, N_i\}$ ,  $m \in \{0, \ldots, N_j\}$ , and we write  $\mathcal{T} \in \text{AAS}.$ 

We will use the notation  $ATJ_i \equiv ATJ_i(\mathcal{T})$  to refer to the set of all *i*-orthogonal abstract T-junction extensions within the mesh T, i.e.

(4.2) 
$$
\text{ATJ}_i = \bigcup_{n=0}^{N_i} \text{ATJ}_i(n),
$$

in which case a mesh is AAS if ATJ<sub>i</sub> ∩ ATJ<sub>j</sub> =  $\varnothing$  for  $i \neq j$ . Note also that if  $n \notin$ conv v<sub>j</sub> (A), then  $S_j(n) \cap \text{supp}_{\Omega} B_{\mathbf{A}} = \emptyset$  and A does not contribute to the right-hand side in [\(4.1\).](#page-8-4) Using the notation  $P_{i,n}(\mathbf{E}) = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{i-1} \times \{n\} \times \mathbf{E}_{i+1} \times$  $\cdots \times E_d$  as in [Definition 3.4,](#page-6-3) the above-defined abstract T-junction extensions are also neighborhoods of T-junctions in the following sense.

PROPOSITION 4.2. For any point  $x$  in a non-empty abstract  $T$ -junction extension ATJ<sub>i</sub>(n), there is an anchor  $A \in A_p$  with  $x \in \text{supp}_\Omega B_A$ . Further, there is an iorthogonal T-junction T and its associated cell  $\mathbf{Q} = \text{ascell}(\mathbf{T})$  between x and  $P_{i,n}(\mathbf{A}),$ i.e.

1. the T-junction T intersects the convex hull of  $P_{i,n}(\mathbf{A})$  and  $\{x\}$ , i.e.

<span id="page-9-1"></span>(4.3) 
$$
\overline{T} \cap \text{conv}(P_{i,n}(\mathbf{A}) \cup \{x\}) \neq \varnothing,
$$

2. in pointing direction of T, the associated cell intersects the convex hull of  $\mathbf{A}_{\text{pdir(T)}}$  and  $\{x_{\text{pdir(T)}}\}, i.e.$ 

(4.4) 
$$
\mathbf{Q}_{\text{pdir}(\mathbf{T})} \cap \text{conv}(\mathbf{A}_{\text{pdir}(\mathbf{T})} \cup \{x_{\text{pdir}(\mathbf{T})}\}) \neq \varnothing,
$$

3. there exists a number  $y \in \mathbf{A}_{\text{pdir}(\mathbf{T})}$  with  $y \neq x_{\text{pdir}(\mathbf{T})}$ .

*Proof.* Consider arbitrary  $i \in \{1, ..., d\}, n \in \{0, ..., N_i\}$  with  $ATJ_i(n) \neq \emptyset$  and arbitrary

(4.5) 
$$
x \in \text{ATJ}_{i}(n) = S_{i}(n) \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \in \mathcal{I}_{i}(\mathbf{A})}} \text{supp}_{\Omega} B_{\mathbf{A}} \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \notin \mathcal{I}_{i}(\mathbf{A})}} \text{supp}_{\Omega} B_{\mathbf{A}}.
$$

There are by construction anchors  $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}$  with  $n \in \mathcal{I}_i(\mathbf{A}^{(1)})$  and  $n \notin \mathcal{I}_i(\mathbf{A}^{(2)})$ . The [Definition 3.4](#page-6-3) of global knot vectors yields equivalently  $P_{i,n}(\mathbf{A}^{(1)}) \subset Sk_i$  and  $P_{i,n}(\mathbf{A}^{(2)}) \not\subset \text{Sk}_i.$ 

If  $x \in Sk_i$ , then set  $\mathbf{A} \coloneqq \mathbf{A}^{(2)}$ , otherwise  $\mathbf{A} \coloneqq \mathbf{A}^{(1)}$ . There is a point  $y \in P_{i,n}(\mathbf{A})$ such that  $x \in Sk_i \neq y$  or  $x \notin Sk_i \ni y$ . [Lemma 2.3](#page-5-1) yields an *i*-orthogonal T-junction  $\texttt{T} \in \mathbb{T}_i$  and associated cell  $\texttt{Q}$  with

$$
\overline{\mathbf{T}} \cap \text{conv}\{x, y\} \neq \varnothing,
$$

(4.7) 
$$
\mathbf{Q}_{\text{pdir}(T)} \cap \text{conv}\{x_{\text{pdir}(T)}, y_{\text{pdir}(T)}\}) \neq \varnothing,
$$

$$
(4.8) \t\t ypair(T) \neq xpair(T).
$$

Since  $y \in P_{i,n}(\mathbf{A})$  and  $\text{pdir}(\mathbf{T}) \neq i = \text{odir}(\mathbf{T})$ , this concludes the proof.

<span id="page-9-0"></span>DEFINITION 4.3 (Geometric T-junction extensions and analysis-suitability). Let T be a T-junction with  $Q = \text{ascell}(T)$ ,  $i = \text{odir}(T)$  and  $j = \text{pdir}(T)$ . We then define local knot vectors as follows.

1. For  $k = j$ , we define  $v_j(T) = (\ell_0, \ldots, \ell_{p_j})$  as the vector of  $(p_j + 1)$  consecutive indices from  $\mathcal{I}_i(\mathbf{T})$ , such that

<span id="page-9-2"></span>(4.9) 
$$
\begin{aligned}\n\{\ell_{p_j/2}\} &= \mathbf{T}_j, &\text{if } p_j \text{ is even,} \\
\ell_{\lfloor p_j/2 \rfloor} &= \inf \mathsf{Q}_j, &\ell_{\lceil p_j/2 \rceil} &= \sup \mathsf{Q}_j, &\text{if } p_j \text{ is odd.} \\
10\n\end{aligned}
$$

 $\Box$ 

- 2. For  $k = i$ , the local knot vector is the singleton  $v_i(T) = T_i$ .
- 3. For  $k \notin \{i, j\}$  we define  $v_k(T) = (\ell_0, \ldots, \ell_{p_k+1+c_k}),$  where  $c_k = p_k \mod 2$ , as the vector of  $(p_k + 2 + c_k)$  consecutive indices from  $\mathfrak{I}_k(\mathbf{T})$ , such that

(4.10) 
$$
\mathbf{T}_k = (\ell_{\lceil p_k/2 \rceil}, \ell_{\lceil p_k/2 \rceil+1}).
$$

This means that the local knot vector has  $p_k + 3$  elements if  $p_k$  is odd and  $p_k+2$  if  $p_k$  is even, and  $T_k$  is centered within these elements, cf. the definition of local knot vectors for anchors.

We then call

(4.11) 
$$
\text{GTJ}_i(\mathbf{T}) \coloneqq \bigtimes_{k=1}^d \text{conv}(\mathbf{v}_k(\mathbf{T}))
$$

the geometric T-junction extension (GTJ) of T, and we say that it is an i-orthogonal extension in j-direction. Note that  $\text{GTJ}_i(\mathbf{T}) \not\subset \text{Sk}_i$ .

A mesh T is strongly geometrically analysis-suitable (SGAS), if for any two Tjunctions  $T_1, T_2$  with orthogonal directions  $i_1 = \text{odir}(T_1) \neq \text{odir}(T_2) = i_2$  holds

<span id="page-10-0"></span>(4.12) 
$$
GTI_{i_1}(T_1) \cap GTJ_{i_2}(T_2) = \varnothing.
$$

We call  $\mathcal T$  weakly geometrically analysis-suitable (WGAS), if [\(4.12\)](#page-10-0) holds for any two T-junctions  $T_1, T_2$  with orthogonal directions  $odir(T_1) \neq odir(T_2)$  and pointing directions  $\text{pdir}(T_1) \neq \text{pdir}(T_2)$ .

We will omit the dependency of the orthogonal direction, when clear from the context, e.g. write  $GTJ(T) \equiv GTJ_i(T)$ , for  $odir(T) = i$ .

Note that the latter is a weaker criterion since T-junction extensions with different orthogonal directions but equal pointing direction are allowed to intersect. Later in this paper, we will refer to the set  $GTJ_i \equiv GTJ_i(\mathcal{T})$  as the union of all geometric T-junction extensions for hanging interfaces T with odir(T) = i, i.e.

(4.13) 
$$
GTJ_i := \bigcup_{T \in \mathbb{T}_i} GTJ(T),
$$

(4.14) 
$$
\mathbb{T}_i \coloneqq \{ \mathbf{T} \in \mathfrak{H}^{(d-2)} \mid \text{valence}(\mathbf{T}) < 4, \mathbf{T} \not\subset \partial \Omega, \text{odir}(\mathbf{T}) = i \}.
$$

A mesh is then SGAS if  $\text{GTJ}_i \cap \text{GTJ}_j = \varnothing$  for  $i \neq j$ .

Remark 4.4. Note that the above definition of geometric T-junction extensions is consistent with the literature [\[8\]](#page-24-2) for the 2D case. A T-junction is then given as  $T = \{i\} \times \{j\}$ , where  $\text{pdir}(T) = 1$  corresponds to a T-junction of type  $\vdash$  or  $\neg$ and pdir(T) = 2 corresponds to a T-junction of type  $\perp$  or  $\top$ . In any case, the T-junction extension will be a line along the pointing direction, consisting of  $p_{\text{ndir}}(\tau) + 1$ consecutive indices from the knot vector, as in the 2D case.

In the case  $d = 2$ , SGAS and WGAS are equivalent and sufficient for linear in-dependence, see [\[8\]](#page-24-2). We assume for the rest of this paper that  $d \geq 3$  and that the initial mesh is sufficiently fine in the sense of the assumption below. It is applied in [Lemma 4.8,](#page-12-0) which is used for the theorems in [sections 5](#page-16-0) and [6.](#page-20-1)

<span id="page-10-2"></span><span id="page-10-1"></span>ASSUMPTION 4.5. For any mesh considered below, there are for each cell  $\mathbf{Q} \in \mathcal{H}^{(d)}$ at least three distinct directions  $i \neq j \neq k \neq i$  in each of which Q has an active neighbor cell. E.g., this is fulfilled if the initial mesh contains at least 2 active cells in each of three pairwise distinct directions.

LEMMA 4.6. Let T be a WGAS mesh, E an anchor or T-junction and  $v_{\ell}(E)$  its local knot vector in direction  $\ell \in \{1, ..., d\}$ , then for any  $m \in \text{conv } \mathsf{v}_{\ell}(\mathsf{E})$  holds  $P_{j,m}(\mathsf{E}) \subset$  $\mathrm{Sk}_j$  or  $P_{j,m}(\mathbf{E}) \cap \mathrm{Sk}_j = \varnothing$ .

*Proof.* Since  $Sk_j$  is by construction a closed set,  $P_{j,m}(E) \subset Sk_j$  is sufficient for  $P_{j,m}(E) \subset Sk_j$ , and we only need to show that  $P_{j,m}(E) \subset Sk_j$  or  $P_{j,m}(E) \cap Sk_j = \emptyset$ .

Assume for contradiction a WGAS mesh and  $m \in \text{conv } v_{\ell}(E)$  such that there exist  $x, y \in P_{j,m}(\mathbf{E})$  with  $x \in \mathcal{S}_{kj} \not\ni y$ . Recall from the beginning of [section 2](#page-1-1) that the mesh consists of boxes with integer vertices and hence  $m$  is an integer. By definition of mesh entities we have  $P_{j,n}(\mathbf{E}) \subset \mathcal{S}_{k_j}$  for  $n \in \{\inf \mathbf{E}_j, \sup \mathbf{E}_j\}$  and  $P_{j,n}(\mathbf{E}) \cap \mathcal{S}_{k_j} = \emptyset$  for  $n \in \mathbf{E}_j \setminus \{\inf \mathbf{E}_j, \sup \mathbf{E}_j\}.$  Hence  $m < \inf \mathbf{E}_j$  or  $m > \sup \mathbf{E}_j$ . Without loss of generality, we assume  $m > \sup E_j$ , and we assume further that m is minimal, i.e. that there is no  $\tilde{m} \in (\sup \mathsf{E}_j, m)$  with  $P_{j, \tilde{m}}(\mathsf{E}) \not\subset \mathsf{Sk}_j$  and  $P_{j, \tilde{m}}(\mathsf{E}) \cap \mathsf{Sk}_j \neq \emptyset$ .

[Lemma 2.3](#page-5-1) yields a T-junction T, odir(T) = j,  $Q = \text{ascell}(T)$ , with  $\text{pdir}(T) = k \neq j$ and

$$
(4.15) \qquad \overline{\mathbf{T}} \cap \operatorname{conv}\{x, y\} \neq \varnothing, \quad x_k \neq y_k, \quad \mathbf{Q}_k \cap [\min(x_k, y_k), \max(x_k, y_k)] \neq \varnothing.
$$

From  $k \neq j$  we get  $x_k, y_k \in \mathbf{E}_k$ , and from  $x_k \neq y_k$  we get that  $\mathbf{E}_k$  is not a singleton but an open interval, which yields  $E \cap Sk_k = \emptyset$ . Due to  $\overline{T} \cap conv\{x, y\} \neq \emptyset$ , there is  $z \in E$  such that

<span id="page-11-0"></span>(4.16) 
$$
P_{j,m}(z) = (z_1, \ldots, z_{j-1}, m, z_{j+1}, \ldots, z_d) \in \overline{\mathbf{T}} \cap \text{conv}\{x, y\}.
$$

From  $odir(T) = j$  and  $pdf(T) = k$  we get  $T \in H^{(1)}(\lbrace j, k \rbrace)$ . Further, T is in the boundary of some k-orthogonal mesh entity, which yields  $\overline{T} \subset Sk_k$ . Together with  $E \cap Sk_k = \emptyset$ , we get  $z \notin Sk_k \ni P_{j,m}(z)$ . [Lemma 2.3](#page-5-1) yields another T-junction  $\mathsf{T}'$ , odir $(\mathsf{T}) = k$ ,  $Q' = \text{ascell}(T'), \text{ with}$ 

(4.17) 
$$
\overline{T'} \cap \text{conv}\{z, P_{j,m}(z)\}\neq \emptyset, \quad z_{\text{pdir}(T')} \neq (P_{j,m}(z))_{\text{pdir}(T')},
$$

(4.18) 
$$
\mathbf{Q}'_{\text{pdir}(\mathbf{T}')} \cap \text{conv}[z_{\text{pdir}(\mathbf{T}')} , (P_{j,m}(z))_{\text{pdir}(\mathbf{T}')}] \neq \varnothing.
$$

Since z and  $P_{j,m}(z)$  differ only in direction  $j$ ,  $z_{\text{pdir}(T')} \neq (P_{j,m}(z))_{\text{pdir}(T')}$  yields that pdir(T') = j. Hence we have  $z_j \neq m$  and  $\mathbb{Q}'_j \cap \text{conv}\{z_j, m\} \neq \emptyset$ . From  $\overline{T'} \cap \text{conv}\{z, P_{j,m}(z)\}\neq \emptyset$  we get  $z_{\ell} = (P_{j,m}(z))_{\ell} \in \overline{T'_{\ell}} \subset \text{conv}\,\text{v}_{\ell}(\mathbf{T'})$  for all  $\ell \neq j$ . From [\(4.16\)](#page-11-0) above, we also have  $P_{j,m}(z) \in \overline{\mathbb{T}} \subset \text{GTJ}(\mathbb{T})$ .

This yields by construction of  $T, T'$  two cases listed below.

Case 1:  $v_j(T') \cap (\sup E_j, m) \subset v_j(E) \cap (\sup E_j, m)$ . This leads to  $m \in \text{conv } v_j(T')$ and consequently  $GTJ(T) \cap GTJ(T') \ni P_{j,m}(z)$  which means that  $\mathcal{T} \notin WGS$  in contradiction to the assumption.

Case 2: There is some  $\tilde{m} \in v_j(T') \cap (\sup E_j, m) \setminus v_j(E)$ . This yields  $P_{j, \tilde{m}}(E) \not\subset Sk_j$ , and  $P_{j,m}(z) \in P_{j,m}(T') \subset Sk_j$ , hence  $P_{j,m}(E) \cap Sk_j \neq \emptyset$  in contradiction to the minimality of  $m$ . Д

<span id="page-11-1"></span>LEMMA 4.7. Let  $\mathcal{T} \in \mathbb{W}$ GAS and  $E, F \in \mathcal{T}$  be anchors or T-junctions, and

(4.19) 
$$
m \in v_j(E) \cap \operatorname{conv} v_j(F) \setminus v_j(F).
$$

Then there is a T-junction  $T \in \mathbb{T}_j$  with  $T_j = \{m\}$ ,  $k = \text{pdir}(T)$ ,  $\mathbb{Q} = \text{ascell}(T)$  such that

(4.20) 
$$
\overline{T} \cap P_{j,m}(\text{MBox}(\mathbf{E}, \mathbf{F})) \neq \emptyset
$$
,  $\mathbf{Q}_k \cap \text{MBox}(\mathbf{E}, \mathbf{F})_k \neq \emptyset$ ,  $\mathbf{E}_k \cap \mathbf{F}_k = \emptyset$ ,  
12

with  $MBox(E, F) = \times_{\ell=1}^d MBox(E, F)_{\ell}$  and

(4.21) 
$$
\text{MBox}(E, F)_{\ell} = \begin{cases} E_{\ell} \cap F_{\ell} & E_{\ell} \cap F_{\ell} \neq \varnothing \\ [\sup E_{\ell}, \inf F_{\ell}] & \sup E_{\ell} \leq \inf F_{\ell} \\ [\sup F_{\ell}, \inf E_{\ell}] & \inf E_{\ell} \geq \sup F_{\ell}. \end{cases}
$$

*Proof.* By construction of local knot vectors, we have  $P_{j,m}(E) \subset Sk_j \not\supset P_{j,m}(F)$ . [Lemma 4.6](#page-10-1) yields  $\overline{P_{j,m}(E)} \subset S_k$  and  $P_{j,m}(F) \cap S_k$  ≠  $\varnothing$ . Using [Lemma 2.3,](#page-5-1) there exists for each  $x \in P_{j,m}(E), y \in P_{j,m}(F)$  a (possibly non-unique) j-orthogonal T-junction  $T^{(x,y)} \in \mathbb{T}_j$ , with  $\text{pdir}(T^{(x,y)}) = k^{(x,y)}$ ,  $\mathbb{Q}^{(x,y)} = \text{ascell}(T^{(x,y)})$ , such that

(4.22) 
$$
\overline{T}^{(x,y)} \cap \text{conv}\{x,y\} \neq \varnothing, \quad x_{k^{(x,y)}} \neq y_{k^{(x,y)}},
$$

$$
(4.23) \t\t and \t\mathbf{Q}_{k^{(x,y)}}^{(x,y)} \cap \text{conv}\{x_{k^{(x,y)}}, y_{k^{(x,y)}}\} \neq \varnothing.
$$

We have

(4.24) 
$$
\bigcup_{\substack{\tilde{x}\in \overline{P_{j,m}(\mathbf{E})}\\ \tilde{y}\in P_{j,m}(\mathbf{F})}}\overline{\mathsf{T}^{(\tilde{x},\tilde{y})}\cap \mathrm{conv}\{x,y\}}\neq \varnothing \quad \text{for any } x\in \overline{P_{j,m}(\mathbf{E})}, y\in P_{j,m}(\mathbf{F}),
$$

and hence also for any choice of  $x \in P_{j,m}(\mathbf{E}), y \in P_{j,m}(\mathbf{F})$ , since the union  $\bigcup_{(x,y)} \mathbf{T}^{(x,y)}$ is a closed set. Consider a pair  $(x, y) \in \overline{P_{j,m}(E)} \times \overline{P_{j,m}(F)}$  with

<span id="page-12-1"></span>(4.25) 
$$
\begin{cases} x_j = y_j = m & \ell = j \\ x_\ell = y_\ell \in \mathbb{E}_\ell \cap \mathbb{F}_\ell & \ell \neq j, \ \mathbb{E}_\ell \cap \mathbb{F}_\ell \neq \varnothing \\ x_\ell = \sup \mathbb{E}_\ell, \ y_\ell = \inf \mathbb{F}_\ell & \ell \neq j, \ \mathbb{E}_\ell \cap \mathbb{F}_\ell = \varnothing, \ \sup \mathbb{E}_\ell \leq \inf \mathbb{F}_\ell \\ x_\ell = \inf \mathbb{E}_\ell, \ y_\ell = \sup \mathbb{F}_\ell & \ell \neq j, \ \mathbb{E}_\ell \cap \mathbb{F}_\ell = \varnothing, \ \inf \mathbb{E}_\ell \geq \sup \mathbb{F}_\ell, \end{cases}
$$

which yields a T-junction  $\mathbf{T} \in \mathbb{T}_j$  from the union above, with pdir(T) = k,  $\mathbf{Q}$  = ascell(T), such that  $x_k \neq y_k$  and

(4.26) 
$$
\overline{T} \cap P_{j,m}(\text{MBox}(\mathbf{E}, \mathbf{F})) \supseteq \overline{T} \cap \text{conv}\{x, y\} \neq \varnothing,
$$

(4.27) Q<sup>k</sup> ∩ MBox(E, F)<sup>k</sup> ⊇ Q<sup>k</sup> ∩ conv{xk, yk} 6= ∅.

If  $y \in P_{j,m}(F)$ , this holds for  $T = T^{(x,y)}$  as above. If  $y \in \overline{P_{j,m}(F)} \setminus P_{j,m}(F)$ , then  $T = T^{(\tilde{x}, \tilde{y})}$  for some  $\tilde{x}, \tilde{y}$  close to  $x, y$ .

From  $j = \text{odir}(T) \neq \text{pdir}(T) = k$  we know that k does not match the first case in [\(4.25\).](#page-12-1) Since  $x_k \neq y_k$ , k also does not match the second case, and hence  $\mathbf{E}_k \cap \mathbf{F}_k = \emptyset$ . This concludes the proof.  $\Box$ 

<span id="page-12-0"></span>LEMMA 4.8. Given a WGAS box subdivision of a WGAS mesh using [Algorithm](#page-3-0) 2.1,  $\mathbf{Q} \in \mathcal{T}^{(n)} \in \text{WGAS}, \ \mathcal{T}^{(n+1)} = \text{SUBDIV}(\mathcal{T}^{(n)}, \mathbf{Q}, j) \in \text{WGAS}, \ \text{there is for each new anchor}$  $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$  an old anchor  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n)}$  with  $\text{supp}_{\Omega} B_{\hat{\mathbf{A}}} \subset \text{supp}_{\Omega,\mathcal{J}^{(n)}} B_{\mathbf{A}}$  and  $v_{\ell}(\hat{\mathbf{A}}) = v_{\ell}(\mathbf{A})$  for all  $\ell \neq j$ .

The proof is given in [Appendix A.1.](#page-24-9)

Note that [Lemma 4.8](#page-12-0) does not hold without [Assumption 4.5.](#page-10-2) Consider the 3D mesh in [Figure 7.](#page-13-0) In this example, [Assumption 4.5](#page-10-2) is not fulfilled, as the center cell Q has active neighbor cells in only two directions (the figure shows the active region). For the 2-orthogonal bisection of Q (highlighted in red), the old and new mesh are WGAS as with  $p_1 = 1$  the new T-junctions only intersect with the neighbor cells, but not with Q or with the old T-junctions. Since  $p_2$  is odd, any new anchor  $\hat{A}$  is contained in the closure of the new interface, and  $v_1(A)$  does not coincide with the local knot vector  $v_1(A)$  of any old anchor A, i.e. [Lemma 4.8](#page-12-0) does not hold in this case.

We close this section with two examples illustrated in [Figures 8](#page-14-0) to [10.](#page-16-1) We consider the 3D mesh visualized in [Figure 8,](#page-14-0) with polynomial degrees  $p = (3, 2, 1)$ , and we construct the T-junction extensions of the hanging interfaces via both approaches, the abstract and the geometric one.

The sketches in [Figure 9](#page-15-0) show the slice  $S_3(2)$ , where the thick red line marks 3-orthogonal T-junctions contained in the slice. The faces inside the red line are part of the skeleton, the faces outside the red line are not. In other words, the faces surrounded by the red line were generated by a bisection orthogonal to the third direction. In [Figures 9a,](#page-15-1) [9c](#page-15-2) and [9e](#page-15-3) the scheme of constructing the abstract T-junction extension is displayed, while [Figures 9b,](#page-15-4) [9d](#page-15-5) and [9f](#page-15-6) shows the procedure for geometric T-junction extensions.

For the abstract T-junction extensions, we consider the two sets

(4.28) 
$$
\mathcal{A}_{\mathbf{p}}^{(1)} = \{ \mathbf{A} \in \mathcal{A}_{\mathbf{p}} \colon n \in \mathcal{I}_{3}(\mathbf{A}) \}, \quad \mathcal{A}_{\mathbf{p}}^{(2)} = \{ \mathbf{A} \in \mathcal{A}_{\mathbf{p}} \colon n \notin \mathcal{I}_{3}(\mathbf{A}) \}.
$$

From the polynomial degree  $p = (3, 2, 1)$ , we get  $\kappa = \{1, 3\}$  in  $(3.4)$  and hence the anchors are the edges in the second direction. The projection of  $\mathcal{A}_{\mathbf{p}}^{(1)}$  (resp.  $\mathcal{A}_{\mathbf{p}}^{(2)}$ ) on the slice  $S_3(2)$  is indicated in [Figure 9a](#page-15-1) (resp. [Figure 9c\)](#page-15-2) by solid dots on the lines, meaning that each marked line corresponds to three (resp. two) anchors with identical first and second components. Following [Definition 4.1,](#page-8-5) we indicate  $\bigcup_{\mathbf{A}\in\mathcal{A}_{\mathbf{P}}^{(1)}}$  supp $_{\Omega}B_{\mathbf{A}}$ by dashed lines, and  $\bigcup_{\mathbf{A}\in\mathcal{A}_{\mathbf{p}}^{(2)}}\text{supp}_{\Omega} B_{\mathbf{A}}$  by dashed lines. The intersection of these sets yields the T-junction extension highlighted in [Figure 9e,](#page-15-3) which contains faces in the center region and intersects with cells in the outer region. Note that the spline supports far away from the T-junctions contribute no information in this construction. It is hence sufficient to consider only anchors near T-junctions when checking for AAS in practice.

For the geometric T-junction extensions, we consider the two types

(4.29) 
$$
\mathbb{T}^{(1)} = \{ \mathbf{T} \in \mathcal{H}^{(1)} \mid \text{valence}(\mathbf{T}) < 4, \mathbf{T} \not\subset \partial \Omega, \text{pdir}(\mathbf{T}) = 1, \text{odir}(\mathbf{T}) = 3 \},
$$

(4.30) 
$$
\mathbb{T}^{(2)} = \{ \mathbf{T} \in \mathcal{H}^{(1)} \mid \text{valence}(\mathbf{T}) < 4, \mathbf{T} \not\subset \partial \Omega, \text{pdir}(\mathbf{T}) = 2, \text{odir}(\mathbf{T}) = 3 \}.
$$

<span id="page-13-0"></span>In [Figure 9,](#page-15-0) the set  $\mathbb{T}^{(1)}$  represents the vertical red edges, and  $\mathbb{T}^{(2)}$  represents the horizontal red edges. We build the geometric T-junction extensions separately for the



FIG. 7. In this example, [Assumption](#page-10-2) 4.5 is not fulfilled. For the 2-orthogonal bisection of Q (highlighted in red), the old and new mesh are WGAS, but for any new anchor  $\hat{A}$ ,  $v_1(\hat{A})$  does not coincide with the local knot vector  $v_1(A)$  of any old anchor A, i.e. [Lemma](#page-12-0) 4.8 does not hold in this case.

<span id="page-14-0"></span>

FIG. 8. Example mesh with  $\mathbf{p} = (3, 2, 1)$  and  $(N_1, N_2, N_3) = (17, 13, 4)$  for which the T-junction extensions are investigated in [Figure](#page-15-0) 9. The figure shows only the active region  $AR_p = [2, 15] \times$  $[1, 12] \times [1, 3].$ 

interfaces in  $\mathbb{T}^{(1)}$  and for the interfaces from  $\mathbb{T}^{(2)}$ . The intersection of the associated cells with  $S_3(2)$  are highlighted in lime in [Figures 9b](#page-15-4) and [9d.](#page-15-5)

For any interface  $T = T_1 \times T_2 \times T_3 \in \mathbb{T}^{(1)}$ , we have pdir(T) = 1, hence, the knot vectors are constructed as follows, recall also [Definition 4.3.](#page-9-0) From  $\mathcal{I}_1(T)$  we select the  $p_1 + 1 = 4$  indices, such that  $T_1$  is either the third or the second entry, i.e. on the left side in [Figure 9b](#page-15-4) the index of  $T_1$  is the third entry in  $v_1(T)$  and on the right side, the index of  $T_2$  is the second entry in  $v_1(T)$ . Since odir(T) = 3, we have  $v_3(T) = \{2\}$  for all T-junctions in this example. We construct the knot vector  $v_2(T)$  to be symmetric around T, i.e. it has  $p_2 + 2 = 4$  consecutive entries from  $\mathcal{I}_2(T)$ , where the indices of T<sup>2</sup> are in the middle.

For any interface  $T = T_1 \times T_2 \times T_3 \in \mathbb{T}^{(2)}$ , we have pdir(T) = 2. Since  $p_2 = 2$  is even,  $v_2(T)$  is composed of  $p_2 + 1 = 3$  indices from  $\mathcal{I}_2(T)$  such that  $T_2$  is the second entry. Thus, the local knot vector  $v_2(T)$  is symmetric around  $T_2$ . Further, we build  $v_1(T)$  from  $\mathfrak{I}_1(T)$  by choosing the  $p_1 + 3 = 6$  ( $p_1$  is odd) consecutive indices, such that the bounds of  $T_1$  are the middle entries.

The unions of these T-junction extensions are shown in [Figures 9b](#page-15-4) and [9d](#page-15-5) by dashed lines, and the union of both sets gives the T-junction extension GTJ highlighted in [Figure 9f.](#page-15-6) Note that the geometric T-junction is slightly larger than the abstract T-junction extension.

The second example is the 2D mesh shown in [Figures 10a](#page-16-2) to [10d.](#page-16-3) The hanging interfaces are the two opposing hanging vertices  $\mathbf{T}^{(1)} = \{m\} \times \{n\}$  and  $\mathbf{T}^{(2)} = \{m+1\} \times \{n\}$ {n}. We will demonstrate the behavior of the two introduced T-junction extensions for different degrees. Let  $p_2$  be odd in any case.

In [Figure 10a,](#page-16-2)  $p_1$  and  $p_2$  are odd. The anchors are marked by red bullets. In this setting, all anchors  $\mathbf{A} \in \mathcal{A}_{p}$  have the index n in their knot vector, i.e.  $n \in \mathcal{I}_{2}(\mathbf{A})$ for all  $A \in \mathcal{A}_p$ . Thus, the abstract T-junction extension is empty here. Note that  $ATJ_2 = \emptyset$  if  $p_1$  is odd.

In [Figure 10b,](#page-16-4)  $p_1$  is even and  $p_2$  is odd, and hence the anchors are the horizontal

lines. In this setting, we have two anchors  $\mathbf{A}^{(1)} = (m, m + 1) \times \{n - 1\}$  and  $\mathbf{A}^{(2)} =$  $(m, m + 1) \times \{n + 1\}$ , which are the bottom center and the top center anchor in

<span id="page-15-1"></span><span id="page-15-0"></span>

(a) The set  $\mathcal{A}_{\mathbf{p}}^{(1)}$  and the corresponding support of T-Splines

<span id="page-15-2"></span>

(c) The set  $\mathcal{A}_{\mathbf{p}}^{(2)}$  and the corresponding support of T-Splines

<span id="page-15-3"></span>

(e) The resulting abstract TJunction extension by intersecting the supports



<span id="page-15-4"></span>(b) Construct the extension of lines in  $\mathbb{T}^{(1)}$ 



<span id="page-15-5"></span>(d) Construct the extensions of the lines in  $\mathbb{T}^{(2)}$ 



<span id="page-15-6"></span>(f) The resulting geometric TJunction extension as union of the two extension sets

Fig. 9. Step-by-step construction of abstract and geometric T-junction extensions.

[Figure 10b,](#page-16-4) with  $n \notin \mathcal{I}_2(\mathbf{A}^{(1)}) = \mathcal{I}_2(\mathbf{A}^{(2)})$ , while  $n \in \mathcal{I}_2(\mathbf{A})$  for the remaining six anchors. Thus the abstract T-junction extension will not be empty. The extension is drawn as a dashed line.

In contrast to the case from [Figure 10a,](#page-16-2) we see in [Figure 10c](#page-16-5) that the geometric T-junction extension  $GTJ(T^{(1)})$  is not empty for any  $p_1$  as it is constructed with  $p_1 + 1$ consecutive indices from the global knot vector  $\mathfrak{I}_1(\mathsf{T}^{(1)})$ . The extension for  $p_1 = 1$  is given by  $GTJ(T^{(1)}) = GTJ(T^{(2)}) = [m, m+1] \times \{n\}.$ 

Furthermore, we get for the case  $p_1 = 2$  the extension shown in [Figure 10d,](#page-16-3) which coincides with the abstract T-junction extension shown in [Figure 10b.](#page-16-4)

Both examples indicate that AAS does not imply SGAS in general, since there may be an intersection of T-junction extensions in points that are contained in a geometric, but not in an abstract T-junction extension. Consider for example [Figure 10e,](#page-16-6) where  $p = (3, 3)$  and the anchors are again the vertices of the mesh. As before, there is  $ATJ_2 = \emptyset$ , whereas the T-junction  $T^{(3)} = \{m-1\} \times \{n+1\}$  yields  $ATJ_1(m-1) =$  ${m-1} \times [n-1, n+2] = GTJ(T^{(3)})$ . We again have  $[m-2, m+2] \times \{n\} \subset GTJ$ , and we see  $GTJ_1 \cap GTJ_2 = \{m-1\} \times \{n\} \neq \emptyset$ , as well as  $ATJ_1 \cap ATJ_2 = \emptyset$ . The extensions  $GTJ<sub>1</sub>$  and  $ATJ<sub>1</sub>$  coincide and are drawn with dashed lines, and the extension  $GTJ<sub>2</sub>$  is drawn with dotted lines in [Figure 10e.](#page-16-6)

Lastly, we give an example to point out the differences between WGAS and SGAS meshes. For WGAS meshes, we consider only intersecting T-junction extensions of T-junctions with different pointing and orthogonal direction, whereas SGAS only consider extensions of T-junctions with different orthogonal directions. For the two hanging interfaces  $\mathbf{T}^{(1)} = \{m+1\} \times (n, n+2) \times \{r+1\}$  and  $\mathbf{T}^{(2)} = \{m+2\} \times \{n+1\} \times (r, r+2)$ from [Figure 11](#page-17-0) we have  $\text{pdir}(\mathbf{T}^{(1)}) = \text{pdir}(\mathbf{T}^{(2)}) = 1$  and  $\text{odir}(\mathbf{T}^{(1)}) = 3 \neq 2 = 1$ odir( $T^{(2)}$ ). For any degrees  $p_1, p_2, p_3 \geq 0$ , the intersection of the two geometric extensions will not be empty, i.e.  $GTJ(T^{(1)}) \cap GTJ(T^{(2)}) \neq \emptyset$ , hence the mesh will not be SGAS. But since  $\text{pdir}(\mathbf{T}^{(1)}) = \text{pdir}(\mathbf{T}^{(1)})$ , the intersection is not considered for the weak criterion of geometric analysis-suitability. Thus, the mesh is WGAS but not SGAS. However, we conjecture that the generated splines are linearly independent, see [Conjecture 6.3](#page-20-0) and [Proposition 5.4.](#page-18-0)

<span id="page-16-0"></span>5. Dual-Compatibility. We recall two versions of dual-compatibility, a strong [\[13,](#page-24-7) Definition 5.3.12] and a weak one [\[9,](#page-24-3) Definition 7.2]. Throughout this

<span id="page-16-5"></span><span id="page-16-2"></span><span id="page-16-1"></span>

<span id="page-16-6"></span><span id="page-16-4"></span><span id="page-16-3"></span>Fig. 10. Opposing hanging interfaces

paper, we suppose that knot vectors are non-decreasing.

DEFINITION 5.1 (Overlapping knot vectors and splines). We say that two knot vectors  $\Xi^{(1)} = (\xi_1^{(1)}, \ldots, \xi_{n_1}^{(1)})$  and  $\Xi^{(2)} = (\xi_1^{(2)}, \ldots, \xi_{n_2}^{(2)})$  overlap, if there is a knot vector  $\Xi = (\xi_1, \ldots, \xi_n)$ ,  $n \geq \max\{n_1, n_2\}$ , and numbers  $k^{(1)}, k^{(2)} \in \mathbb{N}_0$  such that

<span id="page-17-1"></span>(5.1) 
$$
\forall i = 1, ..., n_1: \quad \xi_i^{(1)} = \xi_{i+k^{(1)}},
$$

$$
\forall i = 1, ..., n_2: \quad \xi_i^{(2)} = \xi_{i+k^{(2)}}.
$$

We write  $\Xi^{(1)} \bowtie \Xi^{(2)}$ .

Further, for two anchors  $\mathbf{A}^{(1)}$ ,  $\mathbf{A}^{(2)} \in \mathcal{A}_{\mathbf{p}}$  we say that the splines  $B_{\mathbf{A}^{(1)}}$  and  $B_{\mathbf{A}^{(2)}}$ overlap if the local knot vectors  $v_k(A^{(1)})$  and  $v_k(A^{(2)})$  overlap for each k, and we write  $B_{\mathbf{A}^{(1)}} \bowtie B_{\mathbf{A}^{(2)}}$ .

We say that they weakly partially overlap if there is an  $\ell \in \{1, \ldots, d\}$  such that the knot vectors  $v_{\ell}(A^{(1)})$  and  $v_{\ell}(A^{(2)})$  differ and overlap, and we write  $B_{A^{(1)}} \ltimes^w B_{A^{(2)}}$ . We say they strongly partially overlap, if  $\text{supp}(B_{\mathbf{A}^{(1)}}) \cap \text{supp}(B_{\mathbf{A}^{(2)}}) = \varnothing$  or if  $v_k(\mathbf{A}^{(1)})$ and  $v_k(A^{(2)})$  overlap for at least d–1 directions k. We write  $B_{A^{(1)}} \ltimes B_{A^{(2)}}$ .

DEFINITION 5.2 (Dual-Compatibility). Let  $S = \{B_i\}$  be a set of splines. We say that S is weakly (resp. strongly) Dual-Compatible (WDC resp. SDC), if  $B_i \ltimes^w B_j$  (resp.  $B_i \ltimes B_j$ , for  $i \neq j$ . Further, we say that T is WDC (resp. SDC), if the generated spline space is WDC (resp. SDC), and we write  $\mathfrak{T} \in \text{WDC}$  (resp.  $\mathfrak{T} \in \text{SDC}$ ).

*Remark* 5.3. SDC is sufficient for WDC. This is shown as follows. Let  $A^{(1)}$ ,  $A^{(2)} \in$  $\mathcal{A}_{\mathbf{n}}$  be two anchors with  $\mathbf{A}^{(1)} \neq \mathbf{A}^{(2)}$  and  $B_{\mathbf{A}^{(1)}} \ltimes B_{\mathbf{A}^{(2)}}$ .

Case 1: supp $(B_{\mathbf{A}^{(1)}}) \cap \text{supp}(B_{\mathbf{A}^{(2)}}) = \emptyset$ , then there is k with conv v<sub>k</sub>( $\mathbf{A}^{(1)}$ ) ∩ conv  $v_k(A^{(2)}) = \emptyset$ . We choose  $\Xi_k = v_k(A^{(1)}) \cup v_k(A^{(2)})$  as the global knot vector and  $k^{(1)} = 1, k^{(2)} = p_k + 2$  such that  $\Xi_k$ ,  $v_k(\mathbf{A}^{(1)}), v_k(\mathbf{A}^{(2)})$  fulfill condition [\(5.1\)](#page-17-1).

Case 2:  $\text{supp}(B_{\mathbf{A}^{(1)}}) \cap \text{supp}(B_{\mathbf{A}^{(2)}}) \neq \emptyset$  and  $v_k(\mathbf{A}^{(1)}) \bowtie v_k(\mathbf{A}^{(2)})$  for at least  $d-1$ directions k. If  $v_k(A^{(1)}) = v_k(A^{(2)})$  for all these directions, then  $A^{(1)}$  and  $A^{(2)}$  are equal or aligned in the remaining direction  $j$  and hence share the same global knot vector  $\mathcal{I}_j(\mathbf{A}^{(1)}) = \mathcal{I}_j(\mathbf{A}^{(2)})$ . Hence  $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$  or  $v_j(\mathbf{A}^{(1)}) \boxtimes v_j(\mathbf{A}^{(2)})$ .

In both cases, there exists an  $\ell$  such that  $v_{\ell}(A^{(1)})$  and  $v_{\ell}(A^{(2)})$  differ and overlap. This is, from  $B_{\mathbf{A}^{(1)}} \ltimes B_{\mathbf{A}^{(2)}}$  follows  $B_{\mathbf{A}^{(1)}} \ltimes^w B_{\mathbf{A}^{(2)}}$ , and hence SDC implies WDC.

An example for a mesh that is WDC but not SDC is again the mesh from [Fig](#page-17-0)[ure 11.](#page-17-0) The 1-orthogonal Skeleton  $\mathrm{Sk}_1 = \mathrm{S}_1(m) \cup \mathrm{S}_1(m+1) \cup \mathrm{S}_1(m+2) \cup \mathrm{S}_1(m+3)$ consists of slices of the whole domain. Hence all anchors have the same global knot vector  $\mathcal{I}_1(\mathbf{A}) = (m, m + 1, m + 2, m + 3)$ , regardless of the polynomial degrees and corresponding anchor type. Consequently, any two anchors have overlapping local knot vectors in the first direction, i.e.  $v_1(A^{(1)}) \bowtie v_1(A^{(2)})$ . Further, any two anchors that coincide in their first component also coincide in their global knot vectors in the

<span id="page-17-0"></span>

Fig. 11. A mesh that is WGAS and WDC, but neither SGAS nor SDC, for any polynomial degree.

Table 1

<span id="page-18-1"></span>Global knot vectors for all possible configurations of an anchor's first component, for the mesh from [Figure](#page-17-0) 11. The always contained values  $0, \ldots, \lfloor \frac{p_j+1}{2} \rfloor$  and  $N_j - \lfloor \frac{p_j+1}{2} \rfloor, \ldots, N_j$  are hidden by dots.

$A_1$	$\mathfrak{I}_{2}({\bf A})$	$\mathfrak{I}_3({\bf A})$
$\{m\}$	$(\ldots, n, n+2, \ldots)$	$(\ldots, r, r+1, r+2, \ldots)$
$\{m+1\}$	$(\ldots, n, n+2, \ldots)$	$(,r,r+1,r+2,)$
$\{m+2\}$	$(, n, n+1, n+2,)$	$(\ldots,r,r+2,\ldots)$
$\{m+3\}$	$(, n, n+1, n+2,)$	$(\ldots,r,r+2,\ldots)$
$(m, m+1)$	$(\ldots,n,n+2,\ldots)$	$(,r,r+1,r+2,)$
$(m+1, m+2)$	$(\ldots, n, n+2, \ldots)$	$(\ldots,r,r+2,\ldots)$
$(m+2, m+3)$	$(\ldots, n, n+1, n+2, \ldots)$	$(\ldots,r,r+2,\ldots)$

second and third direction, because these global knot vectors depend only on the first component  $A_1$ , see [Table 1.](#page-18-1) Together, any two anchors either overlap and are different in the first direction, or, if their first components coincide, they strongly overlap. This satisfies the WDC criterion.

However, the mesh is not SDC for any degree  $\mathbf{p} = (p_1, p_2, p_3)$  with  $p_1 > 0$ . Consider two anchors  $\mathbf{A}^{(1)} = \mathbf{A}^{(1)}_1 \times \mathbf{A}^{(1)}_2 \times \mathbf{A}^{(1)}_3$  and  $\mathbf{A}^{(2)} = \mathbf{A}^{(2)}_1 \times \mathbf{A}^{(2)}_2 \times \mathbf{A}^{(2)}_3$ , with

(5.2) 
$$
\begin{cases} \mathbf{A}_1^{(1)} = \{m+1\}, \ \mathbf{A}_1^{(2)} = \{m+2\} & \text{if } p_1 \text{ is odd,} \\ \mathbf{A}_1^{(1)} = (m, m+1), \ \mathbf{A}_1^{(2)} = (m+2, m+3) & \text{if } p_1 \text{ is even and } > 0. \end{cases}
$$

For any  $p_1 > 0$ , the supports supp<sub>Ω</sub>  $B_{\mathbf{A}^{(1)}}$  and supp<sub>Ω</sub>  $B_{\mathbf{A}^{(2)}}$  have nonempty intersection, and from [Table 1,](#page-18-1) we get for any  $p_1 > 0$  that

(5.3) 
$$
\mathcal{I}_2(\mathbf{A}^{(1)}) = (\ldots, n, n+2, \ldots), \qquad \mathcal{I}_3(\mathbf{A}^{(1)}) = (\ldots, r, r+1, r+2, \ldots),
$$

(5.4) 
$$
\mathcal{I}_2(\mathbf{A}^{(2)}) = (\ldots, n, n+1, n+2, \ldots), \ \mathcal{I}_3(\mathbf{A}^{(2)}) = (\ldots, r, r+2, \ldots),
$$

i.e. the knot vectors  $v_2(A^{(1)})$  and  $v_2(A^{(2)})$  do not overlap for any  $p_2 \geq 0$ , and neither do  $v_3(A^{(1)})$  and  $v_3(A^{(2)})$  for any  $p_3 \geq 0$ . Thus,  $B_{A^{(1)}}$  and  $B_{A^{(2)}}$  do not strongly partially overlap, and the mesh is not SDC.

Extensive studies on dual-compatible splines are already existent, see e.g. [\[9\]](#page-24-3). Some important properties are stated in the following proposition.

<span id="page-18-0"></span>PROPOSITION 5.4. Let  $\mathcal{S}_{\mathbf{p}} = \{B_{\mathbf{A},\mathbf{p}}\}$  be a set of weakly dual-compatible splines over the set of anchors  $A_{\bf p}$  with multi-degree  ${\bf p}$ . Then, the following holds

- 1. There exists a set of dual-functions  $\lambda_{\mathbf{A},\mathbf{p}}$ , s.t.  $\lambda_{\mathbf{A}^{(1)},\mathbf{p}}(B_{\mathbf{A}^{(2)},\mathbf{p}})=\delta_{\mathbf{A}^{(1)},\mathbf{A}^{(2)}}$ .
- 2. The splines  $B_{\mathbf{A},\mathbf{p}}$  are linearly independent. If the constant function is in the spline space S, then  $\sum_{\mathbf{A}\in\mathcal{A}_{\mathbf{p}}} B_{\mathbf{A}} = 1$ .
- 3. There exists a constant  $C_{\mathbf{p}}$ , s.t. the projection  $\Pi_{\mathbf{p}}: L^2(\widehat{\Omega}) \to \mathcal{S}_{\mathbf{p}}$  given by

(5.5) 
$$
\Pi_{\mathbf{p}}(f)(\zeta) = \sum_{\mathbf{A}\in\mathcal{A}_{\mathbf{p}}}\lambda_{\mathbf{A},\mathbf{p}}(f)B_{\mathbf{A},\mathbf{p}}(\zeta), \text{ for all } f \in L^{2}(\widehat{\Omega}), \zeta \in \widehat{\Omega}
$$

fulfills

(5.6) 
$$
\|\Pi_{\mathbf{p}}(f)\|_{L^2(\mathbf{Q})} \leq C_{\mathbf{p}} \|f\|_{L^2(\mathbf{Q})}, \quad \text{for all } \mathbf{Q} \subset \widehat{\Omega}, \text{ and } f \in L^2(\widehat{\Omega}).
$$

Proof. See [\[9,](#page-24-3) Proposition 7.4, 7.6, and 7.7].

The following lemma and proposition are used in [section 6](#page-20-1) for connections between dual-compatibility and geometric analysis-suitability.

LEMMA 5.5. Let  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$ ,  $\mathbf{T} \in \mathbb{T}_i$  and  $x \in \text{supp}_{\Omega} B_{\mathbf{A}} \cap S_i(n)$ , such that 1. T touches the segment between  $P_{i,n}(\mathbf{A})$  and  $\{x\},$ 

<span id="page-19-3"></span><span id="page-19-2"></span>(5.7) 
$$
\overline{T} \cap \text{conv}(P_{i,n}(\mathbf{A}) \cup \{x\}) \neq \varnothing,
$$

2. in pointing direction of T, the associated cell  $Q = \text{ascell}(T)$  touches the convex hull of  $A$  and  $\{x\}$ ,

<span id="page-19-6"></span>(5.8) 
$$
\mathbf{Q}_{\text{pdir}(T)} \cap \text{conv}(\mathbf{A}_{\text{pdir}(T)} \cup \{x_{\text{pdir}(T)}\}) \neq \varnothing,
$$

3. there exists a number  $y \in \mathbf{A}_{\text{pdir}(T)}$  s.t.  $y \neq x_{\text{pdir}(T)}$ ,

<span id="page-19-4"></span>(5.9) 
$$
\exists y \in \mathbf{A}_{\text{pdir}(T)}: y \neq x_{\text{pdir}(T)}
$$

4. the anchor **A** and *T*-junction overlap for some arbitrary direction  $\ell \neq i$ ,

<span id="page-19-5"></span>(5.10) 
$$
v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathbf{T}), \text{ for some } \ell \neq i
$$

Then  $x_{\ell}$  is contained in the convex hull of the  $\ell$ -th local index vector of T, i.e.  $x_{\ell} \in$ conv  $v_{\ell}(T)$ .

The proof is given in [Appendix A.2.](#page-27-0)

<span id="page-19-0"></span>PROPOSITION 5.6. Let  $\mathcal{T}$  be an SGAS mesh,  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$  and  $\mathcal{T}$  a T-junction with  $\overline{\mathbf{T}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}} \neq \emptyset$ . Then  $v_k(\mathbf{A}) \bowtie v_k(\mathbf{T})$  for all  $k \neq \text{odir}(\mathbf{T})$ .

The proof is given in [Appendix A.3.](#page-29-0)

[Proposition 5.6](#page-19-0) does not hold for WGAS meshes, an example is depicted in [Fig](#page-19-1)[ure 12.](#page-19-1) Since each T-junction has pointing direction 1, the mesh is WGAS. Let  $\mathbf{p} =$  $(3, 3, 3)$ , and choose  $\mathbf{A} = \{m+1\} \times \{n\} \times \{r\}$ , as well as  $\mathbf{T} = \{m+2\} \times (r-2, r) \times \{n+1\}$ . We then get

(5.11) 
$$
v_3(T) = (r-3, r-2, r, r+1)
$$
, and  $v_3(A) = (r-2, r-1, r, r+1, r+2)$ ,

<span id="page-19-1"></span>and we see that  $r - 1 \in v_3(A)$  but  $r - 1 \notin v_3(T)$ , hence  $v_3(T) \not\approx v_3(A)$ .



FIG. 12. A WGAS mesh, where  $v_3(T) \not\bowtie v_3(A)$  in general.

<span id="page-20-1"></span>6. Main Results. In this Section, we focus on the indicated relations from [Figure 1.](#page-1-0) Note that the relations  $SGAS \subset WGAS$  and  $SDC \subset WDC$  are already evident from the previous sections by construction. In [Theorem 6.1,](#page-20-2) we extend the result from [\[13\]](#page-24-7) to arbitrary degrees, i.e. the initial restriction to only odd polynomial degrees can be dropped.

<span id="page-20-2"></span>THEOREM 6.1. All AAS meshes are SDC and vice versa.

<span id="page-20-4"></span>Theorem 6.2. All SGAS T-meshes are AAS.

<span id="page-20-0"></span>CONJECTURE 6.3. All WGAS meshes are WDC.

*Proof of [Theorem](#page-20-2) 6.1.* This is a generalization of  $[13,$  Theorem 5.3.14, we hence follow the original proof and extend necessary steps to the case of arbitrary polynomial degrees.

AAS  $\subseteq$  SDC. We assume for contradiction a mesh  $\mathcal{T} \in \text{AAS} \setminus \text{SDC}$  and let  $\mathcal{A}_{\mathbf{p}}$  be the set of anchors over  $\mathcal T$  with the corresponding set of T-splines  $\{B_{\mathbf A}: {\mathbf A}\in {\mathcal A}_{\mathbf p}\}$ . Since  $\mathcal T\not\in$ SDC there exist two anchors  $\mathbf{A}^{(1)}$ ,  $\mathbf{A}^{(2)} \in A_{\mathbf{p}}$ ,  $\mathbf{A}^{(1)} \neq \mathbf{A}^{(2)}$ , such that  $B_{\mathbf{A}^{(1)}} \times B_{\mathbf{A}^{(2)}}$ . This implies that the corresponding knot vectors do not overlap in at least two directions and that

<span id="page-20-3"></span>(6.1) 
$$
\operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(2)}} \neq \varnothing.
$$

Denote

(6.2) 
$$
m_k = \max\{\min v_k(\mathbf{A}^{(1)}), \min v_k(\mathbf{A}^{(2)})\},\
$$

$$
M_k = \min\{\max v_k(\mathbf{A}^{(1)}), \max v_k(\mathbf{A}^{(2)})\},\
$$
 $k = 1,...,d$ 

then [\(6.1\)](#page-20-3) yields that  $m_k \leq M_k$  for all  $k = 1, \ldots, d$ . Assume without loss of generality that the directions in which the knot vectors of  $A^{(1)}$  and  $A^{(2)}$  do not overlap are the first and second dimension, i.e.,  $v_1(\mathbf{A}^{(1)}) \not\approx v_1(\mathbf{A}^{(2)})$  and  $v_2(\mathbf{A}^{(1)}) \not\approx v_2(\mathbf{A}^{(2)})$ .

Thus, there is an index  $n_1 \in [m_1, M_1]$ , with either  $n_1 \in v_1(\mathbf{A}^{(1)})$  and  $n_1 \notin v_1(\mathbf{A}^{(2)})$ or  $n_1 \notin v_1(\mathbf{A}^{(1)})$  and  $n_1 \in v_1(\mathbf{A}^{(2)})$ .

Case 1:  $n_1 \in v_1(A^{(1)})$  and  $n_1 \notin v_1(A^{(2)})$ . Then we have  ${n_1} \cap \mathcal{I}_1(A^{(2)}) \subset$  $[m_1, M_1] \cap \mathcal{I}_1(\mathbf{A}^{(2)}) \subset \mathbf{v}_1(\mathbf{A}^{(2)})$ , and it follows  $n_1 \notin \mathcal{I}_1(\mathbf{A}^{(2)})$ , while  $n_1 \in \mathbf{v}_1(\mathbf{A}^{(1)}) \subseteq$  $\mathfrak{I}_1(\mathbf{A}^{(1)})$  yields  $n_1 \in \mathfrak{I}_1(\mathbf{A}^{(1)})$ .

*Case 2:*  $n_1 \notin v_1(A^{(1)})$  and  $n_1 \in v_1(A^{(2)})$ . Then we have  ${n_1} \cap T_1(A^{(1)}) \subset$  $[m_1, M_1] \cap \mathcal{I}_1(\mathbf{A}^{(1)}) \subset \mathcal{V}_1(\mathbf{A}^{(1)})$ , and it follows  $n_1 \notin \mathcal{I}_1(\mathbf{A}^{(1)})$ , while  $n_1 \in \mathcal{V}_1(\mathbf{A}^{(2)}) \subseteq$  $\mathfrak{I}_1(\mathbf{A}^{(2)})$  yields  $n_1 \in \mathfrak{I}_1(\mathbf{A}^{(2)})$ .

In both cases, [Definition 4.1](#page-8-5) yields

(6.3) ATJ<sub>1</sub>(n<sub>1</sub>) 
$$
\supset S_1(n_1) \cap \text{supp}_{\Omega} B_{A^{(1)}} \cap \text{supp}_{\Omega} B_{A^{(2)}} = \{n_1\} \times \bigtimes_{k=2}^d [m_k, M_k].
$$

Analogously, there exists  $n_2$ , such that

(6.4) 
$$
\text{ATJ}_2(n_2) \supset [m_1, M_1] \times \{n_2\} \times \bigtimes_{k=3}^d [m_k, M_k].
$$

Together, there is

(6.5) 
$$
\text{ATJ}_1(n_1) \cap \text{ATJ}_2(n_2) \supset \{n_1\} \times \{n_2\} \times \bigtimes_{k=3}^d [m_k, M_k] \neq \emptyset,
$$

which contradicts the assumption that  $\mathcal{T} \in \text{AAS}.$ 

SDC  $\subseteq$  AAS. Assume that  $\mathcal{T} \in$  SDC \ AAS. Then there exist  $i \neq j$  with ATJ<sub>i</sub> ∩  $\text{ATJ}_j \neq \emptyset$ , and there is a point  $e \in \mathbb{N}^d$ , with  $e = (e_1, \ldots, e_d) \in \text{ATJ}_i \cap \text{ATJ}_j$ . Assume without loss of generality that  $i = 1, j = 2$ , Then there exist by definition anchors  ${\bf A}^{(1)}, {\bf A}^{(2)}, {\bf A}^{(3)}, {\bf A}^{(4)} \in {\bf A}_{\bf p}$  with

 $(6.6)$   $e \in S_1(e_1) \cap S_2(e_2) \cap \text{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \text{supp}_{\Omega} B_{\mathbf{A}^{(2)}} \cap \text{supp}_{\Omega} B_{\mathbf{A}^{(3)}} \cap \text{supp}_{\Omega} B_{\mathbf{A}^{(4)}},$ 

(6.7) with  $e_1 \in \mathcal{I}_1(\mathbf{A}^{(1)}) \setminus \mathcal{I}_1(\mathbf{A}^{(2)})$  and  $e_2 \in \mathcal{I}_2(\mathbf{A}^{(3)}) \setminus \mathcal{I}_2(\mathbf{A}^{(4)})$ .

From  $e \in \text{supp}_{\Omega} B_{\mathbf{A}^{(1)}}$  and  $e_1 \in \mathcal{I}_1(\mathbf{A}^{(1)})$  we deduce  $e_1 \in \text{conv}\, \text{v}_1(\mathbf{A}^{(1)}) \cap \mathcal{I}_1(\mathbf{A}^{(1)}) =$  $\text{v}_1(\mathbf{A}^{(1)})$ , and from  $e \in \text{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \text{supp}_{\Omega} B_{\mathbf{A}^{(2)}}$  and  $e_1 \in \mathcal{I}_1(\mathbf{A}^{(1)}) \setminus \mathcal{I}_1(\mathbf{A}^{(2)})$  we deduce  $e_1 \in \text{conv } \text{v}_1(\mathbf{A}^{(2)}) \setminus \mathcal{I}_1(\mathbf{A}^{(2)}) = \text{conv } \text{v}_1(\mathbf{A}^{(2)}) \setminus \text{v}_1(\mathbf{A}^{(2)})$ . Together, this yields that  $v_1(\mathbf{A}^{(1)}) \not\approx v_1(\mathbf{A}^{(2)})$ . Analogously, we have  $e_2 \in v_2(\mathbf{A}^{(3)}) \setminus v_2(\mathbf{A}^{(4)})$  and  $v_2(\mathbf{A}^{(3)}) \not\approx$  $v_2(A^{(4)}).$ 

We show below that there is a pair of splines whose knot vectors do not overlap in two directions. The arguments for non-overlapping knot vectors will be the same as before.

Case 1: If  $e_2 \in v_2(\mathbf{A}^{(1)})$ , and  $e_2 \notin v_2(\mathbf{A}^{(2)})$ , or vice versa, then  $v_2(\mathbf{A}^{(1)}) \not\bowtie v_2(\mathbf{A}^{(2)})$ , hence  $B_{\mathbf{A}^{(1)}}\!\not\ltimes\! B_{\mathbf{A}^{(2)}}.$ 

Case 2: If  $e_2 \in v_2(\mathbf{A}^{(1)}),$  and  $e_1 \notin v_1(\mathbf{A}^{(4)}),$  then  $B_{\mathbf{A}^{(1)}} \nless B_{\mathbf{A}^{(4)}}$ . Case 3: If  $e_2 \notin v_2(\mathbf{A}^{(1)})$ , and  $e_1 \notin v_1(\mathbf{A}^{(3)})$ , then  $B_{\mathbf{A}^{(1)}} \nless B_{\mathbf{A}^{(3)}}$ . Case 4: If  $e_2 \in v_2(\mathbf{A}^{(2)})$ , and  $e_1 \in v_1(\mathbf{A}^{(4)})$ , then  $B_{\mathbf{A}^{(2)}} \not\lt B_{\mathbf{A}^{(4)}}$ .

Case 5: If  $e_2 \notin v_2(\mathbf{A}^{(2)})$ , and  $e_1 \in v_1(\mathbf{A}^{(3)})$ , then  $B_{\mathbf{A}^{(2)}} \nless B_{\mathbf{A}^{(3)}}$ .

In all cases (see [Table 2\)](#page-21-0), the mesh is not strongly dual-compatible.

Remark 6.4. Note that SDC  $\subset$  WDC, and hence from [\[9\]](#page-24-3) we know that the generated splines are linearly independent. However, the reverse direction does not hold, as the mesh illustrated in [Figure 11](#page-17-0) is WDC, but not SDC (and by [Theorem 6.1](#page-20-2) not AAS, and by [Theorem 6.2](#page-20-4) not SGAS).

 $\Box$ 

In [Figures 9](#page-15-0) and [10,](#page-16-1) we indicated that the abstract T-junction extensions are a subset of the geometric T-junction extensions. However, this is not the case in general. Consider e.g. [Figure 13,](#page-22-0) which can be constructed by subdividing the lower left cell recursively. Again, the figure shows only the active region. We consider  $p = (3, 3)$  and obtain the geometric T-junction extensions given in [Figure 13a](#page-22-1) and the

TABLE 2

<span id="page-21-0"></span>The cases considered in the proof of [Theorem](#page-20-2) 6.1 cover all possible configurations. This is a modified version of [\[12,](#page-24-6) Table 1].

	$e_1 \in v_1(\mathbf{A}^{(3)})$ $e_1 \in v_1(\mathbf{A}^{(4)})$ $e_1 \notin v_1(\mathbf{A}^{(4)})$		$e_1 \notin v_1(\mathbf{A}^{(3)})$ $e_1 \in v_1(\mathbf{A}^{(4)})$ $e_1 \notin v_1(\mathbf{A}^{(4)})$	
$\overline{\mathcal{L}}$ $\mathcal{L}_2$ $e_2 \in v_2(\mathbf{A}^{(2)})$	case 4	case 2	case 4	case 2
$\omega$ $e_2 \notin v_2(\mathbf{A}^{(2)})$ $\sqrt{2}$	case $1, 5$	cases $1, 2, 5$	case 1	cases $1, 2$
$\overline{\mathcal{L}}$ $\overline{\mathcal{L}}$ $e_2 \in v_2(\mathbf{A}^{(2)})$	cases $1, 4$	cases 1	cases $1, 3, 4$	cases $1, 3$
₩ $e_2 \notin v_2({\bf A}^{(2)})$ $\hat{\mathcal{C}}_0$	case 5	case 5	case 3	case 3

abstract T-junction extensions shown in [Figure 13b.](#page-22-2) However, the abstract T-junction extensions are a subset of the geometric T-junction extensions if the mesh is analysissuitable. This is shown below.

*Proof of [Theorem](#page-20-4)* 6.2. Let T be SGAS and ATJ<sub>i</sub>  $\neq \emptyset$ , then there is a point  $x \in$  $ATJ_i(n) \neq \emptyset$  for some  $n \in [0, N_i]$ , and [Proposition 4.2](#page-9-1) yields  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}, \mathbf{T} \in \mathbb{T}_i, \mathbf{Q} =$ ascell(T) with  $x \in \text{supp}_{\Omega} B_{\mathbf{A}}$  and

<span id="page-22-3"></span>(6.8) 
$$
\overline{T} \cap \text{conv}(P_{i,n}(\mathbf{A}) \cup \{x\}) \neq \varnothing,
$$

(6.9) 
$$
\mathbf{Q}_{\text{pdir}(\mathbf{T})} \cap \text{conv}(\mathbf{A}_{\text{pdir}(\mathbf{T})} \cup \{x_{\text{pdir}(\mathbf{T})}\}) \neq \varnothing,
$$

(6.10) 
$$
\exists y \in \mathbf{A}_{\text{pdir}(T)}: y \neq x_{\text{pdir}(T)}.
$$

We write  $v_{\ell}(\mathbf{A}) = (a_1^{(\ell)}, \dots, a_{p_{\ell}+2}^{(\ell)})$ . From  $x \in \text{supp}_{\Omega} B_{\mathbf{A}} = \mathsf{X}_{\ell=1}^d [a_1^{(\ell)}, a_{p+2}^{(\ell)}]$  we have  $x_i \in [a_1^{(i)}, a_{p_i+2}^{(i)}]$  and hence  $P_{i,n}(\mathbf{A}) \subset \text{supp}_{\Omega} B_{\mathbf{A}}$ . With  $(6.8)$ , we get  $\overline{\mathbf{T}} \cap \text{supp}_{\Omega} B_{\mathbf{A}} \neq$  $\varnothing$ , and [Proposition 5.6](#page-19-0) yields  $v_{\ell}(A) \bowtie v_{\ell}(T)$  for all  $\ell \neq i$ . With [Lemma 5.5,](#page-19-2) we obtain  $x_\ell \in \text{conv } \mathsf{v}_\ell(\mathsf{T})$  for  $\ell \neq i$ . Moreover, we have by construction  $x_i \in \{x_i\} = \mathsf{T}_i = \mathsf{v}_i(\mathsf{T}) =$ conv v<sub>i</sub>(T). Altogether, for any  $i \in \{1, ..., d\}$  and  $x \in ATJ_i$  there is a T-junction  $T \in \mathbb{T}_i$  with  $x \in \underset{\ell=1}{\times}^d_{\ell=1}$  conv  $v_{\ell}(T) = GTJ(T)$ , and hence  $ATJ_i \subseteq GTJ_i$ . Since  $\mathcal T$  is SGAS, we get for any  $i \neq j$  that

(6.11) 
$$
\text{ATJ}_i \cap \text{ATJ}_j \subseteq \text{GTJ}_i \cap \text{GTJ}_j = \varnothing,
$$

which concludes the proof.

 $\Box$ 

*Proof sketch of [Conjecture](#page-20-0)* 6.3. Assume for contradiction a mesh  $\mathcal{T} \in WGS \setminus$ WDC. T being not WDC means that there exist anchors  $\mathbf{A}^{(1)}$ ,  $\mathbf{A}^{(2)} \in \mathcal{A}_{\mathbf{p}}$  with

<span id="page-22-5"></span>(6.12) 
$$
\forall \ell \in \{1, ..., d\} : v_{\ell}(\mathbf{A}^{(1)}) = v_{\ell}(\mathbf{A}^{(2)}) \vee v_{\ell}(\mathbf{A}^{(1)}) \not\approx v_{\ell}(\mathbf{A}^{(2)}),
$$

<span id="page-22-4"></span>(6.13) and  $\exists j \in \{1, ..., d\} : v_j(\mathbf{A}^{(1)}) \not\bowtie v_j(\mathbf{A}^{(2)})$ .

[Equation \(6.13\)](#page-22-4) and [Lemma 4.7](#page-11-1) yields a T-junction  $\mathbf{T}^{(0)} \in \mathbb{T}_j$  with  $\mathbf{T}_j^{(0)} = \{m^{(0)}\},$ 

<span id="page-22-1"></span><span id="page-22-0"></span>

<span id="page-22-2"></span>FIG. 13. Example for  $\text{GTJ}_i \subset \text{ATJ}_i$ , where  $\mathbf{p} = (3, 3)$ .

 $k^{(0)} = \text{pdir}(\mathbf{T}^{(0)}), \mathbf{Q}^{(0)} = \text{ascell}(\mathbf{T}^{(0)})$  such that

(6.14) 
$$
m^{(0)} \in \text{conv}\, \mathrm{v}_j(\mathbf{A}^{(1)}) \cap \text{conv}\, \mathrm{v}_j(\mathbf{A}^{(2)}),
$$

$$
(6.15) \qquad \overline{\mathbf{T}^{(0)}} \cap P_{j,m^{(0)}}(\text{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})) \neq \varnothing \neq \mathbf{Q}_{k^{(0)}} \cap \text{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})_{k^{(0)}},
$$

(6.16) 
$$
\mathbf{A}_{k^{(0)}}^{(1)} \cap \mathbf{A}_{k^{(0)}}^{(2)} = \varnothing.
$$

From  $\mathbf{A}_{k^{(0)}}^{(1)} \cap \mathbf{A}_{k^{(0)}}^{(2)} = \emptyset$  we conclude that  $\mathbf{A}_{k^{(0)}}^{(1)} \neq \mathbf{A}_{k^{(0)}}^{(2)}$ , and with  $(6.12)$  and [Lemma 4.7,](#page-11-1) we get another T-junction  $T^{(1)} \in \mathbb{T}_{k^{(0)}}$  with  $T_{k^{(0)}}^{(1)} = \{m^{(1)}\}, k^{(1)} =$ pdir( $T^{(1)}$ ),  $Q^{(1)} = \text{ascell}(T^{(1)})$  such that

(6.17) 
$$
m^{(1)} \in \text{conv}_{V_k^{(0)}}(\mathbf{A}^{(1)}) \cap \text{conv}_{V_k^{(0)}}(\mathbf{A}^{(2)}),
$$

$$
(6.18) \qquad \overline{\mathbf{T}^{(1)}} \cap P_{k^{(0)},m^{(1)}}(\text{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})) \neq \varnothing \neq \mathbf{Q}_{k^{(1)}} \cap \text{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})_{k^{(1)}},
$$

(6.19) 
$$
\mathbf{A}_{k^{(1)}}^{(1)} \cap \mathbf{A}_{k^{(1)}}^{(2)} = \varnothing.
$$

The very same arguments repeated over again yield an infinite sequence of T-junctions  $T^{(0)}, T^{(1)}, T^{(2)}, \dots$  such that  $odir(T^{(\ell+1)}) = k^{(\ell)} = pdir(T^{(\ell)})$  and

(6.20) 
$$
\overline{\mathbf{T}^{(\ell+1)}} \cap P_{k^{(\ell)},m^{(\ell+1)}}(\text{MBox}(\mathbf{A}^{(1)},\mathbf{A}^{(2)})) \neq \emptyset \text{ for all } \ell.
$$

Since the number of T-junctions in the neighborhood of  $A^{(1)}$  and  $A^{(2)}$  is finite, this sequence is a cycle  $T^{(0)},..., T^{(K)} = T^{(0)}$ .

Conjecture: There exists  $\ell \in \{0, ..., K-1\}$  with  $GTJ(T^{(\ell)}) \cap GTJ(T^{(\ell+1)}) \neq \emptyset$ . Then the mesh is not WGAS, which would conclude the proof.  $\Box$ 

<span id="page-23-5"></span>7. Conclusions & Outlook. We have generalized the two existing concepts of analysis-suitability, an abstract concept introduced in [\[12\]](#page-24-6) and a geometric concept introduced in [\[7\]](#page-24-1), to arbitrary dimension and degree. We have, except for the WGAS criterion, shown their sufficiency for dual-compatibility and hence linear independence of the T-spline basis, and investigated the implications between all introduced criteria, including counterexamples where an implication does not hold.

Ongoing work includes the implementation of T-splines in two and three dimensions into deal.ii to solve simple elliptic PDEs using T-splines as ansatz functions, including local mesh refinement. Future work includes a proof that WGAS is sufficient for WDC is the three-dimensional case, and the numerical comparison to other approaches.

### REFERENCES

- <span id="page-23-0"></span>[1] Thomas W. Sederberg, Jianmin Zheng, Almaz Bakenov, and Ahmad Nasri. T-splines and t-nurccs. ACM Trans. Graph., 22(3):477–484, July 2003.
- <span id="page-23-1"></span>[2] Thomas W. Sederberg, David L. Cardon, G. Thomas Finnigan, Nicholas S. North, Jianmin Zheng, and Tom Lyche. T-spline simplification and local refinement. ACM Trans. Graph., 23(3):276–283, August 2004.
- <span id="page-23-2"></span>[3] Y. Bazilevs, V.M. Calo, J.A. Cottrell, J.A. Evans, T.J.R. Hughes, S. Lipton, M.A. Scott, and T.W. Sederberg. Isogeometric analysis using t-splines. Computer Methods in Applied Mechanics and Engineering, 199(5):229–263, 2010. Computational Geometry and Analysis.
- <span id="page-23-3"></span>[4] Michael R. Dörfel, Bert Jüttler, and Bernd Simeon. Adaptive isogeometric analysis by local h-refinement with t-splines. Computer Methods in Applied Mechanics and Engineering, 199(5):264–275, 2010. Computational Geometry and Analysis.
- <span id="page-23-4"></span>[5] A. Buffa, D. Cho, and G. Sangalli. Linear independence of the t-spline blending functions associated with some particular t-meshes. Computer Methods in Applied Mechanics and Engineering, 199(23):1437–1445, 2010.
- <span id="page-24-0"></span>[6] Xin Li, Jianmin Zheng, Thomas W. Sederberg, Thomas J.R. Hughes, and Michael A. Scott. On linear independence of t-spline blending functions. Computer Aided Geometric Design, 29(1):63–76, 2012. Geometric Constraints and Reasoning.
- <span id="page-24-1"></span>[7] L. Beirão da Veiga, A. Buffa, D. Cho, and G. Sangalli. Analysis-suitable t-splines are dualcompatible. Computer Methods in Applied Mechanics and Engineering, 249-252:42–51, 2012. Higher Order Finite Element and Isogeometric Methods.
- <span id="page-24-2"></span>[8] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez. Analysis Suitable T-splines of arbitrary degree: Definition, linear independance and approximation properties. Mathematical Models and Methods in Applied Sciences, 23(11):1979–2003, 2013.
- <span id="page-24-3"></span>[9] L. Beirão da Veiga, A. Buffa, G. Sangalli, and R. Vázquez. Mathematical analysis of variational isogeometric methods. Acta Numerica, 23:157–287, 2014.
- <span id="page-24-4"></span>[10] Yongjie Zhang, Wenyan Wang, and Thomas J.R. Hughes. Solid t-spline construction from boundary representations for genus-zero geometry. Computer Methods in Applied Mechanics and Engineering, 249-252:185–197, 2012. Higher Order Finite Element and Isogeometric Methods.
- <span id="page-24-5"></span>[11] Wenyan Wang, Yongjie Zhang, Lei Liu, and Thomas J.R. Hughes. Trivariate solid t-spline construction from boundary triangulations with arbitrary genus topology. Computer-Aided Design, 45(2):351–360, 2013. Solid and Physical Modeling 2012.
- <span id="page-24-6"></span>[12] Philipp Morgenstern. Globally structured three-dimensional analysis-suitable t-splines: Definition, linear independence and m-graded local refinement. SIAM Journal on Numerical Analysis, 54(4):2163–2186, jan 2016.
- <span id="page-24-7"></span>[13] Philipp Morgenstern. Mesh Refinement Strategies for the Adaptive Isogeometric Method. PhD thesis, Friedrich-Wilhelm-University Bonn, June 2017.
- <span id="page-24-8"></span>[14] Robin Görmer and Philipp Morgenstern. Analysis-suitable T-splines of arbitrary degree and dimension. Proceedings in Applied Mathematics and Mechanics, 2021. accepted.

### Appendix A. Minor proofs.

# A.1. [Lemma 4.8.](#page-12-0)

<span id="page-24-9"></span>*Proof.* For any mesh entity **E** in the old mesh  $\mathcal{T}^{(n)}$  with  $\mathbf{E}_{\ell} \subset \overline{\mathbf{Q}_{\ell}}$  for  $\ell \neq j$  and  $E_i = Q_i$ , the subdivision of Q removes E and inserts three children

(A.1)  $\qquad \qquad \mathsf{E}^{(1)} = \mathsf{E}_1 \times \cdots \times \mathsf{E}_{j-1} \times (\inf \mathsf{Q}_j, \min \mathsf{Q}_j) \times \mathsf{E}_{j+1} \times \cdots \times \mathsf{E}_d,$ 

$$
(A.2) \t E(2) = E1 \times \cdots \times Ej-1 \times \{mid\mathbf{Q}_j\} \times Ej+1 \times \cdots \times Ed,
$$

$$
(A.3) \t E(3) = E1 \times \cdots \times Ej-1 \times (\text{mid } Qj, \text{sup } Qj) \times Ej+1 \times \cdots \times Ed,
$$

with mid  $\mathbf{Q}_j = \frac{1}{2} (\inf \mathbf{Q}_j + \sup \mathbf{Q}_j)$ . From the premise of the claim we know that there is no T-junction T with  $\text{pdir}(T) = j$  in the j-orthogonal faces of Q. We define below parents( $\hat{A}$ ) for any new anchor  $A$ .

Case 1:  $p_j$  is odd, i.e., the anchors' j-th components are singletons. For any mesh entity  $E^{(\sup \mathbb{Q}_j)} = E_1 \times \cdots \times E_{j-1} \times {\sup \mathbb{Q}_j} \times E_{j+1} \times \cdots \times E_d$  there is also the entity  $\mathbf{E}^{(\inf \mathbf{Q}_j)} = P_{j,\inf \mathbf{Q}_j}(\mathbf{E}^{(\sup \mathbf{Q}_j)})$  and vice versa. This is shown as follows.

Assume for contradiction that there is an entity  $E^{(1)} \subset \partial \mathbf{Q} \cap S_j(\sup \mathbf{Q}_j)$  without counterpart in  $\partial \mathbf{Q} \cap S_j(\inf \mathbf{Q}_j)$ . For arbitrary  $x^{(1)} \in \mathbf{E}^{(1)}$ , its counterpart  $x^{(2)} =$  $(x_1^{(1)},\ldots,x_{j-1}^{(1)},\inf \mathtt{Q}_j,x_{j+1}^{(1)},\ldots,x_d^{(1)})$  $\mathcal{L}_d^{(1)}$ ) lies in some j-orthogonal entity  $E^{(2)}$  ⊂ ∂Q ∩  $S_j(\inf \mathbf{Q}_j)$ . Since  $\mathbf{E}^{(2)} \neq P_{j,\inf \mathbf{Q}_j}(\mathbf{E}^{(1)})$ , there is  $k \neq j$  with  $\mathbf{E}_k^{(1)}$  $\mathbf{E}_k^{(1)} \neq \mathbf{E}_k^{(2)}$  $\binom{2}{k}$ .

If  $\mathtt{E}_k^{(1)}$  $\mathbf{g}_k^{(1)}$  and  $\mathbf{E}_k^{(2)}$  $\binom{2}{k}$  are both singletons, their inequality implies that they are disjoint in contradiction to  $x_k^{(1)} \in \mathbb{E}_k^{(1)} \cap \mathbb{E}_k^{(2)}$  $\mathbf{E}_k^{(2)} \neq \varnothing$ . Hence  $\mathbf{E}_k^{(1)}$  $\mathbf{g}_k^{(1)}$  and  $\mathbf{E}_k^{(2)}$  $\binom{2}{k}$  are not both singletons.

If  $\mathtt{E}_k^{(1)}$  $\mathbf{E}_k^{(1)}$  is a singleton, then  $\mathbf{E}_k^{(2)}$  $\mathbf{k}^{(2)}$  is an open interval, and we get  $x^{(1)} \in \text{Sk}_k \not\ni x^{(2)}$ .

If similarly  $E_k^{(2)}$  $\mathbf{E}_k^{(2)}$  is a singleton, then  $\mathbf{E}_k^{(1)}$  $k^{(1)}$  is an open interval, and  $x^{(1)} \notin Sk_k \ni x^{(2)}$ . If both  $E_k^{(1)}$  $\mathbf{E}_k^{(1)}$  and  $\mathbf{E}_k^{(2)}$  $\mathbf{E}_k^{(2)}$  are open intervals, then  $\mathbf{E}_k^{(1)}$  $\mathbf{E}_k^{(1)}$   $\neq$   $\mathbf{E}_k^{(2)}$  $\mathbf{E}_k^{(2)}$  yields  $\mathbf{E}_k^{(1)}$  $\mathbf{E}_k^{(1)}\not\subseteq \mathbf{E}_k^{(2)}$  $\binom{2}{k}$  or  ${\mathtt E}_k^{(2)}$  $\mathbf{g}_k^{(2)} \not\subseteq \mathtt{E}_k^{(1)}$  $\mathbf{E}_k^{(1)}$ , and we assume without loss of generality the first, i.e. that  $\mathbf{E}_k^{(1)}$  $\mathbf{E}_k^{(1)} \setminus \mathbf{E}_k^{(2)}$  $\binom{2}{k} \neq \varnothing$ . Since  $E_k^{(1)}$  $\mathbf{g}_k^{(1)}$  and  $\mathbf{E}_k^{(2)}$  $k_k^{(2)}$  are open intervals, there is  $y_k \in \mathbb{E}_k^{(1)} \cap \partial \mathbb{E}_k^{(2)}$  $\binom{2}{k}$ , and the point  $y^{(2)} =$   $(x_1^{(2)},...,x_{k-1}^{(2)},y_k,x_{k+1}^{(2)},...,x_d^{(2)})$  lies in a k-orthogonal entity  $E^{(3)} \subset \partial E^{(2)}$ , hence  $k=1, \mathcal{Y}_k, \mathcal{U}_{k+1}, \ldots, \mathcal{U}_d$  $y^{(2)} \in Sk_k$ , while  $y^{(1)} = (x_1^{(1)}, \ldots, x_{k-1}^{(1)})$  $(x_{k-1}^{(1)}, y_k, x_{k+1}^{(1)}, \ldots, x_d^{(1)})$  $\mathbf{E}^{(1)}\in \mathbf{E}^{(1)}$  satisfies  $y^{(1)} \notin \mathrm{Sk}_k$ .

In all cases, [Lemma 2.3](#page-5-1) yields a T-junction T with  $\overline{T} \cap \overline{Q} \neq \emptyset$ , odir $(T) = k$ , pdir(T) = j, ascell(T)<sub>j</sub>∩ $\overline{\mathbb{Q}_j} \neq \emptyset$ , since  $x^{(1)}$  and  $x^{(2)}$  (or  $y^{(1)}$  and  $y^{(2)}$ , respectively) differ only in the j-th direction. Let  $T_k = \{t\}$ , then  $\overline{Q} \cap S_\ell(t) \subset GTJ(T)$ . By [Assumption 4.5,](#page-10-2) Q has active neighbor cells in at least three directions  $\ell_1 \neq \ell_2 \neq \ell_3 \neq \ell_1$ . One of these directions is j since T is not in the j-th frame region, as  $\overline{T} \cap \overline{Q} \neq \emptyset$  and in particular ascell(T)<sub>j</sub> ∩ $\overline{Q}_i \neq \emptyset$ . At least one of the two remaining directions is not k, without loss of generality  $\ell_1 \neq k$ . The bisection of Q creates or eliminates a j-orthogonal T-junction  $T' \subset \partial \mathbf{Q}$  with  $\text{pdir}(T') = \ell_1$  and  $\emptyset \neq T' \cap \overline{\mathbf{Q}} \cap S_k(t) \subset \text{GTJ}(T') \cap \text{GTJ}(T)$ , and  $\mathfrak{T}^{(n)} \notin \text{WGAS}$  or  $\mathfrak{T}^{(n+1)} \notin \text{WGAS}$  in contradiction to above. This shows the claim that each entity  $E^{(\sup \mathbb{Q}_j)} \subset \partial \mathbb{Q} \cap S_j(\sup \mathbb{Q}_j)$  has a counterpart  $E^{(\inf \mathbb{Q}_j)} \subset \partial \mathbb{Q} \cap S_j(\inf \mathbb{Q}_j)$ .

For each such pair  $E^{(\inf \mathbb{Q}_j)}$ ,  $E^{(\sup \mathbb{Q}_j)}$ , the new mesh contains the entity  $E^{(\min \mathbb{Q}_j)}$  $\mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times \{\text{mid } \mathbf{Q}_j\} \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d$ . This particularly holds for the anchors, i.e.,  $\mathcal{A}_{\mathbf{p}}^{(n)}$  contains pairs  $(\mathbf{A}^{(\inf \mathfrak{q}_j)}, \mathbf{A}^{(\sup \mathfrak{q}_j)})$  that lie in the boundary of Q, and for each such pair,  $A_{\mathbf{p}}^{(n+1)}$  contains an anchor  $\hat{\mathbf{A}} = \mathbf{A}^{(\text{mid } \mathbf{Q}_j)}$ . Consider a new anchor  $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$ . We call  $\mathbf{A}^{(\inf \mathbb{Q}_j)}$ ,  $\mathbf{A}^{(\sup \mathbb{Q}_j)}$  the parent anchors of  $\hat{\mathbf{A}}$  and write  $\text{parents}(\hat{\mathbf{A}}) = \{ \mathbf{A}^{(\inf \mathbf{Q}_j)}, \mathbf{A}^{(\sup \mathbf{Q}_j)} \}.$ 

Case 2:  $p_j$  is even, i.e., the anchors' j-th components are open intervals. The subdivision of Q removes any  $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_d$  with  $\mathbf{A}_j = \mathbf{Q}_j$  and inserts  $\hat{\mathbf{A}}^{(\inf \mathbf{Q}_j)}, \hat{\mathbf{A}}^{(\sup \mathbf{Q}_j)}$ with

$$
(A.4) \qquad \hat{\mathbf{A}}^{(\inf \mathbf{Q}_j)} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_{j-1} \times (\inf \mathbf{Q}_j, \text{mid } \mathbf{Q}_j) \times \mathbf{A}_{j+1} \times \cdots \times \mathbf{A}_d,
$$

$$
(A.5) \qquad \hat{\mathbf{A}}^{(\sup \mathbb{Q}_j)} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_{j-1} \times (\text{mid } \mathbb{Q}_j, \text{sup } \mathbb{Q}_j) \times \mathbf{A}_{j+1} \times \cdots \times \mathbf{A}_d.
$$

We call **A** the parent anchor of  $\hat{\mathbf{A}}^{(\text{inf } \mathbf{Q}_j)}$  and  $\hat{\mathbf{A}}^{(\text{sup } \mathbf{Q}_j)}$  and write parents( $\hat{\mathbf{A}}^{(\text{inf } \mathbf{Q}_j)}$ ) =  ${A}$  and parents $(\hat{\mathbf{A}}^{(\sup \mathbf{Q}_j)}) = {\mathbf{A}}.$ 

In both cases, any new anchor  $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$  is in direction j aligned with its parent  $\mathbf{A} \in \text{parents}(\hat{\mathbf{A}})$ , and hence shares the same global index vector  $\mathcal{I}_i(\hat{\mathbf{A}}) = \mathcal{I}_i(\mathbf{A})$ in the new mesh  $\mathfrak{I}^{(n+1)}$ . In what follows, we use the local index vector  $v_j(A)$  of the old anchor with respect to the old and new mesh, and the local index vector  $v_i(\hat{A})$  of the new anchor with respect to the new mesh. In other directions  $k \neq j$ , the skeleton  $Sk_k$  is unchanged as well as the global and local index vectors  $\mathcal{I}_k(\mathbf{A})$ ,  $v_k(\mathbf{A})$ , and therefore these refer to the new mesh, and to the old mesh where applicable. For the existence of T-junctions below, we refer to the new mesh if not stated otherwise.

The subdivision of Q inserts mid  $Q_i$  to these global index vectors, such that we have by construction  $v_i(\hat{\mathbf{A}}) \subset v_i(\mathbf{A}) \cup \{\text{mid } \mathbb{Q}_i\}$  and, since  $\text{mid } \mathbb{Q}_i \in \text{conv } v_i(\mathbf{A})$ , we have conv  $v_j(\hat{\mathbf{A}}) \subset \text{conv } v_j(\mathbf{A})$ . If  $v_{\ell}(\hat{\mathbf{A}}) = v_{\ell}(\mathbf{A})$  for all  $\ell \neq j$ , this yields  $\text{supp}_{\Omega} B_{\hat{\mathbf{A}}} \subset$  $\text{supp}_{\Omega,\mathcal{T}^{(n)}} B_{\mathbf{A}}.$ 

Assume for contradiction that there is  $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$ ,  $\mathbf{A} \in \text{parents}(\hat{\mathbf{A}})$ , and  $k \neq j$  with  $v_k(\hat{\mathbf{A}}) \neq v_k(\mathbf{A})$ . Since  $\hat{\mathbf{A}}_k = \mathbf{A}_k$ , the middle entries of  $v_k(\hat{\mathbf{A}})$  and  $v_k(\mathbf{A})$ coincide by construction. Consequently, there is some  $m < \inf \mathbb{Q}_k$  or  $m > \sup \mathbb{Q}_k$  with

<span id="page-25-0"></span>
$$
(A.6) \t v_k(\hat{A}) \ni m \in \operatorname{conv} v_k(A) \setminus v_k(A) \quad \text{or} \quad v_k(A) \ni m \in \operatorname{conv} v_k(\hat{A}) \setminus v_k(\hat{A}).
$$

Without loss of generality we assume the latter cases, i.e.,  $m > \sup \mathbb{Q}_k$  and  $v_k(\mathbf{A}) \ni$  $m \in \text{conv}_{k}(\hat{\mathbf{A}}) \setminus v_{k}(\hat{\mathbf{A}})$ . [Lemma 4.7](#page-11-1) yields a T-junction T with odir(T) = k,  $T_{k} = \{m\},$   $\tilde{\mathbf{Q}} = \text{ascell}(\mathbf{T})$  such that

$$
\overline{\mathbf{T}} \cap P_{k,m}(\text{MBox}(\hat{\mathbf{A}}, \mathbf{A})) \neq \varnothing,
$$

$$
\text{(A.8)} \hspace{1cm} \tilde{\mathtt{Q}}_{\mathrm{pdir}(T)} \cap \mathrm{MBox}(\hat{\mathbf{A}}, \mathbf{A})_{\mathrm{pdir}(T)} \neq \varnothing, \quad \hat{\mathbf{A}}_{\mathrm{pdir}(T)} \cap \mathbf{A}_{\mathrm{pdir}(T)} = \varnothing.
$$

Since  $\hat{A}$  and  $A$  differ only in direction j, we have pdir(T) = j, and with  $\text{MBox}(\hat{\mathbf{A}}, \mathbf{A})_i \subset \overline{\mathbb{Q}_i}$ , we get  $\tilde{\mathbb{Q}}_i \cap \overline{\mathbb{Q}_i} \neq \emptyset$ . Similarly, from  $\text{MBox}(\hat{\mathbf{A}}, \mathbf{A}) \subset \overline{\mathbb{Q}}$ , we get  $T_j \cap \overline{\mathbb{Q}_j}$ , and since  $T_j$  is a singleton, this is  $T_j \subset \overline{\mathbb{Q}_j}$ .

Having the existence of  $\texttt{T}$ , there is also a T-junction  $\texttt{T}^{(0)}$  with the same properties as T, which is closest to Q in direction k. We therefore consider the minimal  $m_0$  >  $\sup \mathsf{Q}_k$  such that there is a T-junction  $\mathsf{T}^{(0)} \in \mathbb{T}_k(\hat{\mathsf{T}})$  with

(A.9) 
$$
\overline{T_{\ell}^{(0)}} \cap \hat{A}_{\ell} \neq \varnothing \quad \text{for all } \ell \notin \{k, j\},
$$

<span id="page-26-2"></span>(A.10) 
$$
\text{odir}(\mathbf{T}^{(0)}) = k, \quad \text{pdir}(\mathbf{T}^{(0)}) = j, \quad \mathbf{T}_k^{(0)} = \{m_0\},
$$

(A.11) 
$$
\mathbf{T}_j^{(0)} \subset \overline{\mathbf{Q}_j}, \quad \mathbf{Q}^{(0)} = \text{ascell}(\mathbf{T}^{(0)}), \quad \mathbf{Q}_j^{(0)} \cap \overline{\mathbf{Q}_j} \neq \emptyset.
$$

Case 1:  $\mathbf{v}_j(\mathbf{T}^{(0)}) \cap \overline{\mathbf{Q}_j} \subseteq \mathbf{v}_j(\hat{\mathbf{A}}) \cap \overline{\mathbf{Q}_j}$ . Since  $\mathbf{v}_j(\hat{\mathbf{A}}) \cap \overline{\mathbf{Q}_j} = {\text{inf } \mathbf{Q}_j, \text{mid } \mathbf{Q}_j, \text{sup } \mathbf{Q}_j},$ this leads to mid  $\mathbf{Q}_j \in \text{conv } \mathbf{v}_j(\mathbf{T}^{(0)})$  by construction of local knot vectors.

Since T with  $T_k = \{m\}$  from above is a k-orthogonal T-junction, it is not in the k-th frame region, and  $\sup \mathbb{Q}_k < m < N_k - \lfloor \frac{p_k+1}{2} \rfloor$ , i.e. Q does not touch the k-th frame region in positive direction. Hence for any  $x^{(0)} \in \overline{\mathbb{Q}} \cap S_k(\sup \mathbb{Q}_k) \cap S_j(\text{mid } \mathbb{Q}_j)$ , the subdivision of  $Q$  creates or eliminates a T-junction  $T^{(1)}$  with

# <span id="page-26-0"></span>(A.12)

$$
odir(\mathbf{T}^{(1)}) = j, \quad \text{pdir}(\mathbf{T}^{(1)}) = k, \quad \mathbf{T}_k^{(1)} = \{\text{sup } \mathbf{Q}_k\}, \quad \mathbf{T}_j^{(1)} = \{\text{mid } \mathbf{Q}_j\}, \quad x^{(0)} \in \overline{\mathbf{T}^{(1)}}.
$$

We choose  $x^{(0)}$  such that  $x_k^{(0)} = \sup \mathsf{Q}_k$ ,  $x_j^{(0)} = \text{mid } \mathsf{Q}_j$ , and  $x_\ell^{(0)} \in \mathsf{T}_\ell^{(0)} \cap \hat{\mathbf{A}}_\ell$  for all  $\ell \notin \{k, j\}$ . This yields

<span id="page-26-3"></span>
$$
(A.13) \quad x_{\ell}^{(0)} \in \overline{T_{\ell}^{(1)}} \cap \overline{T_{\ell}^{(0)}} \subseteq v_{\ell}(T^{(1)}) \cap \text{conv } v_{\ell}(T^{(0)}) \neq \varnothing \quad \text{for all } \ell \notin \{k, j\},
$$
  
and 
$$
\text{mid } \mathbb{Q}_{j} \in v_{j}(T^{(1)}) \cap \text{conv } v_{j}(T^{(0)}) \neq \varnothing.
$$

Case 1.1:  $v_k(T^{(1)}) \cap (\sup \mathbb{Q}_k, m_0) \subseteq v_k(\hat{\mathbf{A}}) \cap (\sup \mathbb{Q}_k, m_0)$ . By construction we know that  $\#v_k(\hat{\mathbf{A}}) = p_k + 2$ , and from sup  $\hat{\mathbf{A}}_k \leq \sup \mathbf{Q}_k$  we get that (A.14)

$$
\#\{z\in v_k(\hat{\mathbf{A}})\mid z>\sup\mathsf{Q}_k\}=\#\{z\in v_k(\hat{\mathbf{A}})\mid z\geq \sup\mathsf{Q}_k\}-1\leq \left\lceil \frac{p_k+2}{2}\right\rceil-1=\left\lceil \frac{p_k}{2}\right\rceil.
$$

From  $(A.6)$  and  $m_0 \in (\sup \mathbb{Q}_k, m]$  we know that  $m_0 \in \text{conv } \mathcal{V}_k(\hat{\mathbf{A}})$ , and hence

<span id="page-26-1"></span>
$$
(A.15) \qquad \#(v_k(\mathbf{T}^{(1)}) \cap (\sup \mathbf{Q}_k, m_0)) \leq \#(v_k(\hat{\mathbf{A}}) \cap (\sup \mathbf{Q}_k, m_0)) \leq \lceil \frac{p_k-2}{2} \rceil.
$$

Moreover, from  $(A.12)$  we have  $T_k^{(1)} = {\sup \mathbb{Q}_k}$  and hence by construction

$$
\# \{ z \in v_k(T^{(1)}) \mid z > \sup \mathsf{Q}_k \} = \left\lfloor \frac{p_k + 1}{2} \right\rfloor = \left\lceil \frac{p_k}{2} \right\rceil.
$$

Together with [\(A.15\),](#page-26-1) there exists  $z \in v_k(T^{(1)})$  with  $z \geq m_0$ , and hence  $m_0 \in$ conv v<sub>k</sub>(T<sup>(1)</sup>). Together with [\(A.10\),](#page-26-2) this is  $m_0 \in \text{conv}_{N_k}(\mathbb{T}^{(0)}) \cap \mathbb{v}_k(\mathbb{T}^{(1)}) \neq \emptyset$ , and together with  $(A.13)$ ,  $\mathfrak{T}^{(n)}$  or  $\mathfrak{T}^{(n+1)}$  is not WGAS in contradiction to the assumption.

Case 1.2: There exists some  $m_2 \in v_k(\mathbf{T}^{(1)}) \cap (\sup \mathbf{Q}_k, m_0) \setminus v_k(\hat{\mathbf{A}})$ . [Lemma 4.6](#page-10-1) yields that for any  $x^{(1)} \in P_{k,m_2}(\hat{\mathbf{A}}), y^{(1)} \in \overline{P_{k,m_2}(\mathbf{T}^{(1)})}$  holds  $x^{(1)} \notin S_k$   $\Rightarrow y^{(1)}$ . Choose  $x^{(1)}, y^{(1)}$  such that  $x^{(1)}_{\ell} = y^{(1)}_{\ell}$ <sup>(1)</sup> for all  $\ell \neq j$ . This is possible because  $x_k^{(1)} = y_k^{(1)}$  $k^{(1)}$  holds trivially and  $x_{\ell}^{(0)} \in \mathcal{T}_{\ell}^{(0)} \cap \mathcal{T}_{\ell}^{(1)} \cap \hat{A}_{\ell}$  for  $\ell \notin \{k, j\}$  from  $(A.13)$  and above. [Lemma 2.3](#page-5-1) yields another T-junction  $T^{(2)}$  and  $Q^{(2)} = \text{ascell}(T^{(2)})$  with

(A.17) 
$$
\text{odir}(\mathbf{T}^{(2)}) = k, \quad x_{\text{pdir}(\mathbf{T}^{(2)})}^{(1)} \neq y_{\text{pdir}(\mathbf{T}^{(2)})}^{(1)}
$$
 and hence  $\text{pdir}(\mathbf{T}^{(2)}) = j$ ,

$$
(A.18) \t\t T_k^{(2)} = \{m_2\}, \quad \overline{\mathbf{T}^{(2)}} \cap \text{conv}\{x^{(1)}, y^{(1)}\} \neq \varnothing, \quad \mathbf{Q}_j^{(2)} \cap \overline{\mathbf{Q}_j} \neq \varnothing.
$$

From  $\overline{\mathcal{T}^{(2)}} \cap \text{conv}\{x^{(1)}, y^{(1)}\} \neq \emptyset$  and  $\mathcal{T}_j^{(2)}$  being a singleton, we get

(A.19) 
$$
\mathbf{T}_{j}^{(2)} \subseteq \operatorname{conv}(\overline{\hat{\mathbf{A}}_{j}} \cup \overline{\mathbf{T}_{j}^{(1)}}) \subseteq \overline{\mathbf{Q}_{j}},
$$

(A.20) 
$$
x_{\ell}^{(1)} = y_{\ell}^{(1)} \in \overline{\mathrm{T}^{(2)}}_{\ell} \cap \hat{\mathbf{A}}_{\ell} \neq \varnothing \quad \text{for all } \ell \notin \{k, j\}
$$

in contradiction to the minimality of  $m_0$ . This ends Case 1.

Case 2: There is  $m_1 \in v_j(T^{(0)}) \cap \overline{\mathbb{Q}_j}$  with  $m_1 \notin v_j(\hat{\mathbf{A}}) \cap \overline{\mathbb{Q}_j}$ . [Lemma 4.6](#page-10-1) yields that  $x^{(0)} \notin Sk_j \ni y^{(0)}$  holds for all  $x^{(0)} \in P_{j,m_1}(\hat{\mathbf{A}}), y^{(0)} \in \overline{P_{j,m_1}(\mathbf{T}^{(0)})}$ . We choose  $x^{(0)}, y^{(0)}$ such that  $x_{\ell}^{(0)} = y_{\ell}^{(0)} \in \mathcal{T}_{\ell}^{(0)} \cap \hat{\mathbf{A}}_{\ell}$  for all  $\ell \notin \{k, j\}$ , and  $x_{j}^{(0)} = y_{j}^{(0)} = m_1$ . [Lemma 2.3](#page-5-1) yields  $T^{(2)} \in \mathbb{T}_j$  with

$$
(A.21) \quad \overline{\mathbf{T}^{(2)}} \cap \text{conv}(\overline{P_{j,m_1}(\mathbf{T}^{(0)})} \cup P_{j,m_1}(\hat{\mathbf{A}})) \neq \varnothing, \quad \text{pdir}(\mathbf{T}^{(2)}) = k, \quad \mathbf{T}^{(2)}_j = \{m_1\},
$$

(A.22) 
$$
y_{\ell}^{(0)} \in \mathbf{T}_{\ell}^{(2)} \cap \hat{\mathbf{A}}_{\ell} \neq \varnothing \quad \text{for all } \ell \notin \{k, j\}.
$$

From  $(A.6)$  and  $\sup \mathbb{Q}_k < m_0 \le m$  we get  $m_0 \in \text{conv } \mathsf{v}_k(\mathbf{A})$ .

Case 2.1:  $v_k(T^{(2)}) \cap (\sup \mathbb{Q}_k, m_0) \subseteq v_k(\hat{\mathbf{A}}) \cap (\sup \mathbb{Q}_k, m_0)$ . This leads to  $m_0 \in$ conv $v_k(T^{(2)})$  and hence

$$
(A.23) \qquad m_0 \in \operatorname{conv}_{v_k}(T^{(0)}) \cap v_k(T^{(2)}) \neq \varnothing, \quad m_1 \in \operatorname{conv}_{v_j}(T^{(0)}) \cap v_j(T^{(2)}) \neq \varnothing,
$$

(A.24) 
$$
y_{\ell}^{(0)} \in \text{conv } \mathsf{v}_{\ell}(\mathsf{T}^{(0)}) \cap \mathsf{v}_{\ell}(\mathsf{T}^{(2)}) \neq \varnothing \quad \text{for all } \ell \notin \{k, j\},\
$$

(A.25) 
$$
\text{pdir}(T^{(0)}) \neq \text{odir}(T^{(0)}) = k = \text{pdir}(T^{(2)}) \neq \text{odir}(T^{(2)}),
$$

which means that  $\mathcal{T}^{(n+1)}$  is not WGAS in contradiction to the assumption.

Case 2.2:  $\exists m_2 \in v_k(\mathbb{T}^{(2)}) \cap (\sup \mathbb{Q}_k, m_0) \setminus v_k(\hat{\mathbf{A}}).$  [Lemma 4.6](#page-10-1) yields that for any  $x^{(1)} \in P_{k,m_2}(\hat{\mathbf{A}}), y^{(1)} \in \overline{P_{k,m_2}(\mathbf{T}^{(2)})}$  holds  $x^{(1)} \notin S_k$   $\Rightarrow y^{(1)}$ . Choose  $y^{(1)}$  such that  $y_\ell^{(1)}\in \mathtt{T}^{(2)}_\ell$  $\ell_{\ell}^{(2)}$  for all  $\ell \notin \{k, j\}, y_k^{(1)} = m_2 = x_k^{(1)}$  $\mathbf{r}_{k}^{(1)}$ , and  $\mathbf{T}_{j}^{(2)} = \{y_{j}^{(1)}\}$ . [Lemma 2.3](#page-5-1) yields another T-junction  $T^{(3)}$  and  $Q^{(3)} = \text{ascell}(T^{(3)})$  with

$$
(A.26) \qquad \text{odir}(\mathbf{T}^{(3)}) = k, \quad \mathbf{T}_k^{(3)} = \{m_2\}, \quad \mathbf{T}_j^{(3)} \subset \overline{\mathbf{Q}_j}, \quad \mathbf{Q}_j^{(3)} \cap \overline{\mathbf{Q}_j} \neq \varnothing
$$

<span id="page-27-0"></span>in contradiction to the minimality of  $m_0$ . This ends Case 2.2 and concludes the proof.

## A.2. [Lemma 5.5.](#page-19-2)

Proof. We set

$$
(A.27) \quad \text{partsupp}(\mathbf{A}, x, \ell) := \begin{cases} [\min_{\mathbf{v}_{\ell}} (\mathbf{A}), \inf_{\mathbf{A}_{\ell}} \mathbf{A}_{\ell}] \cup \mathbf{A}_{\ell} & \text{if } x_{\ell} < y \text{ for all } y \in \mathbf{A}_{\ell}, \\ \mathbf{A}_{\ell} & \text{if } x_{\ell} \in \mathbf{A}_{\ell}, \\ \mathbf{A}_{\ell} \cup [\sup_{\mathbf{A}_{\ell}} \mathbf{A}_{\ell}, \max_{\mathbf{v}_{\ell}} \mathbf{A}_{\ell}] & \text{if } x_{\ell} > y \text{ for all } y \in \mathbf{A}_{\ell}. \end{cases}
$$

Then we have by construction for  $p_\ell \geq 1$  that

$$
(A.28) \tpartsupp(\mathbf{A}, x, \ell) \supseteq \text{conv}(\mathbf{A}_{\ell} \cup \{x_{\ell}\}).
$$

The combination of [\(5.7\)](#page-19-3) and [\(A.28\)](#page-28-0) yields

$$
(A.29) \t\t \t\t \mathcal{D} \neq \overline{T_{\ell}} \cap conv(\mathbf{A}_{\ell} \cup \{x_{\ell}\}) \subseteq \overline{T_{\ell}} \cap partsupp(\mathbf{A}, x, \ell).
$$

<span id="page-28-1"></span>We distinguish eight cases illustrated in [Table 3.](#page-28-1)

<span id="page-28-2"></span><span id="page-28-0"></span>TABLE 3 overview of cases in the proof of [Lemma](#page-19-2) 5.5.

	$i \neq \ell = \text{pdir}(T)$ $i \neq \ell \neq \text{pdir}(T)$			
		$p_{\ell}$ odd   $p_{\ell}$ even   $p_{\ell}$ odd   $p_{\ell}$ even		
		$x_{\ell} \in \mathbf{A}_{\ell}$   case 1   case 2   case 3   case 4		
$x_{\ell} \notin \mathbf{A}_{\ell}$   case 8			case $7 \mid \text{case } 6 \mid \text{case } 5$	

Case 1:  $x_\ell \in \mathbf{A}_\ell, i \neq \ell = \text{pdir}(T), \text{ and } p_\ell \text{ is odd. Since } p_\ell \text{ is odd, } \mathbf{A}_\ell \text{ is a singleton,}$ i.e.  $\mathbf{A}_{\ell} = \{x_{\ell}\}\text{, in contradiction to the existence of } y \in \mathbf{A}_{\ell} \text{ with } y \neq x_{\ell} \text{ from (5.9)}$  $\mathbf{A}_{\ell} = \{x_{\ell}\}\text{, in contradiction to the existence of } y \in \mathbf{A}_{\ell} \text{ with } y \neq x_{\ell} \text{ from (5.9)}$  $\mathbf{A}_{\ell} = \{x_{\ell}\}\text{, in contradiction to the existence of } y \in \mathbf{A}_{\ell} \text{ with } y \neq x_{\ell} \text{ from (5.9)}$ above.

Case 2:  $x_{\ell} \in \mathbf{A}_{\ell}$ ,  $i \neq \ell = \text{pdir}(T)$ , and  $p_{\ell}$  is even. Then  $\mathbf{A}_{\ell}$  is an open interval and  $\mathbf{T}_{\ell} = \{t\}$  is a singleton. From [\(A.29\)](#page-28-2) we obtain

$$
(A.30) \t t \in \text{partsupp}(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \subset \text{conv } \text{v}_{\ell}(\mathbf{A}) \setminus \text{v}_{\ell}(\mathbf{A}).
$$

From the definition of local index vectors we also know  $t \in v_{\ell}(T)$ , which yields  $v_{\ell}(A) \not\approx$  $v_{\ell}(T)$  in contradiction to [\(5.10\).](#page-19-5)

Case 3:  $x_{\ell} \in \mathbf{A}_{\ell}$ ,  $i \neq \ell \neq$  pdir(T), and  $p_{\ell}$  is odd. Then  $T_{\ell}$  is an open interval, and we have partsupp $(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} = \{x_{\ell}\}.$  Hence  $x_{\ell} \in v_{\ell}(\mathbf{A})$ . [Equation \(A.29\)](#page-28-2) yields  $x_{\ell} \in \overline{\mathrm{T}_{\ell}} \subseteq \mathrm{conv\,} \mathrm{v}_{\ell}(\mathrm{T}).$ 

Case 4:  $x_{\ell} \in \mathbf{A}_{\ell}$ ,  $i \neq \ell \neq$  pdir(T), and  $p_{\ell}$  is even. Then  $T_{\ell}$  and  $\mathbf{A}_{\ell}$  are open intervals, and [\(A.29\)](#page-28-2) yields that  $\overline{T_{\ell}} \cap A_{\ell} \neq \emptyset$ . Together with [\(5.10\),](#page-19-5) we have  $x_{\ell} \in$  $\mathbf{A}_{\ell} = \mathbf{T}_{\ell} \subset \operatorname{conv} \mathbf{v}_{\ell}(\mathbf{T}).$ 

Case 5:  $x_{\ell} \notin \mathbf{A}_{\ell}$ ,  $i \neq \ell \neq$  pdir(T), and  $p_{\ell}$  is even. Assume without loss of generality that  $x_{\ell} > y$  for all  $y \in A_{\ell}$ . In this case, we have partsupp $(A, x, \ell) =$  $\mathbf{A}_\ell \cup [\sup \mathbf{A}_\ell, \max \mathbf{v}_\ell(\mathbf{A})]$  with  $\mathbf{A}_\ell$  being an open interval and  $x_\ell \in [\sup \mathbf{A}_\ell, \max \mathbf{v}_\ell(\mathbf{A})].$ Also  $T_\ell$  is an open interval and [\(5.10\)](#page-19-5) and [\(A.29\)](#page-28-2) yield that either  $T_\ell = A_\ell$  or inf  $T_\ell \in$  $[\sup A_\ell, \max v_\ell(A)],$  this is,  $\overline{T_\ell} \cap [\sup A_\ell, \max v_\ell(A)] \neq \emptyset$ . The knot vector  $v_\ell(T)$ contains  $\frac{p_\ell}{2} + 1$  entries that are not smaller than sup  $T_\ell$  and  $\frac{p_\ell}{2} + 1$  entries that are not greater than inf  $T_\ell$ . The interval  $[\sup \mathbf{A}_\ell, \max v_\ell(\mathbf{A})]$  contains  $\frac{p_\ell}{2} + 1$  entries of  $v_\ell(\mathbf{A})$ . Together with [\(5.10\),](#page-19-5) all entries of  $v_{\ell}(A) \cap [\sup A_{\ell}, \max v_{\ell}(A)]$  match with entries of  $v_{\ell}(T)$ , and we get  $x_{\ell} \in [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] \subseteq \text{conv } v_{\ell}(T)$ .

Case 6:  $x_\ell \notin \mathbf{A}_\ell, i \neq \ell \neq \text{pdir}(T)$ , and  $p_\ell$  is odd. Then  $\mathbf{A}_\ell$  is a singleton  $\mathbf{A}_\ell = \{s\}.$ Assume without loss of generality that  $x_{\ell} > s$ , then we have partsupp $(\mathbf{A}, x, \ell) =$  $\mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] = [s, \max v_{\ell}(\mathbf{A})]$  which contains  $\lceil \frac{p_{\ell}}{2} \rceil + 1$  entries of  $v_{\ell}(\mathbf{A})$ . As in case 5 above, we have  $\overline{T_{\ell}} \cap [s, \max_{\ell} v_{\ell}(\mathbf{A})] \neq \emptyset$ , and  $v_{\ell}(T)$  containing  $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries that are  $\geq \sup \mathsf{T}_{\ell}$  and  $\lceil \frac{p_{\ell}}{2} \rceil + 1$  entries that are  $\leq \inf \mathsf{T}_{\ell}$ . Together with [\(5.10\),](#page-19-5) we get  $x_\ell \in [s, \max v_\ell(\mathbf{A})] \subseteq \text{conv } v_\ell(\mathbf{T}).$ 

Case 7:  $x_{\ell} \notin \mathbf{A}_{\ell}$ ,  $i \neq \ell = \text{pdir}(T)$ , and  $p_{\ell}$  is even. Then  $\mathbf{A}_{\ell}$  is an open interval, and we assume without loss of generality  $x_{\ell} > y$  for all  $y \in \mathbf{A}_{\ell}$ , obtaining partsupp $(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$  with  $x_{\ell} \in [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$ . Since  $\ell = \text{pdir}(T), T_{\ell}$  is a singleton  $T_{\ell} = \{t\} = \overline{T_{\ell}}, \text{ and } (A.29)$  $T_{\ell} = \{t\} = \overline{T_{\ell}}, \text{ and } (A.29)$  yields  $t \in \text{partsupp}(\mathbf{A}, x, \ell).$ Together with  $t \in v_{\ell}(\mathbf{T})$  and  $(5.10)$ , we get that  $t \in v_{\ell}(\mathbf{A}) \cap [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$ . The partial index vector  $v_{\ell}(A) \cap [\sup A_{\ell}, \max v_{\ell}(A)]$  contains  $\frac{p_{\ell}}{2} + 1$  entries of  $v_{\ell}(A)$ , while  $v_{\ell}(T)$  contains  $\frac{p_{\ell}}{2} + 1$  entries that are  $\leq t$  and  $\frac{p_{\ell}}{2} + 1$  entries that are  $\leq t$ . As in previous cases, we obtain with [\(5.10\)](#page-19-5) that  $v_{\ell}(A) \cap [\sup A_{\ell}, \max v_{\ell}(A)] \subset v_{\ell}(T)$  and consequently  $x_\ell \in [\sup \mathbf{A}_\ell, \max v_\ell(\mathbf{A})] \subset \text{conv } v_\ell(\mathbf{T}).$ 

Case 8:  $x_{\ell} \notin \mathbf{A}_{\ell}$ ,  $i \neq \ell = \text{pdir}(\mathbf{T})$ , and  $p_{\ell}$  is odd. Then  $\mathbf{A}_{\ell}$  is a singleton  $\mathbf{A}_{\ell} = \{s\}$ . Assume without loss of generality that  $x_{\ell} > s$ , then we have partsupp $(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \cup$  $[\sup A_\ell, \max v_\ell(A)] = [s, \max v_\ell(A)]$  which contains  $\lceil \frac{p_\ell}{2} \rceil + 1$  entries of  $v_\ell(A)$ . Since  $\ell = \text{pdir}(T), T_{\ell}$  is a singleton  $T_{\ell} = \{t\} = \overline{T_{\ell}},$  and  $(A.29)$  yields  $t \in \text{partsupp}(\mathbf{A}, x, \ell) =$  $[s, \max v_{\ell}(\mathbf{A})].$  Moreover,  $t \in \partial \mathbf{Q}_{\ell} = \{\inf \mathbf{Q}_{\ell}, \sup \mathbf{Q}_{\ell}\} \subseteq v_{\ell}(\mathbf{T})$  for the associated cell  $\mathbf{Q} = \text{ascell}(\mathbf{T})$  from the definition [\(4.9\).](#page-9-2) By construction of the knot vector we have  $\mathsf{Q}_\ell \subset \text{conv } \mathsf{v}_\ell(T) \setminus \mathsf{v}_\ell(T)$ , and with  $(5.10)$  we obtain  $\mathsf{Q}_\ell \cap \mathsf{v}_\ell(A) = \emptyset$ . Consequently s, max  $v_{\ell}(\mathbf{A}) \notin \mathbb{Q}_{\ell}$  and hence we have either  $\mathbb{Q}_{\ell} \subset [s, \max v_{\ell}(\mathbf{A})]$  or  $\mathbb{Q}_{\ell} \cap [s, \max v_{\ell}(\mathbf{A})] =$  $\emptyset$ . Together with [\(5.8\)](#page-19-6) we have  $\mathbb{Q}_{\ell} \subset [s, \max v_{\ell}(\mathbf{A})],$  and, since  $[s, \max v_{\ell}(\mathbf{A})]$  is closed,  $\overline{\mathsf{Q}_\ell} \subset [s, \max \mathsf{v}_\ell(\mathbf{A})].$  The combination with  $(5.10)$  yields that  $\{\inf \mathsf{Q}_\ell, \sup \mathsf{Q}_\ell\} \subseteq$  $\mathsf{v}_{\ell}(\mathbf{A}) \cap [s, \max \mathsf{v}_{\ell}(\mathbf{A})].$  Since  $\mathsf{v}_{\ell}(\mathsf{T})$  contains  $\lceil \frac{p_{\ell}}{2} \rceil + 1$  entries that are  $\geq \inf \mathsf{Q}_{\ell}$  and  $\lceil \frac{p_\ell}{2} \rceil + 1$  entries that are  $\leq \sup \mathsf{Q}_\ell$ , [\(5.10\)](#page-19-5) yields  $v_\ell(\mathbf{A}) \cap [s, \max v_\ell(\mathbf{A})] \subseteq v_\ell(\mathsf{T})$  and hence  $x_\ell \in [s, \max v_\ell(\mathbf{A})] \subset \text{conv } v_\ell(\mathbf{T}).$ 

We have shown the claim in all cases, which concludes the proof.

 $\Box$ 

# <span id="page-29-0"></span>A.3. [Proposition 5.6.](#page-19-0)

Proof. The proof is by induction over box bisections. As assumed in [section 2,](#page-1-1) T is constructed via symmetric bisections of boxes from an initial tensor-product mesh. For a tensor-product mesh, the claim is trivially true due to the absence of T-junctions. Assume that the claim is true for an SGAS mesh T and consider an SGAS mesh  $\mathcal{T} = \text{SUBDIV}(\mathcal{T}, \mathbf{Q}, j)$  that results from the j-orthogonal bisection of a cell  $\mathbf{Q} \in \mathcal{T}$ . Since this bisection inserts only one j-orthogonal hyperface  $\mathbf{F} = \mathbf{Q}_1 \times \cdots \times \mathbf{Q}_{j-1} \times \{r\} \times \mathbf{Q}_j$  $\mathsf{Q}_{i+1} \times \cdots \times \mathsf{Q}_d$  and lower-dimensional entities that are subsets of other, previously present entities, only the j-orthogonal skeleton  $\text{Sk}_j(\mathfrak{I}) \supsetneq \text{Sk}_j(\mathfrak{I})$  is modified, while all other *i*-orthogonal skeletons  $Sk_i(\mathcal{T}) = Sk_i(\mathcal{T}), i \neq j$ , remain unchanged. Hence for any anchor or T-junction that exist in both meshes, the local knot vectors (or knot vectors, resp.) remain unchanged in all directions  $i \neq j$ . In the following, all knot vectors are understood with respect to the refined mesh  $\tilde{\mathcal{T}}$ .

Assume for contradiction that in the new mesh  $\hat{\mathcal{T}}$ , there exist  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)}$  and a T-junction T with  $\overline{T} \cap \text{supp}_{\Omega} B_{\mathbf{A}} \neq \emptyset$ , and  $v_k(\mathbf{A}) \not\approx v_k(T)$  for some  $k \neq \text{odir}(T)$ . The non-overlapping  $v_k(A) \not\approx v_k(T)$  means that there is  $m \in \{0, \ldots, N_k\}$  with

<span id="page-29-1"></span>
$$
(A.31) \t v_k(\mathbf{A}) \ni m \in conv(v_k(\mathbf{T})) \setminus v_k(\mathbf{T}) \t or \t v_k(\mathbf{T}) \ni m \in conv(v_k(\mathbf{A})) \setminus v_k(\mathbf{A}).
$$

[Lemma 4.6](#page-10-1) yields that for any  $x \in P_{k,m}(\mathbf{T}), y \in P_{k,m}(\mathbf{A})$  holds  $x \in Sk_k \not\ni y$  or  $x \notin Sk_k \ni y$ . [Lemma 2.3](#page-5-1) yields a T-junction  $T' \in \mathbb{T}_k$  and associated cell  $\mathsf{Q} = \text{ascell}(T')$ with

<span id="page-29-2"></span>(A.32)  $\overline{T'} \cap \text{conv}(P_{k,m}(\mathbf{A}) \cup \{x\}) \neq \emptyset,$ 

$$
\text{(A.33)} \qquad \qquad \mathsf{Q}_{\text{pdir}(T')} \cap \text{conv}(\mathbf{A}_{\text{pdir}(T')} \cup \{x_{\text{pdir}(T')}\}) \neq \varnothing,
$$

<span id="page-29-3"></span>
$$
\exists y' \in \mathbf{A}_{\mathrm{pdir}(\mathbf{T}')} \colon y' \neq x_{\mathrm{pdir}(\mathbf{T}')}.
$$

We know that there is  $z \in \text{supp}_{\Omega} B_{\mathbf{A}} \cap \overline{T} \neq \emptyset$ . We deduce from  $(A.31)$  that  $\min v_k(\mathbf{A}) \leq m \leq \max v_k(\mathbf{A})$  and hence

(A.35) 
$$
P_{k,m}(z) = (z_1, ..., z_{k-1}, m, z_{k+1}, ..., z_d) \in \text{supp}_{\Omega} B_{\mathbf{A}} \cap P_{k,m}(\mathbf{T}).
$$
  
30

We choose  $x = P_{k,m}(z)$  in  $(A.32)$  and obtain

<span id="page-30-5"></span>
$$
\begin{aligned} \text{(A.36)} \qquad \qquad & \varnothing \neq \overline{\mathbf{T}'} \cap \text{conv}(P_{k,m}(\mathbf{A}) \cup \{x\}) \\ & \subset \overline{\mathbf{T}'} \cap \text{conv}(P_{k,m}(\mathbf{A}) \cup \text{supp}_{\Omega} B_{\mathbf{A}}) = \overline{\mathbf{T}'} \cap \text{supp}_{\Omega} B_{\mathbf{A}}. \end{aligned}
$$

*Case 1*: odir(T) = j and **A** is old, i.e.  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \cap \mathcal{A}_{\mathbf{p}}^{(n)}$ . For all old anchors and T-junctions from  $T$ , the knot vectors in directions other than  $j$  are unchanged, and the claim is still true. Hence T is a new T-junction with  $T_j = \{r\}$ . Since odir(T') =  $k \neq j$ ,  $T'$  is an old T-junction and we have  $v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(T')$  in the old mesh T for all  $\ell \neq k$ , and consequently

<span id="page-30-0"></span>(A.37) 
$$
v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathbf{T}') \text{ in } \hat{\mathcal{T}} \text{ for all } \ell \notin \{j,k\}.
$$

The combination of  $(A.32)$ – $(A.34)$  and  $(A.37)$  and [Lemma 5.5](#page-19-2) yields  $x_\ell \in \text{conv } v_\ell(T)$ for all  $\ell \notin \{j, k\}$ . By construction, we also have

(A.38) 
$$
x_k \in \{x_k\} = \{m\} = T'_k = v_k(T') = \text{conv } v_k(T').
$$

Moreover, we have

<span id="page-30-1"></span>
$$
(A.39) \t x \in P_{k,m}(T) \subseteq GTJ(T).
$$

and hence

(A.40) 
$$
x_{\ell} \in \operatorname{conv} \mathsf{v}_{\ell}(\mathsf{T}) \quad \text{for all } \ell \in \{1, \ldots, d\}.
$$

If  $v_j(A) \bowtie v_j(T')$  in  $\hat{\mathcal{T}}$ , then from [Lemma 5.5](#page-19-2) there also holds  $x_\ell \in \text{conv } v_\ell(T')$  for  $\ell = i$  and hence

(A.41) 
$$
x \in \bigtimes_{\ell=1}^d \operatorname{conv}_{\mathbf{V}\ell}(\mathbf{T}') = \operatorname{GTJ}(\mathbf{T}'),
$$

and the combination of  $(A.39)$  and  $(A.41)$  yields that the mesh  $\hat{\mathcal{T}}$  is not SGAS.

<span id="page-30-2"></span>If on the other hand  $v_j(A) \not\approx v_j(T')$  in  $\hat{\mathcal{T}}$ , then there is  $s \in \{0, \ldots, N_j\}$  with

$$
(A.42) \t v_j(\mathbf{A}) \ni s \in \operatorname{conv}(v_j(\mathbf{T}')) \setminus v_j(\mathbf{T}') \text{ or } v_j(\mathbf{T}') \ni s \in \operatorname{conv}(v_j(\mathbf{A})) \setminus v_j(\mathbf{A}).
$$

Since T' is an old T-junction with  $v_j(A) \bowtie v_j(T')$  in the old mesh T, and the only entry added to any global knot vector by the subdivision of  $\mathbb{Q}$  is r, we obtain  $s = r$ and hence

<span id="page-30-4"></span>
$$
(A.43) \t\t r \in conv v_j(\mathbf{A}) \cap conv v_j(\mathbf{T}').
$$

Since the mesh is supposed to be SGAS, we have  $GTJ(T) \cap GTJ(T') = \emptyset$  and hence there is  $\ell \in \{1, \ldots, d\}$  with

<span id="page-30-3"></span>
$$
(A.44) \t\t conv v_{\ell}(T) \cap conv v_{\ell}(T') = \varnothing.
$$

Then  $\ell = j$ , since for  $\ell \neq j$  we already found that  $x_{\ell} \in \text{conv}\, v_{\ell}(T') \cap \text{conv}\, v_{\ell}(T) \neq \emptyset$ . By definition of T-junction extensions, we have conv  $v_i(T) = \{r\}$ . Together with  $(A.44)$ , this yields  $r \notin \text{conv } \mathbf{v}_j(\mathbf{T}')$  in contradiction to  $(\mathbf{A}.43)$ .

Case 2:  $odir(T) \neq j$  and  $A \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \cap \mathcal{A}_{\mathbf{p}}^{(n)}$ . Then T is an old T-junction since all new T-junctions are j-orthogonal. Note that  $\hat{\mathcal{T}} \in \text{SGAS eliminates the possibility of}$  k-orthogonal T-junctions,  $k \neq j$ , being subdivided, e.g. subdividing cell Q in [Figure 4](#page-5-0) is prohibited. Since the claim was true in  $\mathcal T$  and only j-orthogonal knot vectors have been affected by the bisection, we have  $k = j$ . Since we have  $v_i(\mathbf{A}) \bowtie v_i(\mathbf{T})$  in T and  $v_j(A) \not\bowtie v_j(T)$  in  $\hat{\mathcal{T}}$ , there is a new T-junction T' that satisfies [\(A.32\).](#page-29-2) For new T-junctions T' that satisfy [\(A.36\),](#page-30-5) we have shown in case 1 that the claim  $v_{\ell}(A) \bowtie$  $v_{\ell}(T')$  holds for all  $\ell \neq j$ . Again, [Lemma 5.5](#page-19-2) yields  $x_{\ell} \in \text{conv } v_{\ell}(T')$  for all  $\ell \neq j$ . Moreover,  $x_j = z_j \in \overline{T_j} \subseteq \text{conv } \text{v}_j(T)$ . We again obtain  $x \in \text{GTJ}(T) \cap \text{GTJ}(T') \neq \emptyset$ , which concludes this case.

Case 3:  $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$ . [Lemma 4.8](#page-12-0) yields an old anchor  $\tilde{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \cap \mathcal{A}_{\mathbf{p}}^{(n)}$ <br>with  $\text{supp}_{\Omega} B_{\mathbf{A}} \subseteq \text{supp}_{\Omega} B_{\tilde{\mathbf{A}}}$  and  $v_{\ell}(\mathbf{A}) = v_{\ell}(\tilde{\mathbf{A}})$  fo  $\sup_{\Omega} B_{\mathbf{A}} \subseteq \overline{\mathbf{T}} \cap \sup_{\Omega} B_{\mathbf{A}} \neq \emptyset$  and the cases 1 and 2 prove the claim.

Case 3.1:  $odir(T) = j$ . Similar to case 1, T' is an old T-junction and we have  $\mathbf{v}_{\ell}(\tilde{\mathbf{A}}) \bowtie \mathbf{v}_{\ell}(\mathbf{T}')$  in the old mesh  $\mathcal{T}$  for all  $\ell \neq k$ , and consequently

<span id="page-31-0"></span>
$$
(A.45) \t v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathbf{T}') \text{ in } \hat{\mathcal{T}} \text{ for all } \ell \notin \{j,k\}.
$$

The combination of  $(A.32)$ – $(A.34)$  and  $(A.45)$  and [Lemma 5.5](#page-19-2) yields  $x_\ell \in \text{conv } v_\ell(T)$ for all  $\ell \notin \{j, k\}$ . The remaining arguments follow as is case 1.

Case 3.2: odir(T)  $\neq j$ . Then T is an old T-junction and  $k = j$  as in case 2. We have  $v_i(\tilde{\mathbf{A}}) \bowtie v_i(\mathbf{T})$  in  $\mathcal{T}$  and  $v_i(\mathbf{A}) \not\bowtie v_i(\mathbf{T})$  in  $\hat{\mathcal{T}}$ , and we have  $v_i(\mathbf{A}) = v_i(\tilde{\mathbf{A}}) \cup \{r\} \setminus \{s\}$ with  $s \in \{\inf v_i(\tilde{\mathbf{A}}), \sup v_i(\tilde{\mathbf{A}})\}\$ . This leads to  $v_i(\mathbf{A}) \ni \{r\} \in \text{conv } v_i(\mathbf{T}) \setminus v_i(\mathbf{T})\$ . Hence there is a new T-junction  $T'$  that satisfies  $(A.32)$ , and the arguments of case 2 follow similarly.  $\Box$