Truly Low-Space Element Distinctness and Subset Sum via Pseudorandom Hash Functions

Lijie Chen[1], Ce Jin[1], R. Ryan Williams[1], and Hongxun Wu[2]

[1] MIT
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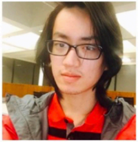
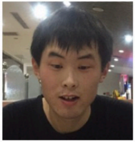

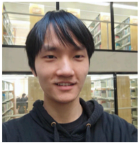

Element Distinctness

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42	3	23	1	12	30	42	15

- INPUT: n positive integers $a_1, a_2, \ldots, a_n$ with $a_i \in [m], m \leq \text{poly}(n)$[1].

[1] Here $[m] = \{1, 2, \ldots, m\}$.

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- INPUT: n positive integers $a_1, a_2, \ldots, a_n$ with $a_i \in [m]$, $m \leq \text{poly}(n)$[1].
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- With linear space, we can simply sort the integers.

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- Here we consider the **low-space** regime where $S = O(\text{polylog}\, n)$.
- Brute force takes $T = O(n^2)$ time.

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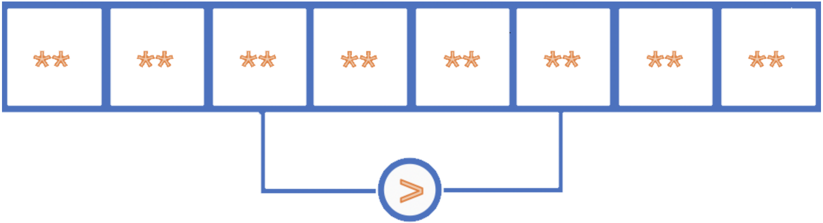
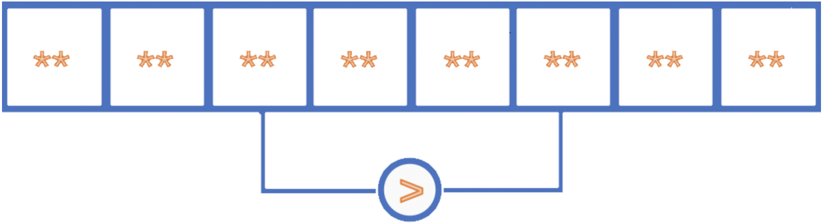
Comparison Model



- No direct access to the INPUT a.
- Each query (i, j) returns one of "$a_i < a_j$", "$a_i = a_j$", "$a_i > a_j$".

Comparison Model



Theorem (Borodin et al., 1987) (Yao, 1988)

When space $S = O(\text{polylog}\, n)$, Element Distinctness requires $T \geq n^{2-o(1)}$ time in comparison model.

More generally, $TS \geq n^{2-o(1)}$ (Yao, 1988).

RAM model

- Random access to read-only input. Allow arbitrary arithmetic and bit operations.
- Surprisingly, in RAM model, one can bypass the $n^{2-o(1)}$ barrier! (Beame, Clifford, and Machmouchi, 2013)

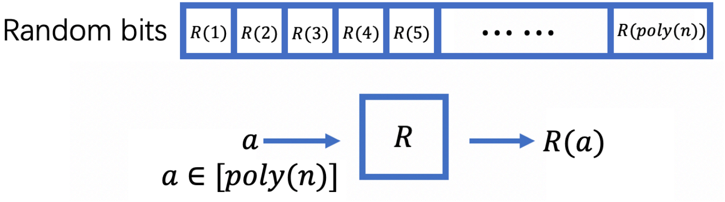
RAM model

Theorem (Beame, Clifford, and Machmouchi, 2013)

Assuming a *Random Oracle*, Element Distinctness can be solved in $S = O(\text{polylog}\, n)$ space and $T = \tilde{O}\left(n^{1.5}\right)$ time in RAM model.

More generally, $T^2 S = \tilde{O}(n^3)$.

Random Oracle Model



- Random access to poly(n) random bits which do not count into space complexity.

Our Results - 1

Our Result: Element Distinctness

~~Assuming a *Random Oracle*~~, Element Distinctness can be solved in $S = O(\text{polylog}\, n)$ space and $T = \tilde{O}\left(n^{1.5}\right)$ time in RAM model.

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- We construct a pseudorandom hash function family with $O(\text{polylog } n)$ seed length to replace the Random Oracle.
- In order to explain our result, let's first review BCM algorithm.

Sketch of BCM - Part 1

Pollard's ρ type algorithm [BCM13]

Assuming a *Random Oracle*, Element Distinctness can be solved in $S = O(\text{polylog}\, n)$ space and $T = \tilde{O}\left(n^{1.5}\right)$ time in RAM model.

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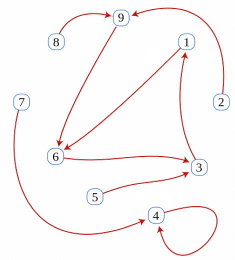
- Implicitly define the directed functional graph $\mathcal{G}_R$ with
  - vertex set $\{1, 2, \ldots, n\}$
  - one outgoing edge $x \mapsto R(a_x)$ for each vertex.



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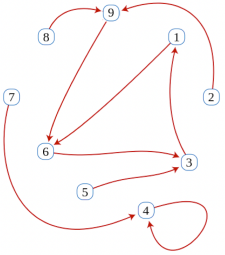
  - vertex set $\{1, 2, \ldots, n\}$
  - one outgoing edge $x \mapsto R(a_x)$ for each vertex.
- If $a_x = a_y$, x and y must point to the same vertex in $\mathcal{G}_R$.



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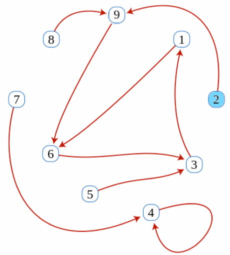

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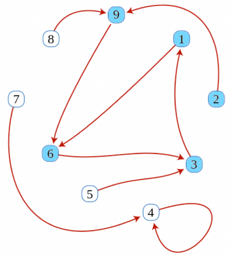
- The vertices reachable from s form a ρ-shape.



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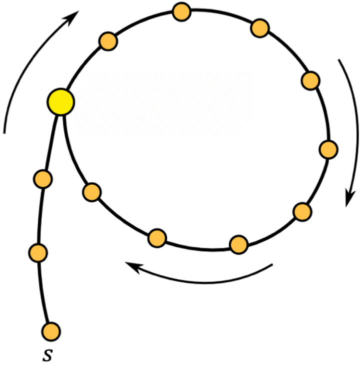

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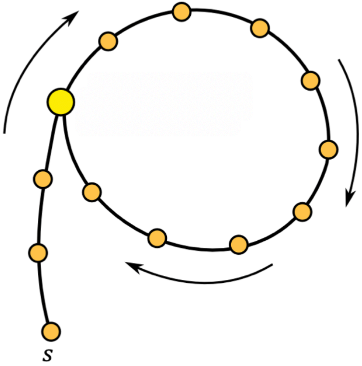
- Perform Floyd's cycle finding from s.



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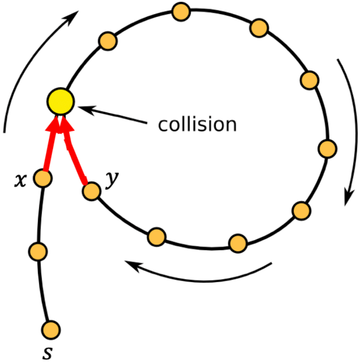
- It takes $O(\log n)$ space and returns $x \neq y$ s.t. $R(a_x) = R(a_y)$.



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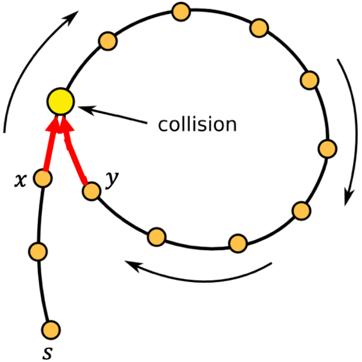
  - a hash collision : $a_x \neq a_y$ but $R(a_x) = R(a_y)$.



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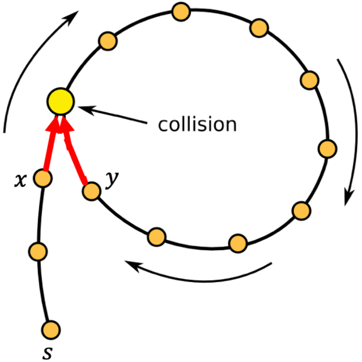
  - a "real" collision : $a_x = a_y$.



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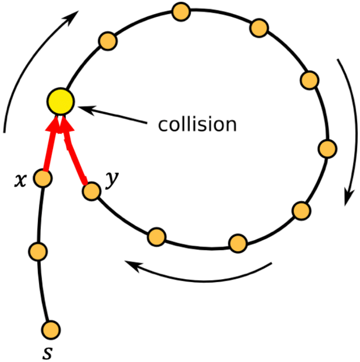
  - a "real" collision : $a_x = a_y$.
- For any "real" collision (x, y), it is found iff x, y are reachable from s.

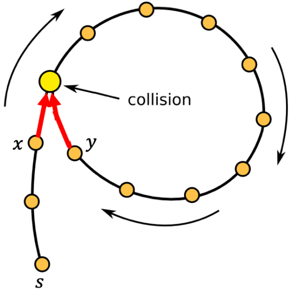


Sketch of BCM - Part 2

Birthday Paradox Type Properties [BCM13]

Let $s \in [n]$ be a uniform random starting point. In functional graph $\mathcal{G}_R$,

W.l.o.g. assume that there is only one pair of $x < y$, $a_x = a_y$.



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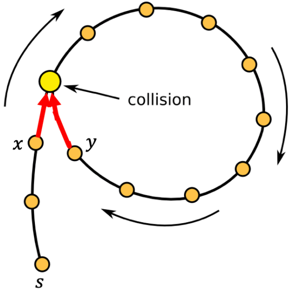

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- $\Pr_{R,s}[u, v \text{ are reachable from } s] \ge \Omega(1/n), \ \forall u, v \in [n]$

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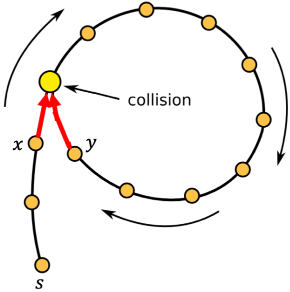

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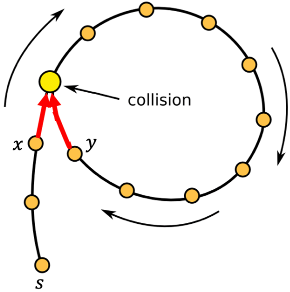

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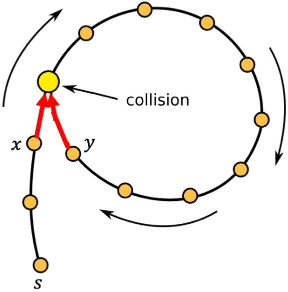

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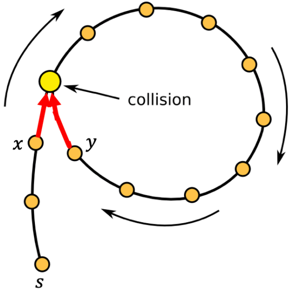
- So each cycle-finding takes $O(\sqrt{n})$ time.
- For the "real" collision, we find it with probability $\Omega(1/n)$.
- Repeat $\tilde{O}(n)$ times. In total, $\tilde{O}(n^{1.5})$ time.



Our Results - 2

Our Main Lemma

There exsits a family $\{h_{\text{seed}}\}$ of pseudorandom hash functions with **seed length** $O(\log^3 n \log\log n)$, such that functional graph $\mathcal{G}_h : x \mapsto h_{\text{seed}}(a_x)$ satisfies

- $\mathop{\mathbb{E}}_{s,seed}[\#\text{vertices reachable from } s] \leq O(\sqrt{n})$
- $\Pr_{s,seed}[u, v \text{ are reachable from } s] \geq \Omega(1/n),\ \forall u, v \in [n]$

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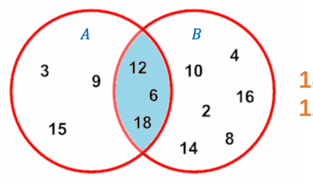
Out Results - 3

Set Intersection: Given two integer sets A, B, print all the elements in $A \cap B$ in any order. Each element is allowed to be printed multiple times.



18, 6, 12,
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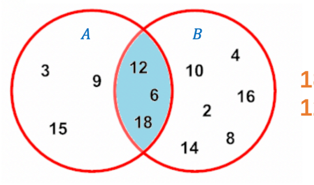

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Our RAM Upper Bound

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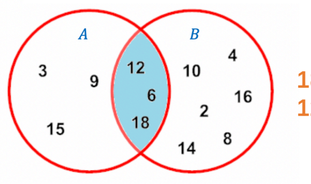

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RAM Lower bound (Patt-Shamir and Peleg, 1993) (Dinur, 2020)

$O(\text{polylog}\, n)$ space algorithms for Set Intersection require $\tilde{\Omega}(n^{1.5})$ time.

Our Results - 4

Subset Sum: Given n integers $a_1, a_2, \ldots, a_n$ and target t, decide whether a subset of them sum up to t.

Low-space Subset Sum (Bansal, Garg, Nederlof, and Vyas, 2017)

Assuming a *Random Oracle*, Subset Sum and Knapsack can be solved by a Monte Carlo algorithm in $2^{0.87n}$ time, with $O(\text{poly}(n))$ space.

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Our Result

~~Assuming a *Random Oracle*~~, Subset Sum and Knapsack can be solved by a Monte Carlo algorithm in $2^{0.87n}$ time, with $O(\text{poly}(n))$ space.

Constructing Pseudorandom Hash Function

Analysis for Random Oracle

Birthday Paradox Type Properties [BCM13]

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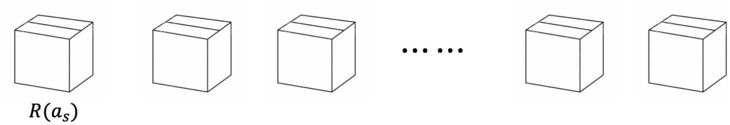

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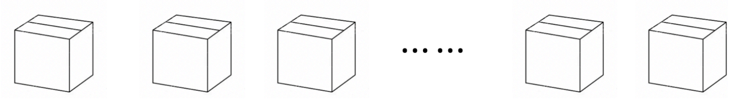

$R(a_s)$

First step, $s \to v_1 = R(a_s)$.
For any $x \in [n]$,

$$\Pr_{s,R}[R(a_s) = x] = \frac{1}{n}.$$

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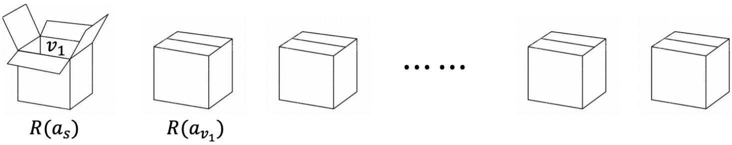

Second step, $v_1 \to v_2 = R(a_{v_1})$.
Given $a_s \neq a_{v_1}$, for any $x \in [n]$,

$$\Pr_{s,R}[R(a_{v_1}) = x \mid R(a_s) = v_1] = \frac{1}{n}.$$

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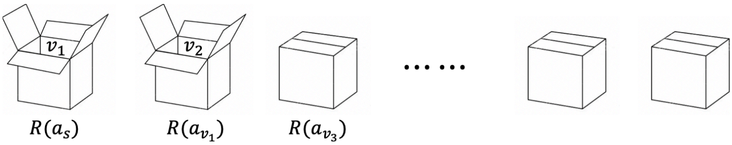

Third step, $v_2 \to v_3 = R(a_{v_2})$.

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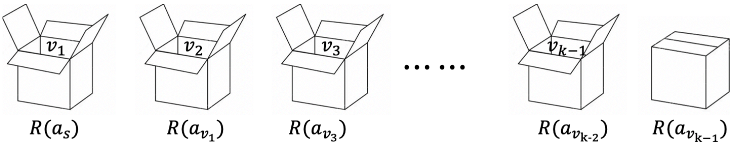

k-th step, $v_{k-1} \to v_k = R(a_{v_{k-1}})$.

Analysis for Random Oracle

Birthday Paradox Type Properties [BCM13]

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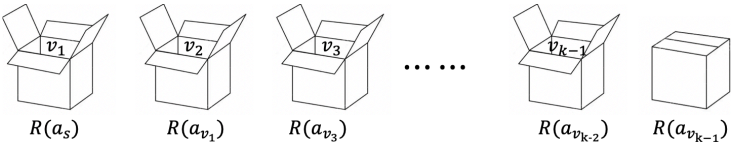

- After opening $k-1$ boxes, the k-th one still has to be random.

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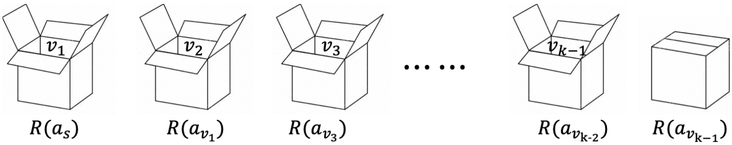

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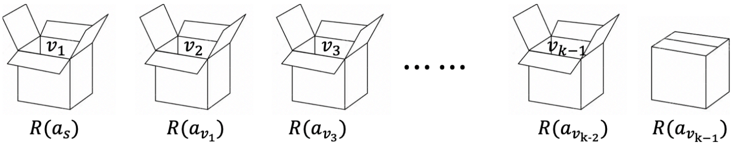

- After opening $k-1$ boxes, the k-th one still has to be random.
- Standard Birthday Paradox.
- Difficulty: $\sqrt{n}$-wise independence.

Our Construction via Iterative Restriction

Our Construction of hash function $h : [m] \to [n]$

Let $\ell = \Theta(\log n)$. Our construction has ℓ independent levels.
For the i-th level, we sample two hash functions r_i, g_i.

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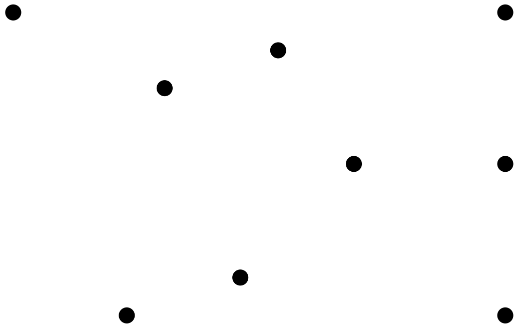
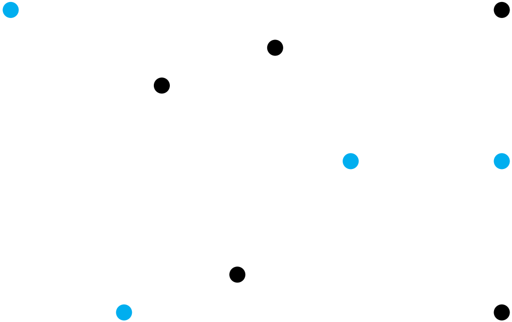
- $g_i(a)$ acts as a "filter".

Our Construction via Iterative Restriction



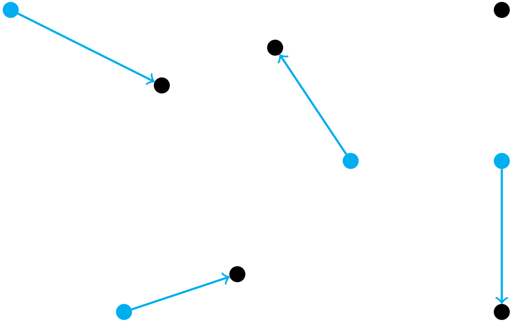
Initially, the functional graph $\mathcal{G}_h$ is empty.

Our Construction via Iterative Restriction

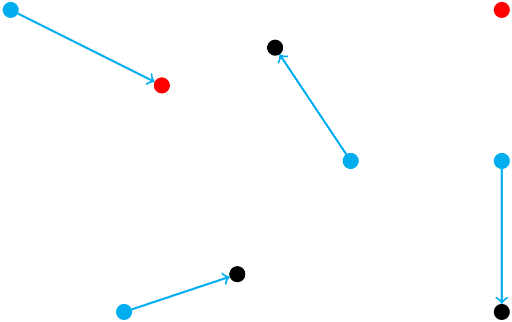


In the 1st level, we select the vertices x with $g_1(a_x) = 1$ ($n/2$ vertices in expectation).

Our Construction via Iterative Restriction

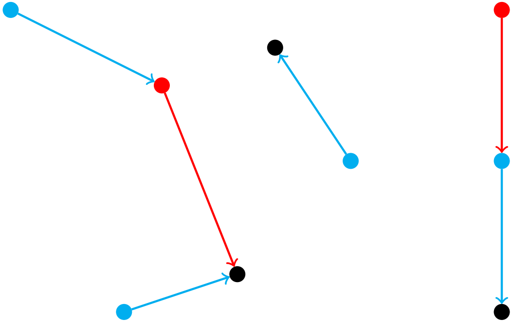


We sample their outgoing edges $x \to r_1(a_x)$ using r_1.



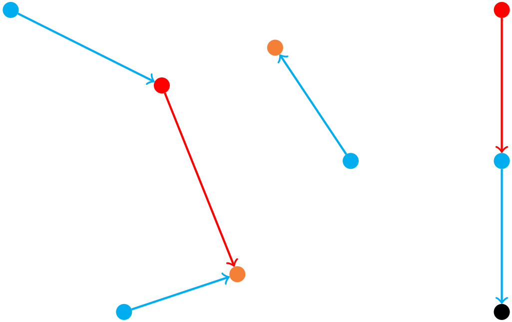
In the 2nd level, we select the remaining vertices x with $g_2(a_x) = 1$ ($n/4$ vertices in expectation).

Our Construction via Iterative Restriction



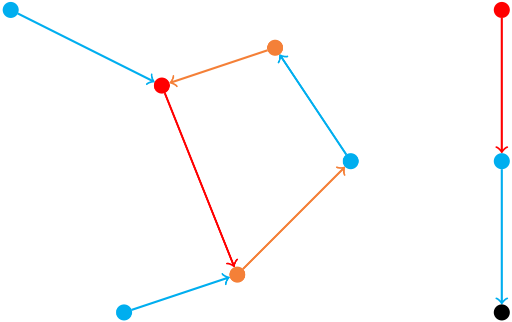
We sample their outgoing edges $x \to r_2(a_x)$ using r_2.

Our Construction via Iterative Restriction



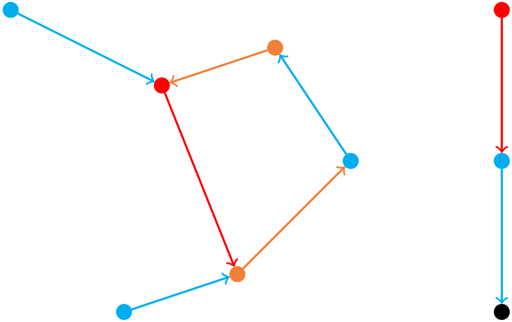
In the 3rd level, we select the remaining vertices x with $g_3(a_x) = 1$ ($n/8$ vertices in expectation).

Our Construction via Iterative Restriction



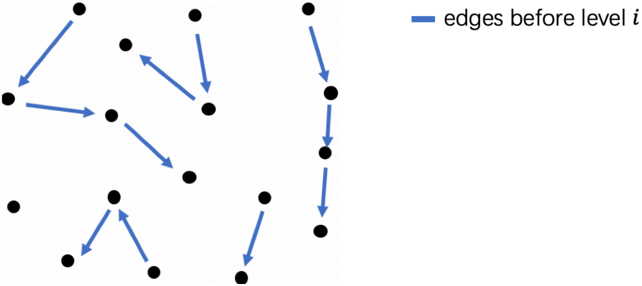
We sample their outgoing edges $x \to r_3(a_x)$ using r_3.

Our Construction via Iterative Restriction



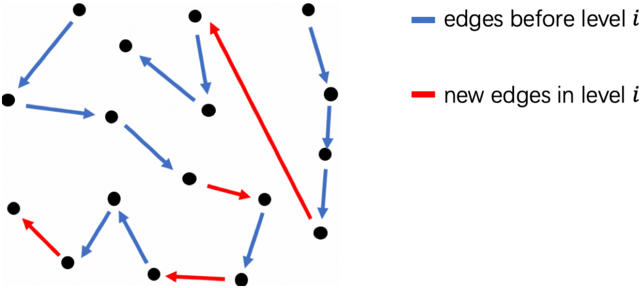
We repeat this for $\ell = \Theta(\log n)$ levels.
Each vertex x got its outgoing edge at level $i^* = \min\{i \mid g_i(a_x) = 1\}$.

Intuition for $\Theta(\log n)$-wise independence



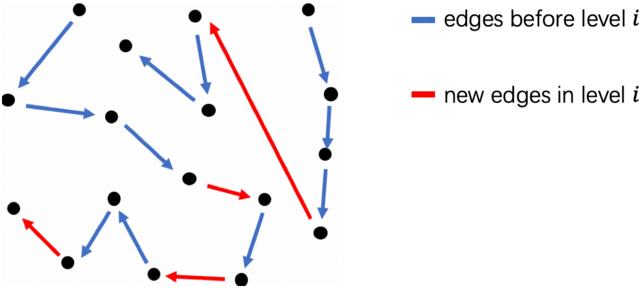
- Remaining vertex are those with no blue outgoing edge.

Intuition for $\Theta(\log n)$-wise independence



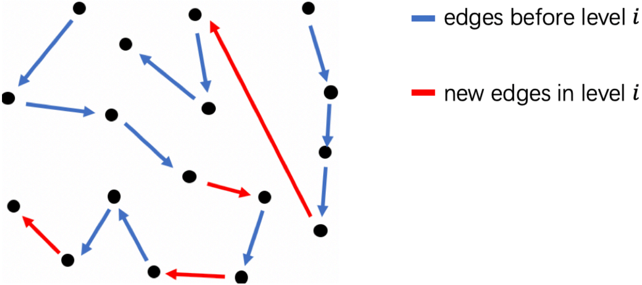
- Remaining vertex are those with no blue outgoing edge.
- Each remaining vertex x has red outgoing edge w.p. $\frac{1}{2}$ $(g_i(a_x) = 1)$.

Intuition for $\Theta(\log n)$-wise independence



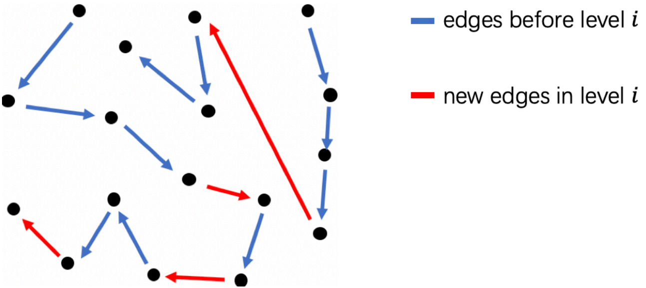
- Remaining vertex are those with no blue outgoing edge.
- Each remaining vertex x has red outgoing edge w.p. $\frac{1}{2}$ $(g_i(a_x) = 1)$.
- W.h.p. a path in this graph contains $O(\log n)$ many red edges.

Intuition for $\Theta(\log n)$-wise independence



- Roughly speaking, this is why we need $\Theta(\log n)$-wise independence per level.

Intuition for $\Theta(\log n)$-wise independence



- Roughly speaking, this is why we need $\Theta(\log n)$-wise independence per level.
- The actual proof is more complicated (40 pages).

Open Problems

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- **Time-space Tradeoffs**
  In this work, we only solved the case when $S = O(\text{polylog}\, n)$. Can we extend it to the full tradeoff?
- **Shorter Seed Length**
  In this work, our seed length is $O(\log^3 n \log\log n)$. Can this be improved?
- **Shorter Paper Length**
  Can we obtain a simpler analysis?

Q & A

Questions?

Thank you!

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