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MULTIVARIATE ANALYSIS-SUITABLE T-SPLINES OF **ARBITRARY DEGREE**

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Abstract. This paper defines analysis-suitable T-splines for arbitrary degree (including even and mixed degrees) and arbitrary dimension. We generalize the concept of anchor elements known from the two-dimensional setting, extend existing concepts of analysis-suitability and show their sufficiency for linearly independent T-Splines.

Key words. multivariate T-splines, Analysis-Suitability, Dual-Compatibility

AMS subject classifications. 65D07, 65D99, 65K99

1. Introduction. T-splines were introduced in 2003 in computer-aided design as a new realization for B-splines on non-uniform meshes [1] with local mesh refinement [2]. Shortly after, Isogeometric Analysis was introduced, and T-splines were applied as ansatz functions for Galerkin schemes with promising results [3, 4], but were proven to lack linear independence in certain cases [5], which actually excludes them from the application in a Galerkin method. The issue was solved in 2012 [6], proving that linear independence is guaranteed if meshline extensions at the hanging nodes, called T-junction extensions, do not intersect. This criterion is referred to as analysis-suitability in the literature, however we denote it as geometric analysissuitability in this paper for distinction against abstract analysis-suitability below. Still in 2012, the introduction of dual-compatibility and its equivalence to analysissuitability [7] provided new insight on the linear independence of T-splines, and in 2013, analysis-suitability was generalized to T-splines of arbitrary polynomial degree [8], still restricted to the two-dimensional case, while dual-compatibility could easily be generalized to higher dimensions [9, Definition 7.2]. At that time, techniques for the construction of 3D T-spline meshes from boundary representations were introduced [10, 11], motivating the theoretical research on T-splines in three space dimensions, but in particular the linear independence of higher-dimensional T-splines was only characterized through the dual-compatibility criterion, until in 2016, an abstract version of analysis-suitability in three dimensions [12] was introduced and, in 2017, generalized to arbitrary dimension [13], but only for odd polynomial degrees. Throughout this paper, we refer to this version as *abstract analysis-suitability* (AAS), and to its equivalent strong version of dual-compatibility as SDC, while we abbreviate the weaker version from [9] with WDC.

This paper generalizes abstract analysis-suitability from [13] to arbitrary degrees and geometric analysis-suitability from [6] to arbitrary dimensions. We investigate the sufficiency for linearly independent spline bases as well as the relations and implications between all above-mentioned versions of analysis-suitability and dualcompatibility (see Figure 1 for a visualization of the results).

The paper is organized as follows. In section 2, we investigate T-junctions in the high-dimensional setting, i.e. hanging (d-2)-dimensional interfaces in d-dimensional box meshes. In section 3, we generalize the concept of anchor elements from [8] to arbitrary dimension, as outlined in [14]. This allows a straight-forward generalization of [13] to arbitrary degrees in section 4. The generalization of T-junction extensions

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FIG. 1. Nesting behavior of the mesh classes considered in this paper.

from [8] is more technical, but yields geometric criteria for linearly independent splines that can easily be visualized and checked. We define a weak and a strong version of geometric analysis-suitability (WGAS and SGAS, respectively). For the strong version, we prove sufficiency for linearly independence of the T-splines, for the weak version we conjecture it, see Conjecture 6.3, providing two incomplete proofs. Section 5 recalls the concept of dual-compatibility, which is already available for arbitrary degree and dimension [9, 13] and does not need further generalization. In section 6, we show that the equivalence of AAS and SDC is valid analogously to the odd-degree case from [13]. We further show that SGAS implies AAS and argument, however with incomplete proof, that WGAS implies WDC. To apply results of dual-compatible splines such as linear independence or projection properties, it is hence sufficient that the considered mesh is analysis-suitable in the geometric or abstract sense. Conclusions and outlook to future work are given in section 7.

2. T-junctions in high-dimensional box meshes. We consider a box-shaped open index domain $\Omega = \bigotimes_{k=1}^{d} (0, N_k)$, with $N_k \in \mathbb{N}$ for $k = 1, \ldots, d$ and an associated parametric domain $\widehat{\Omega} = \bigotimes_{k=1}^{d} (\xi_0^{(k)}, \xi_{N_k}^{(k)})$, with global p_k -open knot vectors $\Xi^{(k)} = \{\xi_0^{(k)}, \dots, \xi_{N_k}^{(k)}\}\$, for polynomial degrees $p_k \in \mathbb{N}$. Let \mathcal{T} be a mesh of Ω , consisting of open axis-parallel boxes with integer vertices, and constructed via symmetric bisections of boxes from an initial tensor-product mesh, which is described in detail in Algorithm 2.1. We assume to obtain integer vertices from Algorithm 2.1, i.e. that for the bisection of a cell Q in direction j we get $m = \frac{1}{2}(\inf Q_j + \sup Q_j) \in \mathbb{N}$. This excludes for example mesh configurations as shown in Figure 2. Further, \mathcal{T} contains all lower-dimensional entities such as hyperfaces, faces, edges and vertices of these boxes. For k = 1, ..., d, we denote by $\mathcal{H}^{(k)}$ the set of open k-dimensional mesh entities of \mathcal{T} , e.g. by $\mathcal{H}^{(0)}$ the set of nodes, by $\mathcal{H}^{(1)}$ the set of one-dimensional edges without start and end point, by $\mathcal{H}^{(2)}$ the set of two-dimensional faces without the boundary edges, and so on, such that the union $\overline{\Omega} = \bigcup \mathcal{T}$, with $\mathcal{T} = \bigcup_{j=0}^{d} \mathcal{H}^{(j)}$, is disjoint. The union of all element boundaries $Sk = \bigcup_{q \in \mathcal{H}^{(d)}} \partial q = \bigcup_{j=0}^{d-1} \mathcal{H}^{(j)} = \overline{\Omega} \setminus \mathcal{H}^{(d)}$ is called the *skeleton*



FIG. 2. Two examples of excluded mesh configurations. 2



FIG. 3. A 3-dimensional mesh, refined in the front corner (top left), and the corresponding 1-orthogonal, 2-orthogonal and 3-orthogonal skeleton (top right, bottom left, bottom right, respectively).

of \mathfrak{T} . Note that this includes not only the 1-dimensional edges, but also the faces and hyperfaces up to dimension d-1. For an index set $\kappa = \{\kappa_1, \ldots, \kappa_\iota\} \subset \{1, \ldots, d\}$ and a *d*-dimensional (volumetric) element $\mathbb{Q} = \mathbb{Q}_1 \times \cdots \times \mathbb{Q}_d \in \mathcal{H}^{(d)}$ composed from open intervals $\mathbb{Q}_1, \ldots, \mathbb{Q}_d$, we denote the $(d - \iota)$ -dimensional, κ -orthogonal interfaces by $\mathbb{H}^{(\kappa)}(\mathbb{Q})$, i.e.

(2.1)
$$\mathbf{H}^{(\kappa)}(\mathbf{Q}) \coloneqq \{ \widetilde{\mathbf{Q}} = \widetilde{\mathbf{Q}}_1 \times \cdots \times \widetilde{\mathbf{Q}}_d \mid \widetilde{\mathbf{Q}}_j \subsetneq \partial \mathbf{Q}_j \text{ for } j \in \kappa, \, \widetilde{\mathbf{Q}}_j = \mathbf{Q}_j \text{ for } j \notin \kappa \},$$

where the components \widetilde{Q}_j are either singleton sets or open intervals with start and end points in $\{0, \ldots, N_j\}$.

The global set of κ -orthogonal mesh entities is denoted by $\mathbb{H}^{(\kappa)} = \bigcup_{\mathbb{Q} \in \mathcal{H}^{(d)}} \mathbb{H}^{(\kappa)}(\mathbb{Q})$, with $\mathbb{H}^{(\emptyset)}(\mathbb{Q}) = \{\mathbb{Q}\}$ and $\mathbb{H}^{(\emptyset)} = \mathcal{H}^{(d)}$. For singleton index sets, we write $\mathbb{H}^{(j)} := \mathbb{H}^{(\{j\})}$, and we call $\mathrm{Sk}_j := \bigcup_{E \in \mathbb{H}^{(j)}} \overline{E}$ the *j*-orthogonal skeleton of \mathcal{T} . Note that it is composed of (d-1)-dimensional hyperfaces, see Figure 3 for an example.

For polynomial degrees $\mathbf{p} = (p_1, \ldots, p_d) \in \mathbb{N}^d$, we split the index domain Ω into an *active region* AR_{**p**} and a *frame region* FR_{**p**}, with

(2.2)
$$\operatorname{AR}_{\mathbf{p}} \coloneqq \bigotimes_{k=1}^{d} \left[\left\lfloor \frac{p_{k}+1}{2} \right\rfloor, N_{k} - \left\lfloor \frac{p_{k}+1}{2} \right\rfloor \right] \text{ and } \operatorname{FR}_{\mathbf{p}} \coloneqq \overline{\Omega \setminus \operatorname{AR}_{\mathbf{p}}}.$$

Consider two cells $\mathbb{Q}^{(1)}$, $\mathbb{Q}^{(2)} \in \mathcal{H}^{(d)}$ that share a common face $\mathbb{P} \in \mathcal{H}^{(d-1)}$, $\partial \mathbb{Q}^{(1)} \cap \partial \mathbb{Q}^{(2)} = \overline{\mathbb{P}}$. The *j*-orthogonal subdivision of $\mathbb{Q}^{(1)}$, i.e. the bisection of $\mathbb{Q}^{(1)}_j$, for some direction *j* which is not orthogonal to \mathbb{P} , removes all mesh entities $\mathbb{E} = \mathbb{E}_1 \times \cdots \times \mathbb{E}_d$ with $\mathbb{E}_j = \mathbb{Q}^{(1)}_j$ and inserts child entities $\mathbb{E}^{(1)}, \mathbb{E}^{(2)}, \mathbb{E}^{(3)}$ including the children $\mathbb{Q}^{(1,1)}$ and $\mathbb{Q}^{(1,2)}$ of $\mathbb{Q}^{(1)}$, with mid $\mathbb{Q}^{(1)}_j = \frac{1}{2}(\inf \mathbb{Q}^{(1)}_j + \sup \mathbb{Q}^{(1)}_j)$. This procedure is summarized

Algorithm 2.1 Subdivision of a cell.

procedure SUBDIV($\mathcal{T}, \mathbf{Q}, j$) assert that $Q \subset AR_{\mathbf{p}}$ $D \leftarrow \overline{\mathsf{Q}}$ for all $\ell = 1, \ldots, d, \ \ell \neq j$ do if $\min D_{\ell} = \lfloor \frac{p_k + 1}{2} \rfloor$ then \triangleright If D touches the frame region, then $D_{\ell} \leftarrow D_{\ell} \cup \left[0, \left[\frac{p_k + 1}{2} \right] \right]$ \triangleright extend it to the end of the domain. end if \triangleright See Remark 3.2 for an explanation. if $\max D_{\ell} = N_{\ell} - |\frac{p_k + 1}{2}|$ then $D_{\ell} \leftarrow D_{\ell} \cup \left[N_{\ell} - \bar{\left[\frac{p_k+1}{2}\right]}, N_{\ell}\right]$ end if end for for all $E \in \mathcal{T}$, $E \subset D$, $E_j = Q_j$ do $m \leftarrow \frac{1}{2}(\inf \mathbf{Q}_j + \sup \mathbf{Q}_j)$ $\mathbf{E}^{(1)} \leftarrow \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times (\inf \mathbf{Q}_j, m) \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d$ $\mathbf{E}^{(2)} \leftarrow \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times \{m\} \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d$
$$\begin{split} \mathbf{E}^{(3)} &\leftarrow \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times (m, \sup \mathbf{Q}_j) \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d \\ \mathcal{T} &\leftarrow \mathcal{T} \setminus \{\mathbf{E}\} \cup \{\mathbf{E}^{(1)}, \mathbf{E}^{(2)}, \mathbf{E}^{(3)}\} \qquad \triangleright \text{ Since } \end{split}$$
 \triangleright Since D is a superset of Q, end for \triangleright at least Q is subdivided. return Tend procedure

in Algorithm 2.1, where additional subdivisions are done, whenever the cell to be subdivided touches the frame region, see Remark 3.2 for an explanation. Since the children inherit all but the *j*-th component of $Q^{(1)}$, they satisfy $\partial Q^{(1,1)} \cap \partial Q^{(2)} \neq \emptyset$ and $\partial Q^{(1,2)} \cap \partial Q^{(2)} \neq \emptyset$. Furthermore, we see that $Q^{(1,1)}$ and $Q^{(1,2)}$ share a face $\mathbf{F} = \mathbf{Q}_1^{(1)} \times \cdots \times \mathbf{Q}_{j-1}^{(1)} \times \{ \min \mathbf{Q}_j^{(1)} \} \times \mathbf{Q}_{j+1}^{(1)} \times \cdots \times \mathbf{Q}_d^{(1)} \in \mathcal{H}^{(d-1)}$. By subdividing $Q^{(1)}$ we have thus generated an interface $\overline{\mathbf{I}} = \overline{\mathbf{F}} \cap \partial \mathbf{Q}^{(2)}$, $\mathbf{I} \subset \mathcal{H}^{(d-2)}$, that is in the boundary of exactly three cells $\mathbf{Q}^{(2)}, \mathbf{Q}^{(1,1)}$, and $\mathbf{Q}^{(1,2)}$. We classify this type of entities in the following definition.

DEFINITION 2.1 (T-junctions). We call an interface $T \in \mathcal{H}^{(d-2)}$ with $T \nsubseteq \partial\Omega$ a hanging interface or T-junction if it has valence $|\{F \in \mathcal{H}^{(d-1)} \mid T \subset \partial F\}| < 4$, or equivalently, if it is in the boundary of a cell $Q = Q_1 \times \cdots \times Q_d \in \mathcal{T}$ without being connected to any of its vertices, $T \subset \partial Q$, $\overline{T} \cap \partial Q_1 \times \cdots \times \partial Q_d = \emptyset$. We then call Q the associated cell of T and write $Q = \operatorname{ascell}(T)$. Since $T = T_1 \times \cdots \times T_d \in \mathcal{H}^{(d-2)}$, there are two unique and distinct directions $i, j \in \{1, \ldots, d\}$ such that T_i, T_j are singletons, $T \in H^{\{i,j\}}, T_i \subsetneq Q_i$ and $T_j \subsetneq \partial Q_j$. We call i the orthogonal direction and j the pointing direction of T, and write $\operatorname{odir}(T) = i$, $\operatorname{pdir}(T) = j$.

PROPOSITION 2.2. For any T-junction T, the above-defined $\operatorname{ascell}(T)$, $\operatorname{odir}(T)$ and $\operatorname{pdir}(T)$ are unique.

Proof. Consider any (d-2)-dimensional mesh entity $T \in \mathcal{H}^{(d-2)}$ that is not contained in the boundary of Ω . Then T is of the form $T = T_1 \times \cdots \times T_d$ and there exist exactly two indices $i, j \in \{1, \ldots, d\}$ such that T_i and T_j are singletons and all other components $T_k, i \neq k \neq j$, are open intervals. Since T is a mesh entity of a *d*-dimensional box mesh constructed via refinement of a tensor-product mesh as assumed above, there is by construction a (possibly non-unique) cell $Q \in \mathcal{H}^{(d)}$ with $Q = Q_1 \times \cdots \times Q_d$ and

(2.3) $\mathbf{T}_i \subset \partial \mathbf{Q}_i, \quad \mathbf{T}_j \subset \partial \mathbf{Q}_j, \text{ and } \mathbf{T}_k \subseteq \mathbf{Q}_k \text{ for } i \neq k \neq j.$

Q is bounded by $2 \cdot d$ (or more, in case of T-junctions in its boundary) hyperfaces, and for each $k \in \{1, \ldots, d\}$ and $n_k \in \partial Q_k = \{\inf Q_k, \sup Q_k\}$, there is a hyperface $F \in \mathcal{H}^{(d-1)}$ with $F = F_1 \times \cdots \times F_d$, $F_k = \{n_k\}$ and $F_\ell \subseteq Q_\ell$ for $\ell \neq k$. In particular, there are two such hyperfaces $F^{(i)}, F^{(j)}$ with $F_i^{(i)} = T_i$ and $F_j^{(j)} = T_j$. $F^{(i)}$ neighbors T in positive (resp. negative) *j*-th direction if $T_j = \inf F_j^{(i)}$ (resp. $T_j = \sup F_j^{(i)}$), and $F^{(j)}$ neighbors T in positive (resp. negative) *i*-th direction if $T_i = \inf F_i^{(j)}$ (resp. $T_i = \sup F_i^{(j)}$). Together, T is neighbored by at least two (d-1)-dimensional interfaces in different directions. We assume without loss of generality that T has neighbor interfaces in positive *i*-th and *j*-th direction, i.e. that $T_j = \inf F_j^{(i)}$ and $T_i = \inf F_i^{(j)}$. Let $h_{\mathcal{T}} = \min\{\sup Q_k - \inf Q_k \mid Q = Q_1 \times \cdots \times Q_d \in \mathcal{H}^{(d)}, \ k \in \{1, \ldots, d\}\}$ be the

Let $h_{\mathfrak{T}} = \min\{\sup \mathbb{Q}_k - \inf \mathbb{Q}_k \mid \mathbb{Q} = \mathbb{Q}_1 \times \cdots \times \mathbb{Q}_d \in \mathcal{H}^{(d)}, k \in \{1, \ldots, d\}\}$ be the minimal mesh size. If there is no neighbor interface in negative *i*-th direction, then for any point $x \in \mathbb{T}$ and $0 < \varepsilon < h_{\mathfrak{T}}$, the point $x - \varepsilon e_i$ (with e_i being the *i*-th unit vector) is in the interior of some cell $\tilde{\mathbb{Q}} \in \mathcal{H}^{(d)}$, as well as the points $x - \varepsilon e_i + \varepsilon e_j$ and $x - \varepsilon e_i - \varepsilon e_j$, since there is no *j*-orthogonal hyperface separating them.

If similarly T has no neighbor interface in negative *j*-th direction, then the points $x - \varepsilon e_j + \varepsilon e_i$ and $x - \varepsilon e_j - \varepsilon e_i$ are in the interior of \tilde{Q} .

If T does not have neighbor interfaces neither in negative *i*-th nor in negative *j*-th direction, then the three points $x^{(1)} = x - \varepsilon e_i - \varepsilon e_j$, $x^{(2)} = x - \varepsilon e_i + \varepsilon e_j$, $x^{(3)} = x - \varepsilon e_j + \varepsilon e_i$ are in \tilde{Q} , but the midpoint $\frac{1}{2}(x^{(2)} + x^{(3)}) = x \notin \tilde{Q}$ since $x \in T \subset \partial \tilde{Q}$ and \tilde{Q} is open. This means that \tilde{Q} is not convex in contradiction to the assumption that $\mathcal{H}^{(d)}$ consists of open axis-aligned (and hence convex) boxes.

Together, any $T \in H^{(\{i,j\})}$ is neighbored by at least three and at most four (d-1)dimensional faces. Thus, all T-junctions have valence 3. Let j be the unique direction in which there is no neighbor interface, and let $s \in \{-1, 1\}$ indicate whether there is no neighbor face in negative (s=-1) or positive (s=1) j-th direction. Then odir(T) = i, pdir(T) = j, and ascell(T) is the unique neighbor cell containing the point $x + s\varepsilon e_j$ for any $x \in T$.

We give brief examples for odir(T) and pdir(T) for a hanging interface T in 2D and 3D, see also Figure 4 for related sketches. For 2D, let $T = \{n\} \times \{m\}$ be a hanging node, and assume that it is of type \perp or \top . Then there is an associated cell ascell(T) = $Q = Q_1 \times Q_2$ such that the integer *n* is in the interior of Q_1 and *m* is the upper or lower bound of Q_2 , i.e.

 $n \in \mathbb{Q}_1$ and $m \in \{\inf \mathbb{Q}_2, \sup \mathbb{Q}_2\}$, or equivalently $\{n\} \subsetneq \mathbb{Q}_1$ and $m \subsetneq \partial \mathbb{Q}_2$.

We hence have odir(T) = 1 and pdir(T) = 2. Similarly, for T-junctions of type \vdash or \dashv we have odir(T) = 2 and pdir(T) = 1.

As a 3D example, consider a hanging edge of the type $T = \{n\} \times (\underline{m}, \overline{m}) \times \{\ell\}$ with an associated cell ascell(T) = $Q = Q_1 \times Q_2 \times Q_3$ such that

 $\ell \in \mathsf{Q}_3 \quad \text{and} \quad n \in \{\inf \mathsf{Q}_1, \sup \mathsf{Q}_1\}, \quad \text{or equivalently} \quad \{\ell\} \subsetneq \mathsf{Q}_3 \quad \text{and} \quad \{n\} \subsetneq \partial \mathsf{Q}_1,$

which yields odir(T) = 3 and pdir(T) = 1.

The above-defined properties of T-junctions are essential for the analysis-suitability described in section 4. Each T-junction is extended in its pointing direction and, for d > 2, by a larger amount in all other directions except the orthogonal direction,



FIG. 4. Examples for T-junctions and associated cells in 2D (left) and 3D (right).

and T-junction extensions with different pointing/orthogonal direction are required to be disjoint. Details are given in section 4 below. We end this section with a Lemma used for the proofs in section 6, using the notation conv Z for the convex hull of a set Z.

LEMMA 2.3. If two points x, y are aligned in *i*-direction, and $x \in Sk_i \not\ni y$, then there is an *i*-orthogonal *T*-junction and its associated cell between these points, *i.e.*

 $\begin{array}{ll} (2.4) \quad \forall x, y \in \overline{\Omega}, i \in \{1, \dots, d\}, x_i = y_i, x \in \operatorname{Sk}_i \not\ni y \quad \exists \, \mathsf{T}, \operatorname{odir}(\mathsf{T}) = i, \mathsf{Q} = \operatorname{ascell}(\mathsf{T}) :\\ \overline{\mathsf{T}} \cap \operatorname{conv}\{x, y\} \neq \varnothing, \quad x_{\operatorname{pdir}(\mathsf{T})} \neq y_{\operatorname{pdir}(\mathsf{T})}, \quad \mathsf{Q}_{\operatorname{pdir}(\mathsf{T})} \cap \operatorname{conv}\{x_{\operatorname{pdir}(\mathsf{T})}, y_{\operatorname{pdir}(\mathsf{T})}\} \neq \varnothing. \end{array}$

Note that this implies $T_i = \{x_i\} = \{y_i\}.$

Proof. Define the function $f: [0,1] \to \{0,1\}$ with

(2.5)
$$f(t) = \begin{cases} 1 & \text{if } (1-t)x + ty \in \mathrm{Sk}_i \\ 0 & \text{otherwise.} \end{cases}$$

Since Sk_i is a finite union of closed sets, Sk_i is closed as well. Consequently, the value of f at jump locations is always 1. Since f(0) = 1 and f(1) = 0, there is at least one jump location $t^* \in (0, 1)$ with $f(t^*) = 1$ and $f(t^* + \varepsilon) = 0$ for arbitrarily small $\varepsilon > 0$. This means that $x^{(t^*)} = (1 - t^*)x + t^*y \in \overline{F}$ for some *i*-orthogonal face $F \in H^{(i)}$, while $x^{(t^* + \varepsilon)}$ is not in $\overline{F'}$ for any $F' \in H^{(i)}$.

Moreover, since $x^{(t^*+\varepsilon)} \in \operatorname{conv}\{x, y\} \subset \overline{\Omega}$, we have $x^{(t^*+\varepsilon)} \in \overline{\mathbb{Q}}$ for some cell \mathbb{Q} such that any vertex v of \mathbb{Q} satisfies $v_i \neq x_i$, since otherwise \mathbb{Q} has an *i*-orthogonal hyperface in Sk_i and $x^{(t^*+\varepsilon)}$ lies in Sk_i in contradiction to $f(t^*+\varepsilon) = 0$. Since $\overline{\mathbb{Q}}$ is closed and $x^{(t^*+\varepsilon)} \in \overline{\mathbb{Q}}$ holds for arbitrarily small ε , we also have $x^{(t^*)} \in \overline{\mathbb{Q}}$. However, $f(t^*) = 1$ tells us that also $x^{(t^*)} \in \overline{\mathbb{Q}'}$ holds for a different cell \mathbb{Q}' that has the *i*-orthogonal hyperface F in its boundary. Hence $x^{(t^*)} \in \partial \mathbb{Q}$. The fact that $x^{(t^*)} \in \overline{\mathbb{F}}$ but $x^{(t^*+\varepsilon)} \notin \overline{\mathbb{F}}$ means that $x^{(t^*)} \in \partial \mathbb{F}$ and hence that $x^{(t^*)} \in \overline{\mathbb{T}} \subset \partial \mathbb{F}$ for some *i*-orthogonal entity $\mathbb{T} \in \mathcal{H}^{(d-2)}$.

If $x^{(t^*)} \in \mathbb{T} \in \mathbb{H}^{(i,j)} \subset \mathcal{H}^{(d-2)}$, then T, F, Q are unique, $\mathbb{T} \subset \partial \mathbb{Q}$ and T is a T-junction with \mathbb{Q} = ascell(T) since it is a (d-2)-dimensional entity in the boundary of Q without being connected to any of its vertices. From $x^{(t^*+\varepsilon)} \notin \mathbb{T}$ we conclude that $x^{(t^*)}$ and $x^{(t^*+\varepsilon)}$ differ in the *i*-th or *j*-th component. From $x_i = y_i$ we get $x_i^{(t^*)} = x_i^{(t^*+\varepsilon)}$ and hence $x_j^{(t^*)} \neq x_j^{(t^*+\varepsilon)}$ with pdir(T) = *j*, which yields $x_j \neq y_j$. Moreover, $x^{(t^*+\varepsilon)} \in \mathbb{Q}$ yields $x_j^{(t^*+\varepsilon)} \in \mathbb{Q}_j$, and $x^{(t^*+\varepsilon)} \in \operatorname{conv}\{x, y\}$ yields $x_j^{(t^*+\varepsilon)} \in \operatorname{conv}\{x_j, y_j\}$ where $\operatorname{conv}\{x_j, y_j\}$ is $[x_j, y_j]$ or $[y_j, x_j]$. We thus have $\mathbb{Q}_j \cap \operatorname{conv}\{x_j, y_j\} \neq \emptyset$. If otherwise $x^{(t^*)} \in \partial \mathbb{T}$, then we consider a perturbation $u \in \mathbb{R}^d$ such that the

If otherwise $x^{(t^*)} \in \partial \mathbf{T}$, then we consider a perturbation $u \in \mathbb{R}^d$ such that the same construction with $\tilde{x} = x + \tilde{\varepsilon}u$ and $\tilde{y} = y + \tilde{\varepsilon}u$, for any sufficiently small $\tilde{\varepsilon} > 0$, yields $\tilde{x}^{(t^*)} \in \mathbf{T}$ for some $\mathbf{T} \in \mathcal{H}^{(d-2)}$ with $x^{(t^*)} \in \partial \mathbf{T}$. The claim follows for any $\tilde{\varepsilon} > 0$ and remains true for $\tilde{\varepsilon} \to 0$.

3. Multivariate T-splines. This section explains the construction of multivariate T-splines, following the construction in [9].

DEFINITION 3.1 (admissible meshes). We define for k = 1, ..., d and $n = 0, ..., N_k$ the slice

(3.1)
$$S_k(n) \coloneqq \sum_{j=1}^{k-1} [0, N_j] \times \{n\} \times \sum_{j=k+1}^d [0, N_j] = \{(x_1, \dots, x_d) \in \overline{\Omega} \mid x_k = n\},$$

and the k-th frame region

(3.2)
$$\operatorname{FR}_{\mathbf{p}}^{(k)} \coloneqq \left\{ x \in \overline{\Omega} \mid x_k \in \left[0, \lfloor \frac{p_k + 1}{2} \rfloor\right] \cup \left[N_k - \lfloor \frac{p_k + 1}{2} \rfloor, N_k\right] \right\}.$$

A T-mesh T is called admissible if for k = 1, ..., d, there is no T-junction T with odir(T) = k or pdir(T) = k in the k-th frame region, and

(3.3)
$$S_k(n) \subseteq Sk_k \quad \text{for } n = 0, \dots, \lfloor \frac{p_k + 1}{2} \rfloor \text{ and } n = N_k - \lfloor \frac{p_k + 1}{2} \rfloor, \dots, N_k.$$

Remark 3.2. Algorithm 2.1 preserves admissibility in the above sense. When subdividing a cell that touches the k-th frame region, T-junctions with pointing direction k are avoided by extending the refinement to the domain boundary. Further, since only cells in the active region can be subdivided, no k-orthogonal T-junction can be created in the k-th frame region.

For the definition of anchors and knot vectors, we follow the ideas of [9]. Anchors are defined as a certain type of mesh entities, e.g. edges or faces in a certain direction, and the knot vectors and sets are constructed by ray tracing these entities along the mesh. Using the above introduced sets $\mathbb{H}^{(\kappa)}$, the anchors can be generalized to arbitrary dimensions.

DEFINITION 3.3 (anchors). Let $\mathbf{p} = (p_1, \ldots, p_d)$ be the vector of polynomial degrees of the T-splines. The set of anchors is then defined by

$$(3.4) \qquad \mathcal{A}_{\mathbf{p}} := \{ \mathbf{A} \in \mathbb{H}^{(\kappa)} \mid \mathbf{A} \subset AR_{\mathbf{p}} \} \quad with \ \kappa = \{ \ell \in \{1, \dots, d\} \mid p_{\ell} \ odd \}.$$

Similar to the literature [6, 8, 9], we assign to each anchor a knot vector in each axis direction. This is achieved by fixing the anchor's *j*-th component to an index n and checking for which indices n the result is part of the skeleton.

DEFINITION 3.4 (global and local knot vectors). For any mesh entity $\mathbf{E} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_d$ and $j \in \{1, \ldots, d\}$, we define the projection $P_{j,n}(\mathbf{E}) = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times \{n\} \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d$ of \mathbf{E} on the slice $S_j(n)$, and the global knot vector

(3.5)
$$\mathfrak{I}_{j}(\mathsf{E}) \coloneqq \left(n \in \mathbb{N} \mid P_{j,n}(\mathsf{E}) \subset \mathrm{Sk}_{j}\right)$$

with entries in non-decreasing order. The local knot vector $\mathbf{v}_j(\mathbf{A})$ for an anchor $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_d$ is given by the $p_j + 2$ consecutive indices $\ell_0, \ldots, \ell_{p_j+1} \in \mathfrak{I}_j(\mathbf{A})$, such that $\ell_k = \inf \mathbf{A}_j$ for $k = \lfloor \frac{p_j+1}{2} \rfloor$. This is, if p_j is odd, the singleton \mathbf{A}_j contains the middle entry of $\mathbf{v}_j(\mathbf{A})$, and if p_j is even, the two middle entries of $\mathbf{v}_j(\mathbf{A})$ are the boundary values of \mathbf{A}_j .

Note that we treat global and local knot vectors as ordered sets in the sense that $n \in v_j(\mathbf{A})$ means that $v_j(\mathbf{A})$ has a component equal to n. As a consequence of Definition 3.1, any global knot vector $\mathcal{I}_j(\mathbf{E})$ in an admissible mesh contains the values $n = 0, \ldots, \lfloor \frac{p_j+1}{2} \rfloor$ and $n = N_j - \lfloor \frac{p_j+1}{2} \rfloor, \ldots, N_j$.



(a) $\mathbf{p} \mod 2 \equiv (1,1,1)$ (b) $\mathbf{p} \mod 2 \equiv (0,1,1)$ (c) $\mathbf{p} \mod 2 \equiv (0,1,0)$ (d) $\mathbf{p} \mod 2 \equiv (0,0,1)$

FIG. 5. Different anchor types on a cell in \mathbb{R}^3 for various degrees **p**. Since the specific degree of **p** is not of interest for the anchor elements, we just consider different parities of **p**.

An example of different anchor elements for 3D is given in Figure 5. Each example illustrates the anchor entities of a cell in the active region of the mesh. Note that $\mathbf{H}^{(\kappa)}$ determines the anchor type, where each direction in κ is fixed to singletons. In Figure 5a, the polynomial degree is odd in every direction, hence, we get $\kappa = \{1, 2, 3\}$ and $\mathbf{H}^{(\kappa)}$ corresponds to the vertices of the mesh inside the active region. In Figure 5b the polynomial degrees in the second and third coordinate are odd. It follows, $\kappa = \{2, 3\}$, from which we infer $\mathbf{H}^{(\kappa)}$ as the entities with singletons in its second and third direction, i.e. lines along the x-axis. In Figure 5c, resp. 5d, we have $\kappa = \{3\}$, resp. $\kappa = \{2\}$, hence the set $\mathbf{H}^{(\kappa)}$ are faces with singletons in direction 3, resp. 2.

Figure 6 shows two examples for the construction of local knot vectors in 3D. In each example, we show for two anchors the construction of one local knot vector. The anchors are faces, and the local knot vector in direction 1 is constructed for the anchors highlighted in light blue. By tracing the anchor along the first direction, we highlight the projections that lie in the skeleton.

Figure 6a considers the case $\mathbf{p} \mod 2 = (1, 0, 0)$, i.e. anchors consist of singletons in their first coordinate, $\mathbf{A} = \{\bar{m}\} \times (n_1, n_2) \times (l_1, l_2)$. We collect the global knot vector of each anchor by tracing it along direction 1 and including the indices mfor which $P_{1,m}(\mathbf{A})$ is in the skeleton of the mesh, i.e. we check for each m if $\{m\} \times (n_1, n_2) \times (l_1, l_2) \subset \text{Sk}_1$ and include m in $\mathcal{I}_1(\mathbf{A})$ if this is the case. We then pick the consecutive $p_1 + 2$ indices from $\mathcal{I}_1(\mathbf{A})$ as the local knot vector $\mathbf{v}_1(\mathbf{A})$. For the anchor $\mathbf{A}^{(1)}$ at the top of Figure 6a, we get $\mathbf{v}_1(\mathbf{A}^{(1)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 1, \bar{m} + 2)$, and for the anchor $\mathbf{A}^{(2)}$ at the bottom, we get $\mathbf{v}_1(\mathbf{A}^{(2)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 2, \bar{m} + 3)$.

In Figure 6b we consider anchors with singletons in their second coordinate, i.e. $\mathbf{A} = (m_1, m_2) \times \{\bar{n}\} \times (\ell_1, \ell_2)$. Fixing the first coordinate to some index m, we test $\{m\} \times \{\bar{n}\} \times (\ell_1, \ell_2) \subset \text{Sk}_1$. For the anchor at the top, we then get $v_1(\mathbf{A}^{(1)}) = (m_1 - 2, m_1 - 1, m_1, m_2, m_2 + 1, m_2 + 2)$ and for the anchor at the bottom $v_1(\mathbf{A}^{(2)}) = (m_1 - 2, m_1 - 1, m_1, m_2 + 1, m_2 + 2, m_2 + 3)$.

DEFINITION 3.5 (T-spline). For $p_j \in \mathbb{N}$, we denote by $B_{\mathbf{v}_j(\mathbf{A})}: \widehat{\Omega} \to \mathbb{R}$ the univariate B-spline function of degree p_j that is returned by the Cox-deBoor recursion with knot vector $\xi_{\mathbf{v}_j(\mathbf{A})} = (\xi_{\ell_0}^{(j)}, \ldots, \xi_{\ell_{p_j+1}}^{(j)})$. We assume that $\xi_{\ell_0}^{(j)} < \xi_{\ell_{p_j+1}}^{(j)}$ is always fulfilled. The T-spline function associated with the anchor \mathbf{A} is defined as

(3.6)
$$B_{\mathbf{A}}(\zeta_1,\ldots,\zeta_d) \coloneqq \prod_{j=1}^d B_{\mathbf{v}_j(\mathbf{A})}(\zeta_j), \quad for \ (\zeta_1,\ldots,\zeta_d) \in \widehat{\Omega},$$

and the corresponding T-spline space is given by $\mathfrak{S}_{\mathfrak{T},\mathbf{p}}(\widehat{\Omega}) = \operatorname{span}\{B_{\mathbf{A}} \mid \mathbf{A} \in \mathcal{A}_{\mathbf{p}}\}.$



(a) Example for $\mathbf{p} = (3, 2, 2)$. The illustrated local knot vectors are $v_1(\mathbf{A}^{(1)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 1, \bar{m} + 2)$ and $v_1(\mathbf{A}^{(2)}) = (\bar{m} - 2, \bar{m} - 1, \bar{m}, \bar{m} + 2, \bar{m} + 3)$.



 $\mathbf{v}_1(\mathbf{A}^{(1)}) = (m_1 - 2, m_1 - 1, m_1, m_2, m_2 + 1, m_2 + 2)$ and $\mathbf{v}_1(\mathbf{A}^{(2)}) = (m_1 - 2, m_1 - 1, m_1, m_2 + 1, m_2 + 2, m_2 + 3).$

FIG. 6. Construction of $v_1(\mathbf{A})$ for the given anchors marked in light blue for various degrees \mathbf{p} .

The index support of $B_{\mathbf{A}}$ will be denoted by $\operatorname{supp}_{\Omega} B_{\mathbf{A}} = \bigotimes_{k=1}^{d} \operatorname{conv} v_{k}(\mathbf{A})$, where $\operatorname{conv} v_{k}(\mathbf{A}) = \operatorname{conv}(\ell_{0}, \ldots, \ell_{p_{k}+1}) = [\ell_{0}, \ell_{p_{k}+1}]$ is the closed interval from the first to the last entry of $v_{k}(\mathbf{A})$.

4. Analysis-Suitability. We introduce below two versions of analysis-suitability. As shown in section 6, both are sufficient criteria for the linear independence of the T-splines associated with the considered mesh, and we conjecture that the geometric version can be weakened, see Conjecture 6.3.

DEFINITION 4.1 (Abstract T-junction extensions and analysis-suitability). We define for all j = 1, ..., d and $n = 0, ..., N_j$ the abstract T-junction extension

(4.1)
$$\operatorname{ATJ}_{j}(n) = \operatorname{S}_{j}(n) \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \in \mathcal{I}_{j}(\mathbf{A})}} \operatorname{supp}_{\Omega} B_{\mathbf{A}} \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \notin \mathcal{I}_{j}(\mathbf{A})}} \operatorname{supp}_{\Omega} B_{\mathbf{A}}$$

We call the mesh \mathfrak{T} abstractly analysis-suitable (AAS) if the abstract T-junction extensions do not intersect in different directions, i.e. if $\operatorname{ATJ}_i(n) \cap \operatorname{ATJ}_j(m) = \emptyset$ for any $i \neq j$ and $n \in \{0, \ldots, N_i\}$, $m \in \{0, \ldots, N_j\}$, and we write $\mathfrak{T} \in AAS$. We will use the notation $\operatorname{ATJ}_i \equiv \operatorname{ATJ}_i(\mathcal{T})$ to refer to the set of all *i*-orthogonal abstract T-junction extensions within the mesh \mathcal{T} , i.e.

(4.2)
$$\operatorname{ATJ}_{i} = \bigcup_{n=0}^{N_{i}} \operatorname{ATJ}_{i}(n),$$

in which case a mesh is AAS if $\operatorname{ATJ}_i \cap \operatorname{ATJ}_j = \emptyset$ for $i \neq j$. Note also that if $n \notin \operatorname{conv} v_j(\mathbf{A})$, then $S_j(n) \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}} = \emptyset$ and \mathbf{A} does not contribute to the righthand side in (4.1). Using the notation $P_{i,n}(\mathbf{E}) = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{i-1} \times \{n\} \times \mathbf{E}_{i+1} \times \cdots \times \mathbf{E}_d$ as in Definition 3.4, the above-defined abstract T-junction extensions are also neighborhoods of T-junctions in the following sense.

PROPOSITION 4.2. For any point x in a non-empty abstract T-junction extension $\operatorname{ATJ}_i(n)$, there is an anchor $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$ with $x \in \operatorname{supp}_{\Omega} B_{\mathbf{A}}$. Further, there is an i-orthogonal T-junction T and its associated cell $\mathbf{Q} = \operatorname{ascell}(\mathbf{T})$ between x and $P_{i,n}(\mathbf{A})$, *i.e.*

1. the T-junction T intersects the convex hull of $P_{i,n}(\mathbf{A})$ and $\{x\}$, i.e.

(4.3)
$$\overline{\mathsf{T}} \cap \operatorname{conv}(P_{i,n}(\mathbf{A}) \cup \{x\}) \neq \emptyset$$

in pointing direction of T, the associated cell intersects the convex hull of A_{pdir(T)} and {x_{pdir(T)}}, i.e.

(4.4)
$$\mathbf{Q}_{\mathrm{pdir}(\mathbf{T})} \cap \mathrm{conv}(\mathbf{A}_{\mathrm{pdir}(\mathbf{T})} \cup \{x_{\mathrm{pdir}(\mathbf{T})}\}) \neq \emptyset,$$

3. there exists a number $y \in \mathbf{A}_{pdir(T)}$ with $y \neq x_{pdir(T)}$.

Proof. Consider arbitrary $i \in \{1, \ldots, d\}, n \in \{0, \ldots, N_i\}$ with $ATJ_i(n) \neq \emptyset$ and arbitrary

(4.5)
$$x \in \operatorname{ATJ}_{i}(n) = \operatorname{S}_{i}(n) \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \in \mathcal{I}_{i}(\mathbf{A})}} \operatorname{supp}_{\Omega} B_{\mathbf{A}} \cap \bigcup_{\substack{\mathbf{A} \in \mathcal{A}_{\mathbf{p}} \\ n \notin \mathcal{I}_{i}(\mathbf{A})}} \operatorname{supp}_{\Omega} B_{\mathbf{A}}.$$

There are by construction anchors $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}$ with $n \in \mathcal{J}_i(\mathbf{A}^{(1)})$ and $n \notin \mathcal{J}_i(\mathbf{A}^{(2)})$. The Definition 3.4 of global knot vectors yields equivalently $P_{i,n}(\mathbf{A}^{(1)}) \subset \mathrm{Sk}_i$ and $P_{i,n}(\mathbf{A}^{(2)}) \not\subset \mathrm{Sk}_i$.

If $x \in \operatorname{Sk}_i$, then set $\mathbf{A} := \mathbf{A}^{(2)}$, otherwise $\mathbf{A} := \mathbf{A}^{(1)}$. There is a point $y \in P_{i,n}(\mathbf{A})$ such that $x \in \operatorname{Sk}_i \not\ni y$ or $x \notin \operatorname{Sk}_i \ni y$. Lemma 2.3 yields an *i*-orthogonal T-junction $T \in \mathbb{T}_i$ and associated cell \mathbb{Q} with

$$(4.6) \qquad \qquad \overline{\mathsf{T}} \cap \operatorname{conv}\{x, y\} \neq \varnothing,$$

(4.7)
$$\mathbf{Q}_{\mathrm{pdir}(\mathbf{T})} \cap \mathrm{conv}\{x_{\mathrm{pdir}(\mathbf{T})}, y_{\mathrm{pdir}(\mathbf{T})}\}) \neq \emptyset,$$

(4.8) $y_{\text{pdir}(\mathsf{T})} \neq x_{\text{pdir}(\mathsf{T})}.$

Since $y \in P_{i,n}(\mathbf{A})$ and $pdir(\mathbf{T}) \neq i = odir(\mathbf{T})$, this concludes the proof.

DEFINITION 4.3 (Geometric T-junction extensions and analysis-suitability). Let T be a T-junction with Q = ascell(T), i = odir(T) and j = pdir(T). We then define local knot vectors as follows.

1. For k = j, we define $v_j(T) = (\ell_0, \ldots, \ell_{p_j})$ as the vector of $(p_j + 1)$ consecutive indices from $\mathcal{I}_j(T)$, such that

(4.9)
$$\{\ell_{p_j/2}\} = \mathsf{T}_j, \qquad if \ p_j \ is \ even, \\ \ell_{\lfloor p_j/2 \rfloor} = \inf \mathsf{Q}_j, \quad \ell_{\lceil p_j/2 \rceil} = \sup \mathsf{Q}_j, \quad if \ p_j \ is \ odd.$$

- 2. For k = i, the local knot vector is the singleton $v_i(T) = T_i$.
- 3. For $k \notin \{i, j\}$ we define $v_k(T) = (\ell_0, \ldots, \ell_{p_k+1+c_k})$, where $c_k = p_k \mod 2$, as the vector of $(p_k + 2 + c_k)$ consecutive indices from $J_k(T)$, such that

(4.10)
$$\mathbf{T}_{k} = (\ell_{\lceil p_{k}/2 \rceil}, \ell_{\lceil p_{k}/2 \rceil+1}).$$

This means that the local knot vector has $p_k + 3$ elements if p_k is odd and p_k+2 if p_k is even, and T_k is centered within these elements, cf. the definition of local knot vectors for anchors.

We then call

(4.11)
$$\operatorname{GTJ}_{i}(\mathsf{T}) \coloneqq \bigotimes_{k=1}^{d} \operatorname{conv}(\mathsf{v}_{k}(\mathsf{T}))$$

the geometric T-junction extension (GTJ) of T, and we say that it is an i-orthogonal extension in j-direction. Note that $\operatorname{GTJ}_i(T) \not\subset \operatorname{Sk}_i$.

A mesh \mathcal{T} is strongly geometrically analysis-suitable (SGAS), if for any two Tjunctions T_1, T_2 with orthogonal directions $i_1 = \operatorname{odir}(T_1) \neq \operatorname{odir}(T_2) = i_2$ holds

(4.12)
$$\operatorname{GTJ}_{i_1}(\mathsf{T}_1) \cap \operatorname{GTJ}_{i_2}(\mathsf{T}_2) = \varnothing.$$

We call \mathfrak{T} weakly geometrically analysis-suitable (WGAS), if (4.12) holds for any two T-junctions T_1, T_2 with orthogonal directions $\operatorname{odir}(T_1) \neq \operatorname{odir}(T_2)$ and pointing directions $\operatorname{pdir}(T_1) \neq \operatorname{pdir}(T_2)$.

We will omit the dependency of the orthogonal direction, when clear from the context, e.g. write $GTJ(T) \equiv GTJ_i(T)$, for odir(T) = i.

Note that the latter is a weaker criterion since T-junction extensions with different orthogonal directions but equal pointing direction are allowed to intersect. Later in this paper, we will refer to the set $\text{GTJ}_i \equiv \text{GTJ}_i(\mathcal{T})$ as the union of all geometric T-junction extensions for hanging interfaces T with odir(T) = i, i.e.

(4.13)
$$\operatorname{GTJ}_i \coloneqq \bigcup_{\mathsf{T} \in \mathbb{T}_i} \operatorname{GTJ}(\mathsf{T}),$$

(4.14)
$$\mathbb{T}_i := \{ \mathsf{T} \in \mathcal{H}^{(d-2)} \mid \text{valence}(\mathsf{T}) < 4, \mathsf{T} \not\subset \partial\Omega, \text{ odir}(\mathsf{T}) = i \}.$$

A mesh is then SGAS if $\text{GTJ}_i \cap \text{GTJ}_j = \emptyset$ for $i \neq j$.

Remark 4.4. Note that the above definition of geometric T-junction extensions is consistent with the literature [8] for the 2D case. A T-junction is then given as $T = \{i\} \times \{j\}$, where pdir(T) = 1 corresponds to a T-junction of type \vdash or \dashv and pdir(T) = 2 corresponds to a T-junction of type \perp or \top . In any case, the T-junction extension will be a line along the pointing direction, consisting of $p_{pdir(T)}+1$ consecutive indices from the knot vector, as in the 2D case.

In the case d = 2, SGAS and WGAS are equivalent and sufficient for linear independence, see [8]. We assume for the rest of this paper that $d \ge 3$ and that the initial mesh is sufficiently fine in the sense of the assumption below. It is applied in Lemma 4.8, which is used for the theorems in sections 5 and 6.

ASSUMPTION 4.5. For any mesh considered below, there are for each cell $\mathbf{Q} \in \mathcal{H}^{(d)}$ at least three distinct directions $i \neq j \neq k \neq i$ in each of which \mathbf{Q} has an active neighbor cell. E.g., this is fulfilled if the initial mesh contains at least 2 active cells in each of three pairwise distinct directions.

LEMMA 4.6. Let \mathfrak{T} be a WGAS mesh, \mathbf{E} an anchor or T-junction and $\mathbf{v}_{\ell}(\underline{\mathbf{E}})$ its local knot vector in direction $\ell \in \{1, \ldots, d\}$, then for any $m \in \operatorname{conv} \mathbf{v}_{\ell}(\mathbf{E})$ holds $\overline{P_{j,m}(\mathbf{E})} \subset$ Sk_{j} or $P_{j,m}(\mathbf{E}) \cap \operatorname{Sk}_{j} = \emptyset$.

Proof. Since Sk_j is by construction a closed set, $P_{j,m}(\mathsf{E}) \subset \mathrm{Sk}_j$ is sufficient for $\overline{P_{j,m}(\mathsf{E})} \subset \mathrm{Sk}_j$, and we only need to show that $P_{j,m}(\mathsf{E}) \subset \mathrm{Sk}_j$ or $P_{j,m}(\mathsf{E}) \cap \mathrm{Sk}_j = \emptyset$.

Assume for contradiction a WGAS mesh and $m \in \operatorname{conv} v_{\ell}(\mathbf{E})$ such that there exist $x, y \in P_{j,m}(\mathbf{E})$ with $x \in \operatorname{Sk}_j \not\supseteq y$. Recall from the beginning of section 2 that the mesh consists of boxes with integer vertices and hence m is an integer. By definition of mesh entities we have $P_{j,n}(\mathbf{E}) \subset \operatorname{Sk}_j$ for $n \in \{\inf E_j, \sup E_j\}$ and $P_{j,n}(\mathbf{E}) \cap \operatorname{Sk}_j = \emptyset$ for $n \in E_j \setminus \{\inf E_j, \sup E_j\}$. Hence $m < \inf E_j$ or $m > \sup E_j$. Without loss of generality, we assume $m > \sup E_j$, and we assume further that m is minimal, i.e. that there is no $\tilde{m} \in (\sup E_j, m)$ with $P_{j,\tilde{m}}(\mathbf{E}) \not\subseteq \operatorname{Sk}_j$ and $P_{j,\tilde{m}}(\mathbf{E}) \cap \operatorname{Sk}_j \neq \emptyset$.

Lemma 2.3 yields a T-junction T, dir(T) = j, Q = ascell(T), with $pdir(T) = k \neq j$ and

$$(4.15) \qquad \overline{\mathsf{T}} \cap \operatorname{conv}\{x, y\} \neq \emptyset, \quad x_k \neq y_k, \quad \mathsf{Q}_k \cap [\min(x_k, y_k), \max(x_k, y_k)] \neq \emptyset.$$

From $k \neq j$ we get $x_k, y_k \in E_k$, and from $x_k \neq y_k$ we get that E_k is not a singleton but an open interval, which yields $E \cap Sk_k = \emptyset$. Due to $\overline{T} \cap conv\{x, y\} \neq \emptyset$, there is $z \in E$ such that

(4.16)
$$P_{j,m}(z) = (z_1, \dots, z_{j-1}, m, z_{j+1}, \dots, z_d) \in \overline{\mathsf{T}} \cap \operatorname{conv}\{x, y\}.$$

From $\operatorname{odir}(\mathsf{T}) = j$ and $\operatorname{pdir}(\mathsf{T}) = k$ we get $\mathsf{T} \in \operatorname{H}^{(j)}(\{j, k\})$. Further, T is in the boundary of some k-orthogonal mesh entity, which yields $\overline{\mathsf{T}} \subset \operatorname{Sk}_k$. Together with $\mathsf{E} \cap \operatorname{Sk}_k = \emptyset$, we get $z \notin \operatorname{Sk}_k \ni P_{j,m}(z)$. Lemma 2.3 yields another T-junction T' , $\operatorname{odir}(\mathsf{T}) = k$, $\mathsf{Q}' = \operatorname{ascell}(\mathsf{T}')$, with

(4.17)
$$\overline{\mathbf{T}'} \cap \operatorname{conv}\{z, P_{j,m}(z)\} \neq \emptyset, \quad z_{\operatorname{pdir}(\mathbf{T}')} \neq (P_{j,m}(z))_{\operatorname{pdir}(\mathbf{T}')},$$

(4.18)
$$\mathbf{Q}'_{\mathrm{pdir}(\mathbf{T}')} \cap \operatorname{conv}[z_{\mathrm{pdir}(\mathbf{T}')}, (P_{j,m}(z))_{\mathrm{pdir}(\mathbf{T}')}] \neq \emptyset$$

Since z and $P_{j,m}(z)$ differ only in direction j, $z_{\text{pdir}(\mathbb{T}')} \neq (P_{j,m}(z))_{\text{pdir}(\mathbb{T}')}$ yields that $\text{pdir}(\mathbb{T}') = j$. Hence we have $z_j \neq m$ and $\mathbb{Q}'_j \cap \text{conv}\{z_j, m\} \neq \emptyset$. From $\overline{\mathbb{T}'} \cap \text{conv}\{z, P_{j,m}(z)\} \neq \emptyset$ we get $z_\ell = (P_{j,m}(z))_\ell \in \overline{\mathbb{T}'_\ell} \subset \text{conv} v_\ell(\mathbb{T}')$ for all $\ell \neq j$. From (4.16) above, we also have $P_{j,m}(z) \in \overline{\mathbb{T}} \subset \text{GTJ}(\mathbb{T})$.

This yields by construction of T,T^\prime two cases listed below.

Case 1: $v_j(T') \cap (\sup E_j, m) \subset v_j(E) \cap (\sup E_j, m)$. This leads to $m \in \operatorname{conv} v_j(T')$ and consequently $\operatorname{GTJ}(T) \cap \operatorname{GTJ}(T') \ni P_{j,m}(z)$ which means that $\mathfrak{T} \notin WGAS$ in contradiction to the assumption.

Case 2: There is some $\tilde{m} \in v_j(T') \cap (\sup E_j, m) \setminus v_j(E)$. This yields $P_{j,\tilde{m}}(E) \not\subset Sk_j$, and $P_{j,\tilde{m}}(z) \in P_{j,\tilde{m}}(T') \subset Sk_j$, hence $P_{j,\tilde{m}}(E) \cap Sk_j \neq \emptyset$ in contradiction to the minimality of m.

LEMMA 4.7. Let $\mathcal{T} \in WGAS$ and $E, F \in \mathcal{T}$ be anchors or T-junctions, and

(4.19)
$$m \in v_j(\mathbf{E}) \cap \operatorname{conv} v_j(\mathbf{F}) \setminus v_j(\mathbf{F}).$$

Then there is a T-junction $T \in T_j$ with $T_j = \{m\}, k = pdir(T), Q = ascell(T)$ such that

(4.20)
$$\overline{\mathsf{T}} \cap P_{j,m}(\mathrm{MBox}(\mathsf{E},\mathsf{F})) \neq \emptyset, \quad \mathsf{Q}_k \cap \mathrm{MBox}(\mathsf{E},\mathsf{F})_k \neq \emptyset, \quad \mathsf{E}_k \cap \mathsf{F}_k = \emptyset,$$

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(4.21)
$$\operatorname{MBox}(\mathsf{E},\mathsf{F})_{\ell} = \begin{cases} \mathsf{E}_{\ell} \cap \mathsf{F}_{\ell} & \mathsf{E}_{\ell} \cap \mathsf{F}_{\ell} \neq \emptyset \\ [\sup \mathsf{E}_{\ell}, \inf \mathsf{F}_{\ell}] & \sup \mathsf{E}_{\ell} \leq \inf \mathsf{F}_{\ell} \\ [\sup \mathsf{F}_{\ell}, \inf \mathsf{E}_{\ell}] & \inf \mathsf{E}_{\ell} \geq \sup \mathsf{F}_{\ell}. \end{cases}$$

Proof. By construction of local knot vectors, we have $P_{j,m}(\mathbf{E}) \subset \mathrm{Sk}_j \not\supseteq P_{j,m}(\mathbf{F})$. Lemma 4.6 yields $\overline{P_{j,m}(\mathbf{E})} \subset \mathrm{Sk}_j$ and $P_{j,m}(\mathbf{F}) \cap \mathrm{Sk}_j \neq \emptyset$. Using Lemma 2.3, there exists for each $x \in \overline{P_{j,m}(\mathbf{E})}, y \in P_{j,m}(\mathbf{F})$ a (possibly non-unique) *j*-orthogonal T-junction $\mathbf{T}^{(x,y)} \in \mathbb{T}_j$, with $\mathrm{pdir}(\mathbf{T}^{(x,y)}) = k^{(x,y)}, \mathbf{Q}^{(x,y)} = \mathrm{ascell}(\mathbf{T}^{(x,y)})$, such that

(4.22)
$$\overline{\mathsf{T}}^{(x,y)} \cap \operatorname{conv}\{x,y\} \neq \emptyset, \quad x_{k^{(x,y)}} \neq y_{k^{(x,y)}}$$

(4.23) and
$$\mathbb{Q}_{k(x,y)}^{(x,y)} \cap \operatorname{conv}\{x_{k(x,y)}, y_{k(x,y)}\} \neq \emptyset$$

We have

(4.24)
$$\bigcup_{\substack{\tilde{x}\in\overline{P_{j,m}(\mathsf{E})}\\\tilde{y}\in P_{j,m}(\mathsf{F})}} \overline{\mathrm{T}^{(\tilde{x},\tilde{y})}}\cap\mathrm{conv}\{x,y\}\neq\varnothing\quad\text{for any }x\in\overline{P_{j,m}(\mathsf{E})},y\in P_{j,m}(\mathsf{F}),$$

and hence also for any choice of $x \in \overline{P_{j,m}(\mathsf{E})}, y \in \overline{P_{j,m}(\mathsf{F})}$, since the union $\bigcup_{(x,y)} \overline{\mathsf{T}^{(x,y)}}$ is a closed set. Consider a pair $(x,y) \in \overline{P_{j,m}(\mathsf{E})} \times \overline{P_{j,m}(\mathsf{F})}$ with

(4.25)
$$\begin{cases} x_j = y_j = m & \ell = j \\ x_\ell = y_\ell \in \mathsf{E}_\ell \cap \mathsf{F}_\ell & \ell \neq j, \ \mathsf{E}_\ell \cap \mathsf{F}_\ell \neq \emptyset \\ x_\ell = \sup \mathsf{E}_\ell, \ y_\ell = \inf \mathsf{F}_\ell & \ell \neq j, \ \mathsf{E}_\ell \cap \mathsf{F}_\ell = \emptyset, \ \sup \mathsf{E}_\ell \leq \inf \mathsf{F}_\ell \\ x_\ell = \inf \mathsf{E}_\ell, \ y_\ell = \sup \mathsf{F}_\ell & \ell \neq j, \ \mathsf{E}_\ell \cap \mathsf{F}_\ell = \emptyset, \ \inf \mathsf{E}_\ell \geq \sup \mathsf{F}_\ell, \end{cases}$$

which yields a T-junction $T \in T_j$ from the union above, with pdir(T) = k, Q = ascell(T), such that $x_k \neq y_k$ and

(4.26)
$$\overline{\mathsf{T}} \cap P_{j,m}(\operatorname{MBox}(\mathsf{E},\mathsf{F})) \supseteq \overline{\mathsf{T}} \cap \operatorname{conv}\{x,y\} \neq \emptyset,$$

$$(4.27) \qquad \qquad \mathsf{Q}_k \cap \mathrm{MBox}(\mathsf{E},\mathsf{F})_k \supseteq \mathsf{Q}_k \cap \mathrm{conv}\{x_k,y_k\} \neq \varnothing.$$

If $y \in P_{j,m}(F)$, this holds for $T = T^{(x,y)}$ as above. If $y \in \overline{P_{j,m}(F)} \setminus P_{j,m}(F)$, then $T = T^{(\tilde{x},\tilde{y})}$ for some \tilde{x}, \tilde{y} close to x, y.

From $j = \text{odir}(T) \neq \text{pdir}(T) = k$ we know that k does not match the first case in (4.25). Since $x_k \neq y_k$, k also does not match the second case, and hence $E_k \cap F_k = \emptyset$. This concludes the proof.

LEMMA 4.8. Given a WGAS box subdivision of a WGAS mesh using Algorithm 2.1, $\mathbf{Q} \in \mathcal{T}^{(n)} \in \text{WGAS}, \ \mathcal{T}^{(n+1)} = \text{SUBDIV}(\mathcal{T}^{(n)}, \mathbf{Q}, j) \in \text{WGAS}, \text{ there is for each new anchor}$ $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)} \text{ an old anchor } \mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n)} \text{ with } \operatorname{supp}_{\Omega} B_{\hat{\mathbf{A}}} \subset \operatorname{supp}_{\Omega, \mathcal{T}^{(n)}} B_{\mathbf{A}} \text{ and}$ $v_{\ell}(\hat{\mathbf{A}}) = v_{\ell}(\mathbf{A}) \text{ for all } \ell \neq j.$

The proof is given in Appendix A.1.

Note that Lemma 4.8 does not hold without Assumption 4.5. Consider the 3D mesh in Figure 7. In this example, Assumption 4.5 is not fulfilled, as the center cell Q has active neighbor cells in only two directions (the figure shows the active region). For the 2-orthogonal bisection of Q (highlighted in red), the old and new mesh are

WGAS as with $p_1 = 1$ the new T-junctions only intersect with the neighbor cells, but not with \mathbf{Q} or with the old T-junctions. Since p_2 is odd, any new anchor $\hat{\mathbf{A}}$ is contained in the closure of the new interface, and $v_1(\hat{\mathbf{A}})$ does not coincide with the local knot vector $v_1(\mathbf{A})$ of any old anchor \mathbf{A} , i.e. Lemma 4.8 does not hold in this case.

We close this section with two examples illustrated in Figures 8 to 10. We consider the 3D mesh visualized in Figure 8, with polynomial degrees $\mathbf{p} = (3, 2, 1)$, and we construct the T-junction extensions of the hanging interfaces via both approaches, the abstract and the geometric one.

The sketches in Figure 9 show the slice $S_3(2)$, where the thick red line marks 3-orthogonal T-junctions contained in the slice. The faces inside the red line are part of the skeleton, the faces outside the red line are not. In other words, the faces surrounded by the red line were generated by a bisection orthogonal to the third direction. In Figures 9a, 9c and 9e the scheme of constructing the abstract T-junction extension is displayed, while Figures 9b, 9d and 9f shows the procedure for geometric T-junction extensions.

For the abstract T-junction extensions, we consider the two sets

(4.28)
$$\mathcal{A}_{\mathbf{p}}^{(1)} = \{ \mathbf{A} \in \mathcal{A}_{\mathbf{p}} \colon n \in \mathfrak{I}_{3}(\mathbf{A}) \}, \quad \mathcal{A}_{\mathbf{p}}^{(2)} = \{ \mathbf{A} \in \mathcal{A}_{\mathbf{p}} \colon n \notin \mathfrak{I}_{3}(\mathbf{A}) \}.$$

From the polynomial degree $\mathbf{p} = (3, 2, 1)$, we get $\kappa = \{1, 3\}$ in (3.4) and hence the anchors are the edges in the second direction. The projection of $\mathcal{A}_{\mathbf{p}}^{(1)}$ (resp. $\mathcal{A}_{\mathbf{p}}^{(2)}$) on the slice $S_3(2)$ is indicated in Figure 9a (resp. Figure 9c) by solid dots on the lines, meaning that each marked line corresponds to three (resp. two) anchors with identical first and second components. Following Definition 4.1, we indicate $\bigcup_{\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(1)}} \sup_{\mathcal{B}_{\mathbf{A}}} B_{\mathbf{A}}$ by dashed lines. The intersection of these sets yields the T-junction extension highlighted in Figure 9e, which contains faces in the center region and intersects with cells in the outer region. Note that the spline supports far away from the T-junctions contribute no information in this construction. It is hence sufficient to consider only anchors near T-junctions when checking for AAS in practice.

For the geometric T-junction extensions, we consider the two types

$$(4.29) \qquad \mathbb{T}^{(1)} = \{ \mathsf{T} \in \mathcal{H}^{(1)} \mid \text{valence}(\mathsf{T}) < 4, \, \mathsf{T} \not\subset \partial\Omega, \, \text{pdir}(\mathsf{T}) = 1, \, \text{odir}(\mathsf{T}) = 3 \},\$$

$$(4.30) \qquad \mathbb{T}^{(2)} = \{ \mathtt{T} \in \mathcal{H}^{(1)} \mid \text{valence}(\mathtt{T}) < 4, \, \mathtt{T} \not\subset \partial\Omega, \, \text{pdir}(\mathtt{T}) = 2, \, \text{odir}(\mathtt{T}) = 3 \}.$$

In Figure 9, the set $\mathbb{T}^{(1)}$ represents the vertical red edges, and $\mathbb{T}^{(2)}$ represents the horizontal red edges. We build the geometric T-junction extensions separately for the



FIG. 7. In this example, Assumption 4.5 is not fulfilled. For the 2-orthogonal bisection of Q (highlighted in red), the old and new mesh are WGAS, but for any new anchor \hat{A} , $v_1(\hat{A})$ does not coincide with the local knot vector $v_1(A)$ of any old anchor A, i.e. Lemma 4.8 does not hold in this case.



FIG. 8. Example mesh with $\mathbf{p} = (3, 2, 1)$ and $(N_1, N_2, N_3) = (17, 13, 4)$ for which the T-junction extensions are investigated in Figure 9. The figure shows only the active region $AR_{\mathbf{p}} = [2, 15] \times [1, 12] \times [1, 3]$.

interfaces in $\mathbb{T}^{(1)}$ and for the interfaces from $\mathbb{T}^{(2)}$. The intersection of the associated cells with $S_3(2)$ are highlighted in lime in Figures 9b and 9d.

For any interface $T = T_1 \times T_2 \times T_3 \in \mathbb{T}^{(1)}$, we have pdir(T) = 1, hence, the knot vectors are constructed as follows, recall also Definition 4.3. From $\mathcal{I}_1(T)$ we select the $p_1 + 1 = 4$ indices, such that T_1 is either the third or the second entry, i.e. on the left side in Figure 9b the index of T_1 is the third entry in $v_1(T)$ and on the right side, the index of T_2 is the second entry in $v_1(T)$. Since odir(T) = 3, we have $v_3(T) = \{2\}$ for all T-junctions in this example. We construct the knot vector $v_2(T)$ to be symmetric around T, i.e. it has $p_2 + 2 = 4$ consecutive entries from $\mathcal{I}_2(T)$, where the indices of T_2 are in the middle.

For any interface $T = T_1 \times T_2 \times T_3 \in \mathbb{T}^{(2)}$, we have pdir(T) = 2. Since $p_2 = 2$ is even, $v_2(T)$ is composed of $p_2 + 1 = 3$ indices from $\mathcal{I}_2(T)$ such that T_2 is the second entry. Thus, the local knot vector $v_2(T)$ is symmetric around T_2 . Further, we build $v_1(T)$ from $\mathcal{I}_1(T)$ by choosing the $p_1 + 3 = 6$ (p_1 is odd) consecutive indices, such that the bounds of T_1 are the middle entries.

The unions of these T-junction extensions are shown in Figures 9b and 9d by dashed lines, and the union of both sets gives the T-junction extension GTJ highlighted in Figure 9f. Note that the geometric T-junction is slightly larger than the abstract T-junction extension.

The second example is the 2D mesh shown in Figures 10a to 10d. The hanging interfaces are the two opposing hanging vertices $T^{(1)} = \{m\} \times \{n\}$ and $T^{(2)} = \{m+1\} \times \{n\}$. We will demonstrate the behavior of the two introduced T-junction extensions for different degrees. Let p_2 be odd in any case.

In Figure 10a, p_1 and p_2 are odd. The anchors are marked by red bullets. In this setting, all anchors $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$ have the index n in their knot vector, i.e. $n \in \mathcal{I}_2(\mathbf{A})$ for all $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$. Thus, the abstract T-junction extension is empty here. Note that $\operatorname{ATJ}_2 = \emptyset$ if p_1 is odd.

In Figure 10b, p_1 is even and p_2 is odd, and hence the anchors are the horizontal

lines. In this setting, we have two anchors $\mathbf{A}^{(1)} = (m, m+1) \times \{n-1\}$ and $\mathbf{A}^{(2)} = (m, m+1) \times \{n+1\}$, which are the bottom center and the top center anchor in



(a) The set $\mathcal{A}_{\mathbf{p}}^{(1)}$ and the corresponding support of T-Splines



(c) The set $\mathcal{A}^{(2)}_{\mathbf{p}}$ and the corresponding support of T-Splines



(e) The resulting abstract TJunction extension by intersecting the supports



(b) Construct the extension of lines in $\mathbb{T}^{(1)}$



(d) Construct the extensions of the lines in $\mathbb{T}^{(2)}$



(f) The resulting geometric TJunction extension as union of the two extension sets

FIG. 9. Step-by-step construction of abstract and geometric T-junction extensions.

Figure 10b, with $n \notin \mathcal{I}_2(\mathbf{A}^{(1)}) = \mathcal{I}_2(\mathbf{A}^{(2)})$, while $n \in \mathcal{I}_2(\mathbf{A})$ for the remaining six anchors. Thus the abstract T-junction extension will not be empty. The extension is drawn as a dashed line.

In contrast to the case from Figure 10a, we see in Figure 10c that the geometric T-junction extension $\operatorname{GTJ}(\mathsf{T}^{(1)})$ is not empty for any p_1 as it is constructed with $p_1 + 1$ consecutive indices from the global knot vector $\mathcal{J}_1(\mathsf{T}^{(1)})$. The extension for $p_1 = 1$ is given by $\operatorname{GTJ}(\mathsf{T}^{(1)}) = \operatorname{GTJ}(\mathsf{T}^{(2)}) = [m, m+1] \times \{n\}$.

Furthermore, we get for the case $p_1 = 2$ the extension shown in Figure 10d, which coincides with the abstract T-junction extension shown in Figure 10b.

Both examples indicate that AAS does not imply SGAS in general, since there may be an intersection of T-junction extensions in points that are contained in a geometric, but not in an abstract T-junction extension. Consider for example Figure 10e, where $\mathbf{p} = (3,3)$ and the anchors are again the vertices of the mesh. As before, there is $\operatorname{ATJ}_2 = \emptyset$, whereas the T-junction $\operatorname{T}^{(3)} = \{m-1\} \times \{n+1\}$ yields $\operatorname{ATJ}_1(m-1) = \{m-1\} \times [n-1, n+2] = \operatorname{GTJ}(\operatorname{T}^{(3)})$. We again have $[m-2, m+2] \times \{n\} \subset \operatorname{GTJ}$, and we see $\operatorname{GTJ}_1 \cap \operatorname{GTJ}_2 = \{m-1\} \times \{n\} \neq \emptyset$, as well as $\operatorname{ATJ}_1 \cap \operatorname{ATJ}_2 = \emptyset$. The extensions GTJ_1 and ATJ_1 coincide and are drawn with dashed lines, and the extension GTJ_2 is drawn with dotted lines in Figure 10e.

Lastly, we give an example to point out the differences between WGAS and SGAS meshes. For WGAS meshes, we consider only intersecting T-junction extensions of T-junctions with different pointing and orthogonal direction, whereas SGAS only consider extensions of T-junctions with different orthogonal directions. For the two hanging interfaces $T^{(1)} = \{m+1\} \times (n, n+2) \times \{r+1\}$ and $T^{(2)} = \{m+2\} \times \{n+1\} \times (r, r+2)$ from Figure 11 we have $pdir(T^{(1)}) = pdir(T^{(2)}) = 1$ and $odir(T^{(1)}) = 3 \neq 2 = odir(T^{(2)})$. For any degrees $p_1, p_2, p_3 \geq 0$, the intersection of the two geometric extensions will not be empty, i.e. $GTJ(T^{(1)}) \cap GTJ(T^{(2)}) \neq \emptyset$, hence the mesh will not be SGAS. But since $pdir(T^{(1)}) = pdir(T^{(1)})$, the intersection is not considered for the weak criterion of geometric analysis-suitability. Thus, the mesh is WGAS but not SGAS. However, we conjecture that the generated splines are linearly independent, see Conjecture 6.3 and Proposition 5.4.

5. Dual-Compatibility. We recall two versions of dual-compatibility, a strong [13, Definition 5.3.12] and a weak one [9, Definition 7.2]. Throughout this



FIG. 10. Opposing hanging interfaces

paper, we suppose that knot vectors are non-decreasing.

DEFINITION 5.1 (Overlapping knot vectors and splines). We say that two knot vectors $\Xi^{(1)} = (\xi_1^{(1)}, \ldots, \xi_{n_1}^{(1)})$ and $\Xi^{(2)} = (\xi_1^{(2)}, \ldots, \xi_{n_2}^{(2)})$ overlap, if there is a knot vector $\Xi = (\xi_1, \ldots, \xi_n)$, $n \ge \max\{n_1, n_2\}$, and numbers $k^{(1)}, k^{(2)} \in \mathbb{N}_0$ such that

(5.1)
$$\begin{aligned} \forall i = 1, \dots, n_1 \colon & \xi_i^{(1)} = \xi_{i+k^{(1)}}, \\ \forall i = 1, \dots, n_2 \colon & \xi_i^{(2)} = \xi_{i+k^{(2)}}. \end{aligned}$$

We write $\Xi^{(1)} \bowtie \Xi^{(2)}$.

Further, for two anchors $\mathbf{A}^{(1)}, \mathbf{A}^{(2)} \in \mathcal{A}_{\mathbf{p}}$ we say that the splines $B_{\mathbf{A}^{(1)}}$ and $B_{\mathbf{A}^{(2)}}$ overlap if the local knot vectors $\mathbf{v}_k(\mathbf{A}^{(1)})$ and $\mathbf{v}_k(\mathbf{A}^{(2)})$ overlap for each k, and we write $B_{\mathbf{A}^{(1)}} \bowtie B_{\mathbf{A}^{(2)}}$.

We say that they weakly partially overlap if there is an $l \in \{1, \ldots, d\}$ such that the knot vectors $v_{\ell}(\mathbf{A}^{(1)})$ and $v_{\ell}(\mathbf{A}^{(2)})$ differ and overlap, and we write $B_{\mathbf{A}^{(1)}} \ltimes^{w} B_{\mathbf{A}^{(2)}}$. We say they strongly partially overlap, if $\operatorname{supp}(B_{\mathbf{A}^{(1)}}) \cap \operatorname{supp}(B_{\mathbf{A}^{(2)}}) = \emptyset$ or if $v_k(\mathbf{A}^{(1)})$ and $v_k(\mathbf{A}^{(2)})$ overlap for at least d-1 directions k. We write $B_{\mathbf{A}^{(1)}} \ltimes B_{\mathbf{A}^{(2)}}$.

DEFINITION 5.2 (Dual-Compatibility). Let $S = \{B_i\}$ be a set of splines. We say that S is weakly (resp. strongly) Dual-Compatible (WDC resp. SDC), if $B_i \ltimes^{\mathsf{w}} B_j$ (resp $B_i \ltimes B_j$), for $i \neq j$. Further, we say that \mathfrak{T} is WDC (resp. SDC), if the generated spline space is WDC (resp. SDC), and we write $\mathfrak{T} \in \text{WDC}$ (resp. $\mathfrak{T} \in \text{SDC}$).

Remark 5.3. SDC is sufficient for WDC. This is shown as follows. Let $\mathbf{A}^{(1)}, \mathbf{A}^{(2)} \in \mathcal{A}_{\mathbf{p}}$ be two anchors with $\mathbf{A}^{(1)} \neq \mathbf{A}^{(2)}$ and $B_{\mathbf{A}^{(1)}} \ltimes B_{\mathbf{A}^{(2)}}$.

Case 1: $\operatorname{supp}(B_{\mathbf{A}^{(1)}}) \cap \operatorname{supp}(B_{\mathbf{A}^{(2)}}) = \emptyset$, then there is k with $\operatorname{conv} v_k(\mathbf{A}^{(1)}) \cap \operatorname{conv} v_k(\mathbf{A}^{(2)}) = \emptyset$. We choose $\Xi_k = v_k(\mathbf{A}^{(1)}) \cup v_k(\mathbf{A}^{(2)})$ as the global knot vector and $k^{(1)} = 1, k^{(2)} = p_k + 2$ such that $\Xi_k, v_k(\mathbf{A}^{(1)}), v_k(\mathbf{A}^{(2)})$ fulfill condition (5.1).

Case 2: $\operatorname{supp}(B_{\mathbf{A}^{(1)}}) \cap \operatorname{supp}(B_{\mathbf{A}^{(2)}}) \neq \emptyset$ and $v_k(\mathbf{A}^{(1)}) \bowtie v_k(\mathbf{A}^{(2)})$ for at least d-1 directions k. If $v_k(\mathbf{A}^{(1)}) = v_k(\mathbf{A}^{(2)})$ for all these directions, then $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are equal or aligned in the remaining direction j and hence share the same global knot vector $\mathcal{J}_j(\mathbf{A}^{(1)}) = \mathcal{J}_j(\mathbf{A}^{(2)})$. Hence $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$ or $v_j(\mathbf{A}^{(1)}) \bowtie v_j(\mathbf{A}^{(2)})$.

In both cases, there exists an ℓ such that $v_{\ell}(\mathbf{A}^{(1)})$ and $v_{\ell}(\mathbf{A}^{(2)})$ differ and overlap. This is, from $B_{\mathbf{A}^{(1)}} \ltimes B_{\mathbf{A}^{(2)}}$ follows $B_{\mathbf{A}^{(1)}} \ltimes^{\mathrm{w}} B_{\mathbf{A}^{(2)}}$, and hence SDC implies WDC.

An example for a mesh that is WDC but not SDC is again the mesh from Figure 11. The 1-orthogonal Skeleton $Sk_1 = S_1(m) \cup S_1(m+1) \cup S_1(m+2) \cup S_1(m+3)$ consists of slices of the whole domain. Hence all anchors have the same global knot vector $\mathcal{I}_1(\mathbf{A}) = (m, m+1, m+2, m+3)$, regardless of the polynomial degrees and corresponding anchor type. Consequently, any two anchors have overlapping local knot vectors in the first direction, i.e. $v_1(\mathbf{A}^{(1)}) \bowtie v_1(\mathbf{A}^{(2)})$. Further, any two anchors that coincide in their first component also coincide in their global knot vectors in the



FIG. 11. A mesh that is WGAS and WDC, but neither SGAS nor SDC, for any polynomial degree.

TABLE 1

Global knot vectors for all possible configurations of an anchor's first component, for the mesh from Figure 11. The always contained values $0, \ldots, \lfloor \frac{p_j+1}{2} \rfloor$ and $N_j - \lfloor \frac{p_j+1}{2} \rfloor, \ldots, N_j$ are hidden by dots.

\mathbf{A}_1	$\mathfrak{I}_2(\mathbf{A})$	$\mathfrak{I}_3(\mathbf{A})$
$\{m\}$	$(\ldots, n, n+2, \ldots)$	$(\ldots, r, r+1, r+2, \ldots)$
$\{m+1\}$	$(\ldots, n, n+2, \ldots)$	$(\ldots, r, r+1, r+2, \ldots)$
$\{m+2\}$	$(\ldots,n,n+1,n+2,\ldots)$	$(\ldots, r, r+2, \ldots)$
$\{m+3\}$	$(\ldots,n,n+1,n+2,\ldots)$	$(\ldots, r, r+2, \ldots)$
(m, m+1)	$(\ldots, n, n+2, \ldots)$	$(\ldots, r, r+1, r+2, \ldots)$
(m+1,m+2)	$(\ldots, n, n+2, \ldots)$	$(\ldots, r, r+2, \ldots)$
(m+2,m+3)	$\left (\ldots, n, n+1, n+2, \ldots) \right $	$(\ldots, r, r+2, \ldots)$

second and third direction, because these global knot vectors depend only on the first component \mathbf{A}_1 , see Table 1. Together, any two anchors either overlap and are different in the first direction, or, if their first components coincide, they strongly overlap. This satisfies the WDC criterion.

However, the mesh is not SDC for any degree $\mathbf{p} = (p_1, p_2, p_3)$ with $p_1 > 0$. Consider two anchors $\mathbf{A}^{(1)} = \mathbf{A}_1^{(1)} \times \mathbf{A}_2^{(1)} \times \mathbf{A}_3^{(1)}$ and $\mathbf{A}^{(2)} = \mathbf{A}_1^{(2)} \times \mathbf{A}_2^{(2)} \times \mathbf{A}_3^{(2)}$, with

(5.2)
$$\begin{cases} \mathbf{A}_{1}^{(1)} = \{m+1\}, \ \mathbf{A}_{1}^{(2)} = \{m+2\} & \text{if } p_{1} \text{ is odd,} \\ \mathbf{A}_{1}^{(1)} = (m, m+1), \ \mathbf{A}_{1}^{(2)} = (m+2, m+3) & \text{if } p_{1} \text{ is even and } > 0. \end{cases}$$

For any $p_1 > 0$, the supports $\operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}}$ and $\operatorname{supp}_{\Omega} B_{\mathbf{A}^{(2)}}$ have nonempty intersection, and from Table 1, we get for any $p_1 > 0$ that

(5.3) $\mathcal{I}_2(\mathbf{A}^{(1)}) = (\dots, n, n+2, \dots), \qquad \mathcal{I}_3(\mathbf{A}^{(1)}) = (\dots, r, r+1, r+2, \dots),$

(5.4)
$$J_2(\mathbf{A}^{(2)}) = (\dots, n, n+1, n+2, \dots), \ J_3(\mathbf{A}^{(2)}) = (\dots, r, r+2, \dots),$$

i.e. the knot vectors $v_2(\mathbf{A}^{(1)})$ and $v_2(\mathbf{A}^{(2)})$ do not overlap for any $p_2 \ge 0$, and neither do $v_3(\mathbf{A}^{(1)})$ and $v_3(\mathbf{A}^{(2)})$ for any $p_3 \ge 0$. Thus, $B_{\mathbf{A}^{(1)}}$ and $B_{\mathbf{A}^{(2)}}$ do not strongly partially overlap, and the mesh is not SDC.

Extensive studies on dual-compatible splines are already existent, see e.g. [9]. Some important properties are stated in the following proposition.

PROPOSITION 5.4. Let $S_{\mathbf{p}} = \{B_{\mathbf{A},\mathbf{p}}\}\$ be a set of weakly dual-compatible splines over the set of anchors $\mathcal{A}_{\mathbf{p}}$ with multi-degree \mathbf{p} . Then, the following holds

- 1. There exists a set of dual-functions $\lambda_{\mathbf{A},\mathbf{p}}$, s.t. $\lambda_{\mathbf{A}^{(1)},\mathbf{p}}(B_{\mathbf{A}^{(2)},\mathbf{p}}) = \delta_{\mathbf{A}^{(1)},\mathbf{A}^{(2)}}$.
- 2. The splines $B_{\mathbf{A},\mathbf{p}}$ are linearly independent. If the constant function is in the spline space S, then $\sum_{\mathbf{A}\in\mathcal{A}_{\mathbf{p}}}B_{\mathbf{A}}=1$.
- 3. There exists a constant $C_{\mathbf{p}}$, s.t. the projection $\Pi_{\mathbf{p}} \colon L^2(\widehat{\Omega}) \to \mathbb{S}_{\mathbf{p}}$ given by

(5.5)
$$\Pi_{\mathbf{p}}(f)(\zeta) = \sum_{\mathbf{A} \in \mathcal{A}_{\mathbf{p}}} \lambda_{\mathbf{A},\mathbf{p}}(f) B_{\mathbf{A},\mathbf{p}}(\zeta), \quad for \ all \ f \in L^{2}(\widehat{\Omega}), \zeta \in \widehat{\Omega}$$

fulfills

(5.6)
$$\|\Pi_{\mathbf{p}}(f)\|_{L^{2}(\mathbf{Q})} \leq C_{\mathbf{p}} \|f\|_{L^{2}(\mathbf{Q})}, \quad \text{for all } \mathbf{Q} \subset \widehat{\Omega}, \text{ and } f \in L^{2}(\widehat{\Omega})$$

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Proof. See [9, Proposition 7.4, 7.6, and 7.7].

The following lemma and proposition are used in section 6 for connections between dual-compatibility and geometric analysis-suitability.

LEMMA 5.5. Let $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$, $\mathbf{T} \in \mathbb{T}_i$ and $x \in \operatorname{supp}_{\Omega} B_{\mathbf{A}} \cap S_i(n)$, such that 1. T touches the segment between $P_{i,n}(\mathbf{A})$ and $\{x\}$,

(5.7)
$$\overline{\mathsf{T}} \cap \operatorname{conv}(P_{i,n}(\mathbf{A}) \cup \{x\}) \neq \emptyset,$$

2. in pointing direction of T, the associated cell $Q = \operatorname{ascell}(T)$ touches the convex hull of A and $\{x\}$,

(5.8)
$$\mathbf{Q}_{\mathrm{pdir}(\mathbf{T})} \cap \mathrm{conv}(\mathbf{A}_{\mathrm{pdir}(\mathbf{T})} \cup \{x_{\mathrm{pdir}(\mathbf{T})}\}) \neq \emptyset,$$

3. there exists a number $y \in \mathbf{A}_{pdir(T)}$ s.t. $y \neq x_{pdir(T)}$,

$$(5.9) \qquad \qquad \exists y \in \mathbf{A}_{\mathrm{pdir}(\mathsf{T})} \colon y \neq x_{\mathrm{pdir}(\mathsf{T})}$$

4. the anchor **A** and *T*-junction overlap for some arbitrary direction $\ell \neq i$,

(5.10)
$$v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathbf{T}), \text{ for some } \ell \neq \ell$$

Then x_{ℓ} is contained in the convex hull of the ℓ -th local index vector of T, i.e. $x_{\ell} \in \text{conv } v_{\ell}(T)$.

The proof is given in Appendix A.2.

PROPOSITION 5.6. Let \mathfrak{T} be an SGAS mesh, $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}$ and \mathtt{T} a T-junction with $\overline{\mathtt{T}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}} \neq \emptyset$. Then $v_k(\mathbf{A}) \bowtie v_k(\mathtt{T})$ for all $k \neq \operatorname{odir}(\mathtt{T})$.

The proof is given in Appendix A.3.

Proposition 5.6 does not hold for WGAS meshes, an example is depicted in Figure 12. Since each T-junction has pointing direction 1, the mesh is WGAS. Let $\mathbf{p} = (3, 3, 3)$, and choose $\mathbf{A} = \{m+1\} \times \{n\} \times \{r\}$, as well as $T = \{m+2\} \times (r-2, r) \times \{n+1\}$. We then get

(5.11)
$$v_3(T) = (r-3, r-2, r, r+1)$$
, and $v_3(A) = (r-2, r-1, r, r+1, r+2)$,

and we see that $r - 1 \in v_3(\mathbf{A})$ but $r - 1 \notin v_3(\mathbf{T})$, hence $v_3(\mathbf{T}) \not\bowtie v_3(\mathbf{A})$.



FIG. 12. A WGAS mesh, where $v_3(T) \bowtie v_3(\mathbf{A})$ in general.

6. Main Results. In this Section, we focus on the indicated relations from Figure 1. Note that the relations $SGAS \subset WGAS$ and $SDC \subset WDC$ are already evident from the previous sections by construction. In Theorem 6.1, we extend the result from [13] to arbitrary degrees, i.e. the initial restriction to only odd polynomial degrees can be dropped.

THEOREM 6.1. All AAS meshes are SDC and vice versa.

THEOREM 6.2. All SGAS T-meshes are AAS.

CONJECTURE 6.3. All WGAS meshes are WDC.

Proof of Theorem 6.1. This is a generalization of [13, Theorem 5.3.14], we hence follow the original proof and extend necessary steps to the case of arbitrary polynomial degrees.

AAS \subseteq SDC. We assume for contradiction a mesh $\mathcal{T} \in$ AAS \SDC and let $\mathcal{A}_{\mathbf{p}}$ be the set of anchors over \mathcal{T} with the corresponding set of T-splines $\{B_{\mathbf{A}} : \mathbf{A} \in \mathcal{A}_{\mathbf{p}}\}$. Since $\mathcal{T} \notin$ SDC there exist two anchors $\mathbf{A}^{(1)}, \mathbf{A}^{(2)} \in \mathcal{A}_{\mathbf{p}}, \mathbf{A}^{(1)} \neq \mathbf{A}^{(2)}$, such that $B_{\mathbf{A}^{(1)}} \not\ltimes B_{\mathbf{A}^{(2)}}$. This implies that the corresponding knot vectors do not overlap in at least two directions and that

(6.1)
$$\operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(2)}} \neq \emptyset.$$

Denote

(6.2)
$$m_k = \max\{\min v_k(\mathbf{A}^{(1)}), \min v_k(\mathbf{A}^{(2)})\}, \\ M_k = \min\{\max v_k(\mathbf{A}^{(1)}), \max v_k(\mathbf{A}^{(2)})\}, \quad k = 1, \dots, d$$

then (6.1) yields that $m_k \leq M_k$ for all k = 1, ..., d. Assume without loss of generality that the directions in which the knot vectors of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ do not overlap are the first and second dimension, i.e., $v_1(\mathbf{A}^{(1)}) \not\bowtie v_1(\mathbf{A}^{(2)})$ and $v_2(\mathbf{A}^{(1)}) \not\bowtie v_2(\mathbf{A}^{(2)})$.

Thus, there is an index $n_1 \in [m_1, M_1]$, with either $n_1 \in v_1(\mathbf{A}^{(1)})$ and $n_1 \notin v_1(\mathbf{A}^{(2)})$ or $n_1 \notin v_1(\mathbf{A}^{(1)})$ and $n_1 \in v_1(\mathbf{A}^{(2)})$. *Case 1:* $n_1 \in v_1(\mathbf{A}^{(1)})$ and $n_1 \notin v_1(\mathbf{A}^{(2)})$. Then we have $\{n_1\} \cap \mathcal{I}_1(\mathbf{A}^{(2)}) \subset$

Case 1: $n_1 \in v_1(\mathbf{A}^{(1)})$ and $n_1 \notin v_1(\mathbf{A}^{(2)})$. Then we have $\{n_1\} \cap \mathfrak{I}_1(\mathbf{A}^{(2)}) \subset [m_1, M_1] \cap \mathfrak{I}_1(\mathbf{A}^{(2)}) \subset v_1(\mathbf{A}^{(2)})$, and it follows $n_1 \notin \mathfrak{I}_1(\mathbf{A}^{(2)})$, while $n_1 \in v_1(\mathbf{A}^{(1)}) \subseteq \mathfrak{I}_1(\mathbf{A}^{(1)})$ yields $n_1 \in \mathfrak{I}_1(\mathbf{A}^{(1)})$.

Case 2: $n_1 \notin v_1(\mathbf{A}^{(1)})$ and $n_1 \in v_1(\mathbf{A}^{(2)})$. Then we have $\{n_1\} \cap \mathcal{I}_1(\mathbf{A}^{(1)}) \subset [m_1, M_1] \cap \mathcal{I}_1(\mathbf{A}^{(1)}) \subset v_1(\mathbf{A}^{(1)})$, and it follows $n_1 \notin \mathcal{I}_1(\mathbf{A}^{(1)})$, while $n_1 \in v_1(\mathbf{A}^{(2)}) \subseteq \mathcal{I}_1(\mathbf{A}^{(2)})$ yields $n_1 \in \mathcal{I}_1(\mathbf{A}^{(2)})$.

In both cases, Definition 4.1 yields

(6.3)
$$\operatorname{ATJ}_{1}(n_{1}) \supset \operatorname{S}_{1}(n_{1}) \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(2)}} = \{n_{1}\} \times \bigotimes_{k=2}^{d} [m_{k}, M_{k}]$$

Analogously, there exists n_2 , such that

(6.4)
$$\operatorname{ATJ}_2(n_2) \supset [m_1, M_1] \times \{n_2\} \times \bigotimes_{k=3}^d [m_k, M_k].$$

Together, there is

(6.5)
$$\operatorname{ATJ}_1(n_1) \cap \operatorname{ATJ}_2(n_2) \supset \{n_1\} \times \{n_2\} \times \bigotimes_{k=3}^d [m_k, M_k] \neq \emptyset,$$

which contradicts the assumption that $\mathcal{T} \in AAS$.

 $SDC \subseteq AAS$. Assume that $\mathcal{T} \in SDC \setminus AAS$. Then there exist $i \neq j$ with $ATJ_i \cap$ $ATJ_j \neq \emptyset$, and there is a point $e \in \mathbb{N}^d$, with $e = (e_1, \ldots, e_d) \in ATJ_i \cap ATJ_j$. Assume without loss of generality that i = 1, j = 2. Then there exist by definition anchors $\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, \mathbf{A}^{(3)}, \mathbf{A}^{(4)} \in \mathcal{A}_{\mathbf{p}}$ with

$$(6.6) \quad e \in \mathcal{S}_1(e_1) \cap \mathcal{S}_2(e_2) \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(2)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(3)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(4)}},$$

(6.7) with
$$e_1 \in \mathcal{J}_1(\mathbf{A}^{(1)}) \setminus \mathcal{J}_1(\mathbf{A}^{(2)})$$
 and $e_2 \in \mathcal{J}_2(\mathbf{A}^{(3)}) \setminus \mathcal{J}_2(\mathbf{A}^{(4)}).$

From $e \in \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}}$ and $e_1 \in \mathcal{I}_1(\mathbf{A}^{(1)})$ we deduce $e_1 \in \operatorname{conv} v_1(\mathbf{A}^{(1)}) \cap \mathcal{I}_1(\mathbf{A}^{(1)}) =$ $\mathbf{v}_1(\mathbf{A}^{(1)})$, and from $e \in \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(1)}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}^{(2)}}$ and $e_1 \in \mathcal{I}_1(\mathbf{A}^{(1)}) \setminus \mathcal{I}_1(\mathbf{A}^{(2)})$ we deduce $e_1 \in \operatorname{conv} \mathbf{v}_1(\mathbf{A}^{(2)}) \setminus \mathcal{I}_1(\mathbf{A}^{(2)}) = \operatorname{conv} \mathbf{v}_1(\mathbf{A}^{(2)}) \setminus \mathbf{v}_1(\mathbf{A}^{(2)})$. Together, this yields that $\mathbf{v}_1(\mathbf{A}^{(1)}) \not\bowtie \mathbf{v}_1(\mathbf{A}^{(2)})$. Analogously, we have $e_2 \in \mathbf{v}_2(\mathbf{A}^{(3)}) \setminus \mathbf{v}_2(\mathbf{A}^{(4)})$ and $\mathbf{v}_2(\mathbf{A}^{(3)}) \not\bowtie$ $v_2(\mathbf{A}^{(4)}).$

We show below that there is a pair of splines whose knot vectors do not overlap in two directions. The arguments for non-overlapping knot vectors will be the same as before.

Case 1: If $e_2 \in v_2(\mathbf{A}^{(1)})$, and $e_2 \notin v_2(\mathbf{A}^{(2)})$, or vice versa, then $v_2(\mathbf{A}^{(1)}) \not\bowtie v_2(\mathbf{A}^{(2)})$, hence $B_{\mathbf{A}^{(1)}} \not \ltimes B_{\mathbf{A}^{(2)}}$.

Case 2: If $e_2 \in v_2(\mathbf{A}^{(1)})$, and $e_1 \notin v_1(\mathbf{A}^{(4)})$, then $B_{\mathbf{A}^{(1)}} \not\ltimes B_{\mathbf{A}^{(4)}}$. Case 3: If $e_2 \notin v_2(\mathbf{A}^{(1)})$, and $e_1 \notin v_1(\mathbf{A}^{(3)})$, then $B_{\mathbf{A}^{(1)}} \not\ltimes B_{\mathbf{A}^{(3)}}$.

Case 4: If $e_2 \in v_2(\mathbf{A}^{(2)})$, and $e_1 \in v_1(\mathbf{A}^{(4)})$, then $B_{\mathbf{A}^{(2)}} \not \ltimes B_{\mathbf{A}^{(4)}}$.

Case 5: If $e_2 \notin v_2(\mathbf{A}^{(2)})$, and $e_1 \in v_1(\mathbf{A}^{(3)})$, then $B_{\mathbf{A}^{(2)}} \not\ltimes B_{\mathbf{A}^{(3)}}$.

In all cases (see Table 2), the mesh is not strongly dual-compatible.

Remark 6.4. Note that SDC \subset WDC, and hence from [9] we know that the generated splines are linearly independent. However, the reverse direction does not hold, as the mesh illustrated in Figure 11 is WDC, but not SDC (and by Theorem 6.1 not AAS, and by Theorem 6.2 not SGAS).

In Figures 9 and 10, we indicated that the abstract T-junction extensions are a subset of the geometric T-junction extensions. However, this is not the case in general. Consider e.g. Figure 13, which can be constructed by subdividing the lower left cell recursively. Again, the figure shows only the active region. We consider $\mathbf{p} = (3,3)$ and obtain the geometric T-junction extensions given in Figure 13a and the

TABLE 2

The cases considered in the proof of Theorem 6.1 cover all possible configurations. This is a modified version of [12, Table 1].

	$e_1 \in \mathbf{v}_1(\mathbf{A}^{(3)})$		$e_1 \notin \mathbf{v}_1(\mathbf{A}^{(3)})$	
	$e_1 \in \mathbf{v}_1(\mathbf{A}^{(4)})$	$e_1 \notin \mathbf{v}_1(\mathbf{A}^{(4)})$	$e_1 \in \mathbf{v}_1(\mathbf{A}^{(4)})$	$e_1 \notin \mathbf{v}_1(\mathbf{A}^{(4)})$
$\overbrace{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\mathbb{C}}}{\overset{\widetilde{\mathbb{C}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\tilde{\mathbb{C}}}}}}{\overset{\widetilde{\tilde{\tilde{\mathbb{C}}}}}}{\overset{\widetilde{\tilde{\tilde{\mathbb{C}}}}}{\overset{\widetilde{\tilde{\tilde{\tilde{\mathbb{C}}}}}}}{\overset{\widetilde{\tilde{\tilde{\tilde{\mathbb{C}}}}}}}{\overset{\widetilde{\tilde{\tilde{\tilde{\tilde{\mathbb{C}}}}}}}{\overset{\widetilde{\tilde{\tilde{\tilde{\tilde{\tilde{\tilde{\tilde{\tilde}}}}}}}}}}}}}$	case 4	case 2	case 4	case 2
$\psi e_2 \notin \mathbf{v}_2(\mathbf{A}^{(2)})$	case 1, 5	cases 1, 2, 5	case 1	cases 1, 2
$\overbrace{\underbrace{\widetilde{\mathcal{C}}}_{\mathcal{S}^{V}}}^{\widetilde{\mathcal{C}}} e_2 \in \mathrm{v}_2(\mathbf{A}^{(2)})$	cases 1, 4	cases 1	cases 1, 3, 4	cases 1, 3
$v_{\mathfrak{V}}$ $e_2 \notin v_2(\mathbf{A}^{(2)})$	case 5	case 5	case 3	case 3

abstract T-junction extensions shown in Figure 13b. However, the abstract T-junction extensions are a subset of the geometric T-junction extensions if the mesh is analysis-suitable. This is shown below.

Proof of Theorem 6.2. Let \mathfrak{T} be SGAS and $\operatorname{ATJ}_i \neq \emptyset$, then there is a point $x \in \operatorname{ATJ}_i(n) \neq \emptyset$ for some $n \in [0, N_i]$, and Proposition 4.2 yields $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}, \mathbf{T} \in \mathbb{T}_i, \mathbb{Q} = \operatorname{ascell}(\mathbf{T})$ with $x \in \operatorname{supp}_{\Omega} B_{\mathbf{A}}$ and

(6.8)
$$\overline{\mathsf{T}} \cap \operatorname{conv}(P_{i,n}(\mathbf{A}) \cup \{x\}) \neq \varnothing$$

(6.9)
$$\mathbf{Q}_{\mathrm{pdir}(\mathbf{T})} \cap \mathrm{conv}(\mathbf{A}_{\mathrm{pdir}(\mathbf{T})} \cup \{x_{\mathrm{pdir}(\mathbf{T})}\}) \neq \emptyset,$$

(6.10)
$$\exists y \in \mathbf{A}_{\mathrm{pdir}(\mathtt{T})} \colon \quad y \neq x_{\mathrm{pdir}(\mathtt{T})}.$$

We write $v_{\ell}(\mathbf{A}) = (a_1^{(\ell)}, \ldots, a_{p_{\ell}+2}^{(\ell)})$. From $x \in \operatorname{supp}_{\Omega} B_{\mathbf{A}} = \bigotimes_{\ell=1}^{d} [a_1^{(\ell)}, a_{p+2}^{(\ell)}]$ we have $x_i \in [a_1^{(i)}, a_{p_i+2}^{(i)}]$ and hence $P_{i,n}(\mathbf{A}) \subset \operatorname{supp}_{\Omega} B_{\mathbf{A}}$. With (6.8), we get $\overline{\mathsf{T}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}} \neq \emptyset$, and Proposition 5.6 yields $v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathsf{T})$ for all $\ell \neq i$. With Lemma 5.5, we obtain $x_{\ell} \in \operatorname{conv} v_{\ell}(\mathsf{T})$ for $\ell \neq i$. Moreover, we have by construction $x_i \in \{x_i\} = \mathsf{T}_i = \mathsf{v}_i(\mathsf{T}) = \operatorname{conv} \mathsf{v}_i(\mathsf{T})$. Altogether, for any $i \in \{1, \ldots, d\}$ and $x \in \operatorname{ATJ}_i$ there is a T-junction $\mathsf{T} \in \mathbb{T}_i$ with $x \in \bigotimes_{\ell=1}^d \operatorname{conv} v_{\ell}(\mathsf{T}) = \operatorname{GTJ}(\mathsf{T})$, and hence $\operatorname{ATJ}_i \subseteq \operatorname{GTJ}_i$. Since \mathcal{T} is SGAS, we get for any $i \neq j$ that

(6.11)
$$\operatorname{ATJ}_i \cap \operatorname{ATJ}_j \subseteq \operatorname{GTJ}_i \cap \operatorname{GTJ}_j = \emptyset,$$

which concludes the proof.

Proof sketch of Conjecture 6.3. Assume for contradiction a mesh $\mathcal{T} \in WGAS \setminus WDC$. \mathcal{T} being not WDC means that there exist anchors $\mathbf{A}^{(1)}, \mathbf{A}^{(2)} \in \mathcal{A}_{\mathbf{p}}$ with

(6.12)
$$\forall \ell \in \{1, \dots, d\} : v_{\ell}(\mathbf{A}^{(1)}) = v_{\ell}(\mathbf{A}^{(2)}) \lor v_{\ell}(\mathbf{A}^{(1)}) \not\bowtie v_{\ell}(\mathbf{A}^{(2)}),$$

(6.13) and $\exists j \in \{1, \ldots, d\} : \mathbf{v}_j(\mathbf{A}^{(1)}) \not\bowtie \mathbf{v}_j(\mathbf{A}^{(2)}).$

Equation (6.13) and Lemma 4.7 yields a T-junction $T^{(0)} \in \mathbb{T}_j$ with $T_j^{(0)} = \{m^{(0)}\},\$



FIG. 13. Example for $\text{GTJ}_i \subset \text{ATJ}_i$, where $\mathbf{p} = (3, 3)$.

 $k^{(0)} = pdir(\mathbf{T}^{(0)}), \ \mathbf{Q}^{(0)} = ascell(\mathbf{T}^{(0)})$ such that

(6.14)
$$m^{(0)} \in \operatorname{conv} v_j(\mathbf{A}^{(1)}) \cap \operatorname{conv} v_j(\mathbf{A}^{(2)}),$$

(6.15) $\overline{\mathsf{T}^{(0)}} \cap P_{j,m^{(0)}}(\mathrm{MBox}(\mathbf{A}^{(1)},\mathbf{A}^{(2)})) \neq \emptyset \neq \mathsf{Q}_{k^{(0)}} \cap \mathrm{MBox}(\mathbf{A}^{(1)},\mathbf{A}^{(2)})_{k^{(0)}},$

(6.16)
$$\mathbf{A}_{k^{(0)}}^{(1)} \cap \mathbf{A}_{k^{(0)}}^{(2)} = \varnothing.$$

From $\mathbf{A}_{k^{(0)}}^{(1)} \cap \mathbf{A}_{k^{(0)}}^{(2)} = \emptyset$ we conclude that $\mathbf{A}_{k^{(0)}}^{(1)} \neq \mathbf{A}_{k^{(0)}}^{(2)}$, and with (6.12) and Lemma 4.7, we get another T-junction $\mathbf{T}^{(1)} \in \mathbb{T}_{k^{(0)}}$ with $\mathbf{T}_{k^{(0)}}^{(1)} = \{m^{(1)}\}, k^{(1)} = pdir(\mathbf{T}^{(1)}), \mathbf{Q}^{(1)} = ascell(\mathbf{T}^{(1)})$ such that

(6.17)
$$m^{(1)} \in \operatorname{conv} v_{k^{(0)}}(\mathbf{A}^{(1)}) \cap \operatorname{conv} v_{k^{(0)}}(\mathbf{A}^{(2)})$$

(6.18)
$$T^{(1)} \cap P_{k^{(0)}, m^{(1)}}(\operatorname{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})) \neq \emptyset \neq \mathbb{Q}_{k^{(1)}} \cap \operatorname{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})_{k^{(1)}}$$

(6.19)
$$\mathbf{A}_{k^{(1)}}^{(1)} \cap \mathbf{A}_{k^{(1)}}^{(2)} = \emptyset.$$

The very same arguments repeated over again yield an infinite sequence of T-junctions $T^{(0)}, T^{(1)}, T^{(2)}, \ldots$ such that $\operatorname{odir}(T^{(\ell+1)}) = k^{(\ell)} = \operatorname{pdir}(T^{(\ell)})$ and

(6.20)
$$\overline{\mathsf{T}^{(\ell+1)}} \cap P_{k^{(\ell)}, m^{(\ell+1)}}(\mathrm{MBox}(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})) \neq \emptyset \quad \text{for all } \ell.$$

Since the number of T-junctions in the neighborhood of $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ is finite, this sequence is a cycle $\mathbf{T}^{(0)}, \ldots, \mathbf{T}^{(K)} = \mathbf{T}^{(0)}$.

Conjecture: There exists $\ell \in \{0, \ldots, K-1\}$ with $\operatorname{GTJ}(\mathsf{T}^{(\ell)}) \cap \operatorname{GTJ}(\mathsf{T}^{(\ell+1)}) \neq \emptyset$. Then the mesh is not WGAS, which would conclude the proof.

7. Conclusions & Outlook. We have generalized the two existing concepts of analysis-suitability, an abstract concept introduced in [12] and a geometric concept introduced in [7], to arbitrary dimension and degree. We have, except for the WGAS criterion, shown their sufficiency for dual-compatibility and hence linear independence of the T-spline basis, and investigated the implications between all introduced criteria, including counterexamples where an implication does not hold.

Ongoing work includes the implementation of T-splines in two and three dimensions into deal.ii to solve simple elliptic PDEs using T-splines as ansatz functions, including local mesh refinement. Future work includes a proof that WGAS is sufficient for WDC is the three-dimensional case, and the numerical comparison to other approaches.

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Appendix A. Minor proofs.

A.1. Lemma 4.8.

Proof. For any mesh entity \mathbf{E} in the old mesh $\mathcal{T}^{(n)}$ with $\mathbf{E}_{\ell} \subset \overline{\mathbf{Q}_{\ell}}$ for $\ell \neq j$ and $E_i = Q_i$, the subdivision of Q removes E and inserts three children

 $\mathbf{E}^{(1)} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{i-1} \times (\inf \mathbf{Q}_i, \min \mathbf{Q}_i) \times \mathbf{E}_{i+1} \times \cdots \times \mathbf{E}_d,$ (A.1)

(A.2)
$$\mathbf{E}^{(2)} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times \{ \operatorname{mid} \mathbf{Q}_j \} \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d,$$

(A.3)
$$\mathbf{E}^{(3)} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times (\operatorname{mid} \mathbf{Q}_j, \sup \mathbf{Q}_j) \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d,$$

with mid $\mathbf{Q}_i = \frac{1}{2} (\inf \mathbf{Q}_i + \sup \mathbf{Q}_i)$. From the premise of the claim we know that there is no T-junction T with pdir(T) = j in the *j*-orthogonal faces of Q. We define below parents($\hat{\mathbf{A}}$) for any new anchor $\hat{\mathbf{A}}$.

Case 1: p_j is odd, i.e., the anchors' *j*-th components are singletons. For any mesh entity $\mathsf{E}^{(\sup \mathsf{Q}_j)} = \mathsf{E}_1 \times \cdots \times \mathsf{E}_{j-1} \times \{\sup \mathsf{Q}_j\} \times \mathsf{E}_{j+1} \times \cdots \times \mathsf{E}_d$ there is also the entity $\mathsf{E}^{(\inf \mathsf{Q}_j)} = P_{j,\inf \mathsf{Q}_j}(\mathsf{E}^{(\sup \mathsf{Q}_j)})$ and vice versa. This is shown as follows.

Assume for contradiction that there is an entity $\mathbf{E}^{(1)} \subset \partial \mathbf{Q} \cap \mathbf{S}_j(\sup \mathbf{Q}_j)$ with-out counterpart in $\partial \mathbf{Q} \cap \mathbf{S}_j(\inf \mathbf{Q}_j)$. For arbitrary $x^{(1)} \in \mathbf{E}^{(1)}$, its counterpart $x^{(2)} = (x_1^{(1)}, \ldots, x_{j-1}^{(1)}, \inf \mathbf{Q}_j, x_{j+1}^{(1)}, \ldots, x_d^{(1)})$ lies in some *j*-orthogonal entity $\mathbf{E}^{(2)} \subset \partial \mathbf{Q} \cap \mathbf{S}_j(\inf \mathbf{Q}_j)$. Since $\mathbf{E}^{(2)} \neq P_{j,\inf \mathbf{Q}_j}(\mathbf{E}^{(1)})$, there is $k \neq j$ with $\mathbf{E}_k^{(1)} \neq \mathbf{E}_k^{(2)}$.

If $\mathbf{E}_{k}^{(1)}$ and $\mathbf{E}_{k}^{(2)}$ are both singletons, their inequality implies that they are disjoint in contradiction to $x_{k}^{(1)} \in \mathbf{E}_{k}^{(1)} \cap \mathbf{E}_{k}^{(2)} \neq \emptyset$. Hence $\mathbf{E}_{k}^{(1)}$ and $\mathbf{E}_{k}^{(2)}$ are not both singletons. If $\mathbf{E}_{k}^{(1)}$ is a singleton, then $\mathbf{E}_{k}^{(2)}$ is an open interval, and we get $x^{(1)} \in \mathrm{Sk}_{k} \not\ni x^{(2)}$.

If $\mathbf{E}_k^{(1)} \otimes \mathbf{E}_k^{(2)}$ is a singleton, then $\mathbf{E}_k^{(1)}$ is an open interval, and we get $x \in \mathbf{C}$ or $\mathbf{E}_k^{(2)} \neq x^{(1)}$. If similarly $\mathbf{E}_k^{(2)}$ is a singleton, then $\mathbf{E}_k^{(1)}$ is an open interval, and $x^{(1)} \notin \mathbf{S}\mathbf{k}_k \ni x^{(2)}$. If both $\mathbf{E}_k^{(1)}$ and $\mathbf{E}_k^{(2)}$ are open intervals, then $\mathbf{E}_k^{(1)} \neq \mathbf{E}_k^{(2)}$ yields $\mathbf{E}_k^{(1)} \not\subseteq \mathbf{E}_k^{(2)}$ or $\mathbf{E}_k^{(2)} \not\subseteq \mathbf{E}_k^{(1)}$, and we assume without loss of generality the first, i.e. that $\mathbf{E}_k^{(1)} \setminus \mathbf{E}_k^{(2)} \neq \emptyset$. Since $\mathbf{E}_k^{(1)}$ and $\mathbf{E}_k^{(2)}$ are open intervals, there is $y_k \in \mathbf{E}_k^{(1)} \cap \partial \mathbf{E}_k^{(2)}$, and the point $y^{(2)} =$

 $(x_1^{(2)},\ldots,x_{k-1}^{(2)},y_k,x_{k+1}^{(2)},\ldots,x_d^{(2)})$ lies in a k-orthogonal entity $\mathbf{E}^{(3)} \subset \partial \mathbf{E}^{(2)}$, hence $y^{(2)} \in \mathrm{Sk}_k$, while $y^{(1)} = (x_1^{(1)},\ldots,x_{k-1}^{(1)},y_k,x_{k+1}^{(1)},\ldots,x_d^{(1)}) \in \mathbf{E}^{(1)}$ satisfies $y^{(1)} \notin \mathrm{Sk}_k$. In all cases, Lemma 2.3 yields a T-junction T with $\overline{\mathsf{T}} \cap \overline{\mathsf{Q}} \neq \emptyset$, $\mathrm{odir}(\mathsf{T}) = k$,

In all cases, Lemma 2.3 yields a T-junction T with $T \cap \mathbb{Q} \neq \emptyset$, $\operatorname{odir}(T) = k$, pdir(T) = j, $\operatorname{ascell}(T)_j \cap \overline{\mathbb{Q}_j} \neq \emptyset$, since $x^{(1)}$ and $x^{(2)}$ (or $y^{(1)}$ and $y^{(2)}$, respectively) differ only in the *j*-th direction. Let $T_k = \{t\}$, then $\overline{\mathbb{Q}} \cap S_\ell(t) \subset \operatorname{GTJ}(T)$. By Assumption 4.5, \mathbb{Q} has active neighbor cells in at least three directions $\ell_1 \neq \ell_2 \neq \ell_3 \neq \ell_1$. One of these directions is *j* since T is not in the *j*-th frame region, as $\overline{T} \cap \overline{\mathbb{Q}} \neq \emptyset$ and in particular ascell $(T)_j \cap \overline{\mathbb{Q}}_j \neq \emptyset$. At least one of the two remaining directions is not *k*, without loss of generality $\ell_1 \neq k$. The bisection of \mathbb{Q} creates or eliminates a *j*-orthogonal T-junction $T' \subset \partial \mathbb{Q}$ with $\operatorname{pdir}(T') = \ell_1$ and $\emptyset \neq T' \cap \overline{\mathbb{Q}} \cap S_k(t) \subset \operatorname{GTJ}(T') \cap \operatorname{GTJ}(T)$, and $T^{(n)} \notin \operatorname{WGAS}$ or $T^{(n+1)} \notin \operatorname{WGAS}$ in contradiction to above. This shows the claim that each entity $\mathbb{E}^{(\sup \mathbb{Q}_j)} \subset \partial \mathbb{Q} \cap S_j(\sup \mathbb{Q}_j)$ has a counterpart $\mathbb{E}^{(\inf \mathbb{Q}_j)} \subset \partial \mathbb{Q} \cap S_j(\inf \mathbb{Q}_j)$.

For each such pair $\mathbf{E}^{(\inf \mathbf{q}_j)}, \mathbf{E}^{(\sup \hat{\mathbf{q}}_j)}$, the new mesh contains the entity $\mathbf{E}^{(\min \hat{\mathbf{q}}_j)} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_{j-1} \times \{\min \mathbf{q}_j\} \times \mathbf{E}_{j+1} \times \cdots \times \mathbf{E}_d$. This particularly holds for the anchors, i.e., $\mathcal{A}_{\mathbf{p}}^{(n)}$ contains pairs $(\mathbf{A}^{(\inf \mathbf{q}_j)}, \mathbf{A}^{(\sup \mathbf{q}_j)})$ that lie in the boundary of \mathbf{Q} , and for each such pair, $\mathcal{A}_{\mathbf{p}}^{(n+1)}$ contains an anchor $\hat{\mathbf{A}} = \mathbf{A}^{(\min \mathbf{q}_j)}$. Consider a new anchor $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$. We call $\mathbf{A}^{(\inf \mathbf{q}_j)}, \mathbf{A}^{(\sup \mathbf{q}_j)}$ the parent anchors of $\hat{\mathbf{A}}$ and write parents $(\hat{\mathbf{A}}) = \{\mathbf{A}^{(\inf \mathbf{q}_j)}, \mathbf{A}^{(\sup \mathbf{q}_j)}\}$.

Case 2: p_j is even, i.e., the anchors' *j*-th components are open intervals. The subdivision of Q removes any $\mathbf{A} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_d$ with $\mathbf{A}_j = Q_j$ and inserts $\hat{\mathbf{A}}^{(\inf Q_j)}, \hat{\mathbf{A}}^{(\sup Q_j)}$ with

(A.4)
$$\hat{\mathbf{A}}^{(\inf \mathbf{Q}_j)} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_{j-1} \times (\inf \mathbf{Q}_j, \min \mathbf{Q}_j) \times \mathbf{A}_{j+1} \times \cdots \times \mathbf{A}_d,$$

(A.5)
$$\hat{\mathbf{A}}^{(\sup \mathbf{Q}_j)} = \mathbf{A}_1 \times \cdots \times \mathbf{A}_{j-1} \times (\operatorname{mid} \mathbf{Q}_j, \sup \mathbf{Q}_j) \times \mathbf{A}_{j+1} \times \cdots \times \mathbf{A}_d.$$

We call **A** the parent anchor of $\hat{\mathbf{A}}^{(\inf \mathbf{Q}_j)}$ and $\hat{\mathbf{A}}^{(\sup \mathbf{Q}_j)}$ and write parents $(\hat{\mathbf{A}}^{(\inf \mathbf{Q}_j)}) = \{\mathbf{A}\}$ and parents $(\hat{\mathbf{A}}^{(\sup \mathbf{Q}_j)}) = \{\mathbf{A}\}$.

In both cases, any new anchor $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$ is in direction j aligned with its parent $\mathbf{A} \in \text{parents}(\hat{\mathbf{A}})$, and hence shares the same global index vector $\mathcal{I}_j(\hat{\mathbf{A}}) = \mathcal{I}_j(\mathbf{A})$ in the new mesh $\mathcal{T}^{(n+1)}$. In what follows, we use the local index vector $\mathbf{v}_j(\hat{\mathbf{A}})$ of the old anchor with respect to the old and new mesh, and the local index vector $\mathbf{v}_j(\hat{\mathbf{A}})$ of the new anchor with respect to the new mesh. In other directions $k \neq j$, the skeleton Sk_k is unchanged as well as the global and local index vectors $\mathcal{I}_k(\mathbf{A})$, $\mathbf{v}_k(\mathbf{A})$, and therefore these refer to the new mesh, and to the old mesh where applicable. For the existence of T-junctions below, we refer to the new mesh if not stated otherwise.

The subdivision of \mathbb{Q} inserts $\operatorname{mid} \mathbb{Q}_j$ to these global index vectors, such that we have by construction $v_j(\hat{\mathbf{A}}) \subset v_j(\mathbf{A}) \cup {\operatorname{mid} \mathbb{Q}_j}$ and, since $\operatorname{mid} \mathbb{Q}_j \in \operatorname{conv} v_j(\mathbf{A})$, we have $\operatorname{conv} v_j(\hat{\mathbf{A}}) \subset \operatorname{conv} v_j(\mathbf{A})$. If $v_\ell(\hat{\mathbf{A}}) = v_\ell(\mathbf{A})$ for all $\ell \neq j$, this yields $\operatorname{supp}_{\Omega} B_{\hat{\mathbf{A}}} \subset \operatorname{supp}_{\Omega,\mathcal{T}^{(n)}} B_{\mathbf{A}}$.

Assume for contradiction that there is $\hat{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$, $\mathbf{A} \in \text{parents}(\hat{\mathbf{A}})$, and $k \neq j$ with $v_k(\hat{\mathbf{A}}) \neq v_k(\mathbf{A})$. Since $\hat{\mathbf{A}}_k = \mathbf{A}_k$, the middle entries of $v_k(\hat{\mathbf{A}})$ and $v_k(\mathbf{A})$ coincide by construction. Consequently, there is some $m < \inf \mathbf{Q}_k$ or $m > \sup \mathbf{Q}_k$ with

(A.6)
$$\mathbf{v}_k(\hat{\mathbf{A}}) \ni m \in \operatorname{conv} \mathbf{v}_k(\mathbf{A}) \setminus \mathbf{v}_k(\mathbf{A}) \text{ or } \mathbf{v}_k(\mathbf{A}) \ni m \in \operatorname{conv} \mathbf{v}_k(\hat{\mathbf{A}}) \setminus \mathbf{v}_k(\hat{\mathbf{A}}).$$

Without loss of generality we assume the latter cases, i.e., $m > \sup Q_k$ and $v_k(\mathbf{A}) \ni m \in \operatorname{conv} v_k(\hat{\mathbf{A}}) \setminus v_k(\hat{\mathbf{A}})$. Lemma 4.7 yields a T-junction T with $\operatorname{odir}(T) = k$, $T_k = \{m\}$,

 $\tilde{Q} = \operatorname{ascell}(T)$ such that

(A.7)
$$\overline{\mathbf{T}} \cap P_{k,m}(\operatorname{MBox}(\hat{\mathbf{A}}, \mathbf{A})) \neq \emptyset,$$

(A.8)
$$\tilde{\mathbf{Q}}_{\mathrm{pdir}(\mathbf{T})} \cap \mathrm{MBox}(\tilde{\mathbf{A}}, \mathbf{A})_{\mathrm{pdir}(\mathbf{T})} \neq \emptyset, \quad \tilde{\mathbf{A}}_{\mathrm{pdir}(\mathbf{T})} \cap \mathbf{A}_{\mathrm{pdir}(\mathbf{T})} = \emptyset.$$

Since $\hat{\mathbf{A}}$ and \mathbf{A} differ only in direction j, we have $\operatorname{pdir}(\mathbf{T}) = j$, and with $\operatorname{MBox}(\hat{\mathbf{A}}, \mathbf{A})_j \subset \overline{\mathbf{Q}}_j$, we get $\tilde{\mathbf{Q}}_j \cap \overline{\mathbf{Q}}_j \neq \emptyset$. Similarly, from $\operatorname{MBox}(\hat{\mathbf{A}}, \mathbf{A}) \subset \overline{\mathbf{Q}}$, we get $T_j \cap \overline{\mathbf{Q}}_j$, and since T_j is a singleton, this is $T_j \subset \overline{\mathbf{Q}}_j$.

Having the existence of T, there is also a T-junction $T^{(0)}$ with the same properties as T, which is closest to Q in direction k. We therefore consider the minimal $m_0 >$ sup Q_k such that there is a T-junction $T^{(0)} \in \mathbb{T}_k(\hat{T})$ with

(A.9)
$$\overline{\mathsf{T}_{\ell}^{(0)}} \cap \hat{\mathbf{A}}_{\ell} \neq \emptyset \quad \text{for all } \ell \notin \{k, j\},$$

(A.10)
$$\operatorname{odir}(\mathbf{T}^{(0)}) = k, \quad \operatorname{pdir}(\mathbf{T}^{(0)}) = j, \quad \mathbf{T}_k^{(0)} = \{m_0\}$$

(A.11)
$$\mathbf{T}_{j}^{(0)} \subset \overline{\mathbf{Q}_{j}}, \quad \mathbf{Q}^{(0)} = \operatorname{ascell}(\mathbf{T}^{(0)}), \quad \mathbf{Q}_{j}^{(0)} \cap \overline{\mathbf{Q}_{j}} \neq \varnothing.$$

Case 1: $v_j(\mathbf{T}^{(0)}) \cap \overline{\mathbf{Q}_j} \subseteq v_j(\hat{\mathbf{A}}) \cap \overline{\mathbf{Q}_j}$. Since $v_j(\hat{\mathbf{A}}) \cap \overline{\mathbf{Q}_j} = {\inf \mathbf{Q}_j, \operatorname{mid} \mathbf{Q}_j, \sup \mathbf{Q}_j}$, this leads to $\operatorname{mid} \mathbf{Q}_j \in \operatorname{conv} v_j(\mathbf{T}^{(0)})$ by construction of local knot vectors.

Since T with $\overline{T}_k = \{m\}$ from above is a k-orthogonal T-junction, it is not in the k-th frame region, and $\sup \mathbb{Q}_k < m < N_k - \lfloor \frac{p_k+1}{2} \rfloor$, i.e. \mathbb{Q} does not touch the k-th frame region in positive direction. Hence for any $x^{(0)} \in \overline{\mathbb{Q}} \cap S_k(\sup \mathbb{Q}_k) \cap S_j(\operatorname{mid} \mathbb{Q}_j)$, the subdivision of \mathbb{Q} creates or eliminates a T-junction $T^{(1)}$ with

(A.12)

$$\operatorname{odir}(\mathsf{T}^{(1)}) = j, \quad \operatorname{pdir}(\mathsf{T}^{(1)}) = k, \quad \mathsf{T}_k^{(1)} = \{\sup \mathsf{Q}_k\}, \quad \mathsf{T}_j^{(1)} = \{\operatorname{mid} \mathsf{Q}_j\}, \quad x^{(0)} \in \overline{\mathsf{T}^{(1)}}.$$

We choose $x^{(0)}$ such that $x_k^{(0)} = \sup \mathbf{Q}_k$, $x_j^{(0)} = \min \mathbf{Q}_j$, and $x_\ell^{(0)} \in \overline{\mathbf{T}_\ell^{(0)}} \cap \hat{\mathbf{A}}_\ell$ for all $\ell \notin \{k, j\}$. This yields

(A.13)
$$\begin{aligned} x_{\ell}^{(0)} \in \overline{\mathsf{T}_{\ell}^{(1)}} \cap \overline{\mathsf{T}_{\ell}^{(0)}} \subseteq \mathsf{v}_{\ell}(\mathsf{T}^{(1)}) \cap \operatorname{conv} \mathsf{v}_{\ell}(\mathsf{T}^{(0)}) \neq \varnothing \quad \text{for all } \ell \notin \{k, j\}, \\ \text{and} \quad \operatorname{mid} \mathsf{Q}_{j} \in \mathsf{v}_{j}(\mathsf{T}^{(1)}) \cap \operatorname{conv} \mathsf{v}_{j}(\mathsf{T}^{(0)}) \neq \varnothing. \end{aligned}$$

Case 1.1: $v_k(\mathbf{T}^{(1)}) \cap (\sup \mathbf{Q}_k, m_0) \subseteq v_k(\hat{\mathbf{A}}) \cap (\sup \mathbf{Q}_k, m_0)$. By construction we know that $\#v_k(\hat{\mathbf{A}}) = p_k + 2$, and from $\sup \hat{\mathbf{A}}_k \leq \sup \mathbf{Q}_k$ we get that (A.14)

$$\#\{z \in \mathbf{v}_k(\hat{\mathbf{A}}) \mid z > \sup \mathbf{Q}_k\} = \#\{z \in \mathbf{v}_k(\hat{\mathbf{A}}) \mid z \ge \sup \mathbf{Q}_k\} - 1 \le \left\lceil \frac{p_k + 2}{2} \right\rceil - 1 = \left\lceil \frac{p_k}{2} \right\rceil.$$

From (A.6) and $m_0 \in (\sup \mathbf{Q}_k, m]$ we know that $m_0 \in \operatorname{conv} \mathbf{v}_k(\hat{\mathbf{A}})$, and hence

(A.15)
$$\#(\mathbf{v}_k(\mathbf{T}^{(1)}) \cap (\sup \mathbf{Q}_k, m_0)) \le \#(\mathbf{v}_k(\hat{\mathbf{A}}) \cap (\sup \mathbf{Q}_k, m_0)) \le \lceil \frac{p_k - 2}{2} \rceil.$$

Moreover, from (A.12) we have $T_k^{(1)} = {\sup Q_k}$ and hence by construction

(A.16)
$$\#\{z \in \mathbf{v}_k(\mathbf{T}^{(1)}) \mid z > \sup \mathbf{Q}_k\} = \left\lfloor \frac{p_k+1}{2} \right\rfloor = \left\lceil \frac{p_k}{2} \right\rceil.$$

Together with (A.15), there exists $z \in v_k(T^{(1)})$ with $z \ge m_0$, and hence $m_0 \in \operatorname{conv} v_k(T^{(1)})$. Together with (A.10), this is $m_0 \in \operatorname{conv} v_k(T^{(0)}) \cap v_k(T^{(1)}) \neq \emptyset$, and together with (A.13), $T^{(n)}$ or $T^{(n+1)}$ is not WGAS in contradiction to the assumption.

Case 1.2: There exists some $m_2 \in v_k(\mathbb{T}^{(1)}) \cap (\sup \mathbb{Q}_k, m_0) \setminus v_k(\hat{\mathbf{A}})$. Lemma 4.6 yields that for any $x^{(1)} \in P_{k,m_2}(\hat{\mathbf{A}}), y^{(1)} \in \overline{P_{k,m_2}(\mathbb{T}^{(1)})}$ holds $x^{(1)} \notin \operatorname{Sk}_k \ni y^{(1)}$. Choose $x^{(1)}, y^{(1)}$ such that $x_{\ell}^{(1)} = y_{\ell}^{(1)}$ for all $\ell \neq j$. This is possible because $x_k^{(1)} = y_k^{(1)}$ holds trivially and $x_{\ell}^{(0)} \in \overline{\mathsf{T}}_{\ell}^{(0)} \cap \overline{\mathsf{T}}_{\ell}^{(1)} \cap \overline{\hat{\mathbf{A}}_{\ell}}$ for $\ell \notin \{k, j\}$ from (A.13) and above. Lemma 2.3 yields another T-junction $\mathsf{T}^{(2)}$ and $\mathbb{Q}^{(2)} = \operatorname{ascell}(\mathsf{T}^{(2)})$ with

(A.17)
$$\operatorname{odir}(\mathbf{T}^{(2)}) = k, \quad x_{\operatorname{pdir}(\mathbf{T}^{(2)})}^{(1)} \neq y_{\operatorname{pdir}(\mathbf{T}^{(2)})}^{(1)} \text{ and hence } \operatorname{pdir}(\mathbf{T}^{(2)}) = j,$$

(A.18)
$$\mathbf{T}_k^{(2)} = \{m_2\}, \quad \overline{\mathbf{T}^{(2)}} \cap \operatorname{conv}\{x^{(1)}, y^{(1)}\} \neq \emptyset, \quad \mathbf{Q}_j^{(2)} \cap \overline{\mathbf{Q}_j} \neq \emptyset$$

From $\overline{\mathsf{T}^{(2)}} \cap \operatorname{conv}\{x^{(1)}, y^{(1)}\} \neq \emptyset$ and $\mathsf{T}_j^{(2)}$ being a singleton, we get

(A.19)
$$\mathbf{T}_{j}^{(2)} \subseteq \operatorname{conv}\left(\overline{\hat{\mathbf{A}}_{j}} \cup \overline{\mathbf{T}_{j}^{(1)}}\right) \subseteq \overline{\mathbf{Q}_{j}},$$

(A.20)
$$x_{\ell}^{(1)} = y_{\ell}^{(1)} \in \overline{\mathsf{T}^{(2)}}_{\ell} \cap \hat{\mathbf{A}}_{\ell} \neq \emptyset \quad \text{for all } \ell \notin \{k, j\}$$

in contradiction to the minimality of m_0 . This ends Case 1.

Case 2: There is $m_1 \in v_j(\mathsf{T}^{(0)}) \cap \overline{\mathsf{Q}_j}$ with $m_1 \notin v_j(\hat{\mathbf{A}}) \cap \overline{\mathsf{Q}_j}$. Lemma 4.6 yields that $x^{(0)} \notin \mathrm{Sk}_j \ni y^{(0)}$ holds for all $x^{(0)} \in P_{j,m_1}(\hat{\mathbf{A}}), y^{(0)} \in \overline{P_{j,m_1}}(\mathsf{T}^{(0)})$. We choose $x^{(0)}, y^{(0)}$ such that $x_\ell^{(0)} = y_\ell^{(0)} \in \overline{\mathsf{T}_\ell^{(0)}} \cap \hat{\mathbf{A}}_\ell$ for all $\ell \notin \{k, j\}$, and $x_j^{(0)} = y_j^{(0)} = m_1$. Lemma 2.3 yields $\mathsf{T}^{(2)} \in \mathbb{T}_j$ with

(A.21)
$$\overline{\mathsf{T}^{(2)}} \cap \operatorname{conv}(\overline{P_{j,m_1}(\mathsf{T}^{(0)})} \cup P_{j,m_1}(\hat{\mathbf{A}})) \neq \varnothing, \quad \operatorname{pdir}(\mathsf{T}^{(2)}) = k, \quad \mathsf{T}_j^{(2)} = \{m_1\},$$

(A.22)
$$y_{\ell}^{(0)} \in \mathsf{T}_{\ell}^{(2)} \cap \hat{\mathbf{A}}_{\ell} \neq \emptyset \quad \text{for all } \ell \notin \{k, j\}.$$

From (A.6) and $\sup \mathbf{Q}_k < m_0 \leq m$ we get $m_0 \in \operatorname{conv} \mathbf{v}_k(\hat{\mathbf{A}})$.

Case 2.1: $v_k(T^{(2)}) \cap (\sup Q_k, m_0) \subseteq v_k(\hat{\mathbf{A}}) \cap (\sup Q_k, m_0)$. This leads to $m_0 \in \operatorname{conv} v_k(T^{(2)})$ and hence

(A.23) $m_0 \in \operatorname{conv} v_k(\mathsf{T}^{(0)}) \cap v_k(\mathsf{T}^{(2)}) \neq \emptyset, \quad m_1 \in \operatorname{conv} v_j(\mathsf{T}^{(0)}) \cap v_j(\mathsf{T}^{(2)}) \neq \emptyset,$

(A.24)
$$y_{\ell}^{(0)} \in \operatorname{conv} v_{\ell}(\mathsf{T}^{(0)}) \cap v_{\ell}(\mathsf{T}^{(2)}) \neq \emptyset$$
 for all $\ell \notin \{k, j\}$

(A.25)
$$\operatorname{pdir}(\mathsf{T}^{(0)}) \neq \operatorname{odir}(\mathsf{T}^{(0)}) = k = \operatorname{pdir}(\mathsf{T}^{(2)}) \neq \operatorname{odir}(\mathsf{T}^{(2)}),$$

which means that $\mathcal{T}^{(n+1)}$ is not WGAS in contradiction to the assumption.

Case 2.2: $\exists m_2 \in v_k(\mathbf{T}^{(2)}) \cap (\sup \mathbf{Q}_k, m_0) \setminus v_k(\hat{\mathbf{A}})$. Lemma 4.6 yields that for any $x^{(1)} \in \underline{P}_{k,m_2}(\hat{\mathbf{A}}), \ y^{(1)} \in \overline{P}_{k,m_2}(\mathbf{T}^{(2)})$ holds $x^{(1)} \notin \mathrm{Sk}_k \ni y^{(1)}$. Choose $y^{(1)}$ such that $y_\ell^{(1)} \in \overline{\mathsf{T}}_\ell^{(2)}$ for all $\ell \notin \{k, j\}, \ y_k^{(1)} = m_2 = x_k^{(1)}$, and $\mathsf{T}_j^{(2)} = \{y_j^{(1)}\}$. Lemma 2.3 yields another T-junction $\mathsf{T}^{(3)}$ and $\mathsf{Q}^{(3)} = \operatorname{ascell}(\mathsf{T}^{(3)})$ with

(A.26)
$$\operatorname{odir}(\mathsf{T}^{(3)}) = k, \quad \mathsf{T}_k^{(3)} = \{m_2\}, \quad \mathsf{T}_j^{(3)} \subset \overline{\mathsf{Q}_j}, \quad \mathsf{Q}_j^{(3)} \cap \overline{\mathsf{Q}_j} \neq \emptyset$$

in contradiction to the minimality of m_0 . This ends Case 2.2 and concludes the proof.

A.2. Lemma 5.5.

Proof. We set

(A.27) partsupp
$$(\mathbf{A}, x, \ell) \coloneqq \begin{cases} [\min v_{\ell}(\mathbf{A}), \inf \mathbf{A}_{\ell}] \cup \mathbf{A}_{\ell} & \text{if } x_{\ell} < y \text{ for all } y \in \mathbf{A}_{\ell}, \\ \mathbf{A}_{\ell} & \text{if } x_{\ell} \in \mathbf{A}_{\ell}, \\ \mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] & \text{if } x_{\ell} > y \text{ for all } y \in \mathbf{A}_{\ell}. \end{cases}$$

Then we have by construction for $p_{\ell} \ge 1$ that

(A.28)
$$\operatorname{partsupp}(\mathbf{A}, x, \ell) \supseteq \operatorname{conv}(\mathbf{A}_{\ell} \cup \{x_{\ell}\}).$$

The combination of (5.7) and (A.28) yields

(A.29)
$$\varnothing \neq \overline{\mathsf{T}_{\ell}} \cap \operatorname{conv}(\mathbf{A}_{\ell} \cup \{x_{\ell}\}) \subseteq \overline{\mathsf{T}_{\ell}} \cap \operatorname{partsupp}(\mathbf{A}, x, \ell).$$

We distinguish eight cases illustrated in Table 3.

TABLE 3overview of cases in the proof of Lemma 5.5.

	$i \neq \ell = \operatorname{pdir}(\mathbf{T})$		$i \neq \ell \neq \operatorname{pdir}(\mathtt{T})$	
	$p_\ell \text{ odd}$	p_{ℓ} even	$p_\ell \text{ odd}$	p_{ℓ} even
$x_\ell \in \mathbf{A}_\ell$	case 1	case 2	case 3	case 4
$x_\ell \notin \mathbf{A}_\ell$	case 8	case 7	case 6	case 5

Case 1: $x_{\ell} \in \mathbf{A}_{\ell}, i \neq \ell = \text{pdir}(\mathbf{T})$, and p_{ℓ} is odd. Since p_{ℓ} is odd, \mathbf{A}_{ℓ} is a singleton, i.e. $\mathbf{A}_{\ell} = \{x_{\ell}\}$, in contradiction to the existence of $y \in \mathbf{A}_{\ell}$ with $y \neq x_{\ell}$ from (5.9) above.

Case 2: $x_{\ell} \in \mathbf{A}_{\ell}, i \neq \ell = \text{pdir}(\mathbf{T})$, and p_{ℓ} is even. Then \mathbf{A}_{ℓ} is an open interval and $\mathbf{T}_{\ell} = \{t\}$ is a singleton. From (A.29) we obtain

(A.30)
$$t \in \text{partsupp}(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \subset \text{conv} \, \mathbf{v}_{\ell}(\mathbf{A}) \setminus \mathbf{v}_{\ell}(\mathbf{A}).$$

From the definition of local index vectors we also know $t \in v_{\ell}(T)$, which yields $v_{\ell}(A) \not\bowtie v_{\ell}(T)$ in contradiction to (5.10).

Case 3: $x_{\ell} \in \mathbf{A}_{\ell}, i \neq \ell \neq \text{pdir}(\mathbf{T})$, and p_{ℓ} is odd. Then \mathbf{T}_{ℓ} is an open interval, and we have partsupp $(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} = \{x_{\ell}\}$. Hence $x_{\ell} \in v_{\ell}(\mathbf{A})$. Equation (A.29) yields $x_{\ell} \in \overline{\mathbf{T}_{\ell}} \subseteq \text{conv } v_{\ell}(\mathbf{T})$.

Case 4: $x_{\ell} \in \mathbf{A}_{\ell}, i \neq \ell \neq \text{pdir}(\mathsf{T})$, and p_{ℓ} is even. Then T_{ℓ} and \mathbf{A}_{ℓ} are open intervals, and (A.29) yields that $\overline{\mathsf{T}_{\ell}} \cap \mathbf{A}_{\ell} \neq \emptyset$. Together with (5.10), we have $x_{\ell} \in \mathbf{A}_{\ell} = \mathsf{T}_{\ell} \subset \text{conv} v_{\ell}(\mathsf{T})$.

Case 5: $x_{\ell} \notin \mathbf{A}_{\ell}, i \neq \ell \neq \text{pdir}(\mathbf{T})$, and p_{ℓ} is even. Assume without loss of generality that $x_{\ell} > y$ for all $y \in \mathbf{A}_{\ell}$. In this case, we have $\text{partsupp}(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$ with \mathbf{A}_{ℓ} being an open interval and $x_{\ell} \in [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$. Also \mathbf{T}_{ℓ} is an open interval and (5.10) and (A.29) yield that either $\mathbf{T}_{\ell} = \mathbf{A}_{\ell}$ or $\mathbf{inf} \mathbf{T}_{\ell} \in [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$, this is, $\overline{\mathbf{T}_{\ell}} \cap [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] \neq \emptyset$. The knot vector $v_{\ell}(\mathbf{T})$ contains $\frac{p_{\ell}}{2} + 1$ entries that are not smaller than $\sup \mathbf{T}_{\ell}$ and $\frac{p_{\ell}}{2} + 1$ entries that are not greater than $\mathbf{inf} \mathbf{T}_{\ell}$. The interval $[\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$ contains $\frac{p_{\ell}}{2} + 1$ entries of $v_{\ell}(\mathbf{A})$. Together with (5.10), all entries of $v_{\ell}(\mathbf{A}) \cap [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$ match with entries of $v_{\ell}(\mathbf{T})$, and we get $x_{\ell} \in [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] \subseteq \operatorname{conv} v_{\ell}(\mathbf{T})$.

Case 6: $x_{\ell} \notin \mathbf{A}_{\ell}, i \neq \ell \neq \text{pdir}(\mathbf{T})$, and p_{ℓ} is odd. Then \mathbf{A}_{ℓ} is a singleton $\mathbf{A}_{\ell} = \{s\}$. Assume without loss of generality that $x_{\ell} > s$, then we have $\text{partsupp}(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] = [s, \max v_{\ell}(\mathbf{A})]$ which contains $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries of $v_{\ell}(\mathbf{A})$. As in case 5 above, we have $\overline{T}_{\ell} \cap [s, \max v_{\ell}(\mathbf{A})] \neq \emptyset$, and $v_{\ell}(\mathbf{T})$ containing $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries that are $\geq \sup T_{\ell}$ and $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries that are $\leq \inf T_{\ell}$. Together with (5.10), we get $x_{\ell} \in [s, \max v_{\ell}(\mathbf{A})] \subseteq \text{conv } v_{\ell}(\mathbf{T})$.

Case 7: $x_{\ell} \notin \mathbf{A}_{\ell}, i \neq \ell = \text{pdir}(\mathbf{T})$, and p_{ℓ} is even. Then \mathbf{A}_{ℓ} is an open interval, and we assume without loss of generality $x_{\ell} > y$ for all $y \in \mathbf{A}_{\ell}$, obtaining $\text{partsupp}(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$ with $x_{\ell} \in [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})]$. Since

 $\ell = \operatorname{pdir}(\mathsf{T}), \mathsf{T}_{\ell}$ is a singleton $\mathsf{T}_{\ell} = \{t\} = \overline{\mathsf{T}_{\ell}}$, and (A.29) yields $t \in \operatorname{partsupp}(\mathbf{A}, x, \ell)$. Together with $t \in \mathsf{v}_{\ell}(\mathsf{T})$ and (5.10), we get that $t \in \mathsf{v}_{\ell}(\mathbf{A}) \cap [\sup \mathbf{A}_{\ell}, \max \mathsf{v}_{\ell}(\mathbf{A})]$. The partial index vector $\mathsf{v}_{\ell}(\mathbf{A}) \cap [\sup \mathbf{A}_{\ell}, \max \mathsf{v}_{\ell}(\mathbf{A})]$ contains $\frac{p_{\ell}}{2} + 1$ entries of $\mathsf{v}_{\ell}(\mathbf{A})$, while $\mathsf{v}_{\ell}(\mathsf{T})$ contains $\frac{p_{\ell}}{2} + 1$ entries that are $\geq t$ and $\frac{p_{\ell}}{2} + 1$ entries that are $\leq t$. As in previous cases, we obtain with (5.10) that $\mathsf{v}_{\ell}(\mathbf{A}) \cap [\sup \mathbf{A}_{\ell}, \max \mathsf{v}_{\ell}(\mathbf{A})] \subset \mathsf{v}_{\ell}(\mathsf{T})$ and consequently $x_{\ell} \in [\sup \mathbf{A}_{\ell}, \max \mathsf{v}_{\ell}(\mathbf{A})] \subset \operatorname{conv} \mathsf{v}_{\ell}(\mathsf{T})$.

Case 8: $x_{\ell} \notin \mathbf{A}_{\ell}, i \neq \ell = \text{pdir}(\mathbf{T}), \text{ and } p_{\ell} \text{ is odd. Then } \mathbf{A}_{\ell} \text{ is a singleton } \mathbf{A}_{\ell} = \{s\}.$ Assume without loss of generality that $x_{\ell} > s$, then we have $\text{partsupp}(\mathbf{A}, x, \ell) = \mathbf{A}_{\ell} \cup [\sup \mathbf{A}_{\ell}, \max v_{\ell}(\mathbf{A})] = [s, \max v_{\ell}(\mathbf{A})]$ which contains $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries of $v_{\ell}(\mathbf{A})$. Since $\ell = \text{pdir}(\mathbf{T}), \mathbf{T}_{\ell}$ is a singleton $\mathbf{T}_{\ell} = \{t\} = \overline{\mathbf{T}_{\ell}}, \text{ and } (A.29)$ yields $t \in \text{partsupp}(\mathbf{A}, x, \ell) = [s, \max v_{\ell}(\mathbf{A})]$. Moreover, $t \in \partial \mathbf{Q}_{\ell} = \{\inf \mathbf{Q}_{\ell}, \sup \mathbf{Q}_{\ell}\} \subseteq v_{\ell}(\mathbf{T})$ for the associated cell $\mathbf{Q} = \text{ascell}(\mathbf{T})$ from the definition (4.9). By construction of the knot vector we have $\mathbf{Q}_{\ell} \subset \text{conv } v_{\ell}(\mathbf{T}) \setminus v_{\ell}(\mathbf{T}), \text{ and with } (5.10)$ we obtain $\mathbf{Q}_{\ell} \cap v_{\ell}(\mathbf{A}) = \emptyset$. Consequently $s, \max v_{\ell}(\mathbf{A}) \notin \mathbf{Q}_{\ell}$ and hence we have either $\mathbf{Q}_{\ell} \subset [s, \max v_{\ell}(\mathbf{A})]$ or $\mathbf{Q}_{\ell} \cap [s, \max v_{\ell}(\mathbf{A})] = \emptyset$. Together with (5.8) we have $\mathbf{Q}_{\ell} \subset [s, \max v_{\ell}(\mathbf{A})], \text{ and, since } [s, \max v_{\ell}(\mathbf{A})] = \emptyset$. Together with (5.8) we have $\mathbf{Q}_{\ell} \subset [s, \max v_{\ell}(\mathbf{A})], \text{ and, since } [s, \max v_{\ell}(\mathbf{A})]$ is closed, $\overline{\mathbf{Q}_{\ell}} \subset [s, \max v_{\ell}(\mathbf{A})]$. The combination with (5.10) yields that $\{\inf \mathbf{Q}_{\ell}, \sup \mathbf{Q}_{\ell}\} \subseteq v_{\ell}(\mathbf{A}) \cap [s, \max v_{\ell}(\mathbf{A})]$. Since $v_{\ell}(\mathbf{T})$ contains $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries that are $\geq \inf \mathbf{Q}_{\ell}$ and $\lceil \frac{p_{\ell}}{2} \rceil + 1$ entries that are $\leq \sup \mathbf{Q}_{\ell}, (5.10)$ yields $v_{\ell}(\mathbf{A}) \cap [s, \max v_{\ell}(\mathbf{A})] \subseteq v_{\ell}(\mathbf{T})$ and hence $x_{\ell} \in [s, \max v_{\ell}(\mathbf{A})] \subset \operatorname{conv} v_{\ell}(\mathbf{T})$.

We have shown the claim in all cases, which concludes the proof.

A.3. Proposition 5.6.

Proof. The proof is by induction over box bisections. As assumed in section 2, \mathcal{T} is constructed via symmetric bisections of boxes from an initial tensor-product mesh. For a tensor-product mesh, the claim is trivially true due to the absence of T-junctions. Assume that the claim is true for an SGAS mesh \mathcal{T} and consider an SGAS mesh $\hat{\mathcal{T}} = \text{SUBDIV}(\mathcal{T}, \mathbf{Q}, j)$ that results from the *j*-orthogonal bisection of a cell $\mathbf{Q} \in \mathcal{T}$. Since this bisection inserts only one *j*-orthogonal hyperface $\mathbf{F} = \mathbf{Q}_1 \times \cdots \times \mathbf{Q}_{j-1} \times \{r\} \times \mathbf{Q}_{j+1} \times \cdots \times \mathbf{Q}_d$ and lower-dimensional entities that are subsets of other, previously present entities, only the *j*-orthogonal skeleton $\mathrm{Sk}_j(\hat{\mathcal{T}}) \supseteq \mathrm{Sk}_j(\mathcal{T})$ is modified, while all other *i*-orthogonal skeletons $\mathrm{Sk}_i(\hat{\mathcal{T}}) = \mathrm{Sk}_i(\mathcal{T}), i \neq j$, remain unchanged. Hence for any anchor or T-junction that exist in both meshes, the local knot vectors (or knot vectors, resp.) remain unchanged in all directions $i \neq j$. In the following, all knot vectors are understood with respect to the refined mesh $\hat{\mathcal{T}}$.

Assume for contradiction that in the new mesh $\hat{\mathcal{T}}$, there exist $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)}$ and a T-junction T with $\overline{\mathsf{T}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}} \neq \emptyset$, and $v_k(\mathbf{A}) \not\bowtie v_k(\mathsf{T})$ for some $k \neq \operatorname{odir}(\mathsf{T})$. The non-overlapping $v_k(\mathbf{A}) \not\bowtie v_k(\mathsf{T})$ means that there is $m \in \{0, \ldots, N_k\}$ with

$$(A.31) \quad \mathbf{v}_k(\mathbf{A}) \ni m \in \operatorname{conv}(\mathbf{v}_k(\mathsf{T})) \setminus \mathbf{v}_k(\mathsf{T}) \quad \text{or} \quad \mathbf{v}_k(\mathsf{T}) \ni m \in \operatorname{conv}(\mathbf{v}_k(\mathbf{A})) \setminus \mathbf{v}_k(\mathbf{A})$$

Lemma 4.6 yields that for any $x \in P_{k,m}(\mathbf{T})$, $y \in P_{k,m}(\mathbf{A})$ holds $x \in \mathrm{Sk}_k \not\ni y$ or $x \notin \mathrm{Sk}_k \ni y$. Lemma 2.3 yields a T-junction $\mathbf{T}' \in \mathbb{T}_k$ and associated cell $\mathbf{Q} = \mathrm{ascell}(\mathbf{T}')$ with

(A.32) $\overline{\mathsf{T}'} \cap \operatorname{conv}(P_{k,m}(\mathbf{A}) \cup \{x\}) \neq \emptyset,$

(A.33)
$$\mathbf{Q}_{\mathrm{pdir}(\mathbf{T}')} \cap \mathrm{conv}(\mathbf{A}_{\mathrm{pdir}(\mathbf{T}')} \cup \{x_{\mathrm{pdir}(\mathbf{T}')}\}) \neq \emptyset,$$

(A.34)
$$\exists y' \in \mathbf{A}_{\mathrm{pdir}(\mathsf{T}')} \colon y' \neq x_{\mathrm{pdir}(\mathsf{T}')}.$$

We know that there is $z \in \operatorname{supp}_{\Omega} B_{\mathbf{A}} \cap \overline{\mathsf{T}} \neq \emptyset$. We deduce from (A.31) that $\min v_k(\mathbf{A}) \leq m \leq \max v_k(\mathbf{A})$ and hence

(A.35)
$$P_{k,m}(z) = (z_1, \dots, z_{k-1}, m, z_{k+1}, \dots, z_d) \in \operatorname{supp}_{\Omega} B_{\mathbf{A}} \cap P_{k,m}(\mathbf{T})$$

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We choose $x = P_{k,m}(z)$ in (A.32) and obtain

(A.36)
$$\begin{split} & \varnothing \neq \overline{\mathsf{T}'} \cap \operatorname{conv}(P_{k,m}(\mathbf{A}) \cup \{x\}) \\ & \subset \overline{\mathsf{T}'} \cap \operatorname{conv}(P_{k,m}(\mathbf{A}) \cup \operatorname{supp}_{\Omega} B_{\mathbf{A}}) = \overline{\mathsf{T}'} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}}. \end{split}$$

Case 1: $\operatorname{odir}(\mathsf{T}) = j$ and \mathbf{A} is old, i.e. $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \cap \mathcal{A}_{\mathbf{p}}^{(n)}$. For all old anchors and T-junctions from \mathfrak{T} , the knot vectors in directions other than j are unchanged, and the claim is still true. Hence T is a new T-junction with $\mathsf{T}_j = \{r\}$. Since $\operatorname{odir}(\mathsf{T}') = k \neq j$, T' is an old T-junction and we have $v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathsf{T}')$ in the old mesh \mathfrak{T} for all $\ell \neq k$, and consequently

(A.37)
$$v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathbf{T}') \text{ in } \hat{\mathcal{T}} \text{ for all } \ell \notin \{j, k\}$$

The combination of (A.32)–(A.34) and (A.37) and Lemma 5.5 yields $x_{\ell} \in \operatorname{conv} v_{\ell}(\mathsf{T}')$ for all $\ell \notin \{j, k\}$. By construction, we also have

(A.38)
$$x_k \in \{x_k\} = \{m\} = \mathsf{T}'_k = \mathsf{v}_k(\mathsf{T}') = \operatorname{conv} \mathsf{v}_k(\mathsf{T}').$$

Moreover, we have

(A.39)
$$x \in P_{k,m}(\mathsf{T}) \subseteq \mathrm{GTJ}(\mathsf{T})$$

and hence

(A.40)
$$x_{\ell} \in \operatorname{conv} v_{\ell}(\mathsf{T}) \text{ for all } \ell \in \{1, \dots, d\}.$$

If $v_j(\mathbf{A}) \bowtie v_j(\mathbf{T}')$ in $\hat{\mathcal{T}}$, then from Lemma 5.5 there also holds $x_\ell \in \operatorname{conv} v_\ell(\mathbf{T}')$ for $\ell = j$ and hence

(A.41)
$$x \in \bigotimes_{\ell=1}^{d} \operatorname{conv} v_{\ell}(\mathsf{T}') = \operatorname{GTJ}(\mathsf{T}'),$$

and the combination of (A.39) and (A.41) yields that the mesh \hat{T} is not SGAS.

If on the other hand $v_j(\mathbf{A}) \not\bowtie v_j(\mathbf{T}')$ in $\hat{\mathcal{T}}$, then there is $s \in \{0, \dots, N_j\}$ with

(A.42)
$$v_j(\mathbf{A}) \ni s \in \operatorname{conv}(v_j(\mathsf{T}')) \setminus v_j(\mathsf{T}') \text{ or } v_j(\mathsf{T}') \ni s \in \operatorname{conv}(v_j(\mathbf{A})) \setminus v_j(\mathbf{A}).$$

Since T' is an old T-junction with $v_j(\mathbf{A}) \bowtie v_j(\mathbf{T}')$ in the old mesh \mathfrak{T} , and the only entry added to any global knot vector by the subdivision of \mathbb{Q} is r, we obtain s = r and hence

(A.43)
$$r \in \operatorname{conv} v_j(\mathbf{A}) \cap \operatorname{conv} v_j(\mathbf{T}').$$

Since the mesh is supposed to be SGAS, we have $GTJ(T) \cap GTJ(T') = \emptyset$ and hence there is $\ell \in \{1, \ldots, d\}$ with

(A.44)
$$\operatorname{conv} v_{\ell}(\mathsf{T}) \cap \operatorname{conv} v_{\ell}(\mathsf{T}') = \emptyset.$$

Then $\ell = j$, since for $\ell \neq j$ we already found that $x_{\ell} \in \operatorname{conv} v_{\ell}(\mathsf{T}') \cap \operatorname{conv} v_{\ell}(\mathsf{T}) \neq \emptyset$. By definition of T-junction extensions, we have $\operatorname{conv} v_j(\mathsf{T}) = \{r\}$. Together with (A.44), this yields $r \notin \operatorname{conv} v_j(\mathsf{T}')$ in contradiction to (A.43).

Case 2: $\operatorname{odir}(T) \neq j$ and $\mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \cap \mathcal{A}_{\mathbf{p}}^{(n)}$. Then T is an old T-junction since all new T-junctions are *j*-orthogonal. Note that $\hat{\mathcal{T}} \in \operatorname{SGAS}$ eliminates the possibility of

k-orthogonal T-junctions, $k \neq j$, being subdivided, e.g. subdividing cell **Q** in Figure 4 is prohibited. Since the claim was true in \mathcal{T} and only *j*-orthogonal knot vectors have been affected by the bisection, we have k = j. Since we have $v_j(\mathbf{A}) \bowtie v_j(\mathbf{T})$ in \mathcal{T} and $v_j(\mathbf{A}) \bowtie v_j(\mathbf{T})$ in $\hat{\mathcal{T}}$, there is a new T-junction T' that satisfies (A.32). For new T-junctions T' that satisfy (A.36), we have shown in case 1 that the claim $v_\ell(\mathbf{A}) \bowtie v_\ell(\mathbf{T}')$ holds for all $\ell \neq j$. Again, Lemma 5.5 yields $x_\ell \in \text{conv } v_\ell(\mathbf{T}')$ for all $\ell \neq j$. Moreover, $x_j = z_j \in \overline{T_j} \subseteq \text{conv } v_j(\mathbf{T})$. We again obtain $x \in \text{GTJ}(\mathbf{T}) \cap \text{GTJ}(\mathbf{T}') \neq \emptyset$, which concludes this case.

which concludes this case. $Case \ \mathcal{3}: \mathbf{A} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \setminus \mathcal{A}_{\mathbf{p}}^{(n)}$. Lemma 4.8 yields an old anchor $\tilde{\mathbf{A}} \in \mathcal{A}_{\mathbf{p}}^{(n+1)} \cap \mathcal{A}_{\mathbf{p}}^{(n)}$ with $\operatorname{supp}_{\Omega} B_{\mathbf{A}} \subseteq \operatorname{supp}_{\Omega} B_{\tilde{\mathbf{A}}}$ and $v_{\ell}(\mathbf{A}) = v_{\ell}(\tilde{\mathbf{A}})$ for all $\ell \neq j$. Then we have $\overline{\mathbf{T}} \cap \operatorname{supp}_{\Omega} B_{\mathbf{A}} \subseteq \overline{\mathbf{T}} \cap \operatorname{supp}_{\Omega} B_{\tilde{\mathbf{A}}} \neq \emptyset$ and the cases 1 and 2 prove the claim.

Case 3.1: $\operatorname{odir}(T) = j$. Similar to case 1, T' is an old T-junction and we have $v_{\ell}(\tilde{A}) \bowtie v_{\ell}(T')$ in the old mesh \mathcal{T} for all $\ell \neq k$, and consequently

(A.45)
$$v_{\ell}(\mathbf{A}) \bowtie v_{\ell}(\mathbf{T}') \text{ in } \mathcal{T} \text{ for all } \ell \notin \{j, k\}.$$

The combination of (A.32)–(A.34) and (A.45) and Lemma 5.5 yields $x_{\ell} \in \operatorname{conv} v_{\ell}(\mathsf{T}')$ for all $\ell \notin \{j, k\}$. The remaining arguments follow as is case 1.

Case 3.2: $\operatorname{odir}(\mathbf{T}) \neq j$. Then **T** is an old T-junction and k = j as in case 2. We have $v_j(\tilde{\mathbf{A}}) \bowtie v_j(\mathbf{T})$ in \mathfrak{T} and $v_j(\mathbf{A}) \not\bowtie v_j(\mathbf{T})$ in $\hat{\mathfrak{T}}$, and we have $v_j(\mathbf{A}) = v_j(\tilde{\mathbf{A}}) \cup \{r\} \setminus \{s\}$ with $s \in \{\inf v_j(\tilde{\mathbf{A}}), \sup v_j(\tilde{\mathbf{A}})\}$. This leads to $v_j(\mathbf{A}) \ni \{r\} \in \operatorname{conv} v_j(\mathbf{T}) \setminus v_j(\mathbf{T})$. Hence there is a new T-junction **T'** that satisfies (A.32), and the arguments of case 2 follow similarly.