Lecture 5, Systems of Distinct Representatives

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Matchings for a bipartite graph

Definition (matching)

Consider a bipartite graph $G(V, E)$ with vertex set $V = X \cup Y$ (every edge has one endpoint in X and one in Y). A matching in G is a subset $M \subset E$ of the edge set such that no vertex is incident with more than one edge in M.

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Definition (complete matching)

A complete matching from X to Y is a matching such that every vertex in X is incident with an edge in M.

Marriage theorem

Theorem

A necessary and sufficient condition for there to be a complete matching from X to Y in G is that $|\Gamma(A)| \geq |A|$ for every $A \subseteq X$.

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Hall's condition

That $|\Gamma(A)| \geq |A|$ for every $A \subseteq X$ is called Hall's condition, or property H.

Marriage theorem

Proof 1.

- It is obvious that Hall's condtion is necessary.
- We prove the sufficiency of the theorem by induction on $n := |X|$.
- When $n = 1$, it is obviously true.
- Assume the theorem is true for all integer $k < n$, we will prove it is true for n. We consider the following two cases.
    1. For any $A \subset X$, $|\Gamma(A)| \geq |A| + 1$. We pick $x \in X$ and a neighbor $y \in Y$ of x, and put edge $\{x, y\}$ in the matching. The remaining graph $(X \setminus \{X\}, Y \setminus \{y\})$ satisfies the condition and by inductive hypothesis, it has a complete matching M', thus $M = M' \cup \{\{x, y\}\}$.
    2. There is set A with $|\Gamma(A)| = |A|$. By induction, $(A, \Gamma(A))$ has a complete matching M_A. For any $B \subseteq X \setminus A$, $|\Gamma(B) \setminus \Gamma(A)| \geq |B|$. Then by induction $(X \setminus A, Y \setminus \Gamma(A))$ has a complete matching M_B. Let $M = M_A \cup M_B$.

Perfect matching

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Problem 5A

1. Show that a finite regular bipartite graph (regular of degree $d > 0$) has a perfect matching.
2. Suppose G is bipartite with vertices $X \cup Y$. Further assume that every vertex in X has the same degree $s > 0$ and every vertex in Y has the same degree t. Prove: If $|X| \leq |Y|$ (equivalently, if $s \geq t$), then there is a complete matching M of X into Y .

A game of card playing

Example

The game

A parlor trick involving a standard deck of 52 cards is as follows. You are dealt five cards at random. You keep one and put the other four (in a specific order) into an envelope which is taken to your partner in another room. Your partner looks at these and announces the name of the fifth card, that you had retained.

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Solution

- Let X be family of the $\binom{N}{5}$ 5-element subsets of the N cards.
- Let Y be the set of $N(N-1)N(N-2)(N-3)$ ordered 4-tuples of distinct cards.
- Let G be a bipartite graph with vertices $X \cup Y$, and edges between X and Y if $S \in X$ contains $(c_1, c_2, c_3, c_4) \in Y$.
- For $N \leq 124$, $|X| \leq |Y|$. Then by Problem 5A(2), there exists a complete matching from X to Y.

System of distinct representatives

Definition

Consider subsets $A_0, A_1, \cdots, A_{n-1}$ of a finite set S. We shall say that this collection has property H (Hall's condition) if (for all k) the union of any k-tuple of subsets A_i has at least k elements. If the union of some k-tuple of subsets contains exactly k elements ($0 < k < n$), then we call this k-tuple a critical block.

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Definition

A system of distinct representatives (SDR) of the sets $A_0, A_2, \cdots, A_{n-1}$ is a sequence of n distinct elements $a_0, \cdots, a_{n-1}$ with $a_i \in A_i, 0 \leq i \leq n-1$.

Lemma 5.2

A function

Let $m_0 \le m_1 \le \cdots \le m_{n-1}$. We define

$$F_n(m_0, m_1, \cdots, m_{n-1}) \triangleq \prod_{i=0}^{n-1} (m_i - i)_*,$$

where $(a)_* \triangleq \max\{1, a\}$.

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Lemma

For $n \ge 1$, let $f_n : \mathbb{Z}^n \to \mathbb{N}$ be defined by

$$f_n(a_0, a_1, \cdots, a_{n-1}) \triangleq F_n(m_0, m_1, \cdots, m_{n-1})$$

if $(m_0, m_1, \cdots, m_{n-1})$ is a nondecreasing rearrangement of the n-tuple $(a_0, \cdots, a_{n-1})$. Then f_n is nondecreasing with respect to each of the variables a_i.

A lower bound for the number of SDRs

Let $N(A_0, \cdots, A_{n-1})$ be the number of SDRs of $(A_0, \cdots, A_{n-1})$.

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Theorem

Let $(A_0, \cdots, A_{n-1})$ be a sequence of subsets of a set S. Let $m_i \triangleq |A_i| (i = 0, \cdots, n-1)$ and let $m_0 \leq m_1 \leq \cdots \leq m_{n-1}$. If the sequence has property H, then

$$N(A_0, \cdots, A_{n-1}) \geq F_n(m_0, \cdots, m_{n-1})$$

Theorem 5.3

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Proof.

The proof is by induction. Clearly the theorem is true for $n = 1$.

1. *There is no critical block.*
   - *Choose any element a of A_0 as its representative and then remove a from all the other sets.*
   - *This yields $A_1(a), \cdots, A_{n-1}(a)$, and for these sets property H still holds.*

Theorem 5.3

cont.

1. ▶ By the induction hypothesis and by Lemma 5.2, we find

$$\begin{aligned} N(A_0, \cdots, A_{n-1}) &\geq \sum_{a \in A_0} f_{n-1}(|A_1(a)|, \cdots, |A_{n-1}(a)|) \\ &\geq \sum_{a \in A_0} f_{n-1}(m_1 - 1, \cdots, m_{n-1} - 1) \\ &= m_0 f_{n-1}(m_1 - 1, \cdots, m_{n-1} - 1) \\ &= F_n(m_0, m_1, \cdots, m_{n-1}) \end{aligned}$$

2. There is a critical block $(A_{\nu_0}, \cdots, A_{\nu_{k-1}})$ with $\nu_0 < \cdots < \nu_{k-1}$ and $0 < k < n$.
   - ▶ Delete all elements of $(A_{\nu_0}, \cdots, A_{\nu_{k-1}})$ from all the other sets A_i which produces $(A'_{\mu_0}, \cdots, A'_{\mu_{l-1}})$, where $\{\nu_0, \cdots, \nu_{k-1}, \mu_0, \cdots, \mu_{l-1}\} = \{0, 1, \cdots, n-1\}, k + l = n$.

Theorem 5.3

- Now both $(A_{\nu_0}, \cdots, A_{\nu_{k-1}})$ and $(A'_{\mu_0}, \cdots, A'_{\mu_{l-1}})$ satisfy property H and SDRs of the two sequences are always disjoint.
- By the induction hypothesis and the lemma, we have

$$\begin{aligned} N(A_0, \cdots, A_{n-1}) &= N(A_{\nu_0}, \cdots, A_{\nu_{k-1}})N(A'_{\mu_0}, \cdots, A'_{\mu_{l-1}}) \\ &\geq f_k(m_{\nu_0}, \cdots, m_{\nu_{k-1}})f_l(|A'_{\mu_0}|, \cdots, |A'_{\mu_{l-1}}|) \\ &\geq f_k(m_{\nu_0}, \cdots, m_{\nu_{k-1}})f_l(m_{\mu_0} - k, \cdots, m_{\mu_{l-1}} - k) \\ &\geq f_k(m_0, \cdots, m_{k-1})f_l(m_{\mu_0} - k, \cdots, m_{\mu_{l-1}} - k). \end{aligned}$$

- Since

$$m_{\nu_{k-1}} \leq |A_{\nu_0} \cup \cdots \cup A_{\nu_{k-1}}| = k,$$

we have

$$(m_r - r)_* = 1 \quad \text{if } k \leq r \leq \nu_{k-1}$$

and

$$(m_{\mu_i} - k - i)_* = 1 \quad \text{if } \mu_i \leq \nu_{k-1}$$

Theorem 5.3

cont.

- It implies that

$$f_k(m_0, \cdots, m_{k-1}) = \prod_{0 \le i \le \nu_{k-1}} (m_i - i)_*$$

and

$$f_l(m_{\mu_0} - k, \cdots, m_{\mu_l} - k) = \prod_{\nu_{k-1} < j < n} (m_i - i)_*$$

whose product proves the results. $\square$

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The minimum number of lines of A that contain all the 1's of A is equal to the maximum number of 1's in A, no two on a line.

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Proof.

- Let m be the minimum number of lines of A containing all the 1's of A.
- Let M be the maximum number of 1's, no two on a line.
- Clearly $m \geq M$.
- Let the minimum covering by lines consist of r rows and s columns $(r + s = m)$. Without loss of generality, these are the first r rows and the first s columns.

König's theorem

cont.

- Define sets $A_i, 1 \le i \le r$, by $A_i \triangleq \{j > s : a_{ij} = 1\}$.
- We claim A_i's satisfy property H. Assume it is not true. Then some k-tuple of the A_i's contained less than k elements. We could replace the corresponding k rows by $k-1$ columns, still covering all the 1's. Contradiction.
- So the A_i's have an SDR. This means that there are r 1's, no two on a line, in the first r rows and not in the first s columns.
- By the same argument there are s 1's, no two on a line, in the first s columns and not in the first r rows.
- This shows that $M \ge r + s = m$.

$\square$

Birkhoff Theorem

Theorem

Let $A = (a_{ij})$ be an $n \times n$ matrix with nonnegative integers as entries, such that every row and column of A has sum l. Then A is the sum of l permutation matrices.

Proof.

- Define $A_i, 1 \leq i \leq n$, by $A_i \triangleq \{j : a_{ij} > 0\}$.
- We claim that A_i satisfy property H.
- For any k-tuple of A_i, the sum of the corresponding rows of A is kl. Since every column of A has sum l, the nonzero entries in the chosen k rows must be in at least k columns.
- An SDR of the A_i's corresponds to a permutation matrix $P = (p_{ij})$ such that $a_{ij} > 0$ if $p_{ij} = 1$.
- Then $A - P$ is a matrix with both row sum and column sum $l - 1$. The theorem follows by induction on l.