O-minimal spectra, infinitesimal subgroups and cohomology

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Abstract

By recent work on some conjectures of Pillay, each definably compact group G in a saturated o-minimal expansion of an ordered field has a normal "infinitesimal subgroup" G^{00} such that the quotient G/G^{00}, equipped with the "logic topology", is a compact (real) Lie group. Our first result is that the functor $G \mapsto G/G^{00}$ sends exact sequences of definably compact groups into exacts sequences of Lie groups. We then study the connections between the Lie group G/G^{00} and the o-minimal spectrum $\widetilde{G}$ of G. We prove that G/G^{00} is a topological quotient of $\widetilde{G}$. We thus obtain a natural homomorphism Ψ^* from the cohomology of G/G^{00} to the (Čech-)cohomology of $\widetilde{G}$. We show that if G^{00} satisfies a suitable contractibility conjecture then $\widetilde{G^{00}}$ is acyclic in Čech cohomology and Ψ^* is an isomorphism. Finally we prove the conjecture in some special cases.

1 Introduction

We discuss some topics related with Pillay's conjectures [27] for groups definable in an o-minimal expansions of a field. We begin with an example which illustrates the situation in a simplified setting.

Example 1.1. Let $SO(n, \mathbb{R})$ be the special orthogonal group, a compact connected Lie group. Let M be a real closed field properly containing $\mathbb{R}$ and consider the group $SO(n, M)$. Each element of $SO(n, M)$ has coordinates bounded by an element of $\mathbb{R}$ (indeed they are all ≤ 1 in absolute value). Thus there is a surjective group homomorphism $\mathrm{st}\colon SO(n, M) \to SO(n, \mathbb{R})$ where $\mathrm{st}(x)$ is the

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unique $y \in SO(n, \mathbb{R})$ whose coordinates satisfy $|x_i - y_i| < 1/k$ for each positive integer k.

Let $SO(n, M)^{00}$ be the kernel of $\mathrm{st}\colon SO(n, M) \to SO(n, \mathbb{R})$. We call $SO(n, M)^{00}$ the "infinitesimal subgroup" of $SO(n, M)$. The quotient $SO(n, M)/SO(n, M)^{00}$ is isomorphic to $SO(n, \mathbb{R})$ as an abstract group, and also as a topological group provided we give it the following "logic topology" (which is not a quotient topology): a subset $X \subset SO(n, M)/SO(n, M)^{00}$ is closed iff its preimage in $SO(n, M)$ is the intersection of a countable family of semialgebraic sets.

The following provides an intrinsic characterization of $SO(n, M)^{00}$ which is invariant under semialgebraic group isomorphisms over M (unlike the one using the map st).

Proposition 1.2. *$SO(n, M)^{00}$ is the smallest subgroup of $SO(n, M)$ which has index $\leq 2^{\aleph_0}$ and is the intersection of a countable family of semialgebraic sets.*

Proof. By Definition 1.4, Theorem 1.6 and Lemma 2.2. □

It turns out that it is possible to introduce a notion of infinitesimal subgroup and generalize the above observations in a more general context. We consider definable groups $G \subset M^n$ in a big saturated o-minimal expansion M of a real closed field. For a recent survey on definable groups see [20]. The following results give a positive answer to some conjectures of Pillay in [27]. For the definitions of the notions involved (logic-topology, type-definable, bounded index, definably compact) see section 2.

Theorem 1.3. *([3]) Let G be a definable group. Then G has the descending chain condition on type-definable subgroups of bounded index. If $H \lhd G$ is a normal type-definable subgroups of bounded index, then G/H with the logic topology is a compact (real) Lie group.*

Definition 1.4. By the descending chain condition (or by [29]) there exists a smallest type-definable subgroups $G^{00} < G$ of bounded index, which is in fact necessarily normal in G [27, Remark 2.9].

Under the assumption that G is definably compact in the sense of [24] we obtain.

Theorem 1.5. *([16]) Let G be a definably compact definable group. Then the dimension of G/G^{00} as a Lie group equals the o-minimal dimension of G.*

The following result handles the situation that we encountered in Example 1.1.

Theorem 1.6. *([27, Prop. 3.6] and [21, Fact 4.1]) Let G be a definably compact definable group. If M contains as an elementary substructure an o-minimal expansion of the real field $\mathbb{R}$, and if G is defined over $\mathbb{R}$, then $G/G^{00} \simeq G(\mathbb{R})$ as Lie groups.*

The situation of Theorem 1.6 is not the typical one. In fact, even assuming that M contains $\mathbb{R}$ as an elementary substructure, there are definable groups over

M which are not definably isomorphic to groups defined over $\mathbb{R}$ (this happens even for elliptic curves [23]), so in general we cannot naturally identify G/G^{00} with some group defined over $\mathbb{R}$. Nevertheless G/G^{00} is always a compact Lie group.

Let us now discuss the contributions of this note. A point of view which until now has not been stressed is that the correspondence $G \mapsto G/G^{00}$ is a functor from definably compact groups and definable group homomorphisms to compact Lie groups and continuous homomorphism. After giving an exposition of the relevant results from [21, 16], we show that this functor is exact, namely it sends short exact sequences $0 \to H \to G \to B \to 0$ to short exact sequences $0 \to H/H^{00} \to G/G^{00} \to B/B^{00} \to 0$. Many of the arguments are implicit in [16]. The extra ingredient needed is a conjugacy theorem for maximal tori proved in [11] (see also [2] for a different proof). As a side-product we characterize G^{00} as the unique type-definable subgroup of G of bounded index which is divisible and torsion free (this was proved in [16] if G is abelian and definably connected).

The exactness of the functor $G \mapsto G/G^{00}$ may find applications in the attempt to compare the topological invariants of G and G/G^{00}. This idea however is still to be explored.

In the second part of the paper we study the connection between G/G^{00} and the space $\widetilde{G}$ consisting of the types of G (over a small model $M_0 \prec M$) with the spectral topology. The spectral space $\widetilde{G}$ is an o-minimal analogue of the real spectrum of real semialgebraic geometry [6, 5, 25] and it is a quasi-compact normal topological space. Note that $\widetilde{G}$ is not Hausdorff and does not carry a group structure. There is a natural surjective map $\Psi\colon \widetilde{G} \to G/G^{00}$ sending a type p to the coset $gG^{00} \in G/G^{00}$ of a realization g of p. We show that Ψ is a continuous closed map, and therefore it is a quotient map, namely a subset of U of G/G^{00} is open in the logic topology if and only if $\Psi^{-1}(U)$ is open in the spectral topology. Thus we can identify the topological space G/G^{00} (with the logic topology) as a quotient of $\widetilde{G}$ modulo the equivalence relation $\ker(\Psi)$ (with the quotient topology).

The map Ψ induces an homomorphism $\Psi^*\colon \check{H}^n(G/G^{00}) \to \check{H}^n(\widetilde{G})$ in Čech cohomology. In the last part of the paper we show that if G^{00} satisfies a certain contractibility property, then $\widetilde{G^{00}}$ is acyclic in Čech cohomology and Ψ^* is an isomorphism. In this case we also obtain a natural isomorphism $H^n(G/G^{00}) \simeq H^n_{\mathrm{df}}(G)$ where H^n denotes the singular cohomology functor and H^n_{df} is the definable singular cohomology functor studied in [13] and based on the definable singular homology of [32].

2 Logic topology

As in [27, 3], we adopt the model theoretic convention of working in a "universal domain". So let M be a saturated o-minimal L-structure of cardinality κ, where κ is say inaccessible, and strictly greater than the cardinality of the language of M. By a **definable set** in M we mean a subset of some M^n defined by an L-formula with additional parameters from M. By a **type-definable set** in M we mean a subset of M^n which is the intersection of $< \kappa$-many definable sets. If $A \subset M$, we say that X is (type-)definable over A if X can be defined by a formula (set of formulas) with parameters from A. If X is a (possibly type)-definable set and E is a type-definable equivalence relation on X, then we call E **bounded** (or of **bounded index**) if $|X/E| < \kappa$. The real meaning of boundedness is that X/E does not increase when we pass to an elementary extension $M' \succ M$. More precisely the canonical injection $i\colon X/E \to X(M')/E(M')$, $i(a/E) = a/E(M')$, is a bijection [27, Fact 2.1]. If E is a bounded type-definable equivalence relation on X, we put on X/E the **logic topology**, in which a subset $Z \subseteq X/E$ is defined to be closed if its preimage under the natural map $\pi\colon X \to X/E$ is type-definable. Then X/E is a compact (Hausdorff) topological space [27, Lemma 2.5].

In the sequel by a **small** set we mean a set of cardinality $< \kappa$. So if $M_0 \prec M$ is a small model, each type over M_0 is realized in M.

Lemma 2.1. *([3, Remark 1.6]) Let $M_0 \prec M$ be a small model over which X, E are (type)-defined. Then $Z \subset X/E$ is closed iff its preimage under $\pi\colon X \to X/E$ is type-definable over M_0.*

The proof of Lemma 2.1 depends on [18] but there is an easier proof if we further assume, as we will do in the sequel, that M_0 contains a representative from each equivalence class (this can be done keeping M_0 small since E is bounded). In fact we can put on X/E two topologies: the logic topology, and the one which declares a set closed if its preimage in X is type-definable over M_0. The proof in [27, Lemma 2.5] that X/E is compact Hausdorff applies to both topologies (the extra assumption is used to prove the Hausdorff property), and since one is finer than the other they must coincide.

We will be mainly interested in the case when X is a definable group G and E is the equivalence relation induced by the left-cosets of a type-definable subgroup $H < G$. We say that H is bounded if H has a small number of left cosets in G. In this case we can put on G/H the logic topology. So in Example 1.1 the subgroup $SO(n, M)^{00}$ of $SO(n, M)$ is bounded since it has index $2^{\aleph_0}$ (= the cardinality of $SO(n, M)/SO(n, M)^{00} \simeq SO(n, \mathbb{R})$). In general by Theorem 1.3 every normal bounded type-definable subgroup H of a definable group G has index $\leq 2^{\aleph_0}$ since the quotient G/H is a Lie group. (The same conclusion holds if H is not normal since every type-definable bounded subgroup contains a normal type-definable subgroup of bounded index [27].)

By [26] any definable group G has a unique topology which makes it into a "definable manifold". Since we are working over an o-minimal expansion of a field, by the o-minimal version of Robson's embedding theorem [31, Ch. 10, Thm. 1.8], G is definably homeomorphic to a definable submanifold of M^n, where M has the topology generated by the open intervals and M^n the product topology. (The embedding theorem works under very general hypothesis, but for the case of definably compact groups there are simpler proofs, see [1, Thm.

10.7].) So we can assume without loss of generality that G is a submanifold of M^n and the group operation is continuous in the induced topology from M^n. Under this assumption G is **definably compact** in the sense of [24] iff G is closed and bounded in M^n. A definable set $X \subset M^n$ is **definably connected** if X cannot be written as the disjoint union of two definable non-empty open subsets. If G is a definable group (embedded in M^n), G is definably connected iff it has no proper subgroups of finite index [26, Prop. 2.12]. In general if X is a subset of M^n we put on X the topology induced by M^n.

Lemma 2.2. *Let G be a definable group and let $H \lhd G$ be a type-definable normal subgroup of G of bounded index. Then H is the intersection of a countable decreasing sequence Y_i of definable subsets of G. Moreover we can arrange so that $\mathrm{Cl}(Y_{i+1}) \subset \mathrm{Int}(Y_i)$, where Cl and Int denote the closure and the interior in G.*

Proof. Being a Lie group G/H admits a decreasing sequence $O_1 \supset O_2 \supset O_3 \supset \ldots$ of open neighbourhoods of the neutral element e with $\bigcap_i O_i = \overline{\{e\}}$ and $\overline{O_{i+1}} \subset O_i$. The preimages under $\pi\colon G \to G/G^{00}$ of the closed sets $\overline{O_{i+1}}$ and the open sets O_i are respectively type-definable and $\bigvee$-definable (small unions of definable sets). In a saturated model if a type-definable set is contained in a $\bigvee$-definable set, there is a definable set in between. So there are definable sets X_i with $\pi^{-1}(\overline{O_{i+1}}) \subset X_i \subset \pi^{-1}(O_i)$ and we obtain $\bigcap_i X_i = H$. Since π is continuous (Lemma 6.7) $\pi^{-1}(\overline{O_{i+1}})$ is a closed subset of the open set $\pi^{-1}(O_i)$ and $\mathrm{Cl}(X_{i+1}) \subset \pi^{-1}(\overline{O_{i+1}}) \subset \pi^{-1}(O_i) \subset \mathrm{Int}(X_{i-1})$, so we can set $Y_i = X_{2i}$. □

Remark 2.3. If G is defined over $M_0 \prec M$, then G^{00} is type-definable over M_0 ([29] or [16, Prop. 6.1]) and by Lemma 2.1 we can take the sets X_i to be M_0-definable.

3 Generic sets

In this section we recall some results and definitions from [21, 16]. Let G be a definably compact group (in M) and let $\mathrm{Def}(G)$ be the family of all definable subsets of G. A set $X \in \mathrm{Def}(G)$ is **generic** iff finitely many left-translates of X cover G. This is equivalent to require that finitely many right translates of X cover G [16, Prop. 4.2]. If $X \in \mathrm{Def}(G)$ contains a type-definable subgroup of bounded index, then it is easy to see using the saturation of M that X is generic. So in particular if $X \supset G^{00}$, then X is generic. If the union of two definable subsets of G is generic, one of the two is generic [16, Prop. 4.2, Thm. 8.1]. So the non-generic definable subsets of G form an ideal $\mathcal{I}$. Consequently we can introduce an equivalence relation $\sim_{\mathcal{I}}$ on $\mathrm{Def}(G)$ by setting $X \sim_{\mathcal{I}} Y$ iff the symmetric difference $X \triangle Y$ is not generic. Clearly if $g \in G$ and $X \sim_{\mathcal{I}} Y$ then $gX \sim_{\mathcal{I}} gY$. Therefore for each $X \in \mathrm{Def}(G)$ the set

$$\mathrm{Stab}_{\mathcal{I}}(X) = \{g \in G \mid gX \sim_{\mathcal{I}} X\} \tag{1}$$

is a subgroup of G. It is immediate to see that $\mathrm{Stab}_{\mathcal{I}}(X)$ is type-definable. One of the main results of [16] is:

Theorem 3.1. *([16])* $\mathrm{Stab}_{\mathcal{I}}(X)$ *has bounded index in* G*.*

This is a consequence of the following:

Lemma 3.2. *([16, Remark. 4.4, Thm. 8.1]) If* $M_0 \prec M$ *is a model over which* G *is defined, then every generic set* $X \subset G$ *(not necessarily defined over* M_0*) meets* M_0*.*

Most of the results in [16] are stated for simplicity under the assumption that G is definably compact and definably connected, but it is easy to see that in the above results it suffices that G is definably compact. Granted the lemma, to prove Theorem 3.1 one fixes a small model $M_0 \prec M$ over which G is defined. If $g, h \in G$ have the same type over M_0, then the set $gX \triangle hX$ does not meet M_0, and therefore $gX \sim_{\mathcal{I}} hX$. Now $g\,\mathrm{Stab}_{\mathcal{I}}(X) = h\,\mathrm{Stab}_{\mathcal{I}}(X)$ iff $gX \sim_{\mathcal{I}} hX$. So the index of $\mathrm{Stab}_{\mathcal{I}}(X)$ is bounded by the cardinality of $S_G(M_0)$, where $S_G(M_0)$ is the set of types over M_0 of elements of G.

An immediate consequence of Theorem 3.1 is:

Corollary 3.3. *([16])* $G^{00} = \bigcap_{X \in \mathrm{Def}(G)} \mathrm{Stab}_{\mathcal{I}}(X)$*.*

Proof. The $\subset$ inclusion follows from Theorem 3.1. For the opposite inclusion suppose $gX \sim_{\mathcal{I}} X$ for all $X \in \mathrm{Def}(G)$. We want to show $g \in G^{00}$. If not $gG^{00} \cap G^{00} = \emptyset$. Since G^{00} is a small intersection of definable sets, by saturation there is a definable set $X \supset G^{00}$ such that $gX \cap X = \emptyset$. Together with $gX \sim_{\mathcal{I}} X$ this implies $X \in \mathcal{I}$. This is absurd since $X \supset G^{00}$. □

We will later need the following:

Corollary 3.4. *Let* G *be a definably compact group. Then*

$$G^{00} = \bigcap_{X \text{ generic}} XX^{-1}.$$

Proof. Let $X \in \mathrm{Def}(G)$ be generic. Then $G^{00} \subset \mathrm{Stab}_{\mathcal{I}}(X) \subset \{g \in G \mid gX \cap X \neq \emptyset\} = XX^{-1}$. So $G^{00} \subset \bigcap_{X \in \mathrm{Def}(G)} XX^{-1}$. For the other inclusion write $G^{00} = \bigcap_i X_i$ where $\{X_i \mid i \in I\}$ is a downward directed small family of definable sets and note that by saturation $(\bigcap_i X_i)(\bigcap_i X_i)^{-1} = \bigcap_i X_i X_i^{-1}$. □

Remark 3.5. If G is definable over M_0, in Corollary 3.3 and Corollary 3.4 we can restrict X to range over the M_0-definable sets.

The sets $X \in \mathrm{Def}(G)$ whose complement is non-generic are stable under finite intersections. It follows that there is a type $p \in S_G(M)$ containing all the sets $X \in \mathrm{Def}(G)$ whose complement is non-generic. Clearly for all $X \in \mathrm{Def}(G)$, if $X \in p$ then X is generic. We call a type with this property a **generic type**.

So each definably compact group G has a generic type. It is shown in [16] that each generic type $p \in S_G(M)$ induces a left-invariant finitely additive measure on $\mathrm{Def}(G)$ by

$$\mu_p(X) = m\{gG^{00} \in G/G^{00} \mid gX \in p\} \tag{2}$$

where m is the Haar measure on the compact Lie group G/G^{00}. The set $\{gG^{00} \mid gX \in p\}$ is well defined by Corollary 3.3. Indeed if $gG^{00} = hG^{00}$, then $gh^{-1} \in G^{00} \subset \mathrm{Stab}_{\mathcal{I}}(X)$, so $gX \sim_{\mathcal{I}} hX$ and therefore $gX \in p \leftrightarrow hX \in p$. Moreover by [16] $\{gG^{00} \in G/G^{00} \mid gX \in p\}$ is a Borel set, so it has a well defined Haar measure. Thus we have:

Theorem 3.6. *([16]) There is a finitely additive left invariant measure*

$$\mu\colon \mathrm{Def}(G) \to [0,1]^{\mathbb{R}}$$

with $\mu(G) = 1$. Moreover for $X \in \mathrm{Def}(G)$, $\mu(X) > 0$ iff X is generic.

For the last statement note that if X is not generic, then $\{gG^{00} \mid gX \in p\}$ is empty, and therefore $\mu(X) = 0$.

Conjecture 3.7. *There is a unique finitely additive measure $\mu\colon \mathrm{Def}(G) \to [0,1]$ with $\mu(G) = 1$.*

Indeed one may even conjecture that $\mu(X) = m(\pi(X))$ where $\pi(X)$ is the image of $X \subset G$ in G/G^{00}. This would imply in particular that if X has empty interior in G, then $m(\pi(X)) = 0$. By [16, Lemma 10.5] this in turn implies the compact domination conjecture of that paper. A possible attempt to prove the conjecture could be based on the ideas of [15, Thm. A, §13, p. 54] where it is proved that a "measurable group" has a unique measure up to a constant factor.

4 Torsion free subgroups

Lemma 4.1. *Let G be a definably compact definably connected abelian group. Then G^{00} is the unique normal type-definable subgroup of bounded index which is torsion free.*

Proof. G^{00} is normal and divisible by [3, Fact 1.8 and §3, claim 3]. By [21, Proof of 3.15] for each n one can find a generic $X \subset G$ such that $\mathrm{Stab}_{\mathcal{I}}(X)$ has no n-torsion. By [16] $G^{00} = \bigcap_X \mathrm{Stab}_{\mathcal{I}}(X)$, so G^{00} is torsion free. Conversely if H is a torsion free type-definable subgroup of G of bounded index, then $H = G^{00}$ by [3, Cor. 1.2]. □

We next investigate the effect of dropping the assumption that G is abelian and definably connected. Recall that every definable group G has a smallest definable subgroup G^0 of finite index [26]. Then clearly G^0 is the unique definably connected definable subgroup of G. Moreover $G^{00} < G^0$ and we easily obtain:

Lemma 4.2. $(G^0)^{00} = G^{00}$.

Lemma 4.3. *If G is a definably compact group and $H < G$ is a definable subgroup, then $G^{00} \cap H < H^0$.*

Proof. By definable choice (see [31]) there is $D \subset G$ such that $G = \bigsqcup_{d \in D} Hd$ (disjoint union) $= HD$. Let $h \in G^{00} \cap H$ and suppose for a contradiction that $h \notin H^0$. Then $hH^0 \cap H^0 = \emptyset$. By the choice of D it then follows that $hH^0 D \cap H^0 D = \emptyset$. Since H^0 has finite index in H, $H^0 D$ is generic in G. Together with $hH^0 D \cap H^0 D = \emptyset$ this implies that $h \notin \mathrm{Stab}_{\mathcal{I}}(H^0 D)$. So by Corollary 3.3 $h \notin G^{00}$, a contradiction. □

Theorem 4.4. *If G is a definably compact group and H is a definable subgroup, then $H^{00} = G^{00} \cap H$.*

Proof. The special case when H is abelian and definably connected was proved in [16, Proof of Thm. 8.1, p. 33]. The argument is the following. By Lemma 4.1 it suffices to show that $G^{00} \cap H$ is torsion free. Fix a positive integer n. Let $H[n] = \{h \in H \mid h^n = e\} < H$. By definable choice there is a definable $X \subset H$ such that $H = \bigsqcup_{h \in H[n]} hX$ and a definable $D \subset G$ such that $G = \bigsqcup_{d \in D} Hd$. Hence $G = \bigsqcup_{h \in H[n]} hXD$. We have $G^{00} \subset \mathrm{Stab}_{\mathcal{I}}(XD)$ and since the latter set intersects $H[n]$ only in the identity element, $G^{00} \cap H[n] = \{e\}$. Hence $G^{00} \cap H$ has no n-torsion.

The general case can be reduced to the special case as follows. By Lemmas 4.2 and 4.3 we can assume that H is definably connected. Indeed from the definably connected case and the Lemmas we deduce $G^{00} \cap H = G^{00} \cap H^0 = (H^0)^{00} = H^{00}$.

It remains to prove the case when H is definably connected. By [11] or [2, Thm.6.12], H is the union of its maximal definably connected abelian definable subgroups $T < H$ (which are moreover conjugates of each other). By the special case $G^{00} \cap T = H^{00} \cap T = T^{00}$. Since the union of all the T's is H we get $G^{00} \cap H = H^{00} = \bigcup_T T^{00}$. □

Remark 4.5. The proof also shows that if G is a definably compact definably connected group, then

$$G^{00} = \bigcup_T T^{00}$$

where T ranges over the maximal definably connected abelian subgroups of G. Since each T^{00} is divisible and torsion free, we conclude that G^{00} is divisible and torsion free.

We can now eliminate the commutativity assumption in Lemma 4.1.

Theorem 4.6. *Let G be a definably compact definably connected group. Then G^{00} is the unique type-definable subgroup of bounded index which is torsion free.*

Proof. By Remark 4.5 $G^{00} = \bigcup_T T^{00}$ where T ranges over the maximal definably connected abelian definable subgroups of G. Each T^{00} is torsion free. Hence so is their union G^{00}. Conversely suppose H is a torsion free type-definable subgroup of G of bounded index. If T is a maximal definably connected abelian definable subgroups of G, then $H \cap T$ has bounded index in T (as $[T : H \cap T] \leq [G : H]$) and is torsion free. Therefore by Lemma 4.1 $H \cap T = T^{00}$. Taking the union over all T's we obtain $H = G^{00}$. □

Dropping the assumption that G is definably connected we still obtain a nice characterization:

Corollary 4.7. *Let G be a definably compact group. Then G^{00} is the unique type-definable subgroup of bounded index which is divisible and torsion free.*

Proof. Let H be a type-definable subgroup of G of bounded index which is divisible and torsion free. Since divisibility is preserved under quotients, the image of H under $\pi\colon G \to G/G^0$ is divisible. But a divisible finite group must be trivial. So $H < G^0$ and by the previous theorem $H = (G^0)^{00} = G^{00}$. □

Corollary 4.8. *Let G be a definably compact group. Then $(G \times G)^{00} = G^{00} \times G^{00}$.*

This last corollary has the following consequence concerning the measure μ_p induced by a generic type (see equation (2) in section 3).

Proposition 4.9. *Let p be a generic type on G. There is a generic type q on $G \times G$ such that for each pair of definable sets $A, B \subset G$ we have $\mu_q(A \times B) = \mu_p(A)\mu_p(B)$.*

Proof. Choose q so that it contains all the sets whose complement is non-generic and all the sets of the form $A \times B$ with $A, B \in p$. This is possible since this collection of sets has the finite intersection property. Notice that q is generic and for every $A, B \in \mathrm{Def}(G)$ we have $A \times B \in q$ iff $A \in p$ and $B \in p$. In fact one direction is by definition of q, and the other follows easily from the fact that if a set is not in a type its complement does. Now by definition $\mu_q(A \times B)$ is the Haar measure of the set $\{(g,h)(G \times G)^{00} \mid (g,h)(A \times B) \in q\}$. By the choice of q and Corollary 4.8 this set is the cartesian product of $\{gG^{00} \mid gA \in p\}$ and $\{hG^{00} \mid hB \in p\}$. Now use the fact that the Haar measure on $G \times G$ is the product measure of the Haar measure on G [15, (3) p. 263]. □

5 Functorial properties

Let $\varphi\colon G \to H$ be a definable homomorphism of definably compact groups. Then φ induces a morphism $F(\varphi)\colon G/G^{00} \to H/H^{00}$ sending gG^{00} to $\varphi(g)H^{00}$. This is well defined since $\varphi(G^{00}) \subset H^{00}$. (Proof: $\varphi^{-1}(H^{00})$ is a type-definable subgroup of G of bounded index, so it contains G^{00}.) It follows easily from the definitions that $F(\varphi)$ is continuous with respect to the logic topologies. Clearly F preserves compositions and identity maps. So we have:

Proposition 5.1. *$F\colon G \mapsto G/G^{00}$ is a functor from definably compact groups and definable homomorphisms to compact Lie groups and continuous homomorphisms.*

Theorem 5.2. *The functor $G \mapsto G/G^{00}$ is exact.*

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H^{00} & \longrightarrow & G^{00} & \longrightarrow & B^{00} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H & \xrightarrow{i} & G & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H/H^{00} & \longrightarrow & G/G^{00} & \longrightarrow & B/B^{00} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Suppose that the middle row is an exact sequence of definably compact groups and definable homomorphisms. This means that i is injective, π is surjective and the image of i is the kernel of π. So $i(H) \lhd G$ and $B \simeq G/i(H)$.

We must prove that the bottom row is an exact sequence of Lie groups. By a routine diagram chasing argument ("nine Lemma") it suffices to show that the top row is exact. We can assume that $H \lhd G$ and $B = G/H$. In this case proving the exactness of the top row amounts to show that $H^{00} = G^{00} \cap H$ and $B^{00} = \pi(G^{00})$. The first equality is Theorem 4.4. For the second note that $\pi(G^{00})$ is a type definable subgroup of B, and since π is onto it has bounded index in B. So $\pi(G^{00})$ contains B^{00}. For the opposite inclusion note that $\pi^{-1}(B^{00})$ is a type-definable subgroup of bounded index of G, so it contains G^{00}. $\square$

6 Spectral spaces

Definition 6.1. ([25]) Let $M_0 \prec M$ be a small model. Given an M_0-definable set X, let $\widetilde{X} = \widetilde{X}(M_0)$ be the set of types $S_X(M_0)$ (which can be identified with the ultrafilters of M_0-definable subsets of X) with the following **spectral topology**: as a basis of open sets we take the sets of the form $\widetilde{U}$ with U a M_0-definable open subset of X. Note that since M_0 is small and M is saturated, every type $p \in \widetilde{X}$ is realized in M (i.e. there is $x \in M^n$ with $x \in \bigcap_{X \in p} X$).

The spectral topology is coarser than the usual **Stone topology** (or **contructible topology**) where one takes as a basis of open sets all the sets of the form $\widetilde{U}$ with $U \subset X$ an arbitrary M_0-definable set, not necessarily open. The spectral topology generalizes the real spectrum of semialgebraic geometry. More precisely, if M_0 is a real closed field, then the set of n-types over M_0 with the spectral topology coincides with the real spectrum of $M_0[X_1, \dots, X_n]$ (see [6, 25]).

The spectral topology is in general not Hausdorff, but it is quasi-compact (every open covering has a finite subcovering) since it is coarser than the constructible topology (which is compact). In fact the spectral topology makes $\widetilde{X}$ into a **spectral space**, namely $\widetilde{X}$ has a basis of quasi-compact open sets stable under finite intersections and every irreducible closed set is the closure of a unique point (for the verification of these properties in the o-minimal case see [25, Lemma 1.1]). Since we are working over an expansion of a field every definable set X is definably normal, namely every pair of disjoint definable closed sets can be separated by disjoint definable open sets (one can use the M-valued metric $|x - y|$). It then follows:

Lemma 6.2. *([12, Thm. 2.12])* $\widetilde{X}$ *is* **normal**, *namely every pair of disjoint closed sets can be separated by disjoint open sets.*

Proof. Every closed set of $\widetilde{X}$ is of the form $\bigcap_{i \in I} \widetilde{A_i}$ where each A_i is an M_0-definable closed subset of X. Consider two closed disjoint subsets $\bigcap_{i \in I} \widetilde{A_i}$ and $\bigcap_{j \in J} \widetilde{B_j}$ of $\widetilde{X}$, where A_i, B_j are closed M_0-definable subsets of X. By the quasi-compactness of $\widetilde{X}$ there are finite subsets I_0 of I and J_0 of J with $\bigcap_{i \in I_0} \widetilde{A_i} \cap \bigcap_{j \in J_0} \widetilde{B_j} = \emptyset$. Since $A \mapsto \widetilde{A}$ is a boolean algebra isomorphism onto its image, we obtain $\bigcap_{i \in I_0} A_i \cap \bigcap_{j \in J_0} B_j = \emptyset$. Since X is definably normal there are disjoint definable open subsets U, V of X with $\bigcap_{i \in I_0} A_i \subset U$ and $\bigcap_{j \in J_0} B_j \subset V$. But then $\widetilde{U}$ and $\widetilde{V}$ are disjoint open subsets of $\widetilde{X}$ which separate $\bigcap_{i \in I} \widetilde{A_i}$ and $\bigcap_{j \in J} \widetilde{B_j}$. ◻

Definition 6.3. (see [16, Example 2.2]) Let X be a definable set and let E be a bounded type-definable equivalence relation. In this situation we always assume that $M_0 \prec M$ is a small model such that X is defined over M_0, E is type-defined over M_0 and each equivalence class of X/E contains an element from M_0 (so $X/E \simeq X(M_0)/E(M_0)$). Let $\widetilde{X} = S_X(M_0)$ with the spectral topology. There is a natural surjective map

$$\Psi \colon \widetilde{X} \to X/E$$

defined as follows. Let $b \in X$ realize the type p. Then $\Psi(p) = \pi(b)$, where $\pi \colon X \to X/E$ is the natural projection.

Note that Ψ is well defined since if a, b have the same type over M_0, then $\pi(a) = \pi(b)$. (If not by the choice of M_0 there is $a' \in X(M_0)$ with $E(a, a')$ and $\neg E(b, a')$, hence $E(x, a')$ is contained in the type of a but not in the type of b.) Moreover Ψ is continuous with respect to the constructible topology on $\widetilde{X}$ and the logic topology on X/E [16, Example 2.2].

Theorem 6.4. *Assume* $\pi \colon X \to X/E$ *is continuous. Then* $\Psi \colon \widetilde{X} \to X/E$ *is continuous with respect to the spectral topology on* $\widetilde{X}$ *and the logic topology on* X/E*.*

Proof. Let $\pi\colon X \to X/E$ be the projection and let $Z \subset X/E$ be a closed subset. Then by Lemma 2.1 we can write

$$\pi^{-1}(Z) = \bigcap_{i \in I} X_i$$

where I is small and for each $i \in I$ X_i is an M_0-definable set. (Requiring that I is small is actually redundant since the family of all the M_0-definable sets is small.)

Claim 1.

$$\Psi^{-1}(Z) = \bigcap_i \widetilde{X_i}.$$

In fact consider a type $p \in \widetilde{X}$ and let $b \in X$ realize p. Then $p \in \bigcap_i \widetilde{X_i}$ iff $b \in \bigcap_i X_i$ iff $\pi(b) \in Z$ iff $p \in \Psi^{-1}(Z)$, where the last equivalence follows from $\Psi(p) = \pi(b)$ (by definition of Ψ).

Having proved the claim, to finish the proof of the main result we use the assumption that π is continuous. This ensures that $\pi^{-1}(Z) = \bigcap_{i \in I} X_i$ is closed. We can further assume that the family $\{X_i \mid i \in I\}$ is downward directed as otherwise we consider its closure under finite intersections. Since I is small and M is saturated, we can easily conclude:

Claim 2.

$$\overline{\bigcap_i X_i} = \bigcap_i \overline{X_i}$$

In fact it suffices to observe that if a definable open set U is disjoint from $\bigcap_i X_i$, then by saturation U is disjoint from a finite sub-intersection. By the claim we can assume that in the representation $\pi^{-1}(Z) = \bigcap_i X_i$ each X_i is an M_0-definable closed subset of X. But then $\widetilde{X_i}$ is closed in the spectral topology, and therefore so is $\Psi^{-1}(Z) = \bigcap_i \widetilde{X_i}$. Thus Ψ is continuous. □

Lemma 6.5. *Assume $\pi\colon X \to X/E$ is continuous. Then $\Psi\colon \widetilde{X} \to X/E$ is a closed map.*

Proof. Let $Z \subset \widetilde{X}$ be closed. Then Z is quasi-compact, and therefore (since Ψ is continuous) $\Psi(Z) \subset X/E$ is quasi-compact. But X/E is compact (Hausdorff), so the quasi-compact subsets of X/E are actually compact, hence closed. □

Corollary 6.6. *Assume $\pi\colon X \to X/E$ is continuous. Then $\Psi\colon \widetilde{X} \to X/E$ is a quotient map, namely a subset $Z \subset X/E$ is open if and only if $\Psi^{-1}(Z)$ is open.*

Proof. It suffices to show that a subset $Z \subset X/E$ is closed iff $\Psi^{-1}(Z)$ is closed. One implication follows from the continuity of Ψ and the other from the fact that Ψ is a closed surjective map. □

Lemma 6.7. *([27, Lemma 3.2]) Let G be a definable group, and let $H < G$ be a type-definable subgroup of bounded index. Then H is open in G. So the natural map $\pi\colon G \to G/H$ is continuous (indeed the preimage of any subset is open).*

By Lemma 6.7 and Corollary 6.6 we have:

Corollary 6.8. *$\Psi\colon \widetilde{G} \to G/H$ is a continuous closed surjective map, hence a quotient map.*

By the corollary G/H is homeomorphic to $\widetilde{G}/\ker(\Psi)$ where $\ker(\Psi)$ is the equivalence relation $\{(p,q) \mid \Psi(p) = \Psi(q)\}$ and $\widetilde{G}/\ker(\Psi)$ has the quotient topology. Thus going to the spectral space we have managed to understand G/H in terms of the quotient topology.

7 Cohomology

Let $X \subset M^n$ be a definable closed and bounded set. By the triangulation theorem X is definably homeomorphic to the geometrical realization $|K|$ (over M) of a finite simplicial complex K. There are various ways one can define the cohomology group $H^n(X)$ (with coefficients in $\mathbb{Z}$) but it is clear that for any reasonable choice $H^n(X)$ should coincide up to isomorphism with the classical groups $H^n(K)$. One possibility is to use sheaf cohomology. Namely one defines $H^n(X)$ as $H^n(\widetilde{X}, Sh(\mathbb{Z}))$ where $Sh(\mathbb{Z})$ is the sheaf generated by the constant sheaf $\mathbb{Z}$ on $\widetilde{X}$ (see [7, 10, 8, 9, 17] in the semialgebraic case and [12] in the o-minimal case). Since $\widetilde{X}$ is a normal spectral space $H^n(\widetilde{X}, Sh(\mathbb{Z}))$ is naturally isomorphic to the Čech cohomology group $\check{H}^n(X;\mathbb{Z})$ (see [5, Prop. 5]). For simplicity of notation we use cohomology with coefficients in $\mathbb{Z}$, but all our considerations apply to any other coefficient group.

Remark 7.1. Equivalently one can work with sheaves directly on X rather than on $\widetilde{X}$ but then one has to consider X not as topological space but as a site in the sense of Grothendieck, which essentially means that the only admissible open covers are the covers by finite families of definable open sets. In this way the sheaves on X are naturally identified with those on $\widetilde{X}$ [5, §1.3].

Remark 7.2. If X is a definably connected set, the spectral space $\widetilde{X}$ is connected but in general it is not path connected (not even locally). So it would not be satisfactory to use singular cohomology for these spaces.

Remark 7.3. It follows from the above discussion and the results in [12] that

$$\check{H}^n(\widetilde{X};\mathbb{Z}) \simeq H^n_{\mathrm{df}}(X;\mathbb{Z})$$

where H^n_{df} is the definably singular cohomology studied in [13] and based on the definable singular homology of [32].

Definition 7.4. By Corollary 6.8 there is a continuous closed surjective map $\Psi\colon \widetilde{G} \to G/G^{00}$. Let Ψ^* be the induced homomorphism in Čech cohomology:

$$\Psi^*\colon \check{H}^n(G/G^{00};\mathbb{Z}) \to \check{H}^n(\widetilde{G};\mathbb{Z})$$

Notice that G/G^{00} is a very nice space (a compact Lie group), so for this space Čech cohomology coincides with singular cohomology.

Conjecture 7.5. *Ψ^* is an isomorphism.*

Remark 7.6. In the abelian case, by a result of Edmundo and Otero in [13], we have

$$H^n(G/G^{00};\mathbb{Z}) \simeq H^n_{\mathrm{df}}(G;\mathbb{Z})$$

where H^n is singular cohomology and H^n_{df} is definable singular cohomology. By the above discussion, we can equivalently use the Čech cohomology groups, thus obtaining $\check{H}^n(G/G^{00};\mathbb{Z}) \simeq \check{H}^n(\widetilde{G};\mathbb{Z})$. This however does not settle Conjecture 7.5 even in the abelian case. Indeed the proof in [13] only tells us that there is an isomorphism, but does not tell us which function does the job. Notice in particular that there is no obvious way in which the map $\pi\colon G \to G/G^{00}$ can induce an homomorphism $H^n(G/G^{00};\mathbb{Z}) \to H^n_{\mathrm{df}}(G;\mathbb{Z})$.

Lemma 7.7. *([12, Remark 2.17]) Let $f\colon X \to Y$ be an M_0-definable definable function. Then f induces a function $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ by $f(p) = \{Z \mid f^{-1}(Z) \in p\}$ where Z ranges over the M_0-definable subsets of Y. We have $\widetilde{f(X)} = \widetilde{f}\widetilde{X}$. Moreover if f is continuous, $\widetilde{f}$ is continuous. So if f is an homeomorphism, then $\widetilde{f}$ is an homeomorphism.*

Definition 7.8. Let $\widetilde{G^{00}} \subset \widetilde{G}$ be the set of types which are realized by elements in G^{00}. Equivalently, if $G^{00} = \bigcap_i X_i$ with X_i M_0-definable, $\widetilde{G^{00}} = \bigcap_i \widetilde{X_i}$ (this does not depend on the choice of the representation).

Lemma 7.9. *Each fiber $\Psi^{-1}(hG^{00})$ of $\Psi\colon \widetilde{G} \to G/G^{00}$ is homeomorphic to $\widetilde{G^{00}}$.*

Proof. Let $G^{00} = \bigcap_i X_i$ with X_i M_0-definable. Let $p \in \widetilde{G}$ and let $g \in G$ realize p. So by definition $\Psi(p) = gG^{00}$. By the choice of M_0 (see Definition 6.3) each coset of G/G^{00} is of the form hG^{00} with h in M_0. We have the following chain of equivalences: $p \in \Psi^{-1}(hG^{00})$ iff $gG^{00} = hG^{00}$ iff $g \in h(\bigcap_i X_i) = \bigcap_i hX_i$ iff $p \in \bigcap_i \widetilde{hX_i}$. So $\Psi^{-1}(hG^{00}) = \bigcap_i \widetilde{hX_i}$. Let $h\colon G \to G$ be left multiplication by h and let $\widetilde{h}\colon \widetilde{G} \to \widetilde{G}$ be the induced map as in Lemma 7.7. Then $\Psi^{-1}(hG^{00}) = \bigcap_i \widetilde{h}\widetilde{X_i} = \widetilde{h}\bigcap_i \widetilde{X_i} = \widetilde{h}\widetilde{G^{00}}$. So $\Psi^{-1}(hG^{00})$ is homeomorphic via $\widetilde{h}$ to $\widetilde{G^{00}}$. □

Our next goal is to investigate the cohomology of $\widetilde{G^{00}}$.

Lemma 7.10. *$\widetilde{G^{00}}$ is a closed subset of $\widetilde{G}$, so it is quasi-compact.*

Proof. G/G^{00} is a compact Hausdorff space, so in particular its points are closed. By Lemma 7.9 $\widetilde{G^{00}}$ is the preimage of a point of G/G^{00} under a continuous map, so it is closed. □

The fact that $\widetilde{G^{00}}$ is closed also follows from the following lemma.

Lemma 7.11. *Suppose $G^{00} = \bigcap_i X_i$ where $\{X_i \mid i \in I\}$ is a downward directed small family of definable sets. Then $G^{00} = \bigcap_i Cl(X_i)$.*

Proof. By Lemma 2.2 $G^{00} = \bigcap_{n\in\mathbb{N}} Y_i$ with $\mathrm{Cl}(Y_{n+1}) \subset \mathrm{Int}(Y_n)$. Given $n \in \mathbb{N}$ by saturation a finite sub-intersection of the X_i's is contained in Y_n, and since the family is directed there is a single X_{i_n} contained in Y_n. Now $\bigcap_i Cl(X_i) \subset \bigcap_n Cl(X_{i_n}) \subset \bigcap_n \mathrm{Cl}(Y_n) = G^{00}$. □

Čech cohomology enjoys the following continuity property. If Y is a compact subset of a normal space X, then the Čech cohomology of Y is the inductive limit of the Čech cohomologies of its open neighbourhoods in X. A readable proof of this fact can be found in [19, Lemma 73.3]. Exactly the same argument shows:

Lemma 7.12. *If Y is a closed subset of a normal quasi-compact space X, then the Čech cohomology of Y is the inductive limit of the Čech cohomologies of its open neighbourhoods in X.*

A more general statement along the same lines, but using sheaf cohomology, is the following:

Lemma 7.13. *([8, Thm. 3.1]) Let $\mathcal{F}$ be an arbitrary sheaf on the normal spectral space $\widetilde{X}$ and let Y be a quasi-compact subset of $\widetilde{X}$. Then the canonical homomorphism*

$$\varinjlim_{\substack{Y\subset U \\ U \text{ open in } \widetilde{X}}} H^q(U;\mathcal{F}) \to H^q(Y;\mathcal{F}|Y)$$

is an isomorphism for every $q \geq 0$.

Remark 7.14. In the terminology of [4, Def. 10.5] this implies that Y is a **taut** subspace of $\widetilde{X}$ (see [4, Thm. 10.6]).

In [8, Thm. 3.1] the above result is stated under the hypothesis that $\widetilde{X}$ is the the real spectrum of a ring, but as observed in [12] the argument actually works for every normal spectral space.

We plan to apply Lemma 7.12 to the closed (hence quasi-compact) subset $\widetilde{G^{00}}$ of the normal spectral space $\widetilde{G}$. In the computation of the inductive limits, we can restrict ourselves to a cofinal sequence of open neighbourhoods of $\widetilde{G^{00}}$. To produce such cofinal sequences we use the following result.

Lemma 7.15. *Suppose $G^{00} = \bigcap_i X_i$ where $\{X_i \mid i \in I\}$ is a downward directed family of M_0-definable sets. Then each open neighbourhood U of $\widetilde{G^{00}}$ in $\widetilde{G}$ contains one of the sets $\widetilde{X_i}$. So if the X_i's are open, then*

$$\check{H}^n(\widetilde{G^{00}}; \mathbb{Z}) = \varinjlim_i \check{H}^n(\widetilde{X_i}; \mathbb{Z}).$$

Proof. By Lemma 7.11 $G^{00} = \bigcap_i Cl(X_i)$. So $\bigcap_i \widetilde{Cl(X_i)} \subset U$. Since $\widetilde{G}$ is quasi-compact, there is a finite sub-intersection of the closed sets $\widetilde{Cl(X_i)}$ which is contained in the open set U. Since the family $\{X_i\}$ is directed, a single $\widetilde{Cl(X_i)}$ is contained in U. So a fortiori $\widetilde{X_i}$ is contained in U. □

In the next section we consider the conjecture that $\widetilde{G^{00}}$ is acyclic in Čech cohomology. Notice that Lemma 7.15 leaves open even the question whether the Čech cohomology of $\widetilde{G^{00}}$ is finitely generated.

8 Contractibility

Let G be a definably compact group. By Lemma 2.2 G^{00} is the intersection of a decreasing sequence of definable open subsets of G.

Conjecture 8.1. *G^{00} is the intersection $\bigcap_{i \in \mathbb{N}} X_i$ of a decreasing sequence $X_0 \supset X_1 \supset X_2 \supset \dots$ of M_0-definable definably contractible open sets.*

Proposition 8.2. *The conjecture holds for the group $G = SO(n, M)$ of Example 1.1.*

Proof. The idea is to use transfer from $\mathbb{R}$ to M. Let T_e be the tangent plane of $SO(n, \mathbb{R})$ at the neutral element e of the group. Every sufficiently small ball around the origin in T_e is semialgebraically homeomorphic to a neighbourhood of e in $SO(n, \mathbb{R})$ via a projection p along the orthogonal complement of T_e in the ambient space $\mathbb{R}^{n^2}$. So we can fix a positive integer k such that p maps the ball of radius $1/k$ in the tangent plane homeomorphically onto a closed neighbourhood X_k of e in $SO(n, \mathbb{R})$. Moreover there are balls $B_k \subset C_k \subset \mathbb{R}^{n^2}$ centered at e and of rational radius b_k and c_k respectively, such that b_k and c_k tend to 0 with $k \to \infty$ and $B_k \cap SO(n, \mathbb{R}) \subset X_k \subset C_k \cap SO(n, \mathbb{R})$. Once k, b_k, c_k have been fixed, these statements are semialgebraic, so they transfer to $SO(n, M)$ by the Tarski-Seidenberg principle. Since $SO(n, M)^{00}$ is the kernel of the standard part map $\mathrm{st}\colon SO(n, M) \to SO(n, \mathbb{R})$ it follows that $\bigcap_k X_k(M) = SO(n, M)^{00}$. Moreover $X_k(M)$ is semialgebraically contractible since it is semialgebraically homeomorphic to a ball in the tangent plane. □

With the same argument one can prove the conjecture for any definable group with "good reduction" in the sense of [16]. We recall the definition:

Definition 8.3. G has **good reduction** if it is definably isomorphic in M to a group G_1 which can be defined over $\mathbb{R}$ in the following sense: there is a sublanguage L_0 of the language of M which contains $+, \cdot$ and there is an elementary substructure M_0 of $M|L_0$ of the form $M_0 = (\mathbb{R}, +, \cdot, \ldots)$ such that G is defined over M_0.

The arguments in Proposition 8.2 establish the following:

Theorem 8.4. *Suppose G is a definably compact group with good reduction. Then the contractibility conjecture 8.1 holds for G.*

In particular the contractibility conjecture 8.1 holds for every definably simple group G, since such groups have good reductions [22, Thm. 5.1]. Another favorable case is the following.

Theorem 8.5. *If G is a definably compact group with $\dim(G) = 1$, then Conjecture 8.1 holds for G.*

Proof. By Lemma 4.2 we can assume that G is definably connected. By [26] G is abelian and it is a definable manifold without boundary. By [28] (see also [27, Prop. 3.5]) for each $x \in G$ there is a definable linear ordering $<_x$ on $G \setminus \{x\}$ such that the topology of $G \setminus \{x\}$ coincides with the topology generated by the open intervals of $<_x$ (recall that we always assume that G is embedded in some M^n with the induced topology). Moreover G has a unique element x of order 2 and by [27, Prop. 3.5] G^{00} is the intersection of all the $<_x$-intervals $[-a, a] \subset G \setminus \{x\}$ where $a \in G$ ranges over the torsion elements of G (with the exclusion of x). By [1, Thm. 6.5] a one-dimensional definably connected definably compact manifold with non-empty boundary is definably homeomorphic to $[0, 1] \subset M$. A similar argument shows that $G \setminus \{x\}$ is definably homeomorphic to $(0, 1) \subset M$. So the $<_x$-intervals are definably homeomorphic to intervals of $(0, 1)$ and therefore are definably contractible. ☐

Let us now study some consequences of Conjecture 8.1. We need:

Lemma 8.6. *Suppose X is an M_0-definable definably contractible set. Then $\widetilde{X}$ is acyclic in Čech cohomology, namely $\widetilde{X}$ is connected and $\check{H}^n(\widetilde{X}; \mathbb{Z}) = 0$ for all $n > 0$.*

Proof. By the verification of the homotopy axiom in [8, Thm. 4.1] in the semialgebraic case and in [12] in the o-minimal case. ☐

Theorem 8.7. *Let G satisfy the contractibility conjecture 8.1. Then $\widetilde{G^{00}}$ is acyclic in Čech cohomology, namely $\widetilde{G^{00}}$ is connected and $\check{H}^n(\widetilde{G^{00}}; \mathbb{Z}) = 0$ for all $n > 0$.*

Proof. By Lemmas 8.6, 7.15 and 7.9. ☐

Corollary 8.8. *Suppose $\widetilde{G^{00}}$ is acyclic in Čech cohomology. Then*

$$\Psi^*\colon \check{H}^n(G/G^{00};\mathbb{Z}) \simeq \check{H}^n(\widetilde{G};\mathbb{Z}).$$

Proof. The idea is to use the Vietoris-Begle mapping theorem. A simple version of the theorem says that if A is compact and $f\colon A \to B$ is a surjective continuous map with acyclic fibers (with respect to Čech or equivalently Alexander-Spanier cohomology), then f induces an isomorphism in cohomology [30]. We cannot directly apply this version of the theorem since $\widetilde{G}$ is quasi-compact and normal but not Hausdorff. A possible way to handle this problem is to work with the subspace $\widetilde{G}^{\max}$ of the closed points of $\widetilde{G}$, which is compact Hausdorff and has the same Čech cohomology. Alternatively we can use the following more general version of the Vietoris-Begle mapping theorem which works for sheaf cohomology and arbitrary topological spaces (we give the version without family of supports):

Lemma 8.9. *([4, Ch. 2, Thm. 11.7]) Let $f\colon X \to Y$ be a closed continuous surjection, let $\mathcal{B}$ be a sheaf on Y. Also assume that each $f^{-1}(y)$ is connected and taut in X and that $H^p(f^{-1}(y); f^*\mathcal{B}_y) = 0$ for $p > 0$ and all $y \in Y$. Then*

$$f^*\colon H^*(Y;\mathcal{B}) \to H^*(X; f^*\mathcal{B})$$

is an isomorphism.

We can apply Lemma 8.9 to the map $\Psi\colon \widetilde{G} \to G/G^{00}$ and the constant sheaf $\mathcal{B} = Sh(\mathbb{Z})$ over G/G^{00}. Then the inverse image $f^*\mathcal{B}$ is the constant sheaf $Sh(\mathbb{Z})$ over $\widetilde{G}$. We have already remarked that in this case sheaf-cohomology coincide with Čech cohomology (over $\mathbb{Z}$). The hypothesis of Lemma 8.9 are verified by Ψ thanks to Theorem 8.7 and Remark 7.14. This proves Corollary 8.8 □

An attempt to prove the contractibility conjecture 8.1, and therefore the acyclicity of $\widetilde{G^{00}}$, could be based on the following.

Proposition 8.10. *Suppose that for each M_0-definable set $X \subset G$ there is a M_0-cell decomposition of X such that, for each cell C of the decomposition, CC^{-1} is M_0-definably contractible. Then Conjecture 8.1 holds for G.*

Proof. By Lemma 2.2 G^{00} is the intersection of a decreasing sequence of M_0-definable open sets O_i ($i \in \mathbb{N}$). Since $O_i \supset G^{00}$, O_i is generic. Since $G^{00} = \bigcap_i O_i$ is a group, by saturation $\bigcap_i O_i O_i^{-1} = (\bigcap_i O_i)(\bigcap_i O_i)^{-1} = G^{00}$. Built inductively cell decompositions $\mathcal{D}_i$ of O_i refining the previous and such that for all $C \in \mathcal{D}_i$, CC^{-1} is definably contractible. At least one of the cells of O_i is generic. Since each cell of O_i is a union of cells of O_{i+1}, each generic cell of O_i contains a generic cell of O_{i+1}. By König's lemma there is an infinite sequence of generic cells $C_i \subset O_i$ with $C_{i+1} \subset C_i$. Although C_i does not need to contain G^{00}, by Lemma 3.4 we have $G^{00} \subset C_i C_i{}^{-1}$. On the other hand $\bigcap_i C_i C_i^{-1} \subset \bigcap_i O_i O_i^{-1} \subset G^{00}$, so $G^{00} = \bigcap_i C_i C_i^{-1}$ and since by construction $C_i C_i^{-1}$ is definably contractible we are done. □

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References

[1] Intersection theory for o-minimal manifolds. Ann. Pure Appl. Logic 107 (2001), no. 1-3, 87–119

[2] A. Berarducci, Zero-groups and maximal tori, Proceedings of the Logic Colloquium 2004, ASL European Summer Meeting, Torino, 25-31 July 2004, AK Peters, To appear.

[3] A. Berarducci, M. Otero, Y. Peterzil, A. Pillay, A descending chain condition for groups definable in o-minimal structures, Annals of Pure and Applied Logic 134 (2005) 303–313

[4] G. E. Bredon, Sheaf theory. Second edition. Graduate Texts in Mathematics, 170. Springer-Verlag, New York, 1997. xii+502 pp.

[5] M. Carral, M. Coste, Normal spectral spaces and their dimension, J. Pure Appl. Algebra 30 (1983) 227-235

[6] M. Coste, M.-F. Roy, La topologie du Spectre Reel, in: D. W. Dubois and T. Recio, eds., Ordered Fields and Real Algebraic Geometry, Contemporary Mathematics 8, 1982, pp. 27-59

[7] H. Delfs, Kohomologie affiner semialgebraischer Räume. Dissetation Regensburg 1980

[8] H. Delfs, The homotopy axiom in semialgebraic cohomology, Journal für die reine und angewandte Mathematik 355 (1985) 108-128

[9] H. Delfs, Homology of locally semialgebraic spaces. Lecture Notes in Mathematics, 1484. Springer-Verlag, Berlin, 1991. x+136 pp.

[10] H. Delfs, M. Knebusch, On the homology of algebraic varieties over real closed fields. J. Reine Angew. Math. 335 (1982), 122–163.

[11] M. Edmundo, A remark on divisibility of definable groups, Math. Logic Quart. 51 (6) (2005) 639–641

[12] M. Edmundo, G. O. Jones, N. J. Peatfield, Sheaf cohomology in o-minimal structures, Preprint, June 30, 2005, pp. 27.

[13] M. Edmundo, M. Otero, Definably compact abelian groups, Journal of Mathematical Logic, 4, n.2 (2004) 163–180

[14] R. Godement, Topologie algébrique et théorie des faisceaux. Troisiìne édition. Publications de l'Institut de Mathèmatique de l'Université de Strasbourg, XIII. Actualitś Scientifiques et Industrielles, No. 1252. Hermann, Paris, 1973. viii+283 pp.

[15] P. R. Halmos, Measure theory, Springer-Verlag 1974

[16] H. Hrushovski, Y. Peterzil, A. Pillay, Groups, measures and the NIP, Preprint July 15, 2005, pp. 43

[17] M. Knebusch, Semialgebraic topology in the last ten years. Real algebraic geometry (Rennes, 1991), 1–36, Lecture Notes in Math., 1524, Springer, Berlin, 1992.

[18] D. Lascar, A. Pillay, Hyperimaginaries and automorphism groups, J. Symb. Logic 66 (2001) 127-143

[19] J. R. Munkres, Elements of algebraic topology. Addison-Wesley Publishing Company, Menlo Park, CA, 1984. ix+454 pp.

[20] M. Otero, A survey on groups definable in o-minimal structures, Preprint 2006.

[21] Y. Peterzil, A. Pillay, Generic sets in definably compact groups, Preprint 2004

[22] Y. Peterzil, A. Pillay, S. Starchenko, Linear groups definable in o-minimal structures. J. Algebra 247 (2002), no. 1, 1–23.

[23] Y. Peterzil, S. Starchenko, Uniform definability of the Weiestrass $\mathcal{P}$-functions and generalized tori of dimension one, Selecta Mathematica, New Series 10 (2004) 525-550

[24] Y. Peterzil and C. Steinhorn, Definable compactness and definable subgroups of o-minimal groups, J. London Math. Soc. 59 (1999) 769–786

[25] A. Pillay, Sheaves of continuous definable functions, The Journal of Symbolic Logic, vol. 53, n. 4 (1988) 1165 - 1169.

[26] A. Pillay, On groups and rings definable in o-minimal structures, J. Pure and Applied Algebra 53 (1988), 239-255

[27] A.Pillay, Type–definability, compact Lie groups, and o-minimality, Journal of Mathematical Logic 4 (2004) 147-162

[28] V. Razenj, One-dimensional groups over an o-minimal structure. Ann. Pure Appl. Logic 53 (1991), no. 3, 269–277

[29] S. Shelah, Minimal bounded index subgroup for dependent theories. Preprint 2005. ArXiv: math.LO/0603652 v1.

[30] E. H. Spanier, Algebraic topology. Corrected reprint. Springer-Verlag, New York-Berlin, 1981. xvi+528 pp.

[31] L. van den Dries, Tame Topology and o–minimal structures, London Math. Soc. Lect. Notes Series 248, Cambridge Univ. Press 1998. x+180 pp.

[32] A. Woerheide, O-minimal homology, PhD Thesis, University of Illinois at Urbana–Champaign, 1996.