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EXPLICIT BARYCENTRIC FORMULAE FOR OSCULATORY INTERPOLATION AT ROOTS OF CLASSICAL ORTHOGONAL POLYNOMIALS

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Abstract. In this paper we extend the recent results of H. Wang et al. [Math. Comp. 81 (2012) and 83 (2014), pp. 861-877 and 2893-2914, respectively], on barycentric Lagrange interpolation at the roots of Hermite, Laguerre and Jacobi orthogonal polynomials, not only to all classical distributions, but also to osculatory Fejér and Hermite interpolation at the roots $(x_\nu)_1^n$ of orthogonal polynomials $p_n(x)$, generated by these distributions. More precisely, we present comparatively simple unified proofs of representations for barycentric weights of Fejér, Hermite and Lagrange type in terms of values $p_{n-1}(x_\nu)$, $p_n'(x_\nu)$ and Christoffel numbers λ_ν without any additional assumptions on the classical distributions. The first two representations enable us to design a general $O(n^2)$-algorithm to simultaneous computations of barycentric weights and Christoffel numbers, which is based on the stable and efficient divide-and-conquer $O(n^2)$-algorithm for the symmetric tridiagonal eigenproblem due to M. Gu and S. C. Eisenstat [SIAM J. Matrix Anal. Appl. 16 (1995), pp. 172-191]. On the other hand, the third representations can be used to compute all classical barycentric weights in the faster $O(n)$ way proposed for the Lagrange interpolation at the roots of Hermite, Laguerre and Jacobi orthogonal polynomials by H. Wang et al. in the second cited paper. Such an essential accelaration requires one to use the $O(n)$-algorithm of A. Glaser et al. [SIAM J. Sci. Comput. 29 (2007), pp. 1420-1438] to compute the roots x_ν and Christoffel numbers λ_ν by applying the Runge-Kutta and Newton methods to solve the Sturm-Liouville differential problem, which is generic for classical orthogonal polynomials. Finally, in the four special important cases of Jacobi weights $w(x) = (1-x)^\alpha (1+x)^\beta$ with $\alpha = \pm\frac{1}{2}$ and $\beta = \pm\frac{1}{2}$, that is, of the Chebyshev and Szegő weights of the first and second kind, we present explicit representations of the Fejér and Hermite barycentric weights, which yield an $O(1)$-algorithm.

1. Introduction and preliminaries

Let $\{p_n(x)\}_{n\geqslant 0}$ be a finite or infinite sequence of monic polynomials,

$$p_0(x) = 1, \quad p_n(x) = \prod_{\nu=1}^{n} (x - x_\nu), \quad n < n_w, \tag{1.1}$$

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orthogonal with respect to the weighted $L_w^2(a,b)$-inner product

$$\int_a^b p_n(x)\, p_m(x)\, w(x)\, dx = h_n \delta_{n,m}. \tag{1.2}$$

Here $\delta_{n,m}$ denotes the Kronecker delta, $n_w \in \mathsf{N} \cup \{\infty\}$ depends only on the distribution $w(x)\,dx$, and $w(x)$ is a classical weight function on a finite or infinite interval (a,b). The last assumption means that $w(x)$ is a positive solution of the Pearson differential equation

$$\frac{d}{dx}\left[A(x)\, w(x)\right] = B(x)\, w(x), \quad a < x < b,$$

with boundary conditions

$$\lim_{x \downarrow a} A(x)\, w(x) = \lim_{x \uparrow b} A(x)\, w(x) = 0,$$

where polynomial coefficients

$$A(x) = a_2 x^2 + a_1 x + a_0, \quad B(x) = b_1 x + b_0$$

are such that $A(x) > 0$ on (a,b) and $b_1 \neq 0$.

The orthogonal polynomials $p_n(x)$ associated with a classical distribution $w(x)\,dx$ will be called below classical. By the Pearson equation it follows that each polynomial $p_n(x)$, $n = 1, 2, \ldots$, is a solution of the Sturm-Liouville differential equation [20] of the form

$$A(x)\, p_n(x) + B(x)\, p_n(x) = n\left[(n-1)\, a_2 + b_1\right] p_n(x), \quad a < x < b. \tag{1.3}$$

It is well known that there are exactly six different types of classical weight functions and orthogonal polynomials, up to a linear change of variable [20]. These classical weights are listed in Table 1, and unspecified parameters from its last row are defined as follows:

$$\begin{aligned} a_1 &= 2\frac{\mathcal{AB}+\mathcal{CD}}{\mathcal{A}^2+\mathcal{C}^2}, \quad a_0 = \frac{\mathcal{B}^2+\mathcal{D}^2}{\mathcal{A}^2+\mathcal{C}^2}, \quad \zeta = \frac{\mathcal{AD}-\mathcal{BC}}{\mathcal{A}^2+\mathcal{C}^2} > 0, \\ b_0 &= (1-\alpha)\, a_1 + \beta\zeta, \quad E(x) = \frac{1}{\zeta}\left(x + \frac{1}{2}a_1\right). \end{aligned} \tag{1.4}$$

Among classical orthogonal polynomials there are exactly three infinite sequences of orthogonal polynomials of Hermite, Laguerre and Jacobi [45], and exactly three, less known, finite sequences of generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi orthogonal polynomials [20, 22, 28, 29]. The lengths $n_w = \lfloor \frac{1-b_1}{2} \rfloor$ of these finite polynomial sequences $\{p_n(x)\}_{0 \leqslant n < n_w}$ depend only on the leading coefficients $b_1 = \alpha$, $2-\alpha$, $2(1-\alpha)$ of the polynomials $B(x)$, presented in Table 1. Thus the lengths $n_w = n_w(\alpha)$ increase to infinity, whenever $|\alpha| \to \infty$.

Basic properties of the classical orthogonal polynomials were studied in several articles and monographs; cf. Bochner [5], Hahn [14], Krall [23–25], Agarwal and Milovanović [1], Andrews et al. [3], Chihara [7], Nikiforov and Uvarov [34], Koekoek et al. [20], Suetin [43], Koepf and Masjed-Jamei [21, 22], Masjed-Jamei [31], the authors [36–38, 40], Horváth [18], Wang and Xiang [47], and Wang et al. [46]. In particular, Koepf and Masjed-Jamei [21] observed that pseudo-Jacobi distribution generalizes the Student t-distribution, one of the most important distributions in the sampling problems of normal population. According to [21, 31], this distribution also extends the F-distribution. In our papers [36, 37], we gave solutions of the

TABLE 1. The basic types of classical weights $w(x)$.

	Weights	**Polynomial** $A(x)$	**Polynomial** $B(x)$
1	Hermite, e^{-x^2} on $(-\infty,+\infty)$	1	$-2x$
2	Laguerre, $x^{\alpha}e^{-x}$ on $(0,+\infty)$	x	$-x+\alpha+1$ $\alpha>-1$
3	Jacobi, $(1-x)^{\alpha}(1+x)^{\beta}$ on $(-1,1)$	$-x^2+1$	$-(\alpha+\beta+2)x+\beta-\alpha$ $\alpha>-1,\ \beta>-1$
4	Generalized Bessel, $x^{\alpha-2}e^{-\frac{\beta}{x}}$ on $(0,+\infty)$	x^2	$\alpha x+\beta$ $\alpha\notin\{-2,-3,\ldots\}$, $\alpha<-1,\ \beta>0$
5	Jacobi, $\frac{x^{\beta}}{(1+x)^{\alpha+\beta}}$ on $(0,+\infty)$	x^2+x	$(2-\alpha)x+\beta+1$ $\alpha\geqslant 3,\ \beta>-1$
6	Pseudo-Jacobi, $\frac{e^{\beta\arctan E(x)}}{A^{\alpha}(x)}$ on $(-\infty,+\infty)$	$x^2+a_1x+a_0$	$2(1-\alpha)x+b_0$ $\alpha\geqslant\frac{3}{2},\ \beta\in\mathsf{R}$

electrostatic equilibrium problem and the interpolatory Fejér type problem for all classical weight functions. This subject has been continued for some other weighted distributions, generating exceptional Laguerre and Jacobi polynomials, by Horváth [18]. Finally, in the articles [46, 47], Wang et al. expressed explicitly, in terms of Christoffel numbers, the barycentric weights for the Lagrange interpolation at the zeros of Hermite, Laguerre and Jacobi orthogonal polynomials. Next, they applied these results together with the $O(n)$-algorithm of Glaser et al. [10] to implement superfast algorithm to evaluate these weights. However, in the case of Jacobi polynomials they proposed to use the Hale and Townsend $O(n)$-algorithm [15], which is more efficient than the previous algorithm.

In this paper we extend the results of Wang et al. [46,47] not only to all classical weight functions, but also to osculatory interpolation of Fejér and Hermite types at the zeros of orthogonal polynomials generated by these weights. Next, we apply our results to the Lagrange interpolation, which solves the problem considered by Wang et al. [46,47] in full generality.

It should be noted that our simplified and unified approach, proposed to deal with barycentric formulae, has been already applied to solve the electrostatic problem in our paper [36]. It is based on the following fundamental recurrence formula, due to Al-Salam and Chihara [2], for the derivatives of classical orthogonal polynomials

$$A(x)\,p_n(x)=(\delta_n x+\eta_n)\,p_n(x)+\rho_n p_{n-1}(x),\quad n=1,2,\ldots, \tag{1.5}$$

with coefficients δ_n, η_n and ρ_n satisfying

$$\begin{aligned}&\delta_n=na_2,\quad \eta_n=n\frac{(n-1)\,a_1a_2+a_1b_1-a_2b_0}{2(n-1)\,a_2+b_1},\\ &\rho_n=-d_nr_{2n-1},\quad d_n=nr_{n-2}\frac{s_{n-1}(r_{n-1}a_1-a_2b_0)-a_0r_{2n-2}^2}{r_{2n-3}r_{2n-2}^2r_{2n-1}}.\end{aligned} \tag{1.6}$$

Here we assume that $0/0 = 1$ and that

$$r_k = ka_2 + b_1, \quad s_k = ka_1 + b_0.$$

Moreover, we note that coefficients d_n can be also derived from the following three-term recurrence relations [7, 20, 45]:

$$\begin{aligned} &p_0(x) = 1, \quad p_1(x) = x - c_0, \\ &p_{n+1}(x) = (x - c_n)\, p_n(x) - d_n p_{n-1}(x), \quad n = 1, 2, \ldots, \end{aligned} \tag{1.7}$$

for the monic classical orthogonal polynomials, where constant coefficients c_n are defined by

$$c_n = -\frac{2na_1 r_{n-1} - b_0(2a_2 - b_1)}{r_{2n-2} r_{2n}}. \tag{1.8}$$

2. Barycentric formulae for osculatory interpolation

Let $w(x)$ and $w_1(x) = A(x)\, w(x)$ be classical weights on (a, b), and let the classical orthogonal monic polynomials $p_0(x) = 1$ and $p_n(x) = (x - x_1) \cdots (x - x_n)$, $0 < n < n_w$, be associated with the distribution $w(x)\, dx$. Then the derivatives $\{p_n(x)\}_{0<n<n_w}$ are classical orthogonal polynomials, corresponding to the distribution $w_1(x)\, dx$; cf. Hahn [14], Krall [23–25], Agarwal and Milovanović [1], and Mastroianni and Milovanović [32]. Without loss of generality, we may assume that the weight $w(x)$ is basic, i.e., that it is as in Table 1. Otherwise, it suffices to apply an appropriate linear change of the argument x in order to transform a given classical weight to a basic one.

Now we consider the Fejér interpolating positive operator $\mathcal{F}_n$ which, for any continuous function $f \in C(a, b)$, satisfies the interpolating conditions

$$(\mathcal{F}_n f)(x_\nu) = f(x_\nu), \quad (\mathcal{F}_n f)\ (x_\nu) = 0, \quad \nu = 1, 2, \ldots, n,$$

at the roots of a classical orthogonal polynomial $p_n(x)$. It is defined by the formula

$$(\mathcal{F}_n f)(x) = w_1(x) \sum_{\nu=1}^{n} f(x_\nu) \frac{l_\nu^2(x)}{w_1(x_\nu)}, \tag{2.1}$$

where

$$l_\nu(x) = \frac{p_n(x)}{(x - x_\nu)\, p_n(x_\nu)}$$

are the fundamental Lagrange polynomials. The important feature of Fejér operator consists in the fact that its knots are the unique solution of the extremal problem

$$\min_{a<z_1<\ldots<z_n<b} \ \sup_{a<z<b} w_1(z) \sum_{\nu=1}^{n} \frac{l_\nu^2(z)}{w_1(z_\nu)} = 1, \quad l_\nu(z) = \prod_{\substack{k=1 \\ k=\nu}}^{n} \frac{z - z_k}{z_\nu - z_k},$$

which was first proved by Fejér [9] for the Legendre weight $w(x) = 1$, then by Karlin and Studden [19] for Hermite, Laguerre and Jacobi weights, and recently by the authors [37] for all remaining classical weights. It is worth noting that many interesting modifications of the interpolatory problem of Fejér, including among others the non-classical weights, were also studied by Balázs [4], Lau and Studden [26, 27], Lubinsky [30], Szabó [44] and Horváth [16, 17].

It is well known that the Hermite formula (2.1) is unstable and too slow in numerical computations of the Fejér operator $(\mathcal{F}_n f)(x)$. For this purpose, we recommend [39, 46] its barycentric form

$$(\mathcal{F}_n f)(x) = w_1(x)\, p_n^2(x) \sum_{\nu=1}^{n} f(x_\nu) \frac{\gamma_\nu}{(x - x_\nu)^2} \tag{2.2}$$

with the barycentric weights $(\gamma_\nu)_1^n$ independent of $f(x)$.

Theorem 2.1. *The barycentric weights $(\gamma_\nu)_1^n$ of the Fejér operator $\mathcal{F}_n$ satisfy*

$$\gamma_\nu = \frac{1}{A(x_\nu)\, w(x_\nu)\, [p_n(x_\nu)]^2} = \frac{A(x_\nu)}{\rho_n^2 w(x_\nu)\, [p_{n-1}(x_\nu)]^2}. \tag{2.3}$$

Additionally, the constants ρ_n are, for the six basic classical weights, as in Table 2.

Proof. For the proof of the first part of (2.3) it is sufficient to compare the formulae (2.1) and (2.2). Next, we insert roots $x = x_\nu$ of $p_n(x)$ into the Al-Salam and Chihara differentation formula (1.5) to get

$$p_n(x_\nu) = \frac{\rho_n p_{n-1}(x_\nu)}{A(x_\nu)}. \tag{2.4}$$

Hence the second part of (2.3) follows from its first part. Thus it remains to compute the constants $\rho_n = -r_{2n-1} d_n$, $n \geqslant 1$, for all classical weights. During these computations we take occasion to evaluate also the constants $(c_n)_{n\geqslant 0}$, $(d_n)_{n\geqslant 1}$ and

$$m_n = d_1 d_2 \cdots d_n \quad (n \geqslant 1),$$

which are necessary in Section 4. Since $r_k = k a_2 + b_1$ and $s_k = k a_1 + b_0$, we insert coefficients a_j and b_j of polynomials $A(x)$ and $B(x)$ from Table 1 in the formula (1.8) and in the last two formulae in (1.6) to get:

(i) $r_k = -2$, $s_k = 0$, $c_n = 0$, $\rho_n = 2d_n = n$, and $m_n = \frac{n!}{2^n}$, in the case of Hermite weight $w(x)$,

(ii) $r_k = -1$, $s_k = k + \alpha + 1$, $c_n = 2n + \alpha + 1$, $\rho_n = d_n = n(n + \alpha)$, and $m_n = n!\,(1+\alpha)_n$, in the case of Laguerre weight $w(x)$,

(iii) $r_k = -(k + \alpha + \beta + 2)$, $s_k = \beta - \alpha$, $c_n = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}$,
$\rho_n = (2n + 1 + \alpha + \beta)\, d_n = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)^2}$, and
$m_n = \frac{4^n n!(1+\alpha)_n (1+\beta)_n (2+\alpha+\beta)_{n-1}}{(2n+1+\alpha+\beta)(2+\alpha+\beta)_{2n-1}^2}$, whenever $w(x)$ is the Jacobi weight,

(iv) $r_k = k + \alpha$, $s_k = \beta$, $c_n = \frac{\beta(2-\alpha)}{(2n-2+\alpha)(2n+\alpha)}$,
$\rho_n = (1 - 2n - \alpha)\, d_n = \frac{n(n-2+\alpha)\beta^2}{(2n-3+\alpha)(2n-2+\alpha)^2}$, and
$m_n = \frac{(-1)^n n! \beta^{2n} (\alpha)_{n-1}}{(2n-1+\alpha)(\alpha)_{2n-1}^2}$, if $w(x)$ is generalized Bessel weight,

(v) $r_k = k + 2 - \alpha$, $s_k = k + \beta + 1$, $c_n = \frac{\alpha(\beta+1) - 2n(n+1-\alpha)}{(2n-\alpha)(2n+2-\alpha)}$,
$-\rho_n = (2n + 1 - \alpha)\, d_n = \frac{n(n-\alpha)(n+\beta)(n-\alpha-\beta)}{(2n-1-\alpha)(2n-\alpha)^2}$, and
$m_n = \frac{n!(2-\alpha)_{n-1}(1+\beta)_n(1-\alpha-\beta)_n}{(2n+1-\alpha)(2-\alpha)_{2n-1}^2}$, in the case of Jacobi on $(0, +\infty)$ weight,

(vi) $r_k = k + 2(1-\alpha)$, $s_k = (k+1-\alpha)\, a_1 + \beta\zeta$, $c_n = \frac{\alpha\beta\zeta}{2(n-\alpha)(n+1-\alpha)} - \frac{a_1}{2}$,

$\rho_n = (2\alpha - 2n - 1)\, d_n = \frac{n(n-2\alpha)\left[4(n-\alpha)^2+\beta^2\right]\zeta^2}{4(n-\alpha)^2(2n-1-2\alpha)}$, and

$m_n = \frac{(-1)^n n!(2-2\alpha)_{n-1}\prod_{j=1}^{n}\left[4(j-\alpha)^2+\beta^2\right]}{(2n+1-2\alpha)(2-2\alpha)_{2n-1}^2\zeta^{-2n}}$, whenever $w(x)$ is the pseudo-Jacobi weight.

Hence the proof of the Theorem 2.1 is finished. □

Remark 2.1. During computations of the coefficient c_0 or d_1 for the Jacobi weight, the division by zero can take place, whenever $n = \alpha + \beta = 0$ or $n = -(\alpha+\beta) = 1$. Then we have to use the convention $\frac{0}{0} = 1$ in order to get the correct values

$$c_0 = \frac{\beta - \alpha}{2} \quad \text{and} \quad d_1 = \frac{\rho_1}{2} = 2(1+\alpha)(1+\beta).$$

Note that $(\mathcal{F}_n f)(x)$ is a modification of the Hermite interpolating polynomial $(\mathcal{H}_n f)(x)$ of degree at most $2n-1$, defined by the interpolating conditions

$$(\mathcal{H}_n f)(x_\nu) = f(x_\nu), \quad (\mathcal{H}_n f)'(x_\nu) = f'(x_\nu), \quad \nu = 1, 2, \ldots, n,$$

at the roots of the orthogonal polynomial, associated with a classical weight function $w(x)$. It can be expressed, in terms of the fundamental Lagrange polynomials $l_\nu(x)$, as follows [6]:

$$(2.5) \quad (\mathcal{H}_n f)(x) = \sum_{\nu=1}^{n} \left\{ f(x_\nu)\left[1 - 2(x - x_\nu)\, l_\nu'(x_\nu)\right] + f'(x_\nu)(x - x_\nu) \right\} l_\nu^2(x).$$

Hence $(\mathcal{H}_n f)(x)$ has the barycentric form

$$(2.6) \quad (\mathcal{H}_n f)(x) = p_n^2(x) \sum_{\nu=1}^{n} \left\{ f(x_\nu)\left[\frac{\gamma_{\nu,0}}{(x-x_\nu)^2} + \frac{\gamma_{\nu,1}}{x - x_\nu}\right] + f'(x_\nu)\frac{\gamma_{\nu,0}}{x - x_\nu} \right\}.$$

Theorem 2.2. *For the Hermite interpolating operator $(\mathcal{H}_n f)(x)$ at the roots $(x_\nu)_1^n$ of a classical orthogonal monic polynomial $p_n(x)$, the barycentric weights are equal to*

$$(2.7) \quad \begin{aligned} \gamma_{\nu,0} &= \frac{1}{\left[p_n'(x_\nu)\right]^2} = \frac{A^2(x_\nu)}{\rho_n^2\left[p_{n-1}(x_\nu)\right]^2}, \\ \gamma_{\nu,1} &= \frac{B(x_\nu)}{A(x_\nu)\left[p_n'(x_\nu)\right]^2} = \frac{A(x_\nu)\, B(x_\nu)}{\rho_n^2\left[p_{n-1}(x_\nu)\right]^2}. \end{aligned}$$

Moreover, for all six basic classical weights, the constant factors ρ_n are given in Table 2.

Proof. By comparing the right-hand sides of (2.5) and (2.6), we obtain

$$(2.8) \quad \gamma_{\nu,0} = \frac{1}{\left[p_n'(x_\nu)\right]^2} \quad \text{and} \quad \gamma_{\nu,1} = -2 l_\nu'(x_\nu)\, \gamma_{\nu,0}.$$

It follows from formula (2.4) that the second representation (2.7) of $\gamma_{\nu,0}$ hold. To prove the second part of (2.7), we apply the l'Hospital's rule to the quotient

$$l_\nu'(x) = \frac{p_n'(x)(x - x_\nu) - p_n(x)}{(x - x_\nu)^2\, p_n'(x_\nu)}$$

at the point $x = x_\nu$, in order to get

$$l_\nu (x_\nu) = \frac{p_n (x_\nu)}{2p_n (x_\nu)}. \tag{2.9}$$

On the other hand, the Sturm-Liouville differential equation (1.3) yields

$$A (x_\nu) p_n (x_\nu) + B (x_\nu) p_n (x_\nu) = 0.$$

Hence it follows from (2.9) and (2.8) that

$$\gamma_{\nu,1} = \frac{B (x_\nu)}{A (x_\nu) [p_n (x_\nu)]^2}.$$

This combined with (2.4) completes the proof. □

Since the Lagrange interpolating operator

$$(\mathcal{L}_n f) (x) = \sum_{\nu=1}^{n} f (x_\nu) l_\nu (x)$$

has the barycentric form

$$(\mathcal{L}_n f) (x) = p_n (x) \sum_{\nu=1}^{n} f (x_\nu) \frac{\sigma_\nu}{x - x_\nu}, \quad \sigma_\nu = \frac{1}{p_n (x_\nu)},$$

we get the following extension of the results presented in [46] for the infinite sequences of classical orthogonal monic polynomials $p_n (x)$.

Theorem 2.3. *The formulae*

$$\sigma_\nu = \frac{1}{p_n (x_\nu)} = \frac{A (x_\nu)}{\rho_n p_{n-1} (x_\nu)} \tag{2.10}$$

hold for barycentric weights of Lagrange interpolating operator $(\mathcal{L}_n f) (x)$ at the roots $(x_\nu)_1^n$ of arbitrary classical orthogonal polynomial $p_n (x)$. Additionally, the constant coefficients ρ_n are listed in Table 2 for all six basic classical weight functions.

Proof. The second identity in (2.10) follows immediately from the first identity and formula (2.4). □

3. Barycentric weights and Christoffel numbers

Throughout this section we assume that $p_0 (x) = 1$ and $p_n (x) = (x - x_1) \cdots (x - x_n)$, $0 < n < \frac{n_w+1}{2}$, are classical orthogonal monic polynomials corresponding to distributions $w (x) \, dx$. Moreover, we suppose that the leading coefficient b_1 of the polynomial $B (x)$, from the definition of $w (x)$, is such that the inequality $2n-1 < n_w$ holds. Then the Gauss quadrature formula of approximate integration,

$$Q_n (f) = \sum_{\nu=1}^{n} \lambda_\nu f (x_\nu) \approx \int_a^b f (x) w (x) \, dx,$$

with coefficients defined by

$$\lambda_\nu = \int_a^b l_\nu (x) w (x) \, dx, \quad l_\nu (x) = \frac{p_n (x)}{(x - x_\nu) p_n (x_\nu)}, \quad \nu = 1, 2, \ldots, n,$$

is the unique quadrature formula which is exact on the space $\mathcal{P}_{2n-1}$ of all polynomials of degree at most $2n - 1$. These coefficients λ_ν are called the Christoffel

TABLE 2. Constant items of barycentric weights γ_ν, $\gamma_{\nu,0}$, $\gamma_{\nu,1}$ and σ_ν (Theorems 2.1, 2.2 and 2.3), Christoffel numbers λ_ν (Theorem 3.1), and of explicit representations for the barycentric weights of Fejér, Hermite and Lagrange interpolation operators in terms of Christoffel numbers λ_ν (Theorem 3.2), dependent only on n and classical weight functions. Here $i = \sqrt{-1}$, ζ is as in (1.4), and $(v)_k = v(v+1)\cdots(v+k-1)$ denotes the Pochhammer symbol. In addition, it is supposed that $(v)_0 = 1$ and $\rho_1 = 4(1+\alpha)(1+\beta)$, whenever $n = -(\alpha+\beta) = 1$ in the third row; cf. Remark 2.1.

	Constants ρ_n $(n \geqslant 1)$	**Constants** h_{n-1} $(n \geqslant 2)$
1	n	$\frac{(n-1)!\sqrt{\pi}}{2^{n-1}}$
2	$n(n+\alpha)$	$(n-1)!\Gamma(n+\alpha)$
3	$\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n-1+\alpha+\beta)(2n+\alpha+\beta)^2}$	$\frac{2^{2n-1+\alpha+\beta}(n-1)!\Gamma(n+\alpha)\Gamma(n+\beta)}{(n+\alpha+\beta)_{n-1}\Gamma(2n+\alpha+\beta)}$
4	$\frac{n(n-2+\alpha)\beta^2}{(2n-3+\alpha)(2n-2+\alpha)^2}$	$\frac{(-1)^{n-1}(n-1)!(\alpha)_{n-2}\beta^{2n-3+\alpha}\Gamma(1-\alpha)}{(2n-3+\alpha)(\alpha)_{2n-3}^2}$
5	$-\frac{n(n-\alpha)(n+\beta)(n-\alpha-\beta)}{(2n-1-\alpha)(2n-\alpha)^2}$	$\frac{(n-1)!(2-\alpha)_{n-2}(1-\alpha-\beta)_{n-1}\Gamma(\alpha-1)\Gamma(n+\beta)}{(2-\alpha)_{2n-3}^2(2n-1-\alpha)\Gamma(\alpha+\beta)}$
6	$\frac{n(n-2\alpha)\left[(2n-2\alpha)^2+\beta^2\right]}{(2n-1-2\alpha)(2n-2\alpha)^2\zeta^2}$	$\frac{(-1)^{n-1}\pi(n-1)!(2-2\alpha)_{n-2}\Gamma(2\alpha-1)\prod_{j=1}^{n-1}\left[4(j-\alpha)^2+\beta^2\right]}{4^{\alpha-1}(2n-1-2\alpha)(2-2\alpha)_{2n-3}^2\left\|\Gamma\left(\alpha+i\frac{\beta}{2}\right)\right\|^2\zeta^{2\alpha-2n+1}}$

numbers. They have the following representation, which is a consequence of the Christoffel-Darboux identity [45].

Lemma 3.1. *The Christoffel numbers have the representation*

$$\lambda_\nu = \frac{h_{n-1}}{p_n(x_\nu)\,p_{n-1}(x_\nu)}, \qquad \nu = 1, 2, \ldots, n,$$

where constant factors h_{n-1} are defined as in (1.1)–(1.2).

Although the last representation of Christoffel numbers is valid for an arbitrary positive weight function $w(x)$ on (a, b), the next representations are characteristic for the classical weight functions. These representations and their proofs seem to be new at least in the case of generalized Bessel, Jacobi on $(0, +\infty)$ and pseudo-Jacobi classical weight functions.

Theorem 3.1. *Let $(x_\nu)_1^n$ be the zeros of a classical orthogonal monic polynomial $p_n(x)$. Then the Christoffel numbers λ_ν satisfy*

$$\lambda_\nu = \frac{\rho_n h_{n-1}}{A(x_\nu)\left[p_n(x_\nu)\right]^2} = \frac{h_{n-1}A(x_\nu)}{\rho_n\left[p_{n-1}(x_\nu)\right]^2}. \tag{3.1}$$

Here the constants ρ_n and h_{n-1} are as in Table 2, for all six basic classical weights.

Proof. In view of Al-Salam and Chihara identity (1.5) we have

$$p_{n-1}(x_\nu) = \frac{A(x_\nu)\,p_n(x_\nu)}{\rho_n}.$$

Hence one can substitute this expression into Lemma 3.1 to obtain the first part of representation (3.1). Similarly, the second part follows from Lemma 3.1 and formula (2.4). Thus it remains to compute the constants h_{n-1}. For this purpose we note first that the integrals

$$h_0 = \int_a^b w(x)\,dx$$

are equal [8, 20, 45] to

$$\begin{gathered}\sqrt{\pi}, \quad \Gamma(\alpha+1), \quad 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}, \\ \beta^{\alpha-1}\Gamma(1-\alpha), \quad \frac{\Gamma(\alpha-1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta)}, \quad \frac{\pi\Gamma(2\alpha-1)}{4^{\alpha-1}\left|\Gamma\left(\alpha+i\frac{\beta}{2}\right)\right|^2\zeta^{2\alpha-1}}\end{gathered} \tag{3.2}$$

for classical basic weight functions $w(x)$ of Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi, respectively. Further, multiplying the three-term recurrence formula (1.7) by $p_{n-1}(x)\,w(x)\,dx$, and then integrating, we obtain, according to (1.2) and the orthogonality,

$$0 = \int_a^b p_n(x)\left[(x-c_n)\,p_{n-1}(x)\right]w(x)\,dx - d_n h_{n-1} = h_n - d_n h_{n-1}.$$

This establishes the formulae $h_n = d_n h_{n-1}$ and

$$h_n = d_1 d_2 \cdots d_n h_0 = m_n h_0 \quad (n \geqslant 1). \tag{3.3}$$

Thus the formulae (3.1) follow directly from (3.2) combined with representations of m_n, computed in the proof of Theorem 2.1 and presented in Table 3. $\square$

Now we are ready to establish a theorem which includes the main result of Wang et al. [46] on barycentric weights of Lagrange interpolation at roots $(x_\nu)_1^n$ of infinite classical orthogonal polynomials of Hermite, Laguerre and Jacobi. It connects the barycentric weights of Fejér, Hermite and Lagrange interpolation at the roots $(x_\nu)_1^n$ of classical orthogonal polynomials of Hermite, Laguerre, Jacobi, generalized Bessel, Jacobi on $(0,+\infty)$ and pseudo-Jacobi with associated Christoffel numbers. Despite the generality, our proof is simpler, due to resigning from the lowering operator technique in the form proposed by Nikiforov and Uvarov [34].

Theorem 3.2. *Let the roots $(x_\nu)_1^n$ of the classical orthogonal polynomial $p_n(x)$, associated to any classical basic distribution $w(x)\,dx$, be ordered in such a way that $x_1 > x_2 > \ldots > x_n$. Then the barycentric weights of Fejér γ_ν, Hermite $\gamma_{\nu,0}$ and $\gamma_{\nu,1}$, and Lagrange σ_ν have the following representations:*

$$\begin{gathered}\gamma_\nu = \frac{\lambda_\nu}{\rho_n h_{n-1} w(x_\nu)}, \quad \gamma_{\nu,0} = \frac{A(x_\nu)\,\lambda_\nu}{\rho_n h_{n-1}}, \\ \gamma_{\nu,1} = \frac{B(x_\nu)\,\lambda_\nu}{\rho_n h_{n-1}}, \quad \sigma_\nu = (-1)^{\nu-1}\sqrt{\frac{A(x_\nu)\,\lambda_\nu}{\rho_n h_{n-1}}}.\end{gathered} \tag{3.4}$$

Additionally, the constant factors ρ_n and h_{n-1} are as in Table 2.

Proof. From Theorem 3.1 we obtain

$$[p_n(x_\nu)]^2 = \frac{\rho_n h_{n-1}}{A(x_\nu)\lambda_\nu}. \tag{3.5}$$

By inserting this formula into Theorems 2.1 and 2.2 we establish the first three identities in (3.4). To get the fourth identity, we apply the hypotheses $x_1 > x_2 > \ldots > x_n$ and $p_n = (x - x_1)(x - x_2)\cdots(x - x_n)$ to show that

$$\operatorname{sgn}\ p_n(x_\nu) = (-1)^{\nu-1}.$$

Since all terms on both sides of (3.5), perhaps except ρ_n, are evidently positive, it follows that $\rho_n > 0$. Hence, one can set the square root of (3.5) in Theorem 2.3 to complete the proof of (3.4). □

According to Wang et al. [46], the computation of barycentric weights σ_ν, associated with the Hermite, Laguerre and Jacobi orthogonal polynomials, is extremally effective (i.e., is of order $O(n)$), whenever Christoffel numbers are first precomputed by one of the known superfast algorithms of order $O(n)$. However, we should observe that in general this approach has a slight disadvantage. For example, by Theorems 3.1 and 3.2 we should first perform, during computation of λ_ν, the multiplication by $\rho_n h_{n-1}$ in the formula (3.1), and then by $(\rho_n h_{n-1})^{-1}$ in the formulae (3.4) for barycentric weights of Fejér, Hermite and Lagrange type. Clearly, this has to be avoided, e.g., in the way proposed in [46]. An alternative way is proposed in the next section for arbitrary classical distributions.

4. Barycentric weights, Christoffel numbers and tridiagonal symmetric eigenvalue problems

In Theorems 2.1, 2.2, 2.3 and 3.1 we have presented explicit representations of barycentric weights of Fejér, Hermite and Lagrange types, and of Christoffel numbers, in terms of roots $(x_\nu)_1^n$ of the classical orthogonal polynomials $p_n(x)$ and of the last coordinates $p_{n-1}(x)$ of vectors

$$p(x) = (p_0(x), p_1(x), \ldots, p_{n-1}(x))^T \in \mathsf{R}^n$$

at the points $x = x_\nu$, $\nu = 1, 2, \ldots, n$. By the three-term recurrence formulae (1.7), the simultaneous determination of $(x_\nu)_1^n$ and $p_{n-1}(x_\nu)$ is equivalent to the partial solving of the eigenvalue problem

$$Gy = xy, \quad y = (y_0, y_1, \ldots, y_{n-1})^T \in \mathsf{R}^n, \tag{4.1}$$

with the tridiagonal matrix G of the form

$$G = \begin{bmatrix} c_0 & 1 & & & \emptyset \\ d_1 & c_1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & d_{n-2} & c_{n-2} & 1 \\ \emptyset & & & d_{n-1} & c_{n-1} \end{bmatrix}.$$

Its solutions are eigenvalues $x = x_\nu$ and corresponding orthogonal eigenvectors $y = p(x_\nu)$, up to a constant multiple; see [11], [13] and [42].

Theorem 4.1. *Let $M = diag\left(\sqrt{m_0}, \sqrt{m_1}, \ldots, \sqrt{m_{n-1}}\right)$ be the diagonal matrix, and let $H = [h_{\nu,k}]_{\nu,k=0}^{n-1}$ be the tridiagonal symmetric matrix with elements defined by $m_0 = 1$, $h_{0,0} = c_0$ and*

$$m_\nu = d_1 d_2 \cdots d_\nu, \quad h_{\nu,\nu-1} = \sqrt{d_\nu}, \quad h_{\nu,\nu} = c_\nu \quad (\nu = 1, 2, \ldots, n-1),$$

where $(c_\nu)_0^{n-1}$, $(d_\nu)_1^{n-1}$ and $(m_\nu)_1^{n-1}$ are as in Table 3 for any classical weight. Then the barycentric weights of Fejér γ_ν, Hermite $\gamma_{\nu,0}$ and $\gamma_{\nu,1}$, and Lagrange σ_ν have the explicit representations in terms of the solution

$$x = x_\nu, \quad z = \left(p_k(x_\nu)/\sqrt{m_k}\right)_{k=0}^{n-1} \quad (\nu = 1, 2, \ldots, n)$$

of the symmetric tridiagonal eigenvalue problem of the form

$$Hz = xz, \quad z = M^{-1}y. \tag{4.2}$$

Additionally, these representations are identical with the second representations given in formulae (2.3), (2.7), (2.10) *and* (3.1).

Proof. We note that the matrix G of the eigenvalue problem (4.1) is similar to the following symmetric tridiagonal matrix

$$H = M^{-1}GM = \begin{bmatrix} c_0 & \sqrt{d_1} & & & \emptyset \\ \sqrt{d_1} & c_1 & \sqrt{d_2} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{d_{n-2}} & c_{n-2} & \sqrt{d_{n-1}} \\ \emptyset & & & \sqrt{d_{n-1}} & c_{n-1} \end{bmatrix}.$$

Thus the eigenvalue problems (4.1) and (4.2) are equivalent. Hence we apply Theorems 2.1, 2.2, 2.3 and 3.1 to complete the proof. □

To solve numerically the eigenvalue problem (4.2) with a tolerance $\epsilon > 0$, we can use one of the classical, efficient and stable algorithms having complexity of order $O\left(n^2\right)$ [12, 13, 35, 41], e.g., the divide-and-conquer algorithm due to Gu and Eisenstat [13]. It is clear that the computations of barycentric weights and Christoffel numbers from formulae (2.3), (2.7), (2.10) and (3.1) do not increase this order for any classical weight function. This complexity can be improved to $O(n)$ if we extend the remarkable numerical method of Glaser, Liu and Rokhlin [10] of computing the roots x_ν and Christoffel numbers λ_ν by means of the Runge-Kutta and Newton methods applied to the generic Sturm-Liouville differential problem (1.3) for all classical orthogonal polynomials. In view of Koekoek et al. [20] and Table 1 we note that the extension of the Glaser, Liu and Rokhlin $O(n)$-algorithm to generalized Bessel, Jacobi on $(0, +\infty)$ and pseudo-Jacobi classical orthogonal polynomials does not require any new ideas. However, we do not know if the improved $O(n)$-algorithm of Hale and Townsend [15] can be also extended to these polynomials.

It is important that the eigenvalue problem (4.1) has explicit solution for the following four special interesting weights $w(x)$ of Jacobi type:

$$w(x) = (1-x)^\alpha (1+x)^\beta, \quad \alpha = \pm\frac{1}{2}, \quad \beta = \pm\frac{1}{2}. \tag{4.3}$$

TABLE 3. The items c_ν, d_ν and m_ν of tridiagonal symmetric eigenvalue problem (4.2), dependent only on classical weight functions and n. Here ζ and a_1 are as in (1.4), and $(v)_k$ denotes the Pochhammer symbol. Furthermore, we should put $c_0 = (\beta-\alpha)/2$ and $d_1 = 2(1+\alpha)(1+\beta)$ in the third row, whenever α and β are as in Remark 2.1.

	c_ν $(\nu \geqslant 0)$	d_ν $(\nu \geqslant 1)$	m_ν $(\nu \geqslant 1)$
1	0	$\frac{\nu}{2}$	$\frac{\nu!}{2^\nu}$
2	$2\nu+\alpha+1$	$\nu(\nu+\alpha)$	$\nu!\,(1+\alpha)_\nu$
3	$\frac{\beta^2-\alpha^2}{(2\nu+\alpha+\beta)(2\nu+2+\alpha+\beta)}$	$\frac{4\nu(\nu+\alpha)(\nu+\beta)(\nu+\alpha+\beta)}{\left[(2\nu+\alpha+\beta)^2-1\right](2\nu+\alpha+\beta)^2}$	$\frac{4^\nu\nu!(1+\alpha)_\nu(1+\beta)_\nu(2+\alpha+\beta)_{\nu-1}}{(2\nu+1+\alpha+\beta)(2+\alpha+\beta)^2_{2\nu-1}}$
4	$\frac{\beta(2-\alpha)}{(2\nu-2+\alpha)(2\nu+\alpha)}$	$\frac{-\nu(\nu-2+\alpha)\beta^2}{(2\nu-3+\alpha)(2\nu-2+\alpha)^2(2\nu-1+\alpha)}$	$\frac{(-1)^\nu\nu!\beta^{2\nu}(\alpha)_{\nu-1}}{(2\nu-1+\alpha)(\alpha)^2_{2\nu-1}}$
5	$\frac{\alpha(\beta+1)-2\nu(\nu+1-\alpha)}{(2\nu-\alpha)(2\nu+2-\alpha)}$	$\frac{\nu(\nu-\alpha)(\nu+\beta)(\nu-\alpha-\beta)}{(2\nu-\alpha)^2\left[(2\nu-\alpha)^2-1\right]}$	$\frac{\nu!(2-\alpha)_{\nu-1}(1+\beta)_\nu(1-\alpha-\beta)_\nu}{(2\nu+1-\alpha)(2-\alpha)^2_{2\nu-1}}$
6	$\frac{\alpha\beta\zeta}{2(\nu-\alpha)(\nu+1-\alpha)} - \frac{a_1}{2}$	$\frac{-\nu(\nu-2\alpha)\left[4(\nu-\alpha)^2+\beta^2\right]}{4(\nu-\alpha)^2\left[4(\nu-\alpha)^2-1\right]\zeta^{-2}}$	$\frac{\nu!(2-2\alpha)_{\nu-1}\prod_{j=1}^{\nu}\left[4(j-\alpha)^2+\beta^2\right]}{(-1)^\nu(2\nu+1-2\alpha)(2-2\alpha)^2_{2\nu-1}\zeta^{-2\nu}}$

The associated classical orthogonal monic polynomials $p_n(x)$, for $n \geqslant 1$ and

$$\alpha=\beta=-\frac{1}{2},\quad \alpha=\beta=\frac{1}{2},\quad \alpha=-\beta=-\frac{1}{2},\quad \text{and} \quad \alpha=-\beta=\frac{1}{2}, \tag{4.4}$$

have the representations

$$\begin{aligned} t_n(x) &= \frac{1}{2^{n-1}}\cos n\theta, \quad u_n(x) = \frac{1}{2^n}\frac{\sin(n+1)\theta}{\sin\theta}, \\ c_n(x) &= \frac{1}{2^n}\frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}}, \quad s_n(x) = \frac{1}{2^n}\frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}, \end{aligned} \tag{4.5}$$

where $x=\cos\theta$ and $-1<x<1$. These weights and polynomials are called the Chebyshev and Szegő weights/polynomials of the first and second kind, respectively.

Corollary 4.1. *Let $w(x) = \left(1-x^2\right)^{-\frac{1}{2}}$ and $t_n(x) = \frac{1}{2^{n-1}}\cos n\theta$ $(x=\cos\theta)$ be the Chebyshev weight and monic polynomial of the first kind. Then we have*

$$\begin{aligned} \gamma_\nu &= \frac{4^{n-1}\sin\theta_\nu}{n^2} = \frac{4^{n-1}\sin\theta_\nu}{\pi n}\lambda_\nu, \\ \gamma_{\nu,0} &= \frac{4^{n-1}\sin^2\theta_\nu}{n^2} = \frac{4^{n-1}\sin^2\theta_\nu}{\pi n}\lambda_\nu, \\ \gamma_{\nu,1} &= -\frac{4^{n-1}x_\nu}{n^2} = -\frac{4^{n-1}x_\nu}{\pi n}\lambda_\nu, \\ \sigma_\nu &= \frac{(-1)^{\nu-1}2^{n-1}\sin\theta_\nu}{n} = \frac{(-1)^{\nu-1}2^{n-1}\sin\theta_\nu}{\pi}\lambda_\nu, \\ \lambda_\nu &= \frac{\pi}{n}, \quad x_\nu = \cos\theta_\nu, \quad \theta_\nu = \frac{2\nu-1}{2n}\pi. \end{aligned}$$

Additionally, if we denote $T_n(x) = 2^{n-1} t_n(x)$, $n \geqslant 1$, *then we obtain the explicit barycentric operator representations*

$$(\mathcal{F}_n f)(x) = \frac{\sqrt{1-x^2}T_n^2(x)}{n^2} \sum_{\nu=1}^{n} f(x_\nu) \frac{\sin\theta_\nu}{(x-x_\nu)^2},$$

$$(\mathcal{H}_n f)(x) = \frac{T_n^2(x)}{n^2} \sum_{\nu=1}^{n} \left\{ f(x_\nu) \left[\frac{\sin^2\theta_\nu}{(x-x_\nu)^2} - \frac{x_\nu}{x-x_\nu} \right] + f\ (x_\nu) \frac{\sin^2\theta_\nu}{x-x_\nu} \right\},$$

$$(\mathcal{L}_n f)(x) = \frac{T_n(x)}{n} \sum_{\nu=1}^{n} f(x_\nu) \frac{(-1)^{\nu-1}\sin\theta_\nu}{x-x_\nu}$$

for the Fejér, Hermite and Lagrange interpolating operators at the zeros $(x_\nu)_1^n$ *of* $t_n(x)$.

Proof. Since $\alpha = \beta = -\frac{1}{2}$, it follows from Table 1 and (4.5) that

$$A(x) = \sin^2\theta, \quad B(x) = -\cos\theta, \quad w(x) = \frac{1}{\sin\theta}, \quad w_1(x) = \sin\theta,$$

$$t_n(x_\nu) = \frac{(-1)^{\nu-1} n}{2^{n-1}\sin\theta_\nu}, \quad t_0(x_\nu) = 1, \quad t_{n-1}(x_\nu) = \frac{(-1)^{\nu-1}\sin\theta_\nu}{2^{n-2}}.$$

Next, setting $\alpha = \beta = -\frac{1}{2}$ in the third row of Tables 2 and 3, and then simplifying, we obtain $\rho_1 = 1$ and $d_1 = \frac{1}{2}$, by Remark 2.1, and otherwise

$$\rho_n = \frac{n}{2}, \quad d_n = \frac{1}{4}, \quad m_n = \frac{1}{2^{2n-1}},$$
$$h_0 = \Gamma^2\left(\frac{1}{2}\right) = \pi, \quad h_n = \frac{\pi}{2^{2n-1}}.$$

Hence we apply Theorems 2.1, 2.2, 2.3, 3.1 and 3.2 to complete the proof. $\square$

We note that the last barycentric formulae in Corollaries 4.1 and 4.2 have been obtained recently by Wang et al. in [46]. Moreover, the representations of Christoffel numbers λ_ν, presented in all our corollaries for completeness, are well known [45].

Corollary 4.2. *Let* $w(x) = \sqrt{1-x^2}$ *and* $u_n(x) = \frac{1}{2^n}\frac{\sin(n+1)\theta}{\sin\theta}$ $(x = \cos\theta)$ *be the Chebyshev weight and monic polynomial of the second kind. Then we have*

$$\gamma_\nu = \frac{4^n\sin\theta_\nu}{(n+1)^2} = \frac{4^n}{\pi(n+1)\sin\theta_\nu}\lambda_\nu,$$

$$\gamma_{\nu,0} = \frac{4^n\sin^4\theta_\nu}{(n+1)^2} = \frac{4^n\sin^2\theta_\nu}{\pi(n+1)}\lambda_\nu,$$

$$\gamma_{\nu,1} = -\frac{3\cdot 4^n x_\nu\sin^2\theta_\nu}{(n+1)^2} = -\frac{3\cdot 4^n x_\nu}{\pi(n+1)}\lambda_\nu,$$

$$\sigma_\nu = \frac{(-1)^{\nu-1}2^n\sin^2\theta_\nu}{n+1} = \frac{(-1)^{\nu-1}2^n}{\pi}\lambda_\nu,$$

$$\lambda_\nu = \frac{\pi\sin^2\theta_\nu}{n+1}, \quad x_\nu = \cos\theta_\nu, \quad \theta_\nu = \frac{\nu}{n+1}\pi.$$

Furthermore, if $U_n(x) = 2^n u_n(x)$*, then*

$$
\begin{aligned}
(\mathcal{F}_n f)(x) &= \frac{\left(1-x^2\right)^{\frac{3}{2}} U_n^2(x)}{(n+1)^2} \sum_{\nu=1}^{n} f(x_\nu) \frac{\sin\theta_\nu}{(x-x_\nu)^2}, \\
(\mathcal{H}_n f)(x) &= \frac{U_n^2(x)}{(n+1)^2} \sum_{\nu=1}^{n} \left\{ f(x_\nu) \left[\frac{\sin^2\theta_\nu}{(x-x_\nu)^2} - \frac{3x_\nu}{x-x_\nu} \right] \right. \\
&\quad \left. + f\ (x_\nu) \frac{\sin^2\theta_\nu}{x-x_\nu} \right\} \sin^2\theta_\nu, \\
(\mathcal{L}_n f)(x) &= \frac{U_n(x)}{n+1} \sum_{\nu=1}^{n} f(x_\nu) \frac{(-1)^{\nu-1}\sin^2\theta_\nu}{x-x_\nu}.
\end{aligned}
$$

Proof. In view of Table 1 and (4.5) we get

$$
A(x) = \sin^2\theta, \quad B(x) = -3\cos\theta, \quad w(x) = \sin\theta, \quad w_1(x) = \sin^3\theta,
$$

$$
u_n(x_\nu) = \frac{(-1)^{\nu-1}(n+1)}{2^n \sin^2\theta_\nu}, \quad u_{n-1}(x_\nu) = \frac{(-1)^{\nu-1}}{2^{n-1}}.
$$

Next, for $\alpha = \beta = \frac{1}{2}$ we obtain from Tables 2 and 3

$$
\begin{aligned}
\rho_n &= \frac{n+1}{2}, \quad d_n = \frac{1}{4}, \quad m_n = \frac{1}{4^n}, \\
h_0 &= \frac{4\Gamma^2\left(\frac{3}{2}\right)}{\Gamma(3)} = \frac{\pi}{2} \quad \text{and} \quad h_{n-1} = \frac{\pi}{2^{2n-1}}.
\end{aligned}
$$

Hence one can apply Theorems 2.1, 2.2, 2.3, 3.1 and 3.2 to finish the proof. □

In the next corollaries we consider two special cases $\alpha = -\beta = -\frac{1}{2}$ and $\alpha = -\beta = \frac{1}{2}$ of the Jacobi weight functions. It should be mentioned that these weights are of interest in the analytical and numerical solving of singular integral equations with the Cauchy kernel [33].

Corollary 4.3. *Let* $w(x) = \sqrt{\frac{1+x}{1-x}}$ *be the Szegő weight function of the first kind, and let*

$$
c_n(x) = \frac{1}{2^n} \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}}, \quad x = \cos\theta, \quad -1 < x < 1,
$$

be the associated orthogonal monic polynomial. Then we have

$$\begin{aligned}
\gamma_\nu &= \frac{2^{2n-1}\sin\theta_\nu}{\left(n+\frac{1}{2}\right)^2} = \frac{2^{2n-1}\tan\frac{\theta_\nu}{2}}{\pi\left(n+\frac{1}{2}\right)}\lambda_\nu,\\
\gamma_{\nu,0} &= \frac{4^n\sin^2\theta_\nu\cos^2\frac{\theta_\nu}{2}}{\left(n+\frac{1}{2}\right)^2} = \frac{2^{2n-1}\sin^2\theta_\nu}{\pi\left(n+\frac{1}{2}\right)}\lambda_\nu,\\
\gamma_{\nu,1} &= \frac{4^n\left(1-2\cos\theta_\nu\right)\cos^2\frac{\theta_\nu}{2}}{\left(n+\frac{1}{2}\right)^2} = \frac{2^{2n-1}\left(1-2\cos\theta_\nu\right)}{\pi\left(n+\frac{1}{2}\right)}\lambda_\nu,\\
\sigma_\nu &= \frac{(-1)^{\nu-1}2^n\sin\theta_\nu\cos\frac{\theta_\nu}{2}}{n+\frac{1}{2}} = \frac{(-1)^{\nu-1}2^n\sin\frac{\theta_\nu}{2}}{\pi}\lambda_\nu,\\
\lambda_\nu &= \frac{2\pi\cos^2\frac{\theta_\nu}{2}}{n+\frac{1}{2}},\quad x_\nu=\cos\theta_\nu,\quad \theta_\nu=\frac{2\nu-1}{2n+1}\pi.
\end{aligned}$$

Additionally, if we denote $C_n(x) = 2^n c_n(x)$*, then the Fejér, Hermite and Lagrange interpolating operators at the zeros* $(x_\nu)_1^n$ *of* $c_n(x)$ *have the barycentric representations*

$$\begin{aligned}
(\mathcal{F}_n f)(x) &= \frac{(1+x)\sqrt{1-x^2}C_n^2(x)}{2\left(n+\frac{1}{2}\right)^2}\sum_{\nu=1}^n f(x_\nu)\frac{\sin\theta_\nu}{(x-x_\nu)^2},\\
(\mathcal{H}_n f)(x) &= \frac{C_n^2(x)}{\left(n+\frac{1}{2}\right)^2}\sum_{\nu=1}^n\left\{f(x_\nu)\left[\frac{\sin^2\theta_\nu}{(x-x_\nu)^2}+\frac{1-2\cos\theta_\nu}{x-x_\nu}\right]\right.\\
&\qquad\left.+f'(x_\nu)\frac{\sin^2\theta_\nu}{x-x_\nu}\right\}\cos^2\frac{\theta_\nu}{2},\\
(\mathcal{L}_n f)(x) &= \frac{C_n(x)}{n+\frac{1}{2}}\sum_{\nu=1}^n f(x_\nu)\frac{(-1)^{\nu-1}\sin\theta_\nu\cos\frac{\theta_\nu}{2}}{x-x_\nu}.
\end{aligned}$$

Proof. Writing $\alpha=-\frac{1}{2}$ and $\beta=\frac{1}{2}$ in Tables 2 and 3, and then simplifying, we obtain

$$\begin{aligned}
\rho_n &= \frac{1}{2}\left(n+\frac{1}{2}\right),\quad d_n=\frac{1}{4},\quad m_n=\frac{1}{4^n},\\
h_0 &= \frac{2\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2)}=\pi,\quad h_{n-1}=\frac{\pi}{4^{n-1}},\\
c_n &= 0\quad\left(n\geqslant 1;\ c_0=\frac{1}{2}\text{ by Remark 2.1}\right).
\end{aligned}$$

Since we also have

$$\begin{aligned}
&A(x)=\sin^2\theta,\quad B(x)=-2\cos\theta+1,\quad w(x)=\cot\frac{\theta}{2},\\
&w_1(x)=2\sin\theta\cos^2\frac{\theta}{2},\quad c_n'(x_\nu)=\frac{(-1)^{\nu-1}\left(n+\frac{1}{2}\right)}{2^n\sin\theta_\nu\cos\frac{\theta_\nu}{2}},\\
&c_{n-1}(x_\nu)=\frac{(-1)^{\nu-1}\sin\frac{\theta_\nu}{2}}{2^{n-2}},
\end{aligned}$$

the proof follows directly from Theorems 2.1, 2.2, 2.3, 3.1 and 3.2. □

The final explicit barycentric representations are connected with the Szegő polynomials of the second kind. To prove them we proceed similarly as in the proof of Corollary 4.3.

Corollary 4.4. *Let* $w(x) = \sqrt{\frac{1-x}{1+x}}$ *be the Szegő weight of the second kind, and let*

$$s_n(x) = \frac{1}{2^n} \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}}, \quad x = \cos\theta, \quad -1 < x < 1,$$

be the Szegő orthogonal monic polynomial of the second kind. Then the barycentric weights and Christoffel numbers satisfy the formulae

$$\gamma_\nu = \frac{2^{2n-1}\sin\theta_\nu}{\left(n+\frac{1}{2}\right)^2} = \frac{2^{2n-1}\cot\frac{\theta_\nu}{2}}{\pi\left(n+\frac{1}{2}\right)}\lambda_\nu,$$

$$\gamma_{\nu,0} = \frac{4^n\sin^2\theta_\nu\sin^2\frac{\theta_\nu}{2}}{\left(n+\frac{1}{2}\right)^2} = \frac{2^{2n-1}\sin^2\theta_\nu}{\pi\left(n+\frac{1}{2}\right)}\lambda_\nu,$$

$$\gamma_{\nu,1} = -\frac{4^n\left(1+2\cos\theta_\nu\right)\sin^2\frac{\theta_\nu}{2}}{\left(n+\frac{1}{2}\right)^2} = -\frac{2^{2n-1}\left(1+2\cos\theta_\nu\right)}{\pi\left(n+\frac{1}{2}\right)}\lambda_\nu,$$

$$\sigma_\nu = \frac{(-1)^{\nu-1}2^n\sin\theta_\nu\sin\frac{\theta_\nu}{2}}{n+\frac{1}{2}} = \frac{(-1)^{\nu-1}2^n\cos\frac{\theta_\nu}{2}}{\pi}\lambda_\nu,$$

$$\lambda_\nu = \frac{2\pi\sin^2\frac{\theta_\nu}{2}}{n+\frac{1}{2}}, \quad x_\nu = \cos\theta_\nu, \quad \theta_\nu = \frac{2\nu}{2n+1}\pi.$$

Moreover, if $S_n(x) = 2^n s_n(x)$*, then we have the barycentric operator representations*

$$(\mathcal{F}_n f)(x) = \frac{(1-x)\sqrt{1-x^2}S_n^2(x)}{2\left(n+\frac{1}{2}\right)^2}\sum_{\nu=1}^{n} f(x_\nu)\frac{\sin\theta_\nu}{(x-x_\nu)^2},$$

$$(\mathcal{H}_n f)(x) = \frac{S_n^2(x)}{\left(n+\frac{1}{2}\right)^2}\sum_{\nu=1}^{n}\left\{f(x_\nu)\left[\frac{\sin^2\theta_\nu}{(x-x_\nu)^2} - \frac{1+2\cos\theta_\nu}{x-x_\nu}\right]\right.$$
$$\left. + f'(x_\nu)\frac{\sin^2\theta_\nu}{x-x_\nu}\right\}\sin^2\frac{\theta_\nu}{2},$$

$$(\mathcal{L}_n f)(x) = \frac{S_n(x)}{n+\frac{1}{2}}\sum_{\nu=1}^{n} f(x_\nu)\frac{(-1)^{\nu-1}\sin\theta_\nu\sin\frac{\theta_\nu}{2}}{x-x_\nu}.$$

Proof. Since $\alpha = -\beta = \frac{1}{2}$ it follows from Tables 2 and 3 that all constant factors ρ_n, d_n, m_n and h_{n-1} are identical with those in the proof of Corollary 4.3. Moreover, we obtain, by using Table 1 and (4.5),

$$A(x) = \sin^2\theta, \quad B(x) = -2\cos\theta - 1, \quad w(x) = \tan\frac{\theta}{2},$$

$$w_1(x) = 2\sin\theta\sin^2\frac{\theta}{2}, \quad s_n(x_\nu) = \frac{(-1)^{\nu-1}\left(n+\frac{1}{2}\right)}{2^n\sin\theta_\nu\sin\frac{\theta_\nu}{2}},$$

$$s_{n-1}(x_\nu) = \frac{(-1)^{\nu-1}\cos\frac{\theta_\nu}{2}}{2^{n-2}}.$$

Hence we apply Theorems 2.1, 2.2, 2.3, 3.1 and 3.2 to establish the corollary. □

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