Statistical Bochner Integral on Frechet Space

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Abstract: - Probability theory including the Bochner integral is a very important part of modern mathematical concepts including the modern theory of probabilities, especially in the concept of mathematical expectation and dispersion. In this, study a statistical approach form of Bochner's theory is given and extended, but some fundamental properties of statistical integral were previously studied in the Banach case. Our approach formulates an extended integration concept of Bochner. By using the statistical convergence on general locally convex space it is possible to obtain very similar results referring to the Frechet space type. From our results, some interesting comparable outputs to Banach space are carried out. At the end of our research, it is conducted that if a function "f" is Bochner integrable in the classic report then it is statistically Bochner integrable, but conversely, this is not true. Hence, the value of the extension of Bochner integration is a need and is the focus of our work. This extension is given by modifying the model published by Schvabik and Guoju. Mathematically it is substantiated that on the space of Frechet types the space of functions of statistical Bochner integrable is a Frechet space.

Key-Words: - Statistical convergence, st-measurability, st-Bochner integral, statistical Frechet space,st-Cauchy Convergence, st-strong measurable.

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1 Introduction

The subject of statistical convergence has been studied by many researchers since the emergence of the idea of statistical convergence in 1935. Based on statistical convergence a lot of effort is done by many researchers and, [1], on the concept of statistical Bochner integral. The problem of determining a suitable representation for a fractional power of an operator defined in a Banach space X has, in recent years, attracted much attention even in the solution of many engineering problems. A perfect integration theory based on the concept of Riemann-type integral sums was initiated around 1960 by Jaroslav Kurzweil and independently by Ralph Henstock. Much of this theory is presented and extended in the growing number of universities` facts about the Henstock-Kurzweil integral known as the generalized Riemann integral. The relatively new concepts of the Henstock-Kurzweil and McShane integrals based on Riemann-type sums are an interesting challenge also in the study of the integration of function on Banach space value. Further on, the idea of statistical convergence was introduced by Zigmund, [3]. The concept was formalized by Steinhaus, [4], while the concept of statistical convergence of sequences in real numbers was introduced in 1951 by Fast, [5]. Later, the concept was reintroduced by Schoenberg, [6]. The concept of statistical convergence has become an active area of research in recent years. Statistical analogues of limit point results are obtained by Fridy, [7], defining a statistical limit point of x as a number X that is the limit of a subsequence $\{xk(j)\}$ of x such that the set $\{k(j): j \in N\}$ does not have density zero and the convergence of sequences and the concept of Schoenberg about integration. The basic concept of Fridy's is the statistical Cauchy convergence, [8]. On the other hand, we are affected by the work performed by Connor, [9], which gives proof that if a space does not contain a copy of l_1, then every bounded weakly statistically null sequence contains a weakly null subsequence, it is helpful to recall the following result of Rosenthal, [10]. In the study performed by [11], a contribution to the theory of divergent sequences is defined and examined by applying a new method of summation which assigns a general limit $Lim\, x_n$, to certain bounded sequences $x = (x_n)$. This method is analogous to the mean values which are used in the theory of almost periodic functions, furthermore, it is narrowly connected with the limits of S. Lorentz defined the space f of almost convergent sequences, using the idea of Banach limits.

In the study published by authors in [12], a characterization of statistical convergence of sequences in topological groups is obtained and extensions of a decomposition theorem, a completeness theorem, and a Tauberian theorem are given in [13], to the topological group setting are proved. According to [14], a convergence of double sequence spaces in 2-normed spaces and obtained a criterion for double sequences in 2-normed spaces to be statistically Cauchy sequence in 2-normed spaces is given. While in the study given by [15], various kinds of statistical convergence and $\mathcal{I}$-convergence for sequences of functions with values in R or, a metric space is discussed. For real-valued measurable functions defined on a measure space (X, M, μ) it is obtained a statistical version of the Egorov theorem (when μ(X)<∞), a classical result of measure theory. In the study of Maddox, [16], a very interesting way of statistical approach is given. In the study performed by authors in [17], very interesting results presented are easily obtained using a generalized version of the Bochner technique due to theorems on the connection between the geometry of a complete Riemannian manifold and the global behavior of its subharmonic, superharmonic, and convex functions. Hence, the application of such old methods is proven to have a great impact on engineering issues.

Based on the above theories our paper addresses the issue of st-measurability for functions with values in a Frechet space and relates it to the st-measurability of functions in a Banach space. The novelty of our study demonstrates that the essential properties of st-measurability for functions in Banach spaces have corresponding properties for functions with values in Frechet spaces. Furthermore, the study establishes several fundamental properties of the st-Bochner integral in Frechet spaces, which are like those of the st-Bochner integral for functions with values in Banach spaces. Additionally, the paper shows that the set of st-Bochner integrable functions is st-Frechet.

2 Preliminary Approach Used

Let be A_n a subset of the ordered natural set N. It is said to have density $\delta(A)$ if $\delta(A) = \lim_{n\to\infty} \frac{|A_n|}{n}$, where

$A_n = \{k < n : k \epsilon A\}$ and with |A| denotes the cardinality of set A. The finite sets have the density zero and δ(A′)= 1-δ(A) if A'=N-A. If a property Q(k)={k: k∈A} holds for all k∈A with δ(A)=1, we say that property Q holds for almost all k that is a.a.k. Let (X,p) be a semi-normed space, the vectorial sequence $x = (x_k)$ is statistically convergent to the vector(element) L if for each ε >0

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : p(x_k - L) \ge \varepsilon\}| = 0$$

i.e. $p(x_k - L) < \varepsilon$ a.a.k. We write st-limx_k =L.

Lemma1, [18]:

A sequence $x = (x_k)$ is statistically convergent to L if and only if there exists a set K={k_1<k_2<…}⊂N that δ(K)=1 and $\lim_{n\to\infty}(x_{k_n}) = L$

The set K is directed and the sequence (x_{k_n}) is called the essential subsequence of (x_k). The above lemma can be formulated: A sequence (x_k) is statistically convergent to L if and only if there exists an essential subsequence that converges in usual meaning to limes L. We write now $\lim_K x_k = L$ we deal with the generalization of pointwise statistical convergence of functions on semi-normed space.

Definition 2.

A sequence of functions $\{f_k(x)\}$ is said to be pointwise statistically convergent to f if for every ε>0

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n : p(f_k(x) - f(x)) \ge \varepsilon, \forall x \in S\}| = 0$$

i.e., for every x∈S, $p(f_k(x)-f(x)) < \varepsilon$ all allmost k. We write $st - lim f_k(x) = f(x)$ or $f_k \xrightarrow{st} f$ on S. This means that for every δ>0, there exists integer N such that:

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n : \mathrm{p}(\mathrm{f}_k(x)-f(x))\ge\varepsilon,\forall x\in S\}|<\delta$$

For all n>N = (N(ε,δ,x)) and for every ε>0.

We can formulate an immediate corollary of Salat's lemma, [18]. The sequence $\{f_k(x)\}$ where $f_n:S\to X$, (X-a vectorial normed space) is statistically convergent to f(x), if and only if, there exists an essential subsequence (f_{k_n}) of it that is convergent to f(x).

Following the definition of Cauchy sequences introduced by Fridy, [7], and their extension to the functional sequences and from the study performed by authors, [18], the sequence (f_k) is called the statistically Cauchy (or st-Caushy) sequence if, for every ε>0, there exists an integer N(=N(ε,x)) with:

$$\lim_{n\to\infty}\frac{1}{n}|\{k\le n : \mathrm{p}(\mathrm{f}_k(x)-f_N(x))\ge\varepsilon\ \forall \mathrm{x}\in \mathrm{S}\}|=0$$

A vectorial space V is statistically complete if each st-Caushy sequence is convergent by the metric of this space. Further, we denote (S,∑,μ) the probability complete measure space, where S is any set and Σ sigma -algebra of Borel.

Definition 3. A function f: S →X, where X is a vectorial semi-normed space is called a simple function by μ, if there are finite sequence measurable sets $\{E_i\}$, such that $E_i\in S$, i=1,...,n, $E_i\cap E_j=\varnothing$ for i≠j, S = $\bigcup_{i=1}^{n}E_i$ and $f(s) = x_i$ for s $\in E_i$, It represented in a form $=\sum_{i=1}^{n}x_i\chi_{E_i}$, where χ_{E_i} is a characteristic function of E_i.

Let (X,p) be a semi-normed space, the function

f: S→X is called st- Bochner integrable if there exists st- Cauchy sequence of simple functions (f_k) such that :

i) statistically convergent a.e. by to the function f.

ii) $$st-\lim_{k}\int_{s}p(f_k(s)-f_N(s))d\mu=0$$

st-lim $\int_S f_n(s)d\mu$ is called st- Bochner integral and denoted with (sB) $\int_s f(x)d\mu$.

If the function ***f*** is Bochner integrable in the classic definition then it is statistically Bochner integrable, but conversely is not true. This gives the value of the extension of Bochner integration in our article. Let us show this by the modification of one example published by Schvabik and Guoju, [2].

Example 4. Let f : [0,1]→X be the function

$$f=\sum_{k=1}^{\infty}\chi_{]\frac{1}{2^k},\frac{1}{2^{k+1}}]}z_k,\quad f(0)=0$$

where X is a Banach space and (z_k) is a sequence of elements belonging to X and $\|\frac{1}{2^k}z_k\|_X<B,\ \ B>1$.

It is proved that the function f is Bochner integrable if and only if the series is convergent and $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}z_k=(M)\int_0^1 f$. By choosing this form

$$z_k=\begin{cases}2^k & \text{for k = n}^2\\ 1 & \text{for others}\end{cases}$$

then the limes $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}z_k$ does not exist and the function is not Bochner integrable, but this does not prevent the function to be statistically Bochner integrable.

3 Main Results and Discussion

A vectorial space V is called statistically Frechet (shortly st-Frechet) if it is Hausdorf space, statistically complete and its topology is inducted by a countable family of semi-norms (p_k).

Definition 5. Let $(V,(p_k))$ be a statistically Frechet space and the function f : S→V. Then the function f is measurable by μ on $(V, (p_k))$ if there exists a sequence of simple functions (f_n) convergent almost everywhere to function f.

Lemma 6. [18]: Let (S, Σ, μ) be a measurable space and f: S→V is a statistically measurable function by μ on the component (V, p_k) of the Frechet space (V, (p_k)). Then there exists a finite or infinite sequence of the disjoint sets (S_r)∈Σ and

1) $\mu(S\backslash(S_1\cup\ldots\cup S_r)<\frac{1}{2^r}$
2) The set f(S_r) is totally bounded on space(V,p_k).

We can represent the Frechet space only from measurability in every component.

Theorem 7. Let (V,(p_k)) be a st-Frechet space and the function f :S → V. This function is measurable by μ on (V,(p_k)) if and only if it is measurable by each component (V, p_k) of the space (V,(p_k)).

Proof. The first part of the proof is derived from the definition of measurability. Let's use the sequence (T_m) defined for the lemma 6, and fix the natural number k. Since the set f(T_m) is totally bounded (Lemma 6) then from one open cover $(R^k_{\frac{1}{2^m}}, f(s))$ of this set one finds its finite subcover $(R^k_{\frac{1}{2^r}}, f(s^r_i))^p_{i=1}$.

Set $F^k_i = T_j \, \mathrm{H} \, f^{-1}[R^k_{\frac{1}{2^r}}, f(s^r_i)] \, \Xi \, \mathrm{S}$ and further $E^k_1 = F^k_1, E^k_2 = F^k_2 \backslash F^k_1$ … $E^k_i = F^k_i \backslash (F^k_1 \Theta \ldots \Theta F^k_{i-1})$.

The sets E^k_i are disjoint family and $\bigcup^p_{i=1} E^k_i = T_m$. It holds furthermore then

$$("s \, \Xi \, T_m) \quad (\$i_0 \, \Xi \, \{1,2,...,p\}, p_k(f(x) - f(s^r_i) < \frac{1}{2^r})$$

Set the sequence of the simple functions.

$$f_r = \underset{i=1}{\overset{p}{\varepsilon}} f(s^r_i) c_{E^k_i} + 0.c_{S\backslash T_m}$$

then for every s ∈ T_m

$$p_k[f_r(s) - f(s)] < \frac{1}{2^r}$$

When r →∞ the sequence of the simple functions (f_r) converges to f(x) on the arbitrary component (V, p_k). We prove that the function f(x) is measurable in the space (V,(p_k)).

Definition 8. The function f: S → V, (V, (p_k)) st-Frechet space is called statistical Bohner integrable (short sB- integrable) if there exists one sequence of simple functions (f_n) which converges almost everywhere by measure μ to the function f of this space and for every continuous semi -norm p_k holds.

$$\lim_k \int_S p_k(f_n(s) - f(s)) d\mu = 0$$

The sequence (f_n) is called the determinant of the function ***f***.

The following theorem proves that we find a function that constructs the image of (Vp_k) on Freche space to the Banach space.

Theorem 9. Let (V, p_k) be one component of the st-Frechet space (V, (p_k)), and the function f: S → V is sB- integrable on (V, (p_k)). Then for every p_k, the function $\varphi_{p_k} \circ f : S \to V'_{p_k}$ is statistical Bohner integrable on the subspace (V'_{p_k}, p'_k) of the Banach space $(\overline{V}_{p_k}, \overline{p}_k)$.

Proof. Let (f_n) be one determinant sequence of the simple functions of the function f. Since for every p_k and s ∈ S, we can write

$$p_k[f_n(s) - f(s)] = p'_k[\varphi_{p_k} \circ f_n(s) - \varphi_{p_k} \circ f(s)]$$

From this equation derives that the sequence of the simple functions $\varphi_{p_k} \circ f_n$ converges almost everywhere to the function $\varphi_{p_k} \circ f$ of the space

(V'_{p_k}, p'_k). The equation below shows that:

$$\int_S p_k[f_n(s) - f(s)] d\mu = \int_S p_k[\varphi_{p_k} \circ f_n(s) - \varphi_{p_k} \circ f(s)] d\mu$$

the sequence $(\varphi_{p_k} \circ f_n)$ is the determinant of the function $\varphi_{p_k} \circ f$ on the space (V'_{p_k}, p'_k) therefore this function is statistically Bohner integrable.

The function $f: S \to X$ is called statistically strong measurable by μ on set S , (in short form st-measurable) if there exists a sequence of simple functions f_n that for every $x \in S$ and every $\varepsilon > 0$ holds:

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : p(f_k(x) - f(x)) \ge \varepsilon, \forall x \in S\}\right| = 0, \text{ for almost all } x \in S .$$

Theorem 9.(Theorem Egorov). If a function f: S→X is st-measurable by μ, then it is st- strong measurable uniformly almost everywhere on S.

The proof repeated the proof for the Banach space done to Caushi, [1].

Theorem 10. Let X semi-normed space and the function f: S→X is statistically measurable. Then

1) The function f is with a separable value almost everywhere by μ.
2) $f^{-1}(G) \in \Sigma$ for every open set G.

Proof. First, show that the function f is almost separable. The function f(x) is statistically measurable therefore there exists the sequence of simple functions (f_n) and the set $N \subset S$ with μ(N)=0 such that:

$p(f_n(s) - f(s)) < \varepsilon$ almost every n and $s \in S \setminus N$.

Denote $A_k = \{k \le n : p(f_n(s) - f(s)) \ge \varepsilon\}$.

By Egorov, Theorem 9, for every $n \notin A_k$ there exists a subset $E_n \subset S$ with $\mu(E_n) < \varepsilon$ that the sequence of simple functions ($f_n(s)$) converges statistically and uniformly to f(x) on $S \setminus E_n$.

Since f_n are simple functions its value $f_n(S)$ are finite for every $n \in N$. We know that the set $\bigcup_{n \notin \bigcup_k A_k} f_n(S)$ is countable and by considering this fact for each n,

$$f\Big(\bigcup_{n \notin \bigcup_k A_k} (S \setminus E_n)\Big) = \bigcup_{n \notin \bigcup_k A_k} f(S \setminus E_n)$$

then it is easily provided the mathematical expression as below:

$$f\Big(\bigcup_{n \notin \bigcup_k A_k} (S \setminus E_n)\Big) \subset \overline{\bigcup_{n \notin \bigcup_k A_k} f_n(S \setminus E_n)}$$

The set $\overline{\bigcup_{n \notin \bigcup_k A_k} f_n(S \setminus E_n)}$ is separable as the closure of the countable set.

If considering $\mu(E_n) < \frac{1}{n}$ for $n \notin A_k$, then it can be given such as:

$$\bigcap_n E_n = S \setminus \bigcup_n (S \setminus E_n) \text{ for } n \notin A_k$$

and

$$\mu\Big(\bigcap_n E_n\Big) = \mu\Big(S \setminus \bigcup_n (S \setminus E_n)\Big) = 0 .$$

Putting $N = \bigcap_n E_n$ for $n \notin A_k$ we prove the separability of the set $\{f(s), s \in S \setminus N\}$.

(b) Let G be an open set in X. We denote $R(f(s), \frac{1}{n})$ the set $\{y \in X : p(y - f(s)) < \frac{1}{n}\}$ and $G_n = \bigcup_{f(s) \in G} R(f(s), \frac{1}{n})$. It is easy to prove as $G = \bigcup_{n \in N} G_n$ If $s \in S \setminus Z$ then from the equality st-$\lim_{k \in K} f_k(s) = f(s)$ we have that $f_k \in R(f(s), \frac{1}{n})$ for $k \in K$ and k>m. So the $f(s) \in G$ if and only if $f_k \in G_n$ k>m. Hence, the model can be reformulated by the equation:

$$f^{-1}(G) \setminus Z = \bigcup_{n=1, m=1}^{\infty} \bigcap_{k \in K, k>m}^{\infty} (f^{-1}(G_n) \setminus Z)$$

which proves that $f^{-1}(G) \in \Sigma$.

The following lemma is a key lemma to prove that the space (sB(m, V),(q_k)) is statistically-Frechet.

Lemma 11. Let (f_n) be a sequence st-Cauchy of simple functions on the vectorial space (sB(μ, V),(q_k)). Then there exists a subsequence (g_n) of (f_n) convergent to one function f of the Frechet space

$(V'_{p_k},(q_k))$.

Proof. Since the sequence (f_n) is a Cauchy sequence on the space (sB(μ, V),(q_k)) then for every n ∈ K there exists $N(\frac{1}{2^{2r}})$ ∈N such that for every n>N: $q_k(f_n - f_N) < \frac{1}{2^{2r}}$. By the Salat lemma, [11], we can order the elements of the set K as $r_1 < r_2 < \ldots < r_n < \ldots$

By denoting $g_m = f_{r_m}$, then

$$q_k(f_{r_m} - f_{r_N}) = q_k(g_m - g_N) < \frac{1}{2^{2r}}$$

for every m> r_N. Set the series (1)

$$g_1(s) + \sum_{i=1}^{\infty}[g_{i+1}(s) - g_i(s)]$$

for every s∈S. We note

$$M_r = \{s \in S: p_k(g_{i+1}(s) - g_i(s)) \geq \frac{1}{2^r}\}$$

and

$$\frac{1}{2^r}\mu(M_r) = \int_{M_k}\frac{1}{2^r}d\mu \leq \int_{M_k} p_k(g_{i+1}(s) - g_i(s))d\mu$$

$$\leq q_k[g_{i+1}(s) - g_i(s)] < \frac{1}{2^{2r}}$$

Hence,

$$\mu(M_r) < \frac{1}{2^r} .$$

Putting

$$Z_r = M_r \cup M_{r+1} \cup \ldots$$

and it is verified that $Z_{r+1} \subset Z_r$. But, on the other hand, it is known that:

$$\mu(Z_r) \leq \sum_{i=r}^{\infty}\mu(M_i) < \sum_{i=r}^{\infty}\frac{1}{2^i} = \frac{1}{2^{r+1}}$$

and if the set $Z = \prod_{r\in K} Z_r$ then μ(Z) = 0. Let's be an element such that s ∈ S \ Z. Then there exists the set Z_{r_0} that $s \notin Z_{r_0}$ and for $r \geq r_0$

$$\sum_{i=r}^{\infty} p_k[g_{i+1}(x) - g_i(x)] < \sum_{i=r}^{\infty}\frac{1}{2^r}$$

because the seminorms (p_k) is monotone non-decreasing and for i ≥ k

$$p_k[g_{i+1}(s) - g_i(s)] \leq p_i[g_{i+1}(s) - g_i(s)] < \frac{1}{2^i} .$$

as a consequence the series

$$p_k[g_1(s)] + \sum_{i=1}^{\infty} p_k[g_{i+1}(s) - g_i(s)]$$

is convergent and this implies that series (1) is convergent to the function f on the component (V, p_k) of the Frechet space (V, (p_k)) for $k \geq k_0$ and s ∈ S \ Z. Thus

$$\lim_K g_p(s) = \lim_K f_{r_p} = f .$$

For every p ∈ P, we set the function

$$q_p : sBV(\mu, X) \to (V, p_k),\ \ q_p(f) = \int_S p(f(x)d\mu$$

It is easy to see that q_p is a semi-norm on sBV(μ,V) and this implies that the family of $(q_p)_{p\in P}$ determines one locally convex space.

In the following theorem, we prove the (sBV(μ, V), (q_k)) is statistically Frechet.

Theorem 12. The vectorial topological space (sBV(μ, V),(q_k)) is statistically Frechet.

Proof. Let us show that the sequence of semi-norms (q_k) separates the points. Let f be a function of this space and for every k ∈N

$$q_k(f) = \int_S p_k(f(s)d\mu = 0$$

then there exists the set Z_k∈Σ such that μ(Z_k)=0 for which p_k(f(s))=0 for the all s∉Z_k. If one takes

$Z = \bigcup_{k=1}^{\infty} Z_k$ then for every s∉Z and k∈N we obtain that p(f(s)=0. Since the family (p_k) separates the points then f=0 almost everywhere by μ and so are the seminorms q_k.

Let us demonstrate that locally convex space (sBV(μ, V),(q_k)) is statistically Frechet space and let (g_n) be a statistically Cauchy sequence in that space. By the definition of statistical Bohner integral, for every n ∈ K, there exists a simple function f_n such that:

q_n (g_n –f_n)$<\frac{1}{n}$.

Let's consider the sequential Cauchy sequence (g_n) for n > N∈N then

$$q_k(f_n - f_N) \le q_k(f_n - g_n) +$$
$$+ q_k(g_n - g_N) + q_k(g_N - f_N)$$

$$< \frac{1}{n} + p_k(g_n - g_N) + \frac{1}{N} \; .$$

So the sequence (f_n) is statistically Cauchy of simple functions in arbitrary component (sBV(μ, V),q_k) of the space (sBV(μ, V),(q_k)). This implies the sequence (f_n) is statistically Cauchy in the space (sBV(μ, V),(q_k)). By the virtue of Lemma 11, the sequence (f_n) has one subsequence (f_{n_k}) converges almost everywhere to the function f: S→V of the Frechet space (V, (p_k). On the other hand, the sequence (f_{n_k}) is a statistically also Cauchy sequence in the space (sBV(μ, V),(q_k).

The inequalities

$$q_k(g_{n_k} - f) \le q_k(g_{n_k} - f_{n_k}) + q_k(f_{n_k} - f)$$

show that the subsequence (g_{n_k}) of the sequence (g_n) converges on every component (sBV(μ ,V), q_k) of the space (sBV(μ, V), (q_k)). So the sequence (g_n) converges to f on the space (sBV(μ, V),(q_k)). This space is statistically complete by the sequences or statistically Frechet.

4 Conclusions

First, we find another application of statistical Bochner integral in Freshet space in comparison with Banach ones. Our study proves the same results of the classic Bochner integral on Freshet space for the statistic Bochner integrable in more simple and constructive ways. our method on statistical Bochner integral in Freshet space is further explained and extended for each case. The contribution of the methodology in comparison to other researchers which are introduced in a generalized form.

At the end of our research, it is conducted that if a function "***f***" is Bochner integrable in the classic report then it is statistically Bochner integrable, but conversely, this comes not true, at all. Hence, the value of the extension of Bochner integration is a need and is the focus of our work. This extension is given by modifying the model published by Schvabik and Guoju, [2]. Mathematically it is proved that on the space of Frechet types the space of functions of statistical Bochner integrable is a Frechet space.

5 Future work

By considering the concept of Ideal convergence, which is closely related to statistical convergence, it is possible to establish the concept of measurability within the framework of ideals. This, in turn, allows for the construction of a form of ideal Bochner integral for functions that have values in a Frechet space. Probability theory including statistical Bochner integral in Freshet space is a very important part of nowadays mathematical concepts, especially in the concept of mathematical expectation and dispersion. Our future work will be focused on the application of the results in real engineering systems.

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