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FINITE ELEMENT EXTERIOR CALCULUS WITH LOWER-ORDER TERMS

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Abstract. The scalar and vector Laplacians are basic operators in physics and engineering. In applications, they frequently show up perturbed by lower-order terms. The effect of such perturbations on mixed finite element methods in the scalar case is well understood, but that in the vector case is not. In this paper, we first show that, surprisingly, for certain elements there is degradation of the convergence rates with certain lower-order terms even when both the solution and the data are smooth. We then give a systematic analysis of lower-order terms in mixed methods by extending the Finite Element Exterior Calculus (FEEC) framework, which contains the scalar, vector Laplacian, and many other elliptic operators as special cases. We prove that stable mixed discretization remains stable with lower-order terms for sufficiently fine discretization. Moreover, we derive sharp improved error estimates for each individual variable. In particular, this yields new results for the vector Laplacian problem which are useful in applications such as electromagnetism and acoustics modeling. Further, our results imply many previous results for the scalar problem and thus unify them all under the FEEC framework.

1. Introduction

The vector Laplace equation and, more generally, the Hodge Laplace equation associated to a complex arise in many applications. The discretization of such equations is a basic motivation of the Finite Element Exterior Calculus (FEEC) [1,2]. In many applications the equations include variable coefficients and lower-order terms. While the former is included in the standard FEEC framework through weighted inner products, the latter is not, which is the subject of this work. One might expect that lower-order perturbations degrade neither the stability nor the convergence rates of stable Galerkin methods. However, this need not be true. While stable choices of finite elements for the unperturbed Hodge Laplacian remain stable for the perturbed equation, we find that certain lower-order perturbations result in decreased rates of convergence. Other choices of element pairs or perturbations do not lower the convergence rate. The situation is subtle.

First, to fix ideas, we consider a simple example taken from magnetohydrodynamics [11, Chapter 3]: given vector fields f, v on a domain $\Omega \subset \mathsf{R}^3$, find a vector field B satisfying:

$$\operatorname{curl}\operatorname{curl} B - \operatorname{curl}(v \times B) = f, \qquad \operatorname{div} B = 0, \quad \text{in } \Omega,$$
$$B \cdot n = 0, \quad (\operatorname{curl} B - v \times B) \times n = 0, \quad \text{on } \partial\Omega.$$

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Physically, B is the non-dimensionalized magnetic field inside a conductor moving with a velocity field v. The system admits a solution only when $\operatorname{div} f = 0$. In that case, the solution also satisfies the following vector Laplace equation, which is solvable for any data and hence more suitable for discretization:

$$-\operatorname{grad}\operatorname{div} B + \operatorname{curl}\operatorname{curl} B - \operatorname{curl}(v \times B) = f. \tag{1.1}$$

For a mixed method, we introduce $\sigma = \operatorname{curl} B - v \times B$ and solve the coupled system:

$$\sigma - \operatorname{curl} B + v \times B = 0, \qquad \operatorname{curl}\sigma - \operatorname{grad}\operatorname{div} B = f.$$

A common stable choice of mixed elements, at least when $v = 0$, seeks B in the space of Nédélec face elements of the second kind of degree $r \geq 1$ and σ in the space of Nédélec edge elements of the second kind of degree $(r+1)$ [13, 14]. In this case, for the unperturbed problem, that is, when $v = 0$, the convergence for the L^2-error in σ is of optimal order $O(h^{r+2})$ if the solution is smooth enough. However, as we show in Section 3 and verify by numerical computation in Section 7, when v does not vanish, the L^2-convergence for σ is reduced to order $O(h^{r+1})$. A similar phenomenon was observed in mixed methods for the scalar Laplacian by Demlow [6]. But the vector case we study has more surprises. For example, consider the vector Laplacian perturbed by a zeroth-order term: for some real coefficient A,

$$-\operatorname{grad}\operatorname{div} u + \operatorname{curl}\operatorname{curl} u + Au = f, \quad \text{in } \Omega,$$
$$u \times n = 0, \quad \operatorname{div} u = 0, \quad \text{on } \partial\Omega.$$

This problem arises, for example, in electromagnetism where u is the electric field and A is the conductivity coefficient. If we use the same mixed finite element method just considered, here with $\sigma = \operatorname{curl} u$, then the L^2-error in σ is one order suboptimal for a general matrix coefficient A, but optimal if A is a scalar coefficient.

We now summarize convergence rates derived from our main abstract theorems applied to the perturbed Hodge Laplace problem, of which the previous vector Laplace problem is an instance. These are important in and directly relevant to many applications. In particular, our results for the vector cases, that is, 1-forms in 2D and 1- and 2-forms in 3D, are new. On a domain Ω in R^n, a k-form has $\binom{n}{k}$ coefficients, and the meaning of the exterior derivative d and the codifferential δ depend on k. For example, when $n = 3$ and $k = 2$, we have the case considered before with $\delta d + d\delta = -\operatorname{grad}\operatorname{div} + \operatorname{curl}\operatorname{curl}$. For any $0 \leq k \leq n$, the k-form Hodge Laplace equation seeks a k-form u satisfying

$$L_0 u := (d\delta + \delta d)u = f, \tag{1.2}$$

with proper boundary conditions. Our abstract theory applies to equation (1.2) perturbed by general lower-order terms:

$$Lu = [(d + l_3)(\delta + l_2) + (\delta + l_4)(d + l_1) + l_5]u = f,$$

with proper boundary conditions. In Section 3, we allow l_i to be general linear operators, but here we assume that they are multiplication by smooth coefficient fields. For example, l_1 takes a k-form to $(k+1)$-form, thus may be viewed as an $\binom{n}{k+1} \times \binom{n}{k}$ matrix field.

The mixed formulation solves simultaneously for u and for $\sigma = (\delta + l_2)u$, which is a $(k-1)$-form. On a simplicial triangulation of Ω, for any $r \geq 1$, we have four canonical pairs of mixed finite elements for (σ, u):

$$\mathcal{P}_{r+1}\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k, \quad \mathcal{P}_{r+1}^-\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k, \quad \mathcal{P}_r\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k, \quad \mathcal{P}_r^-\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k.$$

In dimension ≤ 3, all these are classical mixed finite elements. For example, the pair consisting of Nédéléc face elements of the second kind and Nédéléc edge elements of the second kind before is $\mathcal{P}_{r+1}\Lambda^1 \times \mathcal{P}_r\Lambda^2$. For more, the Periodic Table of the Finite Elements (`http://femtable.org/`) collects all these elements and their correspondence to classical elements. A main result in FEEC is that the above four pairs lead to stable mixed finite element methods for the unperturbed Hodge Laplace problem. Further, the rates of convergence for the L^2-errors in σ, $d\sigma$, u, and du are optimal, as determined by the approximation properties of the spaces. Thus, for example, if the pair $\mathcal{P}_{r+1}\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k$ is used, the order of convergence for the L^2-error in σ, $d\sigma$, u, and du is $r+2$, $r+1$, $r+1$, and r, respectively, while, for $\mathcal{P}_r^-\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k$ all four L^2-errors converge with order r.

For the perturbed system, our discrete stability result (Theorem 3.2) implies that when the perturbed problem is uniquely solvable at the continuous level (which is the generic case, as we will prove), then the mixed discretization of this problem using any of the four stable pairs before is still stable for sufficiently fine discretization. Further, if we have full elliptic regularity (for example, on a smooth domain [10]), then our improved error estimate (Theorem 3.4) implies that the L^2-convergence rates for the unperturbed Hodge Laplacian still hold for the perturbed problem with a few exceptions, as summarized in Table 1:

TABLE 1. L^2-error rates for FEEC elements solving Hodge Laplace equation.

elements	σ	$d\sigma$	u	du
$\mathcal{P}_{r+1}\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k$	$\begin{cases} r+2, & \text{if no } l_2, l_4, l_5, \\ r+1, & \text{otherwise.} \end{cases}$	$\begin{cases} r+1, & \text{if no } l_4, \\ r, & \text{otherwise.} \end{cases}$	$r+1$	r
$\mathcal{P}_{r+1}^-\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k$	$r+1$	$\begin{cases} r+1, & \text{if no } l_4, \\ r, & \text{otherwise.} \end{cases}$	$r+1$	r
$\mathcal{P}_r\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k$	$\begin{cases} r+1, & \text{if no } l_2, l_5, \\ r, & \text{otherwise.} \end{cases}$	r	r	r
$\mathcal{P}_r^-\Lambda^{k-1} \times \mathcal{P}_r^-\Lambda^k$	r	r	r	r

For example, in our first example, B is a 2-form and equation (1.1) has an l_2 lower-order term, which leads to reduced L^2-error rates for σ. In practice, this suggests that the smaller space $\mathcal{P}_{r+1}^-\Lambda^k$ should be used for $\sigma = \operatorname{curl} B$, because the use of the bigger space $\mathcal{P}_{r+1}\Lambda^k$ does not improve the convergence rates for σ or any other quantity. In our second example, the zeroth-order term is a generic l_5 term; hence it also degrades the L^2-error rate in σ. We observe that some lower-order terms, namely l_1 and l_3 terms, do not degrade the error rates in any of the cases above and that the L^2-convergence rates for u and du are unaffected by the lower-order terms. We also note that, most surprisingly, the lower-order term l_4 has the worst effect on the convergence rates of the error in σ, degrading the L^2-error rates in both σ and $d\sigma$, yet σ has no apparent dependence on l_4.

Historically, the effect of lower-order terms on the convergence of finite element methods was first studied by Schatz [18] for the scalar Laplace equation with Lagrange elements. The key tool in that analysis was the Aubin-Nitsche duality argument, which was introduced to prove L^2-error estimates in both nonmixed [15] and mixed methods [9]. These ideas guide the current techniques. Directly relevant to this work are the studies by Douglas and Roberts [7, 8] and

Demlow [6] on mixed finite element discretization of the scalar Laplace equation, which is the Hodge Laplace equation for n-forms in n dimensions. The primary mixed finite elements for this problem are the Raviart–Thomas (RT) family $\mathcal{P}_r^-\Lambda^{n-1}$ [17] and the BDM family $\mathcal{P}_r\Lambda^{n-1}$ [4, 5]. Douglas and Roberts proved the optimal L^2-convergence rates for both variables when the RT family is used. Demlow showed that for BDM elements, even for constant coefficients and smooth solutions, there is degradation of the convergence rate for σ in the problem $-\operatorname{div}(\operatorname{grad} u + \vec{b}u) + cu = f$, while there is no degradation if the same problem is formulated as $-\operatorname{div}\operatorname{grad} u + \vec{b}\cdot\operatorname{grad} u + cu = f$. All these classical results on non-mixed and mixed finite elements for scalar Laplace problem with lower-order terms can be read directly from Table 1. Our approach here gives a uniform derivation of all classical L^2-estimates for the scalar/vector Laplacian perturbed by lower-order terms under the extended abstract FEEC framework.

The rest of this paper is organized as follows. We first briefly review the basic FEEC framework for the unperturbed abstract Hodge Laplacian in Section 2. Then we lay out our extended abstract framework and state our two main discrete results: the stability theorem (Theorem 3.2) and improved error estimates (Theorem 3.4) in Section 3. After that, we prove the well-posedness theorems at the continuous level in Section 4. Then we prove the two main discrete results in Section 5 and Section 6 respectively. Finally, in Section 7, we show that the estimates are sharp for the Hodge Laplacian case through numerical examples.

2. Review of the abstract FEEC framework

FEEC is an abstract framework for analyzing mixed finite element methods [1,2]. A *Hilbert complex* (W^k, d^k) is a sequence of Hilbert spaces W^k and closed densely defined linear operators $d^k : W^k \to W^{k+1}$ with closed range satisfying $d^{k+1}\circ d^k = 0$. We use $(\,\cdot\,,\,\cdot\,)$ to denote the W-inner product and $\|\cdot\|$ to denote the W-norm. Let d_{k+1}^* be the adjoint of d^k, $V^k = D(d^k)$, and $V_{k+1}^* := D(d_{k+1}^*)$. From this definition, $(d^k u, v) = (u, d_{k+1}^* v)$, for all $u \in V^k$ and $v \in V_{k+1}^*$. One important structure is the *Hodge decomposition*:

$$W^k = \mathsf{Z}^k \oplus (\mathsf{Z}^k)^{\perp W} = \mathsf{B}^k \oplus \mathsf{H}^k \oplus (\mathsf{Z}^k)^{\perp W} = \mathsf{B}^k \oplus \mathsf{H}^k \oplus \mathsf{B}_k^*,$$

where $\mathsf{Z}^k = \ker d^k$, $\mathsf{B}^k = \operatorname{im} d^{k-1}$, $\mathsf{H}^k = \mathsf{Z}^k \cap (\mathsf{B}^k)^\perp$, and $\mathsf{B}_k^* = \operatorname{im} d_{k+1}^* = (\mathsf{Z}^k)^{\perp W}$.

We assume the Hilbert complex satisfies the *compactness property*, that $V^k \cap V_k^*$ is a compact densely embedded subspace of W. This is true for all the cases we are interested in. For example, the de Rham complex on Lipschitz domains in R^n satisfies this property [16].

The abstract Hodge Laplacian is the unbounded operator $L_0^k : W^k \to W^k$ defined by $L_0^k = d^{k-1}d_k^* + d_{k+1}^* d^k$ with the domain $D(L_0^k) = \{u \in V^k \cap V_k^* \mid du \in V_{k+1}^*, d^*u \in V^{k-1}\}$. In the following, we drop the index k when it is clear from the context. For example, $L_0 = dd^* + d^*d$. It is known that for any $f \in W^k$, there exists a unique $u \in D(L_0)$ such that

$$L_0 u = f \mod \mathsf{H}, \quad u \perp \mathsf{H}.$$

Let $K_0 : f \mapsto u$ be the solution operator above. It is known that K_0 is self-adjoint and compact as a map $W \to W$.

The mixed discretization of the abstract Hodge Laplacian is well understood. The problem above can be formulated in the *mixed weak formulation*: given $f \in W$, find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathrm{H}^k$ such that

$$
\begin{aligned}
&(\sigma, \tau) - (u, d\tau) = 0, && \forall \tau \in V^{k-1}, \\
&(d\sigma, v) + (du, dv) + (p, v) = (f, v), && \forall v \in V^k, \\
&(u, q) = 0, && \forall q \in \mathrm{H}^k.
\end{aligned} \tag{2.1}
$$

For each k, let V_h^k be a sequence of discrete subspaces of V^k indexed by h. V_h^k is called *dense* if

$$\forall u \in V^k, \quad \lim_{h \to 0} \inf_{v \in V_h} \|u - v\|_V = 0.$$

Clearly, density is necessary for V_h^k to be good approximations of V^k. However, it is known that this alone is not sufficient for the Galerkin projection to be a convergent method. The key additional properties are the *subcomplex property* that $dV_h^k \subset V_h^{k+1}$ and the existence of V*-bounded cochain projections*, that is, bounded projections $\pi_h^k : V^k \to V_h^k$ satisfying $d\pi_h = \pi_h d$. The main result in FEEC states that the Galerkin projection of system (2.1) using dense discrete subspaces admitting bounded cochain projections is stable. More precisely, Theorem 3.8 of [2] states that the bilinear form associated with the system (2.1),

$$B_0((\sigma, u, p), (\tau, v, q)) := (\sigma, \tau) - (u, d\tau) + (d\sigma, v) + (du, dv) + (p, v) - (u, q),$$

satisfies the inf-sup condition on $(V_h^{k-1} \times V_h^k \times \mathrm{H}_h^k)^2$. This implies quasi-optimal convergence rates in the V-norm. Further if the bounded projections can be extended to W*-bounded cochain projections*, that is, the extension to $\pi_h^k : W^k \to V_h^k$ exists and $\|\pi_h^k\|_{W \to W}$ are bounded uniformly in h, then we get decoupled error estimates for each variable in the W-norm. To state this precisely, we need some notation. We use $a \lesssim b$ to express that $a \leq Cb$ for some generic constant C. Following [2], we let

$$
\begin{gathered}
\delta_0 = \|(I - \pi_h)K_0\|_{W^k \to W^k}, \qquad \mu_0 = \|(I - \pi_h)P_{\mathrm{H}}\|_{W^k \to W^k}, \\
\eta_0 = \max_{j=0,1} \{ \|(I - \pi_h)dK_0\|_{W^{k-j} \to W^{k-j+1}}, \|(I - \pi_h)d^*K_0\|_{W^{k+j} \to W^{k+j-1}} \}, \\
\alpha_0 = \eta_0^2 + \delta_0 + \mu_0,
\end{gathered} \tag{2.2}
$$

where P_{H} is the W-orthogonal projection from W to $\mathrm{H} \subset W$.

All these quantities converge to 0 as $h \to 0$ due to the compactness and density assumption. For example, on smooth or convex polyhedral domains for the de Rham complex, it is known that $\eta_0 = O(h)$, $\delta_0 = O(h^{\min(2, r+1)})$, $\mu_0 = O(h^{r+1})$, where r is the largest degree of complete polynomials in V_h. We use the following notation for the best approximation: for $w \in V^k$,

$$E(w) := \inf_{v \in V_h^k} \|w - v\|.$$

Theorem 3.11 of [2] bounds the W-norm error of each variable in terms of best approximation errors:

$$
\begin{gathered}
\|\sigma - \sigma_h\| \lesssim E(\sigma) + \eta_0 E(d\sigma), \\
\|d(\sigma - \sigma_h)\| \lesssim E(d\sigma), \\
\|u - u_h\| \lesssim E(u) + (\eta_0^2 + \delta_0)[E(d\sigma) + E(p)] + \eta_0[E(du) + E(\sigma)], \\
\|d(u - u_h)\| \lesssim E(du) + \eta_0[E(d\sigma) + E(p)], \\
\|p - p_h\| \lesssim E(p) + \mu_0 E(d\sigma).
\end{gathered} \tag{2.3}
$$

For example, using $\mathcal{P}_{r+1}\Lambda^{k-1} \times \mathcal{P}_r\Lambda^k$ and assuming full elliptic regularity, we have $E(\sigma) = O(h^{r+2})$ and $E(d\sigma) = O(h^{r+1})$ for any $r \geq 1$. Therefore, we get the optimal rates $\|\sigma - \sigma_h\| = O(h^{r+2})$. Similarly, the error rates for all variables can be shown to be optimal for all four canonical FEEC element pairs. Let $P_h : W \to V_h$ be the W-orthogonal projection and $K_{0h} : V_h^k \to V_h^k$ be the discrete solution operator $K_{0h} : f \mapsto u_h$. These convergence estimates can be stated using operators (cf. Corollary 3.17 of [2]):

$$
\begin{aligned}
\|K_0 - K_{0h}P_h\|_{W\to W} &\lesssim \alpha_0, \\
\|dK_0 - dK_{0h}P_h\|_{W\to W} + \|d^*K_0 - d_h^*K_{0h}P_h\|_{W\to W} &\lesssim \eta_0.
\end{aligned} \tag{2.4}
$$

The FEEC approach not only gives optimal error rates but also captures the important structures of the Hodge Laplacian. We have the *discrete Hodge decomposition*:

$$V_h^k = \mathfrak{Z}_h^k \oplus \mathfrak{B}_{k,h}^* = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{B}_{k,h}^*.$$

The map π_h^k ensures that $\mathfrak{Z}_h \subset \mathfrak{Z}$ and $\mathfrak{B}_h \subset \mathfrak{B}$, but the discrete spaces $\mathfrak{H}_h$ and $\mathfrak{B}_h^*$ are generally not subspaces of their continuous counterparts. At the continuous level, we have

$$\forall f \in W, \qquad f = dd^*K_0 f + d^*dK_0 f + P_{\mathfrak{H}} f \in \mathfrak{B} \oplus \mathfrak{B}^* \oplus \mathfrak{H}; \tag{2.5}$$

that is, solving the Hodge Laplacian problem with data f leads to the Hodge decomposition of f. At the discrete level, similarly, we have

$$\forall f \in V_h, \qquad f = dd_h^*K_{0h} f + d_h^*dK_{0h} f + P_{\mathfrak{H}} f \in \mathfrak{B}_h \oplus \mathfrak{B}_h^* \oplus \mathfrak{H}_h. \tag{2.6}$$

This orthogonality is the key ingredient in deriving decoupled W-norm error estimates for each variable above and plays an important role in our analysis as well.

3. Main results

We start by identifying W^k with its dual and form a Gelfand triple $V^k \cap V_k^* \subset W^k \subset (V^k \cap V_k^*)'$. We extend $L_0 = dd^* + d^*d$ to an operator $V \cap V^* \to (V \cap V^*)'$ which is more suitable for studying perturbations: for all $u, v \in V^k \cap V_k^*$,

$$\langle L_0 u, v\rangle := (d^*u, d^*v) + (du, dv).$$

Our main operator $L : V^k \cap V_k^* \to (V^k \cap V_k^*)'$ is obtained by perturbing each abstract differential: for all $u, v \in V^k \cap V_k^*$,

$$\langle Lu, v\rangle := ((d^* + l_2)u, (d^* + l_3^*)v) + ((d + l_1)u, dv) + (l_4 du, v) + (l_5 u, v), \tag{3.1}$$

where $l_i : W \to W$ are bounded linear maps between appropriate levels for $i = 1, \ldots, 5$. More succinctly, we write

$$L = (d + l_3)(d^* + l_2)u + d^*(d + l_1)u + l_4 du + l_5 u.$$

The grouping is convenient for the mixed form later. We prove in Lemma 4.1 of the next section that this L satisfies a Gårding inequality. Then standard techniques in elliptic PDE theory imply that L is invertible up to some arbitrarily small perturbation to l_5. Therefore, generically, it is reasonable to assume that L is a bounded isomorphism.

Let D be the natural domain on which L maps $W^k \to W^k$:

$$D := \{u \in V^k \cap V_k^* \mid (d^* + l_2)u \in V^{k-1}, (d + l_1)u \in V_{k+1}^*\}. \tag{3.2}$$

Our main perturbed problem is: given $f \in W^k$, find $u \in D$ such that

$$Lu = f. \tag{3.3}$$

We reformulate it in the mixed form: given $f \in W^k$ find $(\sigma, u) \in V^{k-1} \times V^k$ satisfying

$$(\sigma, \tau) - (u, d\tau) - (l_2 u, \tau) = 0, \qquad \forall \tau \in V^{k-1}, \tag{3.4a}$$

$$((d + l_3)\sigma, v) + ((d + l_1)u, dv) + (l_4 du, v) + (l_5 u, v) = (f, v), \quad \forall v \in V^k. \tag{3.4b}$$

The first equation (3.4a) is equivalent to $u \in V_k^*$ and $\sigma = (d^* + l_2)u$. The second equation (3.4b) is equivalent to $(d + l_1)u \in V_{k+1}^*$ and $d^*(d + l_1)u = f - (d + l_3)\sigma - l_4 du - l_5 u$. Hence, if (σ, u) solves (3.4), then u solves (3.3). Therefore, it makes sense to use this mixed formulation to solve our problem.

In addition to the assumption that L is a bounded isomorphism, we need more regularity assumptions on l_i to ensure that $L^{-1}(W) \subset D$ so that (3.3) has a solution. One of our main tools for analyzing the discretization is the duality argument, where the dual problem $L'z = g$ has to be solved as well. Here $L' : V^k \cap V_k^* \to (V^k \cap V_k^*)'$ is the dual of L. We collect the conditions under which all of these continuous problems are well-posed in a theorem:

Theorem 3.1 (Continuous well-posedness)**.** *Let (W, d) be a Hilbert complex with the compactness property. Suppose L defined in equation* (3.1) *is a bounded isomorphism and*

$$(d^* + l_2)L^{-1}(W) \subset V^{k-1}, \qquad (d^* + l_3^*)(L')^{-1}(W) \subset V^{k-1}. \tag{3.5}$$

Then both the perturbed problem (3.3) *and its mixed formulation* (3.4) *are well-posed.*

The proof is given in Section 4. The regularity assumption is very mild, without which the perturbed problem does not even make sense. The solution operator to the dual problem $L'z = g$ will be used frequently, so we give it a name. Let

$$K = (L')^{-1} : (V^k \cap V_k^*)' \to V^k \cap V_k^*.$$

Our first discrete result is the following fundamental theorem on mixed methods for problems perturbed by lower-order terms.

Theorem 3.2 (Discrete stability)**.** *Under the assumptions of Theorem* 3.1*, suppose further that $\|(dd^* + d^*d)K\|_{W \to W}$ is bounded and the following operators are compact $W \to W$:*

$$dl_3^* K, \quad (l_1^* d - l_2^* d^*)K.$$

Let V_h^k be a sequence of dense subcomplexes of V^k admitting W-bounded cochain projections. Then the Galerkin projection of the mixed system (3.4) *using the pair $V_h^{k-1} \times V_h^k$ is stable in the sense that there exist positive constants h_0, C_0 such that for any $h \in (0, h_0]$, there exists a unique discrete solution $(\sigma_h, u_h) \in V_h^{k-1} \times V_h^k$ satisfying* (3.4) *for test functions in $V_h^{k-1} \times V_h^k$ and that $\|\sigma_h\|_V + \|u_h\|_V \le C_0 \|f\|$.*

The proof is non-trivial and is given in Section 5. For the de Rham complex, suppose all the lower-order terms are multiplication by smooth coefficients and the domain has $(1+\epsilon)$-regularity such that $\|Kf\|_{H^{1+\epsilon}} \,\square\, \|f\|_{L^2}$ for $\epsilon > 0$. By definition, $L' = (d + l_2^*)(d^* + l_3^*) + (d^* + l_1^*)d + d^* l_4^* + l_5^*$ and $K = (L')^{-1}$, we have

$$(dd^* + d^*d)K = I - (l_2^* d^* + dl_3^* + l_1^* d + d^* l_4^* + l_5^* + l_2^* l_3^*)K$$

is bounded $L^2 \to L^2$. The compactness assumptions are satisfied due to the compact embedding of Sobolev space H^s into L^2 for $s > 0$. This proves the statement on the stability of mixed discretization of Hodge Laplacian in the introduction.

It is well known that stability guarantees optimal error rates in the energy norm:

Corollary 3.3. *Under the assumptions of Theorem* 3.2*, if* $h \le h_0$*, the unique discrete solution* (σ_h, u_h) *satisfies*

$$\|\sigma - \sigma_h\|_V + \|u - u_h\|_V \lesssim \|(I - \pi_h)\sigma\|_V + \|(I - \pi_h)u\|_V.$$

As is common for mixed methods, the energy norm estimate is crude because it couples errors of different variables. For example, for the unperturbed Hodge Laplacian solved with the FEEC pair $\mathcal{P}_{r+1}\Lambda^{k+1} \times \mathcal{P}_r\Lambda^k$, the convergence rate in $\|\sigma\|$ is in fact h^2 higher than that in $\|du\|$ but is lumped together with it. Our next finer discrete result gives the decoupled W-norm estimates for each variable similar to estimates (2.3) for the unperturbed problem. For that, we need more assumptions. To simplify the bookkeeping, we define some approximation quantities:

$$\begin{aligned} \delta &= \max\{\delta_0, \|(I - P_h)l_3^* K\|, \|(I - P_h)(l_5^* - l_2^* l_3^*)K\|, \|(I - P_h)P_{\mathfrak{B}}\, l_4^* K\|\}, \\ \eta &= \max\{\eta_0, \mu_0, \delta, \|(I - P_h)dl_3^* K\|, \|(I - P_h)(l_1^* d - l_2^* d^*)K\|\}, \\ \alpha &= \delta + \eta^2 + \mu_0, \end{aligned} \tag{3.6}$$

where η_0, δ_0, μ_0 are defined in equation (2.2) and all the operator norms are in $\|\cdot\|_{W\to W}$. As before, due to the compactness assumptions and density, all $\delta, \eta, \alpha \to 0$ as $h \to 0$.

Theorem 3.4. *In addition to the assumptions of Theorem* 3.2*, assume that*

$$\|d(l_1^* d - l_2^* d^* + l_5^* - l_2^* l_3^*)K\|_{W\to W}$$

is bounded. Then we have the following improved error estimates:

$$\begin{aligned} \|\sigma - \sigma_h\| \lesssim\ & E(\sigma) + (\eta + \chi_{45}\sqrt{\alpha})E(d\sigma) \\ & + (\chi_{24} + \chi_3\eta + \chi_5\sqrt{\eta})E(u) + (\chi_3\alpha + \chi_{45}\sqrt{\alpha})E(du), \end{aligned}$$

$$\begin{aligned} \|d(\sigma - \sigma_h)\| \lesssim\ & E(d\sigma) + (\chi_3\alpha + \chi_4 + \chi_5\eta)E(du) \\ & + (\chi_{45} + \chi_3\eta + \chi_2\chi_3)E(u) + (\chi_3 + \chi_4\eta)E(\sigma), \end{aligned}$$

$$\begin{aligned} \|u - u_h\| &\lesssim E(u) + \eta E(du) + \eta E(\sigma) + (\alpha + \chi_{45}\sqrt{\alpha})E(d\sigma), \\ \|d(u - u_h)\| &\lesssim E(du) + \eta E(d\sigma) + \chi_{1345}E(u) + (\chi_3 + \chi_{145}\eta)E(\sigma), \end{aligned}$$

where $\chi_{i\ldots j}$ *denotes the presence of lower-order terms. For example,* $\chi_{125} = 1$ *if* $l_1 \ne 0$ *or* $l_2 \ne 0$ *or* $l_5 \ne 0$*, and* $\chi_{125} = 0$ *otherwise.*

The proof is subtle and is given in Section 6.

Corollary 3.5. *Suppose a Hodge Laplacian problem satisfies full* 2*-regularity:*

$$\|Kf\|_{H^{s+2}} \lesssim \|f\|_{H^s}, \quad \text{for all } s \ge 0$$

(for example, on a smooth domain). Then the error estimates for the discretization using FEEC elements are given by Table 1.

Proof. Following the discussion after equation (2.2), we have $\eta_0 = O(h)$, $\delta_0 = O(h^{\min(2,r+1)})$, $\mu_0 = O(h^{r+1})$, where r is the largest degree of complete polynomials in V_h. Using the best approximation estimates for FEEC elements, we see that $\eta = O(h)$ and $\delta = O(h^{\min(2,r+1)})$ as well. Plugging these and the best approximation estimates into Theorem 3.4, we get the rates in Table 1. □

4. Well-posedness at the continuous level

In this section, we establish well-posedness results for the continuous problem and its mixed formulation.

4.1. **Well-posedness for the primal form.** First, we prove that the perturbed bounded operator is almost always an isomorphism.

Lemma 4.1. *Let* (W^k, d) *be a Hilbert complex having the compactness property with domains* V^k. *Let* L *be defined as in* (3.1). *Then,* $L + \lambda I$ *has a bounded inverse for all* $\lambda \in \mathsf{C}$ *except at a discrete subset (so at most countable).*

Proof. Let $M = \max_i \|l_i\|_{W \to W}$ and $\gamma = 4M^2 + M + 1/2$. Then $L + \gamma I$ is coercive on $V^k \cap V_k^*$:

$$\langle Lu, u\rangle + \gamma(u, u) \geq (1/2)[(u, u) + (d^*u, d^*u) + (du, du)].$$

The compactness property ensures that $I : V^k \cap V_k^* \hookrightarrow (V^k \cap V_k^*)'$ is compact, which makes $I(L + \gamma I)^{-1}$ compact on $(V^k \cap V_k^*)'$. Spectral theory then implies that $I + \mu I(L + \gamma I)^{-1}$ has a bounded inverse for all $\mu \in \mathsf{C}$ except at a discrete subset. Then composing with the bounded isomorphism $L + \gamma I$ on the right proves the claim. □

In particular, this shows that either L is invertible or $L + \epsilon I$ is invertible for any small enough non-zero ϵ.

Then, we prove the well-posedness of our main problem.

Lemma 4.2. *Suppose* L *defined in equation* (3.1) *is a bounded isomorphism and* $(d^* + l_2)L^{-1}(W) \subset V^{k-1}$. *Then* $L^{-1}(W) \subset D$, *where* D *is defined in* (3.2). *In particular, problem* (3.3) *has a unique solution.*

Proof. Since L is already an isomorphism, we only need to show that $L^{-1}(W) \subset D$. For any $f \in W$, let $u = L^{-1}f$. Then by definition,

$$((d^*+l_2)u, (d^*+l_3^*)v)+((d+l_1)u, dv)+(l_4du, v)+(l_5u, v) = (f, v), \qquad \forall v \in V^k \cap V_k^*.$$

By assumption, $(d^* + l_2)u \in V^{k-1}$. Thus, we have

$$((d + l_1)u, dv) = (f - (d + l_3)(d^* + l_2)u - l_4du - l_5u, v), \qquad \forall v \in V^k \cap V_k^*.$$

There is no d^*v in the above. By the density of $V^k \cap V_k^*$ in W^k, we conclude that the above holds for all $v \in V^k$. Hence, $(d + l_1)u \in V_{k+1}^*$. Thus $u \in D$ proves the claim. □

We then prove a similar result for the dual problem. Let D' be the natural domain on which L' maps $W^k \to W^k$:

$$D' := \{u \in V^k \cap V_k^* \mid (d^* + l_3^*)u \in V^{k-1}, (d + l_4^*)u \in V_{k+1}^*\}.$$

Using the same argument, we get

Lemma 4.3. *Suppose* L *defined in equation* (3.1) *is a bounded isomorphism and* $(d^* + l_3^*)(L')^{-1}(W) \subset V^{k-1}$. *Then* $(L')^{-1}(W) \subset D'$.

4.2. **Well-posedness for the mixed form.** We then turn to mixed system (3.4). Its associated bilinear form $B : (V^{k-1} \times V^k)^2 \to \mathsf{R}$ is

$$B((\sigma, u), (\tau, v)) := (\sigma, \tau) + (du, dv) + (d\sigma, v) - (u, d\tau) - (l_2 u, \tau) + (l_3\sigma, v) + (l_1 u, dv) + (l_4 du, v) + (l_5 u, v). \tag{4.1}$$

We call this bilinear form *well-posed* if and only if for any $(g, f) \in V'_{k-1} \times V'_k$, there exists a unique solution $(\sigma, u) \in V^{k-1} \times V^k$ satisfying

$$B((\sigma, u), (\tau, v)) = \langle g, \tau\rangle_{V \times V} + \langle f, v\rangle_{V \times V}, \qquad \forall(\tau, v) \in V^{k-1} \times V^k.$$

From the discussion after equation (3.4), we see that the well-posedness of B implies that $L^{-1}(W) \in D$. But it also implies the well-posedness of the dual mixed problem: given any $(g, f) \in V'_{k-1} \times V'_k$, find $(\xi, z) \in V^{k-1} \times V^k$ satisfying

$$B((\rho, w), (\xi, z)) = \langle g, \rho\rangle_{V \times V} + \langle f, w\rangle_{V \times V}, \qquad \forall(\rho, w) \in V^{k-1} \times V^k.$$

A similar argument shows that the well-posedness of B implies $(L')^{-1}(W) \in D'$ as well. We collect these results in a lemma:

Lemma 4.4. *Suppose B defined in* (4.1) *is well-posed and let L be defined as in* (3.1). *Then L is a bounded isomorphism, $L^{-1}(W) \subset D$, and $(L')^{-1}(W) \subset D'$.*

Moreover, the converse is also true.

Lemma 4.5. *Suppose L defined in* (3.1) *is a bounded isomorphism. Then B defined in* (4.1) *is well-posed if and only if condition* (3.5) *holds.*

Proof. The "only if" part is clear from the previous lemmas. We only need to show the "if" part.

First, we show that B satisfies a Gårding-like inequality: there exist positive constants a, b, c depending only on $\|l_i\|_{W\to W}$ such that

$$B((\sigma, u), (\sigma, u + ad\sigma)) \geq b(\|\sigma\|_V + \|u\|_V)^2 - c\|u\|^2, \qquad \forall(\sigma, u) \in V^{k-1} \times V^k. \tag{4.2}$$

Direct computation using Cauchy-Schwarz inequality shows that there exist constants c_1, c_2 depending only on $\|l_i\|_{W\to W}$ such that

$$B((\sigma, u), (\sigma, u)) \geq (1/2)(\|\sigma\|^2 + \|du\|^2) - c_1\|u\|^2,$$
$$B((\sigma, u), (0, d\sigma)) \geq (1/2)\|d\sigma\|^2 - c_2(\|\sigma\|^2 + \|du\|^2 + \|u\|^2).$$

Multiplying the second inequality by any positive $a < 1/(2c_2)$ and adding it to the first inequality, we get the claim.

Second, fix any $(\sigma, u) \in V^{k-1} \times V^k$. We solve a dual problem using u as data: let $z = cKu$ and $\xi = -(d^* + l_3^*)z$. By assumption, $\xi \in V^{k-1}$. Direct computation shows that

$$B((\rho, w), (\xi, z)) = (cu, w), \qquad \forall(\rho, w) \in V^{k-1} \times V^k.$$

Finally, we add (ξ, z) to our choice of test functions in the first step and get

$$B((\sigma, u), (\sigma + \xi, u + ad\sigma + z)) \geq b(\|\sigma\|_V + \|u\|_V)^2.$$

Further, from the definition of (ξ, z), we have

$$\|\sigma + \xi\|_V + \|u + ad\sigma + z\|_V \leq M\|\sigma\|_V + \|u\|_V,$$

where the constant M depends only on a, $\|d(d^* + l_3^*)K\|_{W\to W}$, and $\|dK\|_{W\to W}$.

Thus B satisfies the inf-sup condition. Similarly, for any fixed non-trivial $(\tau, v) \in V^{k-1} \times V^k$, we let $u = L^{-1}v$ and $\sigma = (d^* + l_2)u$. By assumption, $\sigma \in V^{k-1}$. Direct computation shows that

$$B((\sigma, u), (\tau, v)) = (v, v) > 0.$$

It is well-known that these two conditions imply that B is well-posed [3]. □

Given these lemmas, Theorem 3.1 is clearly true.

5. Discrete stability through new projections

In this section, we prove Theorem 3.2. The idea of the proof is similar to the that of the "if" part of Lemma 4.5. First, due to the subcomplex property, the estimate (4.2) still holds at the discrete level with the same constants a, b, c. Second, fix any $(\sigma, u) \in V_h^{k-1} \times V_h^k$. We can again solve a dual problem using u as data: let $z = cKu$ and $\xi = -(d^* + l_3^*)z$. Then we have $\xi \in V^{k-1}$, $\|\xi\|_V + \|z\|_V \,\square\, \|u\|$ independent of h, and

$$B((\rho, w), (\xi, z)) = (cu, w), \qquad \forall (\rho, w) \in V^{k-1} \times V^k.$$

But we can no longer add (ξ, z) to our choice of test functions because (ξ, z) is not discrete. In the rest of this section, we construct a discrete pair $(\xi_h, z_h) \in V_h^{k-1} \times V_h^k$ such that

$$\begin{aligned} &\|\xi_h\|_V + \|z_h\|_V \,\square\, \|\xi\|_V + \|z\|_V \text{ uniformly in } h, \text{ and} \\ &|B((\rho, w), (\xi - \xi_h, z - z_h))| \le \epsilon_h(\|\rho\|_V + \|w\|_V)(\|\xi\|_V + \|z\|_V), \end{aligned} \tag{5.1}$$

for all $(\rho, w) \in V_h^{k-1} \times V_h^k$, where $\epsilon_h \to 0$ as $h \to 0$. Given such a pair, we can add it to our choice of test functions:

$$B((\sigma, u), (\sigma + \xi_h, u + ad\sigma + z_h)) \ge (b - c\epsilon_h)(\|\sigma\|_V + \|u\|_V)^2.$$

Further, there also exists $M > 0$ bounded uniformly in h such that

$$\|\sigma + \xi_h\|_V + \|u + ad\sigma + z_h\|_V \le M(\|\sigma\|_V + \|u\|_V).$$

Choose a sufficiently small h_0 such that $\epsilon_{h_0} < b/c$. Then for all $h < h_0$, the bilinear form $B((\sigma, u), (\tau, v))$ satisfies the inf-sup condition on $V_h^{k-1} \times V_h^k$ with the inf-sup constant bounded uniformly below by $(b - c\epsilon_{h_0})/M$. Since $V_h^{k-1} \times V_h^k$ is of finite dimension, this establishes the well-posedness. Thus Theorem 3.2 is proved.

An obvious choice for (ξ_h, z_h) in (5.1) is the elliptic projection given by

$$B((\rho, w), (\xi_h, z_h)) = B((\rho, w), (\xi, z)), \qquad \forall (\rho, w) \in V_h^{k-1} \times V_h^k.$$

Then $\epsilon_h = 0$. But since we have not proved the well-posedness of B on the discrete level, we neither know a discrete solution exists nor can we show the uniform estimates. The next most obvious choice is obtained using the elliptic projection of the unperturbed problem:

$$B_0((\rho, w, p), (\xi_h, z_h, q_h)) = B_0((\rho, w), (\xi, z)), \qquad \forall (\rho, w, p) \in V_h^{k-1} \times V_h^k \times \mathfrak{H}_h^k.$$

Then we have the existence and uniform bounds. But the second estimate in (5.1) fails. In what follows, we develop two new projection operators to correct the elliptic projection for the unperturbed problem so that both conditions in (5.1) hold. In fact, we do a lot more. Our elaborately chosen (ξ_h, z_h) will not only satisfy (5.1) but also have explicit and optimal error rates in quantities like $\|\xi - \xi_h\|, \|z - z_h\|$, and $|B((\rho, w), (\xi - \xi_h, z - z_h))|$. This is made precise in Theorem 5.3. This result also forms the basis of the improved error estimates later. Moreover, the two new projection operators enjoy many properties making them interesting in their own right.

5.1. **Generalized canonical projection.** Thus far, we have three projections in FEEC: the orthogonal projection P_h, the cochain projection π_h which commutes with d but has no orthogonality property, and the elliptic projection $K_{0h}P_hL_0$ which misses the harmonic part and is only well-defined on the subspace $D(L_0)$. Here, we introduce a new projection operator $\Pi_h : V^k \to V_h^k$ given by

$$\Pi_h := P_{\mathrm{Z}_h} + d_h^* K_{0h} P_h d.$$

In the above and for the rest of this paper, for any subspace X of W, we use the notation $P_X : W \to X$ for the W-orthogonal projection. Among other properties, this Π_h satisfies a commutative property generalizing that of the canonical projection for classical elements like Raviart–Thomas.

Theorem 5.1. *Suppose (W, d) is a Hilbert complex satisfying the compactness property and V_h^k are dense discrete subcomplexes admitting W-bounded cochain projections. Then Π_h is a projection uniformly bounded in the V-norm. Further $d\Pi_h = P_{\mathrm{B}_h} d$. Let η_0, α_0 be defined as in equation* (2.2)*. Then, for any $w \in V$,*

$$\|\Pi_h w\| \lesssim \|w\| + \eta_0 \|dw\|, \qquad \|(I - \Pi_h)w\| \lesssim \|(I - \pi_h)w\| + \eta_0\|dw\|.$$

Moreover, it satisfies "partial orthogonality": for any $w, v \in V$,

$$|((I - \Pi_h)w, v)| \lesssim (\|(I - \pi_h)w\| + \eta_0\|dw\|)(\|(I - \pi_h)v\| + \eta_0\|dv\|) + \alpha_0\|dv\|\|dw\|.$$

Proof. The stability of the unperturbed discrete problem (2.1) implies that Π_h is uniformly bounded in the V-norm. By the subcomplex property, we have $P_h dv = dv$ for $v \in V_h$. Thus,

$$\Pi_h v = P_{\mathrm{Z}_h} v + d_h^* K_{0h} dv = P_{\mathrm{Z}_h} v + P_{\mathrm{B}_h} v = v,$$

showing that Π_h is a projection. We know from equation (2.6) that for the unperturbed problem $dd_h^* K_{0h} = P_{\mathrm{B}_h}$. This proves that $d\Pi_h = dd_h^* K_{0h} P_h d = P_{\mathrm{B}_h} d$. We then prove the first two estimates. Fix any $w \in V$. We split $w - \Pi_h w = (P_{\mathrm{Z}} - P_{\mathrm{Z}_h})w + (P_{\mathrm{B}} w - P_{\mathrm{B}_h} \Pi_h w)$. The second term can be bounded using the error estimates (2.4) for K_{0h}:

$$\|P_{\mathrm{B}} w - P_{\mathrm{B}_h} \Pi_h w\| = \|d^* K_0 dw - d_h^* K_{0h} P_h dw\| \lesssim \eta_0 \|dw\|.$$

We then deal with the first term. The subcomplex property ensures $\mathrm{Z}_h \subset \mathrm{Z}$, and the cochain property of π_h ensures $\pi_h \mathrm{Z} \subset \mathrm{Z}_h$. These two lead to

$$\|(P_{\mathrm{Z}} - P_{\mathrm{Z}_h})w\| \lesssim \|(I - \pi_h)P_{\mathrm{Z}} w\|.$$

But $(I - \pi_h)P_{\mathrm{Z}} w = (I - \pi_h)w - (I - \pi_h)P_{\mathrm{B}} w$ and $P_{\mathrm{B}} = d^* K_0 d$. Thus,

$$\|(P_{\mathrm{Z}} - P_{\mathrm{Z}_h})w\| \lesssim \|(I - \pi_h)w\| + \|(I - \pi_h)d^* K_0 dw\| \lesssim \|(I - \pi_h)w\| + \eta_0\|dw\|.$$

Combining the estimates for the two parts, we get

$$\|(I - \Pi_h)w\| = \|(P_{\mathrm{Z}} - P_{\mathrm{Z}_h})w + (P_{\mathrm{B}} w - P_{\mathrm{B}_h}\Pi_h w)\| \lesssim \|(I - \pi_h)w\| + \eta_0\|dw\|.$$

By the triangle inequality, we get

$$\|\Pi_h w\| \leq \|w\| + \|(I - \Pi_h)w\| \lesssim \|w\| + \eta_0\|dw\|$$

as well. This proves the first two estimates. Finally, for any $w, v \in V$, we have

$$((I - \Pi_h)w, v) = ((P_{\mathrm{Z}} - P_{\mathrm{Z}_h})w, v) + ((P_{\mathrm{B}} - P_{\mathrm{B}_h}\Pi_h)w, (I - \pi_h)v) + ((P_{\mathrm{B}} - P_{\mathrm{B}_h}\Pi_h)w, \pi_h v).$$

The first two terms can be bounded as before. The last term is bounded by

$$(d^* K_0 dw - d_h^* K_{0h} P_h dw, \pi_h v) = ((K_0 - K_{0h}P_h)dw, \pi_h dv) \,\square\, \alpha_0 \|dw\|\|dv\|,$$

where the error estimate (2.4) is used again. This finishes the proof. $\square$

5.2. **Modified elliptic projection.** We modify the unperturbed elliptic projection slightly to accommodate the harmonic forms.

Theorem 5.2. *For any $z \in D(L_0)$, let $z_h = K_{0h}P_h L_0 z + P_{\mathrm{H}_h} P_{\mathrm{H}} z$. Then,*

$$\|z - z_h\| \,\square\, \alpha_0 \|L_0 z\|, \qquad \|d(z - z_h)\| + \|d^* z - d_h^* z_h\| \,\square\, \eta_0 \|L_0 z\|,$$
$$\|P_h(d^* dz + dd^* z) - (d_h^* dz_h + dd_h^* z_h)\| \le \mu_0 \|L_0 z\|.$$

Proof. By equation (2.5), we have $P_{\mathrm{H}} z = K_0 L_0 z$. Thus we have the splitting

$$z - z_h = (P_{\mathrm{H}} z - P_{\mathrm{H}_h} z_h) + (P_{\mathrm{H}} z - P_{\mathrm{H}_h} z_h) = (K_0 - K_{0h}P_h)L_0 z + (I - P_{\mathrm{H}_h})P_{\mathrm{H}} z.$$

The first term has been estimated by (2.4). For the second term, since $P_{\mathrm{H}_h} P_{\mathrm{H}} = P_{\mathrm{Z}_h} P_{\mathrm{H}}$ and $\pi_h \mathrm{Z} \subset \mathrm{Z}_h$, we have

$$\|(I - P_{\mathrm{H}_h})P_{\mathrm{H}} z\|_W \le \|(I - \pi_h)P_{\mathrm{H}} z\|_W \le \|(I - \pi_h)P_{\mathrm{H}}\|_{W \to W} \|z\| = \mu_0 \|z\|,$$

which proves the first estimate. The second estimate follows from (2.4) directly. Finally for the last estimate, we use the continuous and discrete Hodge decomposition (2.5) (2.6), and we get

$$(d_h^* d + dd_h^*) z_h = (d_h^* d + dd_h^*) K_{0h} P_h L_0 z = (P_{\mathrm{B}_h} + P_{\mathrm{B}_h}) P_h L_0 z.$$

Moreover, by definition, $P_{\mathrm{H}} L_0 z = 0$. Thus,

$$P_h(d^* dz + dd^* z - d_h^* dz_h - dd_h^* z_h) = P_h L_0 z - (P_{\mathrm{B}_h} + P_{\mathrm{B}_h}) P_h L_0 z = P_{\mathrm{H}_h} L_0 z.$$

The right-hand side is just $\|p - p_0\|$ for the unperturbed problem with $L_0 z$ as data. By (2.3),

$$\|P_{\mathrm{H}_h} L_0 z\| = \|(P_{\mathrm{H}_h} - P_{\mathrm{H}}) L_0 z\| \,\square\, 0 + \mu_0 \|P_{\mathrm{B}} L_0 z\| \le \mu_0 \|L_0 z\|,$$

which proves the last estimate. $\square$

5.3. **Projection of the dual solution.** We are now ready to construct the discrete pair (ξ_h, z_h) satisfying the conditions (5.1) in the proof of the discrete stability theorem. In fact, we prove a stronger result where the first variable ρ is allowed to be in V instead of V_h and derive explicit error estimates.

Theorem 5.3. *Under the assumption of Theorem 3.2, for any $g \in W^k$, let $z = Kg$, $\xi = -(d^* + l_3^*)z$, $z_h = K_{0h}P_h L_0 z + P_{\mathrm{H}_h} P_{\mathrm{H}} z$, and $\xi_h = -d_h^* z_h - \Pi_h l_3^* z$. Then,*

$$\|z - z_h\| \,\square\, \alpha_0 \|g\|, \qquad \|d(z - z_h)\| \,\square\, \eta_0 \|g\|, \qquad \|\xi - \xi_h\| \,\square\, \eta \|g\|.$$

Further, for any $(\rho, w) \in V^{k+1} \times V_h^k$, we have

$$|B((\rho, w), (\xi - \xi_h, z - z_h))| \,\square\, [\eta\|\rho\| + \alpha\|d\rho\| + (\mu_0 + \chi_{123}\eta + \chi_5\alpha)\|w\| + \chi_4\alpha\|dw\|]\|g\|.$$

Proof. Using the regularity assumption that $\|L_0K\|_{W\to W}$ is bounded, the estimates for $\|z - z_h\|$ and $\|d(z - z_h)\|$ follow directly from Theorem 5.2. From the same theorem, for ξ, we have

$$\|\xi - \xi_h\| \le \|d^*z - d_h^*z_h\| + \|(I - \Pi_h)l_3^*z\| \lesssim \eta_0\|g\| + \|(I - \Pi_h)l_3^*Kg\|.$$

For the second term, using quantities defined in (3.6), we have

$$\|(I - \Pi_h)l_3^*Kg\| \lesssim \|(I - \pi_h)l_3^*Kg\| + \eta\|dl_3^*Kg\| \lesssim \eta\|g\|.$$

The last estimate is just a direct computation using the error estimates in Theorem 5.2, quantities defined in (3.6), and the Cauchy-Schwarz inequality. $\square$

For the proof of Theorem 3.2 at the beginning of this section, we get (ξ_h, z_h) by applying this theorem to $g = cu \in V_h \subset W$. We note that (ξ, z) here is the same as the one defined there. We check that condition (5.1) is satisfied. First,

$$\|z_h\|_V = \|K_{0h}P_hL_0z + P_{\mathfrak{H}_h}P_{\mathfrak{H}}z\|_V \lesssim \|z\|_V,$$
$$\|\xi_h\|_V = \| - d_h^*K_{0h}P_hL_0z - \Pi_hl_3^*z\|_V \lesssim \|d\xi\| + \|z\|_V \lesssim \|\xi\|_V + \|z\|_V,$$

where the constants depend only on the stability constant of the continuous and discrete unperturbed problem, which is either independent of h or bounded uniformly in h. Second, as mentioned before, the compactness assumptions and density, $\alpha, \eta, \mu_0 \to 0$ as $h \to 0$. Thus condition (5.1) is verified. This finishes the proof of Theorem 3.2.

6. Proof of improved error estimates

In this section, we prove Theorem 3.4. To make the notation more compact, we use $e_u := u - u_h$ and $E_u = (I - \pi_h)u$. Corresponding quantities for σ are similarly defined. The Galerkin orthogonality equation reads:

(6.1a) $\quad (e_\sigma, \tau) - (e_u, d\tau) - (l_2e_u, \tau) = 0, \qquad \forall \tau \in V_h^{k-1},$

(6.1b) $\quad ((d + l_3)e_\sigma, v) + ((d + l_1)e_u, dv) + (l_4de_u, v) + (l_5e_u, v) = 0, \quad \forall v \in V_h^k.$

6.1. Preliminary estimates for $\|de_\sigma\|$ and $\|de_u\|$.

Optimal estimates for these two terms can be obtained directly from the error equations (6.1) with carefully chosen test functions.

Lemma 6.1. *For any (σ, u) solving (3.4) and (σ_h, u_h) solving its Galerkin projection,*

$$\|de_\sigma\| \lesssim \|dE_\sigma\| + \chi_3\|e_\sigma\| + \chi_4\|de_u\| + \chi_5\|e_u\|.$$

Proof. Restricting the test function space to $\mathfrak{B}_h$ in equation (6.1b) leads to

$$P_{\mathfrak{B}_h}(de_\sigma + l_3e_\sigma + l_4de_u + l_5e_u) = 0. \tag{6.2}$$

Thus, $de_\sigma = (I - P_{\mathfrak{B}_h})de_\sigma + P_{\mathfrak{B}_h}de_\sigma = (I - P_{\mathfrak{B}_h})de_\sigma - P_{\mathfrak{B}_h}(l_3e_\sigma + l_4de_u + l_5e_u)$. Because π_h maps $\mathfrak{B}$ to $\mathfrak{B}_h$, we have $\|(I - P_{\mathfrak{B}_h})de_\sigma\| \lesssim \|(I - \pi_h)de_\sigma\| = \|dE_\sigma\|$, proving the claim. $\square$

Lemma 6.2. *For any (σ, u) solving (3.4) and (σ_h, u_h) solving its Galerkin projection,*

$$\|de_u\| \lesssim \|dE_u\| + \eta\|de_\sigma\| + \chi_{145}\|e_u\| + \chi_3\|e_\sigma\|.$$

Proof. Let $v_h = P_{\mathrm{B}_h}(\pi_h u - u_h)$ in the second error equation (6.1b). We have

$$(de_u, dv_h) = -(de_\sigma, v_h) - [(l_1 e_u, dv_h) + (l_3 e_\sigma + l_5 e_u, v_h)] - (l_4 de_u, v_h). \tag{6.3}$$

By discrete Poincaré inequality $\|v_h\| \lesssim \|dv_h\|$ for $v_h \in \mathrm{B}_h^*$, the second term in (6.3) becomes

$$|(l_1 e_u, dv_h) + (l_3 e_\sigma + l_5 e_u, v_h)| \lesssim (\chi_{15}\|e_u\| + \chi_3\|e_\sigma\|)\|dv_h\|.$$

Because $de_u = d(u - \pi_h u + \pi_h u - u_h) = dE_u + dv_h$, the last term in (6.3) satisfies

$$|(l_4 de_u, v_h)| = |(l_4 dE_u, v_h) + (l_4 dv_h, v_h)| \lesssim (\|dE_u\| + \|v_h\|)\|dv_h\|.$$

We finally estimate the first term in (6.3). Let $v = P_{\mathrm{B}} v_h$. Then $d(\pi_h v - v_h) = 0$ implies $\pi_h v - v_h \in \mathrm{Z}_h$, so $(v_h - v) \perp (\pi_h v - v_h)$. Thus,

$$\|v - v_h\| \leq \|(I - \pi_h)v\| = \|(I - \pi_h)d^* K dv_h\| \lesssim \eta\|dv_h\|.$$

This implies

$$|(de_\sigma, v_h)| = |(de_\sigma, v - v_h)| \lesssim \eta\|de_\sigma\|\|dv_h\|.$$

Combining all these estimates and $\|de_u\| \leq \|dE_u\| + \|dv_h\|$ gives the estimate in the claim. □

6.2. **Duality lemma.** The optimal W-norm estimates for $\|e_u\|$ and $\|e_\sigma\|$ require more work.

Lemma 6.3. *Under the assumptions of Theorem* 3.4, *for any* $g \in W$,

$$|(\Pi_h e_u, g)| \lesssim [\eta\|e_\sigma\| + \alpha\|de_\sigma\| + (\mu_0 + \chi_{123}\eta + \chi_5\alpha)\|e_u\| + (\mu_0\eta + \chi_{12345}\alpha)\|de_u\|]\|g\|.$$

Proof. Given g, let (ξ, z) and (ξ_h, z_h) be defined as in Theorem 5.3. We have

$$(\Pi_h e_u, g) = B((e_\sigma, \Pi_h e_u), (\xi, z)) = B((e_\sigma, \Pi_h e_u), (\xi_h, z_h)) - B((e_\sigma, \Pi_h e_u), (e_\xi, e_z)).$$

Galerkin orthogonality (6.1) states that $B((e_\sigma, e_u), (\tau, v)) = 0$ for all discrete (τ, v). Thus,

$$(\Pi_h e_u, g) = -B((0, (I - \Pi_h)e_u), (\xi_h, z_h)) - B((e_\sigma, \Pi_h e_u), (e_\xi, e_z)).$$

The second term above can be estimated by the last inequality in Theorem 5.3 and

$$\|\Pi_h e_u\| \lesssim \|e_u\| + \eta\|de_u\|, \qquad \|d\Pi_h e_u\| \lesssim \|de_u\|.$$

The result is the following bound:

$$\begin{aligned}|B((e_\sigma, \Pi_h e_u), (e_\xi, e_z))| \lesssim [\eta\|e_\sigma\| + \alpha\|de_\sigma\| + (\mu_0 + \chi_{123}\eta + \chi_5\alpha)\|e_u\| \\ + (\mu_0\eta + \chi_4\alpha)\|de_u\|]\|g\|.\end{aligned}$$

We bound the first term by splitting it into three parts:

$$\begin{aligned}|B((0, (I - \Pi_h)e_u), (\xi_h, z_h))| &\leq |((I - \Pi_h)e_u, d\xi_h) + (d(I - \Pi_h)e_u, dz_h)| \\ &\quad + |(l_2(I - \Pi_h)e_u, \xi_h) + (l_1(I - \Pi_h)e_u, dz_h) + (l_5(I - \Pi_h)e_u, z_h)| \\ &\quad + |(l_4 d(I - \Pi_h)e_u, z_h)| =: Q_1 + Q_2 + Q_3.\end{aligned}$$

We have $Q_1 = 0$ because $P_{\mathrm{B}_h}\Pi_h = P_{\mathrm{B}_h}$ and $d\Pi_h = P_{\mathrm{B}_h} d$. The second term is

$$Q_2 \leq |((I - \Pi_h)e_u, l_2^*\xi + l_1^* dz + l_5^* z)| + |((I - \Pi_h)e_u, l_2^* e_\xi + l_1^* de_z + l_5^* e_z)|.$$

We note that $l_2^*\xi + l_1^* dz + l_5^* z = [-l_2^* d^* + l_1^* d + (l_5^* - l_2^* l_3^*)]Kg$. The first term above can be bounded by the regularity assumptions and the last estimate in Theorem 5.1. The second term can be bounded using estimates in Theorem 5.3. The result is

$$Q_2 \lesssim [(\chi_{12}\eta + \chi_2\delta + \chi_5\alpha)\|e_u\| + \chi_{125}\alpha\|de_u\|]\|g\|.$$

For Q_3, we have

$$Q_3 = |((I - P_{\mathrm{B}\,h})de_u, l_4^* z_h)| \le |((I - P_{\mathrm{B}\,h})de_u, l_4^* z)| + |((I - P_{\mathrm{B}\,h})de_u, l_4^* e_z)|.$$

For the first term $((I - P_{\mathrm{B}\,h})de_u, l_4^* z) = (de_u, (P_{\mathrm{B}} - P_{\mathrm{B}\,h})l_4^* z)$ can be bounded using (3.6). The second term can be bounded by $|(de_u, l_4^* e_z)|$ and Theorem 5.3. The final result is

$$Q_3 \;\square\; \chi_4 \alpha \|de_u\| \|g\|.$$

Combining all the estimates together, we get the estimate in the claim. $\square$

6.3. **Preliminary estimate for $\|e_u\|$.**

Lemma 6.4. *Under the assumptions of Theorem* 3.4, *we have*

$$\|e_u\| \;\square\; \|E_u\| + \eta \|e_\sigma\| + \alpha \|de_\sigma\| + \eta \|de_u\|.$$

Proof. Letting $g = \Pi_h e_u$ in Lemma 6.3 we get an estimate for $(\Pi_h e_u, \Pi_h e_u)$. Then,

$$\|e_u\| \le \|\Pi_h e_u\| + \|(I - \Pi_h) e_u\| \;\square\; \|\Pi_h e_u\| + \|E_u\| + \eta \|de_u\|.$$

For sufficiently small h, we hide the $\|e_u\|$ term on the right in the left-hand side. The result is the estimate in the claim. $\square$

6.4. **Preliminary estimate for $\|e_\sigma\|$.**

Lemma 6.5. *Under the assumptions of Theorem* 3.4,

$$\|e_\sigma\| \;\square\; \|E_\sigma\| + (\eta + \chi_{45}\sqrt{\alpha})\|de_\sigma\| + (\chi_{24} + \chi_3 \eta + \chi_5 \sqrt{\eta})\|e_u\| + (\chi_3 \alpha + \chi_{45}\sqrt{\alpha})\|de_u\|.$$

Proof. Equation (6.1a) implies

$$\begin{aligned}(e_\sigma, e_\sigma) &= (e_\sigma, (I - \Pi_h)e_\sigma) + (e_\sigma, \Pi_h e_\sigma) \\ &= (e_\sigma, (I - \Pi_h)e_\sigma) + (l_2 e_u, \Pi_h e_\sigma) + (e_u, d\Pi_h e_\sigma).\end{aligned}$$

The first term above is bounded by

$$|(e_\sigma, (I - \Pi_h)e_\sigma)| \le \|e_\sigma\| \|(I - \Pi_h)\sigma\| \;\square\; (\|E_\sigma\| + \eta \|dE_\sigma\|)\|e_\sigma\|.$$

The second term is bounded using $\|\Pi_h e_\sigma\| \;\square\; \|e_\sigma\| + \eta \|de_\sigma\|$ from Theorem 5.1:

$$|(l_2 e_u, \Pi_h e_\sigma)| \;\square\; \chi_2 \|e_u\| (\|e_\sigma\| + \eta \|de_\sigma\|).$$

The last term is estimated by the duality lemma. Because $P_{\mathrm{B}\,h} \Pi_h = P_{\mathrm{B}\,h}$, we have

$$(e_u, d\Pi_h e_\sigma) = (e_u, P_{\mathrm{B}\,h} de_\sigma) = (P_{\mathrm{B}\,h} e_u, P_{\mathrm{B}\,h} de_\sigma) = (\Pi_h e_u, P_{\mathrm{B}\,h} de_\sigma).$$

We apply (6.3) with $g = P_{\mathrm{B}\,h} de_\sigma$ and get

$$\begin{aligned}|(\Pi_h e_u, g)| \;\square\; [\eta \|e_\sigma\| + \alpha \|de_\sigma\| + (\mu_0 + \chi_{123}\eta + \chi_5 \alpha)\|e_u\| \\ + (\mu_0 \eta + \chi_{12345}\alpha)\|de_u\|] \|P_{\mathrm{B}\,h} de_\sigma\|.\end{aligned}$$

From equation (6.2), we have

$$\|P_{\mathrm{B}\,h} de_\sigma\| \;\square\; \chi_3 \|e_\sigma\| + \chi_4 \|de_u\| + \chi_5 \|e_u\|.$$

Combining all these estimates, we get the estimate in the claim. $\square$

6.5. **Proof of improved error estimates theorem.** The estimates in Theorem 3.4 are derived from the four preliminary estimates Lemma 6.1, Lemma 6.2, Lemma 6.4, and Lemma 6.5. Using 1 for quantities which are bounded and ϵ for quantities which go to zero as $h \to 0$, the four preliminary estimates have the following structure:

$$\begin{bmatrix} \|de_\sigma\| \\ \|de_u\| \\ \|e_\sigma\| \\ \|e_u\| \end{bmatrix} \square \begin{bmatrix} 0 & 1 & 1 & 1 \\ \epsilon & 0 & 1 & 1 \\ \epsilon & \epsilon & 0 & 1 \\ \epsilon & \epsilon & \epsilon & 0 \end{bmatrix} \begin{bmatrix} \|de_\sigma\| \\ \|de_u\| \\ \|e_\sigma\| \\ \|e_u\| \end{bmatrix} + \begin{bmatrix} \|dE_\sigma\| \\ \|dE_u\| \\ \|E_\sigma\| \\ \|E_u\| \end{bmatrix}.$$

For example, we can substitute the first line into the second and hide the $\epsilon\|de_u\|$ term in the left-hand side of the second line assuming h is sufficiently small. This, in effect, switches $\|de_\sigma\|$ in the second line to $\|dE_\sigma\|$. Due to the vanishing diagonal and epsilon lower-triangle of the matrix above, this procedure can be repeated to eliminate all unknown error terms like $\|e_u\|$ and switch them with known error terms like $\|E_u\|$. After this linear algebra exercise, we get the estimates in Theorem 3.4.

7. Numerical examples

In this section, we show through numerical examples that the error rates given by Theorem 3.4 in Table 1 are in fact achieved and cannot be improved.

In 3D, there are four cases of the Hodge Laplace problems for differential forms of degree $0, 1, 2, 3$. The 0-form and 3-form cases lead to the scalar Laplace problem in the non-mixed form and mixed form respectively. The numerical results in these two cases are well-known [7, 8, 18] and will not be duplicated here. We focus on the 1-form and 2-form cases.

Let $\Omega = [0, 1]^3$ be the unit cube in R^3. The 1-form mixed Hodge Laplace problem with natural boundary conditions is: given $f \in L^2$, find $u \in D$ satisfying:

$$\begin{aligned} (\operatorname{grad} + l_3)(-\operatorname{div} + l_2)u + \operatorname{curl}(\operatorname{curl} + l_1)u + l_4 \operatorname{curl} u + l_5 u = f, \qquad & \text{in } \Omega, \\ u \cdot n = 0, \qquad (\operatorname{curl} u + l_1 u) \times n = 0, \qquad & \text{on } \partial\Omega. \end{aligned}$$

We choose the following smooth function as the exact solution:

$$u = \begin{bmatrix} \sin \pi x \cos \pi z \\ \cos \pi x \sin \pi y \\ \cos \pi y \sin \pi z \end{bmatrix},$$

with the coefficients

$$l_1 = \begin{bmatrix} \sin \pi y \sin \pi z & \sin \pi z & \sin \pi y \\ \sin \pi z & \sin \pi x \sin \pi z & \sin \pi x \\ \sin \pi y & \sin \pi x & \sin \pi x \sin \pi y \end{bmatrix}, \quad l_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

$$l_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -3 & -3 \\ 1 & 3 & 1 \end{bmatrix}, \quad l_5 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

At the discrete level, since $\sigma = (-\operatorname{div} + l_2)u$ is a 0-form where $\mathcal{P}_r\Lambda^0 = \mathcal{P}_r^-\Lambda^0$, we only have two element pairs: $\mathcal{P}_r\Lambda^0 \times \mathcal{P}_{r-1}\Lambda^1$ and $\mathcal{P}_r\Lambda^0 \times \mathcal{P}_r^-\Lambda^1$.

On the same domain $\Omega = [0,1]^3$ as before, the 2-form mixed Hodge Laplace problem with natural boundary conditions is: given $f \in L^2$, find $u \in D$ satisfying:

$$(\operatorname{curl} + l_3)(\operatorname{curl} + l_2)u - \operatorname{grad}(\operatorname{div} + l_1)u + l_4 \operatorname{div} u + l_5 u. \qquad \text{in } \Omega,$$
$$u \times n = 0, \qquad (\operatorname{div} + l_1)u = 0, \qquad \text{on } \partial\Omega.$$

We choose the following smooth function as the exact solution in this case:

$$u = \begin{bmatrix} (\cos \pi x + 3) \sin \pi y \sin \pi z \\ \sin \pi x (\cos \pi y + 2) \sin \pi z \\ \sin \pi x \sin \pi y (\cos \pi z + 2) \end{bmatrix},$$

with the coefficients

$$l_1 = \begin{bmatrix} \sin \pi x \\ -\sin \pi y \\ 0 \end{bmatrix}, \qquad l_2 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -2 & 0 \\ 1 & 3 & 1 \end{bmatrix},$$
$$l_3 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad l_4 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad l_5 = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

At the discrete level, we have all four canonical pairs:

$$\mathcal{P}_{r+1}\Lambda^1 \times \mathcal{P}_r\Lambda^2, \quad \mathcal{P}^-_{r+1}\Lambda^1 \times \mathcal{P}_r\Lambda^2, \quad \mathcal{P}_r\Lambda^1 \times \mathcal{P}^-_r\Lambda^2, \quad \mathcal{P}^-_r\Lambda^1 \times \mathcal{P}^-_r\Lambda^2.$$

In all the numerical experiments, we obtain a quasi-uniform triangulation of size m for the unit cube Ω by first triangulating it uniformly with an $m \times m \times m$ mesh and then perturbing each interior mesh node randomly within 20% of the mesh size $1/m$ in all three coordinate directions.

All four pairs of the canonical FEEC elements in dimension ≤ 3 of all degrees are supported by the open source finite element package FEniCS [12], in which all our numerical codes are implemented.

For example, for the unperturbed 1-form problem, with $\mathcal{P}_2\Lambda^1 \times \mathcal{P}_1\Lambda^2$, we get

m	$\|\sigma - \sigma_h\|$	rate	$\|d(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|d(u - u_h)\|$	rate
2	2.766e-01		2.362e+00		2.244e-01		1.441e+00	
4	3.940e-02	2.37	8.529e-01	1.24	6.961e-02	1.43	7.434e-01	0.81
8	4.504e-03	3.20	2.312e-01	1.93	1.859e-02	1.95	3.740e-01	1.01
16	5.208e-04	2.98	5.716e-02	1.93	4.659e-03	1.91	1.868e-01	0.96

Thus, for example, $\|\sigma - \sigma_h\|_{L^2} = 2.766 \times 10^{-1}$ on a $2 \times 2 \times 2$ mesh. The rates are computed between the two successive errors. The optimal rates of $3, 2, 2, 1$ for $\sigma, d\sigma, u, du$ are clear. With an l_4 lower-order perturbation, we get:

m	$\|\sigma - \sigma_h\|$	rate	$\|d(\sigma - \sigma_h)\|$	rate	$\|u - u_h\|$	rate	$\|d(u - u_h)\|$	rate
2	4.190e-01		3.522e+00		2.215e-01		1.434e+00	
4	1.259e-01	1.44	1.583e+00	0.96	6.773e-02	1.42	7.335e-01	0.81
8	3.689e-02	1.54	7.393e-01	0.96	1.874e-02	1.61	3.749e-01	0.84
16	9.968e-03	1.86	3.595e-01	1.02	4.799e-03	1.93	1.868e-01	0.99

Clearly the convergence rates for σ and $d\sigma$ in L^2 are reduced by 1 as predicted.

There are too many cases for us to list all the detailed results. We instead only summarize the numerical results here. First, the error rates in Table 1 are guaranteed for all FEEC elements for the Hodge Laplace problem. Second, for each case with a reduction in the error rates predicted in Table 1, there is at least one Hodge Laplace problem with a certain form degree which can only converge at that reduced rate. In this sense, the rates in Table 1 are optimal. However, we note that the rates in Table 1 do no represent an upper bound for all possible cases. For example, when the l_5 term is given by multiplication by a smooth scalar coefficient, we do not observe a reduction of convergence rates in the L^2-error of σ. We also observed that for 1-forms in 3D, an l_2-term given by multiplication by a generic smooth coefficient does not degrade the L^2-error rate in σ.

The full numerical results along with the Python source code used in FEniCS can be found at the companion code repository at

`https://bitbucket.org/lzlarryli/feeclotexp`.

We note that due to the randomness involved (random perturbation applied to the mesh), the error numbers will not be exactly the same as but very close to what we have listed here.

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