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ON THE ROBUSTNESS OF MULTISCALE HYBRID-MIXED METHODS

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Abstract. In this work we prove uniform convergence of the Multiscale Hybrid-Mixed (MHM for short) finite element method for second-order elliptic problems with rough periodic coefficients. The MHM method is shown to avoid resonance errors without adopting oversampling techniques. In particular, we establish that the discretization error for the primal variable in the broken H^1 and L^2 norms are $O(h+\varepsilon^{\delta})$ and $O(h^2+h\,\varepsilon^{\delta})$, respectively, and for the dual variable it is $O(h+\varepsilon^{\delta})$ in the $H(\mathrm{div};\cdot)$ norm, where $0<\delta \leq 1/2$ (depending on regularity). Such results rely on sharpened asymptotic expansion error estimates for the elliptic models with prescribed Dirichlet, Neumann or mixed boundary conditions.

1. Introduction

Flows in porous media, which commonly exhibit multiple scale structures, are usually modeled by a second-order elliptic problem (Darcy equation) with rough discontinuous coefficients. Such a model arises when we consider the simulation of oil reservoirs in a highly heterogeneous and/or fractured media. Multiscale problems necessarily require the use of very fine meshes, which makes their numerical approximation extremely expensive. Since the pioneering work of Babuška and Osborn [9] and its extension to higher dimensions by Hou and Wu [23], multiscale numerical methods have emerged as an attractive "divide and conquer" option to handle heterogeneous problems (see [14, 15, 36], just to cite a few). Overall, the idea relies on basis functions specially designed to upscale submesh oscillations to an overlying coarse mesh. As a result, such numerical methods become precise on coarse meshes. Also interesting, the multiscale basis functions can be locally computed through completely independent problems. This makes the resulting numerical algorithm particularly attractive for use in parallel computing environments.

Recently, a new family of multiscale finite element methods, named the Multiscale Hybrid-Mixed (MHM) method, was presented in [20] and further analyzed in [4]. The framework has since been extended to the linear elasticity model in [19] and the reactive-advective-diffusive problem in [21]. The MHM method has a notably general formulation that recovers some well-established finite element

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methods, such as the ones proposed in [6, 12], under appropriate hypotheses. The method does not require scale separation or periodicity of the media. Moreover, it produces precise numerical primal and dual variables (standing for the pressure and the velocity in porous media problems, respectively), with respect to the mesh parameter h (cf. [4]). Although the primal solution is non-conforming (as the MsFEM with over-sampling [23], for instance), conformity is maintained for the dual variable. Since the velocity variable is often the quantity of interest, such a property is particularly welcome in porous media flow simulations.

In this work we address the important question of the robustness of the MHM method with respect to the fine scale oscillations of the physical coefficient. Specifically, under the assumption that the physical coefficient is periodic with period of order ε, we prove the method converges when both ε and h go to zero. Such a question was beyond the scope of [4], in which the focus was on h-convergence results. It is worth mentioning that the mathematical techniques involved in the present analysis differ completely from those used in [4], requiring in particular the usual periodicity assumption. Moreover, the convergence results are established without involving any kind of oversampling techniques. This is, to the best of our knowledge, a first in the multiscale numerical method literature.

To be more precise, let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, and assume the boundary of $\partial\Omega := \partial\Omega_D \cup \partial\Omega_N$ is Lipschitz. The boundary value problem considered in this work consists of finding u_ε, the solution of

$$\text{(1)} \qquad \begin{cases} -\nabla \cdot (\mathcal{A}_\varepsilon \nabla u_\varepsilon) := -\dfrac{\partial}{\partial x_i}(a_{ij}(\boldsymbol{x}/\varepsilon)\dfrac{\partial}{\partial x_j} u_\varepsilon) \;=\; f \quad \text{in } \Omega\,, \\ u_\varepsilon \;=\; g \quad \text{on } \partial\Omega_D \quad \text{and} \quad \mathcal{A}_\varepsilon \nabla u \cdot \boldsymbol{n} \;=\; b \quad \text{on } \partial\Omega_N\,, \end{cases}$$

where $\mathcal{A}_\varepsilon(\boldsymbol{x}) = \mathcal{A}(\boldsymbol{x}/\varepsilon) = (a_{ij}(\boldsymbol{x}/\varepsilon))$ is a symmetric positive definite matrix. Here $\varepsilon \in (0, 1)$ is the (small) parameter controlling the fine scale oscillations of the physical coefficient, $g \in H^{1/2}(\partial\Omega_D)$, $\boldsymbol{n}$ represents the unit outward normal vector on $\partial\Omega$, $b \in H^{-1/2}(\partial\Omega_N)$, and $f \in L^2(\Omega)$ (these spaces having their usual meaning). Above, and throughout the paper, the indices i, j run from $1, \ldots, d$, even when not explicitly mentioned, and we employ the Einstein summation convention, i.e., repeated indices indicate summation. We also assume that $a_{ij} \in L^\infty_{\text{per}}(Y)$, i.e., $a_{ij} \in L^\infty(\mathbb{R}^d)$ and it is Y-periodic, $Y = (0, 1)^d$, and there exist positive constants γ_a and γ_b such that

$$\text{(2)} \qquad \gamma_a\, |\boldsymbol{\xi}|^2 \;\leq\; a_{ij}(\boldsymbol{y})\xi_i\xi_j \;\leq\; \gamma_b\, |\boldsymbol{\xi}|^2 \quad \text{for all } \boldsymbol{\xi} = \{\xi_i\} \in \mathbb{R}^d \text{ and } \boldsymbol{y} \in Y\,,$$

where $|.|$ represents the Euclidean norm. In the case $\partial\Omega_D = \emptyset$, we also assume that the following compatibility condition holds:

$$\text{(3)} \qquad \int_\Omega f \, d\boldsymbol{x} = \int_{\partial\Omega} b \, d\boldsymbol{s}\,.$$

We employ asymptotic expansion error estimates in our analysis. Such a technique assumes the periodicity hypothesis on a_{ij} and was first adopted in the context of multiscale scheme analysis in the seminal work [22]. It has been used since by several authors [3, 5, 24, 34, 35]. Also, the adoption of asymptotic analysis for multiscale methods has influenced the choice of interpolation spaces, yielding more robust theoretical error estimates [7].

Asymptotic techniques adopted to analyze the MsFEM method [22] highlight that the resulting discrete solution u_h^ε for (1) (with Dirichlet boundary condition) converges as follows:

$$\|u^\varepsilon - u_h^\varepsilon\|_1 \le c_1\, h\, \|f\|_0 + c_2 \left(\frac{\varepsilon}{h}\right)^{1/2},$$

where the constants c_1, c_2 depend on Ω but are independent of h and ε. Here $\|\cdot\|_1$ and $\|\cdot\|_0$ stand for the norms in the $H^1(\Omega)$ and $L^2(\Omega)$ spaces (with their usual meaning), respectively. We observe the presence of the so-called resonance error term $\left(\frac{\varepsilon}{h}\right)^{1/2}$, which indicates that the method may lose convergence when h and ε have the same order of magnitude. Unfortunately, it has been verified numerically that such an estimate is actually sharp, with the resonance error extending from the choice of local boundary conditions used to compute the multiscale basis functions. Strategies to diminish the resonance error have focused on the construction of more involved local boundary conditions using "global information". Examples in this direction are the over-sampling strategy [15, 23] and the limited global information technique [25]. These alternatives result in non-conforming methods with lower resonance errors of order $\frac{\varepsilon}{h}$. When used within a Petrov-Galerkin framework, such a non-conforming approach leads to resonance-free solutions under the prior knowledge of the thickness of boundary layers (cf. [24]) and the assumption $\frac{\varepsilon}{h} < c$, where $c < 1$ is of order one. However, for realistic domains such information is generally not available, and it is particularly difficult to be measured in domains with corners (which is usually the case with finite elements). Moreover, the analyses performed in the aforementioned works assume smooth physical coefficients.

The main result of this work establishes the convergence of the MHM method in the L^2 and the broken H^1 norms for any choice of ε and h under the condition $\varepsilon < c\,h$, where $c < 1$ is of order one. For instance, we prove that under mild regularity conditions, the following estimates hold:

$$\|u^\varepsilon - u_h^\varepsilon\|_{1,h} \le c_1 \left(h + \varepsilon^\delta\right) \|f\|_0 \quad \text{and} \quad \|u^\varepsilon - u_h^\varepsilon\|_0 \le c_2\, h \left(h + \varepsilon^\delta\right) \|f\|_0\,,$$

where $\|.\|_{1,h}$ stands for the broken H^1 norm, $0 < \delta \le 1/2$ (depending on regularity). The established dependence of the error in terms of ε stems from a better approximation result which combines new asymptotic expansion estimates with Galerkin error analysis (see Lemma 7). Next, by choosing an appropriate finite element space, the sharp error estimates with respect to h also emerge (see Theorems 8-9 and Theorem 11). As in [15, 22, 24], the convergence analysis assumes the numerical approximation of the local problems (basis functions) is exact or its associated error is negligible (setting a fine submesh at the second level, for instance). We left the study of the impact of the two-level discretization as well as the influence of high-contrast coefficients on the constants out of the scope of this work (see [13] for a related work).

As we have mentioned, the convergence analysis of the MHM method relies on new asymptotic error estimates. To this end, we first revisit the asymptotic expansion technique to sharpen error estimates for second-order elliptic problems assuming rough coefficients and more diverse boundary conditions (other than the pure Dirichlet case). In fact, Neumann or mixed boundary conditions turn out to be natural choices in Darcy models. Also interesting, the asymptotic estimates obtained here may be used in homogenization problems coming from other applications, e.g., composite materials, nuclear reactors, among others; see for instance [8, 10, 26, 31] and references therein.

Let us highlight the main asymptotic results. We estimate the error between u_ε and its first-order asymptotic expansion approximation, $u_0 - \varepsilon\,\chi^j_\varepsilon \frac{\partial u_0}{\partial x_j}$, where u_0 is the solution of the homogenized problem, χ^j is the solution of the cell problem, and $\chi^j_\varepsilon(\boldsymbol{x}) := \chi^j(\boldsymbol{x}/\varepsilon)$. These terms are precisely defined in Section 2. It is well known that if $u_0 \in W^{2,\infty}(\Omega)$ and $\chi^j \in W^{1,\infty}_{\text{per}}(Y)$, then the estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon\,\chi^j_\varepsilon \frac{\partial u_0}{\partial x_j} \right\|_1 \le c\,\varepsilon^{1/2} \|u_0\|_{2,\infty} \tag{4}$$

holds, where the constant c depends on γ_a, γ_b, χ^j and Ω (cf. [10, 26]). However, the regularity assumption $\chi^j \in W^{1,\infty}_{\text{per}}(Y)$ may not be satisfied. Indeed, such a regularity can be guaranteed if one assumes $a_{ij} \in L^\infty(Y)$ has discontinuities along a $C^{1,\alpha}$ curve, $\alpha > 0$ (see [29, Theorem 1.1] for further details). Observe that the usual case of a piecewise constant coefficient on polygonal subdomains of Y does not fulfill this assumption, and thus (4) may not be used in this case. Generalizations of estimate (4) considering weaker assumptions on χ^j and u^0 have been investigated by several authors in the case of Dirichlet boundary conditions; see for instance [2, 17, 31, 32, 35].

We pursue the idea of building asymptotic estimates under weaker assumptions on χ^j and u_0 than those proposed in [2, 32, 35]. In this work, the regularity assumed on χ^j and u_0 follows the one adopted in [17], although the asymptotic expansion considered here differs from the one used in [17] where more regularity is assumed on Ω. Also, it appears that the asymptotic results in [17] do not seem to have a straightforward application to the analysis of multiscale schemes. Finally, it seems that the problem (1) with a Neumann or mixed boundary condition has not received as much attention as the Dirichlet case, although some results in this direction have been addressed in [31]. In this context, the present asymptotic expansion results improve estimate (4) in two ways (see Theorems 1 and 2). First, we assume $u_0 \in H^2(\Omega)$ and $\chi^j \in W^{1,q}(Y)$, $q > d$, or $u_0 \in W^{1,p}(\Omega)$, $p > d$, and $\chi^j \in H^1(Y)$, which allows the piecewise constant coefficient case to be included in the analysis; see for instance [30, Theorem 1] and [27, Theorem 10.1]. Second, we establish the dependence of the right-hand side of (4) in terms of Ω. Such results turn out to be central to the convergence analysis of multiscale numerical schemes.

The paper is outlined as follows: This section ends with notational conventions. Error estimates for the first-order asymptotic expansion of the exact solution are given in Section 2. Section 3 is dedicated to the numerical analysis of the MHM finite element method, followed by numerical validations in Section 4. Conclusions are drawn in Section 5.

We close this section with some notation used throughout the paper (and also employed above). Let $B \subset \mathbb{R}^d$ be an open set and define

$$\|v\|_{m,\infty,B} := \max_{|\alpha|\le m} \{\text{ess.}\sup_{x\in B} |\partial^\alpha v(\boldsymbol{x})|\} \text{ and } |v|_{m,\infty,B} := \max_{|\alpha|=m} \{\text{ess.}\sup_{x\in B} |\partial^\alpha v(\boldsymbol{x})|\},$$

and for $1 \le q < \infty$

$$\|v\|_{m,q,B} := \left(\int_B \sum_{|\alpha|\le m} |D^\alpha v|^q d\boldsymbol{x} \right)^{1/q} \quad \text{and } |v|_{m,q,B} := \left(\int_B \sum_{|\alpha|=m} |D^\alpha v|^q d\boldsymbol{x} \right)^{1/q}.$$

We define the broken norms related to a partition $\mathcal{T}_h$ of B in elements K by

$$\|v\|_{m,q,h} := \left(\sum_{K\in\mathcal{T}_h} \|v\|^2_{m,q,K} \right)^{1/2},$$

and the norm in the $H(\mathrm{div};B)$ space, i.e., the space of functions belonging to $L^2(B)$ with divergence also in $L^2(B)$, by

$$\|\boldsymbol{\sigma}\|_{\mathrm{div},B} := \left(\int_B |\boldsymbol{\sigma}|^2 d\boldsymbol{x} + \int_B |\nabla\cdot\boldsymbol{\sigma}|^2 d\boldsymbol{x} \right)^{1/2}.$$

Hereafter, we do not make reference to the domain B, or to the coefficient q when $B=\Omega$, or $q=2$, respectively. In what follows, c denotes a generic constant independent of ε and h, although it may change in each occurrence. Also, d_Ω denotes the side of the maximum square (or cube in 3D) contained in Ω, and L_Ω is the length of the boundary of Ω in the 2D case (or area in 3D). Throughout this work we shall assume

$$d_\Omega > c\,\varepsilon\,, \tag{5}$$

where $c>1$ is of order one. Also, we simplify the notation with respect to the norms of χ^j by setting

$$\|\boldsymbol{\chi}\|_{s,p,Y} := \max_{1\le j\le d} \|\chi^j\|_{s,p,Y}.$$

2. Asymptotic expansion error estimates

We start with the weak formulation of (1) which reads: Find $u_\varepsilon \in H^1(\Omega)$ such that

$$\int_\Omega \mathcal{A}_\varepsilon \nabla u_\varepsilon \cdot \nabla\phi \, d\boldsymbol{x} = \int_{\partial\Omega_N} b\,\phi\, d\boldsymbol{s} + \int_\Omega f\,\phi\, d\boldsymbol{x} \quad \text{for all } \phi\in V\,, \tag{6}$$

under the condition $u_\varepsilon|_{\partial\Omega_D} = g$ if $\partial\Omega_D \neq \emptyset$ or $\int_\Omega u_\varepsilon \, d\boldsymbol{x} = 0$ if $\partial\Omega_D = \emptyset$, where

$$V := \begin{cases} \{\phi\in H^1(\Omega) : \phi|_{\partial\Omega_D} = 0\} & \text{if } \partial\Omega_D \neq \emptyset\,, \\ \{\phi\in H^1(\Omega) : \int_\Omega \phi\, d\boldsymbol{x} = 0\} & \text{if } \partial\Omega_D = \emptyset\,. \end{cases} \tag{7}$$

Next, we consider the ansatz

$$u_\varepsilon(\boldsymbol{x}) = u_0(\boldsymbol{x},\boldsymbol{x}/\varepsilon) + \varepsilon\, u_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon) + \varepsilon^2\, u_2(\boldsymbol{x},\boldsymbol{x}/\varepsilon) + \cdots, \tag{8}$$

where the functions $u_j(\boldsymbol{x},\boldsymbol{y})$ are Y-periodic in $\boldsymbol{y}$. Substituting (8) in equation (1) and collecting the terms with the same order in ε define functions u_j. We recall below such a construction for the first terms in (8) (for more details, see [10, 26]).

Let $\chi^j \in H^1_{\mathrm{per}}(Y)$, i.e., $\chi^j \in H^1_{loc}(\mathbb{R}^d)$ and is Y-periodic, be the weak solution with zero mean value on Y of

$$\nabla_y \cdot \mathcal{A}(\boldsymbol{y})\nabla_y \chi^j = \nabla_y\cdot\mathcal{A}(\boldsymbol{y})\nabla_y y_j = \frac{\partial}{\partial y_i} a_{ij}(\boldsymbol{y}), \tag{9}$$

and let $\mathcal{A}_0$ be the symmetric positive definite matrix given by

$$\mathcal{A}_0 := (a^0_{ij}),\ a^0_{ij} = \frac{1}{|Y|}\int_Y a_{lm}(\boldsymbol{y}) \frac{\partial}{\partial y_l}(y_i-\chi^i)\frac{\partial}{\partial y_m}(y_j-\chi^j) d\boldsymbol{y}. \tag{10}$$

By defining $u_0\in H^1(\Omega)$ as the weak solution of

$$\begin{aligned} -\nabla\cdot(\mathcal{A}_0\nabla u_0) &= f \quad \text{in } \Omega\,, \\ u_0 = g \quad \text{on } \partial\Omega_D, &\quad \mathcal{A}_0\nabla u\cdot\boldsymbol{n} = b \quad \text{on } \partial\Omega_N\,, \end{aligned} \tag{11}$$

the first-order corrector u_1 in (8) reads

$$u_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon) := -\chi^j\,(\boldsymbol{x}/\varepsilon)\,\frac{\partial u_0}{\partial x_j}(\boldsymbol{x})\,. \tag{12}$$

Hereafter, we shall denote

$$u^1_\varepsilon(\boldsymbol{x}) = u_0(\boldsymbol{x}) + \varepsilon\, u_1(\boldsymbol{x}, \boldsymbol{x}/\varepsilon). \tag{13}$$

The following theorem provides an estimate for $\|u_\varepsilon - u^1_\varepsilon\|_1$.

Theorem 1. *Let u_ε be the solution of* (1)*, and let u_0 and u^1_ε be defined by* (11)*-*(13)*. Assume* (5) *holds and*

$$u_0 \in H^2(\Omega),\;\; \chi^j \in W^{1,q}_{\mathrm{per}}(Y),\;\; \textit{with } q > d. \tag{14}$$

Then,

$$\|u_\varepsilon - u^1_\varepsilon\|_1 \le \Big[(c(p')(L_\Omega)^{1/2-1/p} \; + c)(1 + L_\Omega)\Big]\; \varepsilon^{1/2-1/p}\; \|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_2, \tag{15}$$

where the last inequality holds for any $2 < p' < K_d$ with

$$\begin{cases} K_d = \infty, & \textit{if} \quad d = 2\,, \\ K_d = 2d/(d-2) & \textit{if} \quad d > 2\,. \end{cases} \tag{16}$$

Also, the constant $c(p')$ depends on p', and $c(p') \to \infty$ when $p' \to K_d$. Finally, the constants c and $c(p')$ may depend on the cone property of Ω, but they do not depend on the size of Ω.

Proof. We start our proof following the ideas in [26, Section 1.4]. Introducing the notation $\boldsymbol{y} = \boldsymbol{x}/\varepsilon$, we obtain

$$\begin{aligned} (\mathcal{A}_\varepsilon \nabla u^1_\varepsilon)_i = a_{ij}(\boldsymbol{y})\frac{\partial u^1_\varepsilon}{\partial x_j} &= \left(a_{ij}(\boldsymbol{y}) + a_{ik}(\boldsymbol{y})\frac{\partial \chi^j(\boldsymbol{y})}{\partial y_k}\right)\frac{\partial u_0}{\partial x_j} + \varepsilon\, a_{ij}(\boldsymbol{y})\chi^k(\boldsymbol{y})\frac{\partial^2 u_0}{\partial x_j \partial x_k} \\ &= a^0_{ij}\frac{\partial u_0}{\partial x_j} + g^j_i(\boldsymbol{y})\frac{\partial u_0}{\partial x_j} + \varepsilon\, a_{ij}(\boldsymbol{y})\chi^k(\boldsymbol{y})\frac{\partial^2 u_0}{\partial x_j \partial x_k}, \end{aligned} \tag{17}$$

where $g^j_i(\boldsymbol{y}) = a_{ij}(\boldsymbol{y}) + a_{ik}(\boldsymbol{y})\frac{\partial \chi^j(y)}{\partial y_k} - a^0_{ij}$. We have from (9) that the vector fields are solenoidal, i.e. $\nabla_{\boldsymbol{y}} \cdot \boldsymbol{g}^j = 0$, as their i-th component is g^j_i. Hence, by Theorem 3.4 and Remark 3.11 from [16] there exists $\boldsymbol{\alpha}^j \in W^{1,q}_{\mathrm{per}}(Y)^3$, $\nabla \cdot \boldsymbol{\alpha}^j = 0$ ($\boldsymbol{\alpha}^j \in W^{1,q}_{\mathrm{per}}(Y)$ in the 2D case), such that

$$\boldsymbol{g}^k = \mathrm{curl}_{\boldsymbol{y}}\boldsymbol{\alpha}^k \text{ with } \|\boldsymbol{\alpha}^k\|_{1,q,Y} \le c\,\|\chi^j\|_{1,q,Y}. \tag{18}$$

Equation (17) yields

$$\mathcal{A}_\varepsilon \nabla u^1_\varepsilon - \mathcal{A}_0 \nabla u_0 = \mathrm{curl}_{\boldsymbol{y}}\boldsymbol{\alpha}^k(\boldsymbol{y})\frac{\partial u_0}{\partial x_j} + \varepsilon\left(a_{ij}(\boldsymbol{y})\chi^k(\boldsymbol{y})\frac{\partial^2 u_0}{\partial x_j \partial x_k}\right)\,, \tag{19}$$

where the last term on the right-hand side of (19) is the vector whose i-th component is $a_{ij}(\boldsymbol{y})\chi^k(\boldsymbol{y})\frac{\partial^2 u_0}{\partial x_j \partial x_k}$. Next, we observe that

$$\mathrm{curl}\left(\boldsymbol{\alpha}^k(\boldsymbol{y})\frac{\partial u_0}{\partial x_k}\right) = \boldsymbol{\alpha}^k(\boldsymbol{y}) \times \nabla\frac{\partial u_0}{\partial x_k} + \mathrm{curl}\,\boldsymbol{\alpha}^k(\boldsymbol{y})\frac{\partial u_0}{\partial x_k}\,.$$

Since $\mathrm{curl}_{\boldsymbol{y}} \boldsymbol{\alpha}^k(\boldsymbol{y}) \frac{\partial u_0}{\partial x_j} = \varepsilon \, \mathrm{curl} \, \boldsymbol{\alpha}^k(\boldsymbol{x}/\varepsilon) \frac{\partial u_0}{\partial x_j}$ it holds from (19) that

$$
\begin{aligned}
\mathcal{A}_\varepsilon \nabla u_\varepsilon^1 - \mathcal{A}_0 \nabla u_0 = \varepsilon \text{ curl} \left(\boldsymbol{\alpha}^k(\boldsymbol{x}/\varepsilon) \frac{\partial u_0}{\partial x_k} \right) & \\
- \varepsilon \, \boldsymbol{\alpha}^k \left(\frac{\boldsymbol{x}}{\varepsilon} \right) \times \nabla \frac{\partial u_0}{\partial x_k} + \varepsilon \, \chi^k \left(\frac{\boldsymbol{x}}{\varepsilon} \right) a_{ij} \left(\frac{\boldsymbol{x}}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial x_j \partial x_k} \, . & \qquad (20)
\end{aligned}
$$

Next, from (18) and Sobolev inequalities we obtain

$$
\begin{aligned}
\left\| \varepsilon \, \boldsymbol{\alpha}^k \left(\frac{\boldsymbol{x}}{\varepsilon} \right) \times \nabla \frac{\partial u_0}{\partial x_k} + \varepsilon \, \chi^k \left(\frac{\boldsymbol{x}}{\varepsilon} \right) a_{ij} \left(\frac{\boldsymbol{x}}{\varepsilon} \right) \frac{\partial^2 u_0}{\partial x_j \partial x_k} \right\|_0 & \qquad (21) \\
\leq \varepsilon \, \left(\|\boldsymbol{\alpha}^k\|_{0,\infty,Y} + \|\boldsymbol{\chi}\|_{0,\infty,Y} \right) \|u_0\|_2 \leq c \, \varepsilon \, \|\boldsymbol{\chi}\|_{1,q,Y} \|u_0\|_2 \, , &
\end{aligned}
$$

where the constant c depends on Y.

From the weak formulation of problems (1) and (11) and the compatibility condition (3) we conclude that for any boundary condition and for all $\phi \in V$

$$
\int_\Omega (\mathcal{A}_\varepsilon \nabla u_\varepsilon^1 - \mathcal{A}_0 \nabla u_0) \cdot \nabla \phi \, d\boldsymbol{x} = \int_\Omega (\mathcal{A}_\varepsilon \nabla u_\varepsilon^1 - \mathcal{A}_\varepsilon \nabla u_\varepsilon) \cdot \nabla \phi \, d\boldsymbol{x}. \qquad (22)
$$

We also observe that

$$
\nabla \cdot \text{curl} \left(\boldsymbol{\alpha}^k(\boldsymbol{x}/\varepsilon) \frac{\partial u_0}{\partial x_k} \right) = 0,
$$

and, therefore, from (20) and (21) we arrive at

$$
\left| \int_\Omega \mathcal{A}_\varepsilon \nabla (u_\varepsilon - u_\varepsilon^1) \cdot \nabla \phi \, d\boldsymbol{x} \right| \leq c \, \varepsilon \, \|\boldsymbol{\chi}\|_{1,q,Y} \|u_0\|_2 |\phi|_1 \quad \text{for all } \phi \in V \, . \qquad (23)
$$

Nevertheless, the function $u_\varepsilon - u_\varepsilon^1 \notin V$ and, hence, we cannot choose $\phi = u_\varepsilon - u_\varepsilon^1$ in (23) to conclude the desired result. To overcome this difficulty, we first introduce a cut-off function τ_ε to define a new approximation of u_ε in $H^1(\Omega)$. In the sequel, we use (23) and a triangle inequality to obtain the desired result. Define τ_ε satisfying

$$
\begin{cases}
\|\nabla \tau_\varepsilon\|_\infty \leq \frac{c}{\varepsilon} \, , \\
\tau_\varepsilon \in C_0^\infty(\Omega) \, , \\
\tau_\varepsilon(x) = 1 \text{ if } \mathrm{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon \, .
\end{cases}
$$

Recalling that $\chi_\varepsilon^j(\boldsymbol{x}) := \chi^j(\boldsymbol{x}/\varepsilon)$, we set $\tilde{u}_\varepsilon^1 \in H^1(\Omega)$, the new approximation of u_ε, as follows:

$$
\tilde{u}_\varepsilon^1 := u_0 + \varepsilon \, \tau_\varepsilon \, \chi_\varepsilon^j \frac{\partial u_0}{\partial x_j} \, .
$$

Now, we measure the error between u_ε^1 and $\tilde{u}_\varepsilon^1$ in the H^1 norm. To this end, we define

$$
\Omega_\varepsilon := \{ \boldsymbol{x} \in \Omega \, : \, \mathrm{dist}(\boldsymbol{x}, \partial\Omega) \leq \varepsilon \} \, ,
$$

and observe that

$$
\begin{aligned}
\|u_\varepsilon^1 - \tilde{u}_\varepsilon^1\|_1^2 &= \left\| \varepsilon \, (1 - \tau_\varepsilon) \chi_\varepsilon^j \frac{\partial u_0}{\partial x_j} \right\|_{0,\Omega_\varepsilon}^2 + \left| \varepsilon \, (1 - \tau_\varepsilon) \chi_\varepsilon^j \frac{\partial u_0}{\partial x_j} \right|_{1,\Omega_\varepsilon}^2 \\
&\leq \left\| \varepsilon \, (1 - \tau_\varepsilon) \chi_\varepsilon^j \frac{\partial u_0}{\partial x_j} \right\|_{0,\Omega_\varepsilon}^2 + c \left(\left\| \varepsilon \, (1 - \tau_\varepsilon) \frac{\partial \chi_\varepsilon^j}{\partial x_k} \frac{\partial u_0}{\partial x_j} \right\|_{0,\Omega_\varepsilon}^2 \right. \qquad (24) \\
&\quad \left. + \left\| \varepsilon \frac{\partial \tau^\varepsilon}{\partial x_k} \chi_\varepsilon^j \frac{\partial u_0}{\partial x_j} \right\|_{0,\Omega_\varepsilon}^2 + \left\| \varepsilon \, (1 - \tau_\varepsilon) \, \chi_\varepsilon^j \frac{\partial^2 u_0}{\partial x_k \partial x_j} \right\|_{0,\Omega_\varepsilon}^2 \right) .
\end{aligned}
$$

The first term on the right-hand side of (24) is bounded using the Sobolev embedding theorem as follows:

$$\begin{aligned}\left\|\varepsilon\,(1-\tau_\varepsilon)\chi^j_\varepsilon\frac{\partial u_0}{\partial x_j}\right\|_{0,\Omega_\varepsilon} &\leq \varepsilon\,\|\boldsymbol{\chi}\|_{0,\infty,\Omega_\varepsilon}\left\|\frac{\partial u_0}{\partial x_j}\right\|_{0,\Omega_\varepsilon}\\ &\leq \varepsilon\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_1\,.\end{aligned}$$

To estimate the remaining terms in (24), we use the Y-periodicity of the function χ^j to get

$$\begin{aligned}\left(\int_{\Omega_\varepsilon}\chi^j(\boldsymbol{x}/\varepsilon)^q\;d\boldsymbol{x}\right)^{1/q} &\leq \left(\frac{L_\Omega}{\varepsilon}\int_{\varepsilon Y}\chi^j(\boldsymbol{x}/\varepsilon)^q\;d\boldsymbol{x}\right)^{1/q}\\ &\leq \left(\varepsilon\,L_\Omega\int_Y\chi^j(\boldsymbol{y})^q\;d\boldsymbol{y}\right)^{1/q}.\end{aligned}\tag{25}$$

The second term on the right-hand side of (24) is estimated as

$$\begin{aligned}\left\|\varepsilon\,(1-\tau^\varepsilon)\frac{\partial\chi^j_\varepsilon}{\partial x_k}\frac{\partial u_0}{\partial x_j}\right\|_{0,\Omega_\varepsilon} &\leq \|(1-\tau^\varepsilon)\|_{0,s,\Omega_\varepsilon}\left\|\varepsilon\,\frac{\partial\chi^j_\varepsilon}{\partial x_k}\right\|_{0,q,\Omega_\varepsilon}\left\|\frac{\partial u_0}{\partial x_j}\right\|_{0,p\;,\Omega_\varepsilon}\\ &\leq |\Omega_\varepsilon|^{\frac{1}{s}}(L_\Omega\,\varepsilon)^{\frac{1}{q}}\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_{2,\Omega_\varepsilon}\\ &\leq c(p')(L_\Omega\,\varepsilon)^{1/2-1/p}\;\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_{2,\Omega_\varepsilon},\end{aligned}$$

where

$$\frac{1}{s}+\frac{1}{p'}+\frac{1}{q}=\frac{1}{2}\,.\tag{26}$$

The constant $c(p')$ depends on p' and its dependence on Ω_ε relies only on the cone property of $\partial\Omega_\varepsilon$. As for the third term on the right-hand side of (24), we observe that

$$\begin{aligned}\left\|\varepsilon\frac{\partial\tau^\varepsilon}{\partial x_k}\chi^j_\varepsilon\frac{\partial u_0}{\partial x_j}\right\|_{0,\Omega_\varepsilon} &\leq \left\|\varepsilon\,\frac{\partial\tau^\varepsilon}{\partial x_k}\right\|_{0,s,\Omega_\varepsilon}\|\chi^j_\varepsilon\|_{0,q,\Omega_\varepsilon}\|u_0\|_{1,p\;,\Omega_\varepsilon}\\ &\leq \left\|\varepsilon\,\frac{\partial\tau^\varepsilon}{\partial x_k}\right\|_{0,\infty,\Omega_\varepsilon}|\Omega_\varepsilon|^{\frac{1}{s}}(L_\Omega\varepsilon)^{\frac{1}{q}}\|\boldsymbol{\chi}\|_{0,q,Y}\left\|\frac{\partial u_0}{\partial x_j}\right\|_{0,p\;,\Omega_\varepsilon}\\ &\leq c(p')(L_\Omega\,\varepsilon)^{1/2-1/p}\;\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_2,\end{aligned}\tag{27}$$

where we used (25) and that p' and s satisfy (26). We now estimate the last term on the right-hand side of (24) as

$$\left\|\varepsilon\,\tau^\varepsilon\,\chi^j_\varepsilon\frac{\partial^2 u_0}{\partial x_k\partial x_j}\right\|_{0,\Omega_\varepsilon}\leq\varepsilon\,\|\chi^j_\varepsilon\|_{0,\infty,\Omega_\varepsilon}\|u_0\|_{2,\Omega_\varepsilon}\leq\varepsilon\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_2\,.$$

Finally, gathering previous contributions, we conclude that

$$\|u^1_\varepsilon-\tilde u^1_\varepsilon\|_1\leq\left(c(p')(L_\Omega)^{1/2-1/p}\;+c\right)\varepsilon^{1/2-1/p}\;\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_2\,,\tag{28}$$

where the constant $c(p')$ satisfies (16).

We now estimate $\|u_\varepsilon - \tilde{u}^1_\varepsilon\|_1$. The ellipticity of $\mathcal{A}_\varepsilon$ in (2) and the triangle inequality yield

$$\left|u_\varepsilon - \tilde{u}^1_\varepsilon\right|_1^2 \le c \left|\int_\Omega \mathcal{A}_\varepsilon \nabla(u_\varepsilon - \tilde{u}^1_\varepsilon)\cdot\nabla(u_\varepsilon - u^1_\varepsilon)\,d\boldsymbol{x}\right| + c\left|\int_\Omega \mathcal{A}_\varepsilon \nabla(u_\varepsilon - \tilde{u}^1_\varepsilon)\cdot\nabla(u^1_\varepsilon - \tilde{u}^1_\varepsilon)\,d\boldsymbol{x}\right|,$$

and from (23) and (28) we arrive at

$$\left|u_\varepsilon - \tilde{u}^1_\varepsilon\right|_1 \le (c(p')(L_\Omega)^{1/2-1/p} + c)\,\varepsilon^{1/2-1/p}\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_2\,.$$

Next, from the Poincaré inequality,

$$\left\|u_\varepsilon - \tilde{u}^1_\varepsilon\right\|_1 \le \left[(c(p')(L_\Omega)^{1/2-1/p} + c)(1+L_\Omega)\right]\,\varepsilon^{1/2-1/p}\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_2 \tag{29}$$

holds, and we obtain (15) from the triangle inequality and from equations (28) and (29). □

The next theorem assumes less regularity on χ^j, and more on u_0.

Theorem 2. *Let u_ε be the solution of* (1)*, and let u_0 and u^1_ε be defined by* (11)-(13)*. Assume* (5) *holds and*

$$u_0 \in W^{2,p}(\Omega),\ \chi^j \in H^1_{\mathrm{per}}(Y),\ \text{with } p > d. \tag{30}$$

Then,

$$\|u_\varepsilon - u^1_\varepsilon\|_1 \le c\,\varepsilon^{1/2}\|\boldsymbol{\chi}\|_{1,Y}\|u_0\|_{2,p}\,. \tag{31}$$

Proof. This result arises following closely the proof of Theorem 1 with straightforward modifications. For instance, estimate (27) now becomes

$$\left\|\varepsilon\,\frac{\partial\tau^\varepsilon}{\partial x_k}\,\chi^j_\varepsilon\,\frac{\partial u_0}{\partial x_j}\right\|_{0,\Omega_\varepsilon} \le \|\chi^j_\varepsilon\|_{0,\Omega_\varepsilon}\left\|\varepsilon\,\frac{\partial\tau^\varepsilon}{\partial x_k}\,\frac{\partial u_0}{\partial x_j}\right\|_{0,\infty,\Omega_\varepsilon} \le (L_\Omega\,\varepsilon)^{\frac{1}{2}}\,\|\boldsymbol{\chi}\|_{0,Y}\|u_0\|_{2,p},$$

where we used (25). □

The next theorem estimates the error between u_ε and u_0 in the L^2 norm.

Theorem 3. *Let u_ε be the solution of* (1)*, and let u_0 and u^ε_1 be defined by* (11)-(13)*. Assume* (5) *holds, $u_0 \in W^{2,p}(\Omega)$ and $\chi^j \in W^{1,q}_{\mathrm{per}}(Y)$, with $p, q > d$. Then,*

$$\|u_\varepsilon - u_0\|_0 \le c\,\varepsilon\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_{2,p}\,. \tag{32}$$

Proof. We introduce the following boundary corrector term $\theta_\varepsilon \in H^1(\Omega)$ as the solution of

$$-\nabla\cdot(\mathcal{A}_\varepsilon\nabla\theta_\varepsilon) = 0 \text{ in } \Omega, \qquad \theta_\varepsilon = -u_1(\boldsymbol{x},\boldsymbol{x}/\varepsilon) \text{ on } \partial\Omega, \tag{33}$$

and observe that $u_0 + \varepsilon\,u_1 + \varepsilon\,\theta_\varepsilon \in H^1(\Omega)$ satisfies the Dirichlet boundary condition in (1). From the definition of θ_ε it holds for all $\phi \in H^1_0(\Omega)$ that

$$\int_\Omega \left(\mathcal{A}_\varepsilon\nabla(u^1_\varepsilon + \varepsilon\,\theta_\varepsilon) - \mathcal{A}_\varepsilon\nabla u_\varepsilon\right)\cdot\nabla\phi\,d\boldsymbol{x} = \int_\Omega (\mathcal{A}_\varepsilon\nabla u^1_\varepsilon - \mathcal{A}_\varepsilon\nabla u_\varepsilon)\cdot\nabla\phi\,d\boldsymbol{x}, \tag{34}$$

where we used notation (13). Hence, from (23) we obtain

$$\left|\int_\Omega \left(\mathcal{A}_\varepsilon\nabla(u^1_\varepsilon + \varepsilon\,\theta_\varepsilon) - \mathcal{A}_\varepsilon\nabla u_\varepsilon\right)\cdot\nabla\phi\,d\boldsymbol{x}\right| \le c\,\varepsilon\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_{2,p}|\phi|_1\,, \tag{35}$$

for all $\phi \in H_0^1(\Omega)$. Now, we take $\phi = u_\varepsilon - u_\varepsilon^1 - \varepsilon\,\theta_\varepsilon$ in (35) to conclude that

$$|u_\varepsilon - u_\varepsilon^1 - \varepsilon\,\theta_\varepsilon|_1 \le c\,\varepsilon\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_{2,p}\,, \tag{36}$$

and the Poincaré inequality yields

$$\|u_\varepsilon - u_\varepsilon^1 - \varepsilon\,\theta_\varepsilon\|_0 \le c\,\varepsilon\,\|\boldsymbol{\chi}\|_{1,q,Y}\|u_0\|_{2,p}\,.$$

Next, the maximum principle guarantees that

$$\|\theta_\varepsilon\|_0 \le c\,(L_\Omega)^d\,\|\boldsymbol{\chi}\|_{0,\infty}|u_0|_{1,\infty} \le c\,(L_\Omega)^d\,\|\boldsymbol{\chi}\|_{1,p}\|u_0\|_{2,p}\,,$$

and the desired result follows observing that

$$\left\|\varepsilon\,\chi_\varepsilon^j\frac{\partial u_0}{\partial x_j}\right\|_0 \le c\,\varepsilon\,\|\boldsymbol{\chi}\|_{0,q,Y}\|u_0\|_{1,p}. \qquad \square$$

3. Convergence analysis of a multiscale method

In this section, we analyze the convergence of the MHM method proposed in [20]. The numerical approximation of u_ε relies on a decomposition of u_ε as a result of the hybridization technique proposed in [33]. For the sake of clarity, we summarize next the main points of the MHM methodology assuming that $b = 0$ in (1) for simplicity (see [20] for further details).

Hereafter, we assume Ω is a polygonal domain and $\{\mathcal{T}_h\}_{h>0}$ is a family of regular triangulations of Ω composed of elements K with boundary ∂K. We denote by $\mathcal{E}^h$ the set of all faces F of elements $K \in \mathcal{T}_h$. For each $F \in \mathcal{E}^h$, we associate a normal $\boldsymbol{n}$, taking care to ensure this is facing outward on $\partial\Omega$, and we introduce the space

$$\Lambda := \{\boldsymbol{\sigma}\cdot\boldsymbol{n}_K\,|_{\partial K}\ :\ \boldsymbol{\sigma} \in H(\mathrm{div};\Omega), \quad \forall K \in \mathcal{T}_h\}\ , \tag{37}$$

where $\boldsymbol{n}_K$ denotes the outward normal vector to ∂K. We equip it with the following norm:

$$\|\mu\|_\Lambda := \inf_{\left\{\substack{\boldsymbol{\sigma}\,\in\, H(\mathrm{div};\,\Omega);\\ \boldsymbol{\sigma}\,\cdot\,\boldsymbol{n}_K|_{\partial K}\,=\,\mu,\ K\,\in\,\mathcal{T}_h}\right\}} \|\boldsymbol{\sigma}\|_{\mathrm{div}}\,. \tag{38}$$

We replace the weak problem (6) by the following one: *Find* $(u_\varepsilon, \lambda_\varepsilon) \in H^1(\mathcal{T}_h) \times \Lambda$ *such that*

$$\begin{aligned} (\mathcal{A}_\varepsilon \nabla u_\varepsilon,\, \nabla v)_{\mathcal{T}_h} + (\lambda_\varepsilon, v)_{\partial\mathcal{T}_h} &= (f,\, v)_{\mathcal{T}_h} \quad \text{for all } v \in H^1(\mathcal{T}_h)\,,\\ (\mu, u_\varepsilon)_{\partial\mathcal{T}_h} &= (\mu, g)_{\partial\Omega_D} \quad \text{for all } \mu \in \Lambda\,, \end{aligned} \tag{39}$$

and we note that the Neumann boundary condition is prescribed as an essential condition, i.e., all μ in Λ vanish on $\partial\Omega_N$. Here the inner (duality) products are given by

$$(\phi,\psi)_{\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\int_K \phi\,\psi\;d\boldsymbol{x} \quad \text{and} \quad (\phi,\psi)_{\partial\mathcal{T}_h} := \sum_{K\in\mathcal{T}_h}\langle\phi,\psi\rangle_{\partial K}\,,$$

where $\langle\phi,\psi\rangle_{\partial K}$ stands for the duality product between the spaces $H^{-1/2}(\partial K)$ and $H^{1/2}(\partial K)$. We recognize problem (39) as the standard hybrid formulation of (6) from which the primal hybrid methods arise [33]. Problem (39) is shown to be well-posed with $u_\varepsilon \in H^1(\Omega)$ also being the solution to (6) and $\lambda_\varepsilon = -\mathcal{A}_\varepsilon\nabla u_\varepsilon\,\cdot\boldsymbol{n}_K\,|_{\partial K}$ for all $K \in \mathcal{T}_h$.

We now characterize the solution of (39) as a collection of solutions of local problems which are pieced together using solutions to a global problem. To this end, we introduce the decomposition

$$H^1(\mathcal{T}_h) \,=\, V_0 \,\oplus\, V_0^{\perp}\,,$$

where V_0 is defined by

$$V_0 := \{v \in L^2(\Omega) \,:\, v\,|_K \text{ is constant on } K \in \mathcal{T}_h\}\,.$$

Notice that the orthogonal complement $V_0^{\perp}$ in $H^1(\mathcal{T}_h)$ corresponds to $V_0^{\perp} \equiv L_0^2(\mathcal{T}_h) \cap H^1(\mathcal{T}_h)$, where $L_0^2(\mathcal{T}_h)$ is the space of functions belonging to $L^2(\Omega)$ with mean value equal to zero in each $K \in \mathcal{T}_h$. Thereby, the exact solution $u_\varepsilon \in H^1(\mathcal{T}_h)$ of (39) admits the expansion

$$u_\varepsilon \,=\, u_\varepsilon^0 + u_\varepsilon^{\perp}\,, \tag{40}$$

in terms of a unique $u_\varepsilon^0 \in V_0$ and $u_\varepsilon^{\perp} := u_\varepsilon - u_\varepsilon^0 \in V_0^{\perp}$.

Next, we observe that problem (39) is equivalent to: *Find* $(u_\varepsilon^0 + u_\varepsilon^{\perp}, \lambda_\varepsilon) \in (V_0 \oplus V_0^{\perp}) \times \Lambda$ *such that*

$$\begin{cases} (\lambda_\varepsilon, v^0)_{\partial\mathcal{T}_h} = (f,\, v^0)_{\mathcal{T}_h} & \text{for all } v^0 \in V_0\,, \\ (\mu, u_\varepsilon^0 + u_\varepsilon^{\perp})_{\partial\mathcal{T}_h} = (\mu, g)_{\partial\Omega_D} & \text{for all } \mu \in \Lambda\,, \end{cases} \tag{41}$$

$$(\mathcal{A}_\varepsilon \nabla u_\varepsilon^{\perp},\, \nabla v^{\perp})_{\mathcal{T}_h} = -(\lambda_\varepsilon, v^{\perp})_{\partial\mathcal{T}_h} + (f,\, v^{\perp})_{\mathcal{T}_h} \quad \text{for all } v^{\perp} \in V_0^{\perp}\,. \tag{42}$$

Thereby, a portion of the solution to problem (39) may be computed locally from λ_ε and f. Indeed, from (42) the component $u_\varepsilon^{\perp}$ of the exact solution can be expanded as

$$u_\varepsilon^{\perp} = T_\varepsilon\, \lambda_\varepsilon + \hat{T}_\varepsilon\, f\,, \tag{43}$$

where $T_\varepsilon : \Lambda \to V_0^{\perp}$ and $\hat{T}_\varepsilon : L^2(\Omega) \to V_0^{\perp}$ are linear bounded operators. They are well defined locally on each $K \in \mathcal{T}_h$ through the (unique) weak solutions of

$$-\nabla \cdot (\mathcal{A}_\varepsilon \nabla T_\varepsilon\, \mu) \,=\, c_K^{\mu} \quad \text{in } K, \quad -\mathcal{A}_\varepsilon \nabla T_\varepsilon\, \mu \cdot \boldsymbol{n}_K \,=\, \mu \quad \text{on } F \subset \partial K\,, \tag{44}$$

and

$$-\nabla \cdot \left(\mathcal{A}_\varepsilon \nabla \hat{T}_\varepsilon\, q\right) \,=\, q - \bar{q}_K \quad \text{in } K, \quad \mathcal{A}_\varepsilon \nabla \hat{T}_\varepsilon\, q \cdot \boldsymbol{n}_K \,=\, 0 \quad \text{on } F \subset \partial K\,, \tag{45}$$

where

$$\bar{q}_K := \frac{1}{|K|}\int_K q\, d\boldsymbol{x} \quad \text{and} \quad c_K^{\mu} := \frac{1}{|K|}\int_{\partial K} \mu\, d\boldsymbol{s}\,. \tag{46}$$

Decomposition (43) provides us a way to eliminate the portion of the solution $u_\varepsilon^{\perp}$ in terms of λ_ε and f. As such, we complete the characterization of the exact solution u_ε by replacing (43) in (41) and solving the resulting global problem: *Find* $(u_\varepsilon^0, \lambda_\varepsilon) \in V_0 \times \Lambda$ *such that*

$$\begin{cases} (\lambda_\varepsilon, v^0)_{\partial\mathcal{T}_h} = (f, v^0)_{\mathcal{T}_h} & \text{for all } v^0 \in V_0\,, \\ (\mu, u_\varepsilon^0 + T_\varepsilon\, \lambda_\varepsilon)_{\partial\mathcal{T}_h} = -(\mu, \hat{T}_\varepsilon\, f)_{\partial\mathcal{T}_h} + (\mu, g)_{\partial\Omega_D} & \text{for all } \mu \in \Lambda\,. \end{cases} \tag{47}$$

Owing to the previous definitions, we establish from (40) and (43) that the exact solution u_ε of (39) (e.g. (6)) can be characterized as follows:

$$u_\varepsilon \,=\, u_\varepsilon^0 + T_\varepsilon \lambda_\varepsilon + \hat{T}_\varepsilon f\,. \tag{48}$$

We use the equivalence between the local-global coupled system (44)-(45) and (47) and the original problem (6) (in the sense that (48) also satisfies (6)) to build the numerical method for (44)-(45) and (47). We start selecting Λ^h as a continuous polynomial subspace of Λ which embeds the space of piecewise constant functions on $\mathcal{E}^h$. The underlying MHM method corresponds to the standard Galerkin method over the space Λ^h: find $(u_\varepsilon^{0,h}, \lambda_\varepsilon^h) \in V_0 \times \Lambda^h$ satisfying

$$(49)\qquad \begin{cases} (\lambda_\varepsilon^h, v^0)_{\partial\mathcal{T}_h} \;=\; (f, v^0)_{\mathcal{T}_h} \quad \text{for all } v^0 \in V_0\,, \\ (\mu^h, u_\varepsilon^{0,h} + T_\varepsilon\lambda_\varepsilon^h)_{\partial\mathcal{T}_h} \;=\; -(\mu^h, \hat{T}_\varepsilon f)_{\partial\mathcal{T}_h} + (\mu^h, g)_{\partial\Omega_D} \quad \text{for all } \mu^h \in \Lambda^h\,, \end{cases}$$

where $T_\varepsilon\lambda_\varepsilon^h$ and $\hat{T}_\varepsilon f$ solve (44) and (45) with $\mu = \lambda_\varepsilon^h$ and $q = f$, respectively. The exact solution u_ε is then approximated as follows:

$$(50)\qquad u_\varepsilon \approx u_\varepsilon^h \;:=\; u_\varepsilon^{0,h} + T_\varepsilon\lambda_\varepsilon^h + \hat{T}_\varepsilon f\,.$$

The assumption that Λ^h includes the space of piecewise constant functions on $\mathcal{E}^h$ yields the well-posedness of the MHM method (49) (see [20, Theorem 3.2]).

Remark 4. Discretization decouples the local problems (44)-(45) from the global one (49). Thereby, a staggered algorithm can be adopted to solve the system. To see this more clearly, it is instructive to consider $T\,\lambda_\varepsilon^h$ in more detail. Suppose $\{\psi_i\}_{i=1}^{\dim\Lambda^h}$ is a basis for Λ^h, and define the set $\{\eta_i\}_{i=1}^{\dim\Lambda^h} \subset V_0^\perp$ such that

$$(51)\qquad -\nabla\cdot(\mathcal{A}_\varepsilon\nabla T_\varepsilon\,\eta_i) \;=\; c_K^{\psi_i} \quad \text{in } K, \quad -\mathcal{A}_\varepsilon\nabla T_\varepsilon\,\eta_i\cdot\boldsymbol{n}_K \;=\; \psi_i \quad \text{on } F\subset\partial K\,,$$

i.e., $\eta_i = T_\varepsilon\,\psi_i$, where ψ_i changes its sign in (51) according to the sign of $\boldsymbol{n}\cdot\boldsymbol{n}_K\,|_F$. Now, given $\lambda_\varepsilon^h \;=\; \sum_{i=1}^{\dim\Lambda^h} c_i\,\psi_i$ in Λ^h, $c_i \in \mathbb{R}$, the linearity of problem (44) implies we may uniquely write

$$T_\varepsilon\,\lambda^h \;=\; \sum_{i=1}^{\dim\Lambda^h} c_i\,T_\varepsilon\,\psi_i \;=\; \sum_{i=1}^{\dim\Lambda^h} c_i\,\eta_i\,.$$

Therefore, the degrees of freedom c_i of λ_ε^h are "inherited" by $T_\varepsilon\,\lambda_\varepsilon^h$. It then follows from (50) that

$$(52)\qquad u_\varepsilon^h = u_\varepsilon^{0,h} + \sum_{i=1}^{\dim\Lambda^h} c_i\,\eta_i + \hat{T}_\varepsilon\,f\,.$$

As a result, the global formulation (49) is responsible for computing the degrees of freedom of $u_\varepsilon^{0,h}$ (one per element) and the c_i's in (52), once the multiscale basis functions η_i and $\hat{T}_\varepsilon f$ are available from the local problems. Also, it is interesting to note that heterogeneous and/or high-contrast aspects of the media automatically impact the design of the basis functions η_i as well as $\hat{T}_\varepsilon\,f$ as they are driven by (51) and (45), respectively.

Observe that $u_\varepsilon^h \notin H^1(\Omega)$, i.e., the MHM method is non-conforming with respect to $H^1(\Omega)$. Also, it is interesting to note that built within the approach is an approximation of the dual variable $\boldsymbol{\sigma}_\varepsilon := -\mathcal{A}_\varepsilon\nabla u_\varepsilon$ through the formula

$$(53)\qquad \boldsymbol{\sigma}_\varepsilon^h \;:=\; -\mathcal{A}_\varepsilon\nabla\left(T_\varepsilon\lambda_\varepsilon^h + \hat{T}_\varepsilon f\right) \in H(\mathrm{div};\Omega)\,.$$

As a result, the MHM method is an $H(\mathrm{div};\Omega)$ conforming approach. Also, it is worth mentioning that, from (47) and (49), λ_ε and λ_ε^h satisfy the compatibility condition

$$\frac{1}{|K|}\int_{\partial K} \lambda_\varepsilon \, d\boldsymbol{s} = \bar{f}_K = \frac{1}{|K|}\int_{\partial K} \lambda_\varepsilon^h \, d\boldsymbol{s}\,, \tag{54}$$

which ensures the consistency of the coupled global-local formulation (44)-(45) and (47) (or of the MHM method (49)).

In summary, the staggered algorithm for computing an approximation to u_ε and $\boldsymbol{\sigma}_\varepsilon$ is:

(i) compute $\hat{T}_\varepsilon f$ from (45) and the multiscale basis $\{\eta_i\}_{i=1}^{\dim \Lambda^h}$ from (51) as a local highly parallelizable preprocessing step;
(ii) compute the degrees of freedom of $u_\varepsilon^{0,h}$ and λ_ε^h from (49), noting that $T\,\lambda_\varepsilon^h$ expands in terms of $\{\eta_i\}_{i=1}^{\dim \Lambda_\varepsilon^h}$ using the degrees of freedom for λ_ε^h;
(iii) build the approximate solution u_ε^h from (52) and $\boldsymbol{\sigma}_\varepsilon^h$ from (53).

We now estimate the error $|u_\varepsilon - u_\varepsilon^h|_{1,h}$, which from (48) and (50) corresponds to measure $|T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h|_{1,h}$. First, notice that $|T_\varepsilon\,\mu|_{1,h}$ actually defines a norm on

$$\left\{\mu \in \Lambda \,:\, (\mu, v^0)_{\partial\mathcal{T}_h} = 0\,, \quad \forall v^0 \in V_0\right\}\,. \tag{55}$$

Indeed, from the definitions of $\|\cdot\|_\Lambda$ in (38) and the operator T_ε in (44), ellipticity condition (2), and the fact that $\nabla\cdot(\mathcal{A}_\varepsilon \nabla T_\varepsilon\,\mu) = 0$ in each $K \in \mathcal{T}_h$, we get

$$\|\mu\|_\Lambda \le c\,|T_\varepsilon\,\mu|_{1,h}\,. \tag{56}$$

To measure $|u_\varepsilon - u_\varepsilon^h|_{1,h}$ we first remark that u_0, the solution of the homogenized problem (11), can also be decomposed in global-local problems as

$$u_0 = u_0^0 + T_0\,\lambda_0 + \hat{T}_0\,f\,, \tag{57}$$

where $T_0 : \Lambda \to V_0^\perp$ and $\hat{T}_0 : L^2(\Omega) \to V_0^\perp$ are linear bounded operators defined such that $T_0\,\mu$ and $\hat{T}_0\,q$ restricted to each K satisfy, respectively,

$$-\nabla\cdot(\mathcal{A}_0\nabla T_0\,\mu) \;=\; c_K^\mu \text{ in } K, \quad -\mathcal{A}_0\nabla T_0\,\mu\cdot\boldsymbol{n}_K \;=\; \mu \text{ on } F\subset\partial K\,, \tag{58}$$

and

$$-\nabla\cdot\left(\mathcal{A}_0\nabla\hat{T}_0\,q\right) \;=\; q - \bar{q}_K \text{ in } K, \quad \mathcal{A}_0\nabla\hat{T}_0\,q\cdot\boldsymbol{n}_K \;=\; 0 \text{ on } F\subset\partial K\,. \tag{59}$$

These local problems are brought together by searching for $(u_0^0, \lambda_0) \in V_0 \times \Lambda$ as the solution of

$$\begin{cases} (\lambda_0, v^0)_{\partial\mathcal{T}_h} \;=\; (f, v^0)_{\mathcal{T}_h} \quad \text{for all } v^0 \in V_0\,, \\ (\mu, u_0^0 + T_0\,\lambda_0)_{\partial\mathcal{T}_h} \;=\; -(\mu, \hat{T}_0\,f)_{\partial\mathcal{T}_h} + (\mu, g)_{\partial\Omega_D} \quad \text{for all } \mu\in\Lambda\,. \end{cases} \tag{60}$$

Observe that the first equation in the system above ensures the consistency of the coupled global-local formulation (58)-(60).

We now set up the asymptotic regime in which the forthcoming convergence results will be proved. Let d_K be the side of the largest square (or cube in 3D) contained in $K \in \mathcal{T}_h$. Hereafter, we shall assume that

$$\inf_{K\in\mathcal{T}_h} d_K > c\,\varepsilon\,, \tag{61}$$

where $c > 1$ is of order one. Observe that such a regime is in accordance with practical standpoints. The following lemma is central to prove the convergence of the MHM method.

Lemma 5. *Assume* (61) *holds, and* $u_0 \in W^{2,p}(\Omega)$ *and* $\chi^j \in W^{1,q}_{per}(Y)$, *with* p *and* q *satisfying either* (14) *or* (30). *Also, let* $\hat{T}_0 f$ *be defined by* (59), *and assume* $\hat{T}_0 f\,|_K \in W^{2,p}(K)$ *for all* $K \in \mathcal{T}_h$. *Then,*

$$\left\| T_\varepsilon \lambda_\varepsilon - T_0\lambda_0 - \varepsilon\, \chi^j_\varepsilon \frac{\partial T_0 \lambda_0}{\partial x_j} \right\|_{1,h} \le c(p')\, \varepsilon^{1/2-1/p}\, \|\boldsymbol{\chi}\|_{1,q,Y} \big(\|u_0\|_{2,p} + \|\hat{T}_0 f\|_{2,p,h}\big),$$

where

$$\begin{cases} 2 < p' < K_d \text{ and } K_d \text{ satisfies } (16) & \text{if } (14) \text{ holds}, \\ p' = \infty & \text{if } (30) \text{ holds}, \end{cases} \tag{62}$$

and the constant $c(p')$ *is such that* $c(p') \to \infty$ *when* $p' \to K_d$.

Proof. We consider only the case when (14) holds, since the other case is proved in a similar manner. From Theorem 1 applied to u_ε it holds that

$$\left\| u_\varepsilon - u_0 - \varepsilon\, \chi^j_\varepsilon \frac{\partial u_0}{\partial x_j} \right\|_1 \le c(p')\, \varepsilon^{1/2-1/p}\, \|\boldsymbol{\chi}\|_{1,q,Y} \|u_0\|_{2,p}. \tag{63}$$

Also, observing that the result in Theorem 1 does not depend on the diameter of Ω, and using the definition of the operator $\hat{T}_\varepsilon$ in (45), it holds from Theorem 1 (with Ω replaced by K and u_ε by $\hat{T}_\varepsilon\, f$) that $\hat{T}_\varepsilon\, f\,|_K$ satisfies

$$\left\| \hat{T}_\varepsilon\, f - \hat{T}_0\, f - \varepsilon\, \chi^j_\varepsilon \frac{\partial \hat{T}_0\, f}{\partial x_j} \right\|_{1,h} \le c(p')\, \varepsilon^{1/2-1/p}\, \|\boldsymbol{\chi}\|_{1,q,Y} \|\hat{T}_0\, f\|_{2,p,h}. \tag{64}$$

Next, we use the characterization of u_0 and u_ε given in (57) and (48), respectively, and the triangle inequality to arrive at

$$\begin{aligned} &\left\| T_\varepsilon \lambda_\varepsilon - T_0\lambda_0 - \varepsilon\, \chi^j_\varepsilon \frac{\partial T_0 \lambda_0}{\partial x_j} \right\|_{1,h} \\ &\quad = \left\| u_\varepsilon - \hat{T}_\varepsilon\, f - u_0 + \hat{T}_0\, f - \varepsilon\, \chi^j_\varepsilon \frac{\partial T_0 \lambda_0}{\partial x_j} \right\|_{1,h} \\ &\quad = \left\| u_\varepsilon - u_0 - \varepsilon\, \chi^j_\varepsilon \frac{\partial u_0}{\partial x_j} - \hat{T}_\varepsilon\, f + \hat{T}_0\, f + \varepsilon\, \chi^j_\varepsilon \frac{\partial \hat{T}_0\, f}{\partial x_j} \right\|_{1,h} \\ &\quad \le \left\| u_\varepsilon - u_0 - \varepsilon\, \chi^j_\varepsilon \frac{\partial u_0}{\partial x_j} \right\|_{1,h} + \left\| \hat{T}_\varepsilon\, f + \hat{T}_0\, f + \varepsilon\, \chi^j_\varepsilon \frac{\partial \hat{T}_0 f}{\partial x_j} \right\|_{1,h} \\ &\quad \le c(p')\, \varepsilon^{1/2-1/p}\, \|\boldsymbol{\chi}\|_{1,q,Y} \left(\|u_0\|_{2,p} + \|T_0 f\|_{2,p,h}\right) \end{aligned}$$

and the result follows. □

Remark 6. The previous lemma assumed that $\hat{T}_0 f|_K \in W^{2,p}(K)$ for all $K \in \mathcal{T}_h$, since the estimates depend on $\|\hat{T}_0 f\|_{2,p,h}$. When $p = 2$, $d = 2$ and $f \in L^2(\Omega)$, the regularity theory for elliptic equations and the assumption $\{\mathcal{T}_h\}_{h>0}$ is regular ensure that (see [18])

$$\hat{T}_0 f|_K \in H^2(K) \text{ and } \|\hat{T}_0 f\|_{2,K} \le c\, \|f\|_{0,K}, \tag{65}$$

where the constant c is independent of K. To guarantee that such an estimate holds in the case $d = 3$, we assume that the elements of $\mathcal{T}_h$ are affine transformations of a finite set of reference elements. Also, in the case that $p > 2$, $d = 2$ or $d = 3$, we may not infer that $\hat{T}_0 f|_K \in W^{2,p}(K)$ if $f \in L^p(\Omega)$ due to the polygonal boundary of K, and then, (65) cannot be used. Moreover, there exist conditions on $K \in \mathcal{T}_h$ such that if $\hat{T}_0 f|_K \in W^{2,p}(K)$, then $\|\hat{T}_0 f\|_{2,p,h} \leq c\,\|f\|_p$ when $p < \infty$ and $d = 2$ (see [18, Theorem 4.3.2.4]). In order to avoid unnecessary technicalities, we shall prove the next results under the condition $p = 2$.

Next, we prove a best approximation result. To this end, we introduce a subset of the discrete space Λ^h which embeds the condition (54); more specifically,

$$\Lambda^h_* := \left\{ \mu \in \Lambda^h \;:\; \int_{\partial K} \mu \, d\boldsymbol{s} = \int_K f \, d\boldsymbol{x}, \quad \forall K \in \mathcal{T}_h \right\}.$$

Lemma 7. *Let u_ε be the exact solution of* (1) *and let u_ε^h be the approximate solution given by* (50). *Assume $u_0 \in H^2(\Omega)$ and $\chi^j \in W^{1,q}_{\text{per}}(Y)$, with $q > d$, $f \in L^2(\Omega)$, and* (61) *and* (65) *hold. Then,*

$$\begin{aligned} |u_\varepsilon - u_\varepsilon^h|_{1,h} \leq\; & c(p')\, \varepsilon^{1/2-1/p}\, \|\boldsymbol{\chi}\|_{1,q,Y} \left(\|u_0\|_2 + \|f\|_0\right) \\ & + \inf_{\mu^h \in \Lambda^h} \Bigg[c(p')\, \varepsilon^{1/2-1/p}\, \|\boldsymbol{\chi}\|_{1,q,Y} \|T_0 \mu^h\|_{2,h} \\ & + \left| T_0\lambda_0 + \varepsilon \chi^j_\varepsilon \frac{\partial T_0 \lambda_0}{\partial x_j} - T_0 \mu^h - \varepsilon \chi^j_\varepsilon \frac{\partial T_0 \mu^h}{\partial x_j} \right|_{1,h} \Bigg], \end{aligned} \tag{66}$$

where we can choose $p' > 2$ satisfying (62), *and the constant $c(p')$ depends on p' ($c(p') \to \infty$ when $p' \to K_d$).*

Proof. We recall from [4, Lemma 3.5] that

$$\begin{aligned} |u_\varepsilon - u_\varepsilon^h|_{1,h} &= |T_\varepsilon \lambda_\varepsilon - T_\varepsilon \lambda_\varepsilon^h|_{1,h} \\ &\leq c \inf_{\mu^h \in \Lambda^h} \|\lambda_\varepsilon - \mu^h\|_\Lambda. \end{aligned} \tag{67}$$

Next, we restrict the infimum in (67) to Λ^h_* and use that $\lambda_\varepsilon - \mu^h$ belongs to the space defined by (55) to get

$$\begin{aligned} |u_\varepsilon - u_\varepsilon^h|_{1,h} &\leq c \inf_{\mu^h \in \Lambda^h} \|\lambda_\varepsilon - \mu^h\|_\Lambda \\ &\leq c \inf_{\mu^h \in \Lambda^h} \|\lambda_\varepsilon - \mu^h\|_\Lambda \\ &\leq c \inf_{\mu^h \in \Lambda^h} |T_\varepsilon \lambda_\varepsilon - T_\varepsilon \mu^h|_{1,h} \\ &\leq c \inf_{\mu^h \in \Lambda^h} \left[|T_\varepsilon \lambda_\varepsilon - T_\varepsilon \lambda_0|_{1,h} + |T_\varepsilon \lambda_0 - T_\varepsilon \mu^h|_{1,h} \right], \end{aligned} \tag{68}$$

where we used (56) and the triangle inequality. The second term on the right-hand side of (68) is also bounded using the triangle inequality as follows:

$$\begin{aligned} |T_\varepsilon \lambda_0 - T_\varepsilon \mu^h|_{1,h} \leq\; & \left| T_\varepsilon \lambda_0 - T_0 \lambda_0 - \varepsilon\, \chi^j_\varepsilon \frac{\partial T_0 \lambda_0}{\partial x_j} \right|_{1,h} \\ & + \left| T_\varepsilon \mu^h - T_0 \mu^h - \varepsilon\, \chi^j_\varepsilon \frac{\partial T_0 \mu^h}{\partial x_j} \right|_{1,h} \\ & + \left| T_0 \lambda_0 + \varepsilon\, \chi^j \partial_j T_0 \lambda_0 - T_0 \mu^h - \varepsilon\, \chi^j_\varepsilon \frac{\partial T_0 \mu^h}{\partial x_j} \right|_{1,h}. \end{aligned} \tag{69}$$

We observe that $T_0\mu^h\,|_K \in H^2(K)$ from a standard regularity argument [18] and characterization (57) and the regularity assumptions on u_0 and on f yield $T_0\lambda_0\,|_K \in H^2(K)$. Hence, from the definition of the operator T_ε (see (44)) and the fact that Theorem 1 does not depend on the size of Ω, we can also use this result applied to $T_\varepsilon\lambda_0\,|_K$ (i.e. replacing Ω by K and u_ϵ by $T_\varepsilon\lambda_0$) to estimate the first term on the right-hand side of (69). The second term on the right-hand side of (69) is also estimated through Theorem 1 applied to $T_\varepsilon\mu^h$. Thus, the term $|T_\varepsilon\lambda_0 - T_\varepsilon\mu^h|_{1,h}$ in (68) is bounded as desired. Similarly, from the triangle inequality the term $|T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_0|_{1,h}$ in (68) is bounded as follows:

(70)
$$|T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_0|_{1,h} \le \left|T_\varepsilon\lambda_\varepsilon - T_0\lambda_0 - \varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\lambda_0}{\partial x_j}\right|_{1,h} + \left|T_0\lambda_0 + \varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\lambda_0}{\partial x_j} - T_\varepsilon\lambda_0\right|_{1,h}.$$

Next, Lemma 5 provides an estimate for the first term in the right-hand side of (70), while the second one is estimated using Theorem 1 as $T_0\lambda_0|_K \in H^2(K)$. The final result follows from summing up all contributions. □

We now choose μ^h in (66) with approximation properties to estimate $|u_\varepsilon - u^h_\varepsilon|_{1,h}$ with respect to h. To this end, we assume $\chi^j \in W^{1,\infty}_{\rm per}(Y)$ and apply Theorem 1.

Theorem 8. *Let u_ε be the solution of* (1) *and let u^h_ε be its numerical approximation defined by* (50)*. Also, let $\boldsymbol{\sigma}_\varepsilon := -\mathcal{A}_\varepsilon\nabla u_\varepsilon$ be the post-processed exact dual variable and let $\boldsymbol{\sigma}^h_\varepsilon$ be its approximation given by* (53)*. Assume $u_0 \in H^2(\Omega)$, $\chi^j \in W^{1,\infty}_{\rm per}(Y)$, $f \in L^2(\Omega)$, and* (61) *and* (65) *hold. Then,*

$$|u_\varepsilon - u^h_\varepsilon|_{1,h} \le c_1(p')\left(\varepsilon^{1/2-1/p} + h\right)\|\boldsymbol{\chi}\|_{1,\infty,Y}\left(\|u_0\|_2 + \|f\|_0\right), \tag{71}$$

$$\|\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}^h_\varepsilon\|_{\rm div} \le c_2(p')\left(\varepsilon^{1/2-1/p} + h\right)\|\boldsymbol{\chi}\|_{1,\infty,Y}\left(\|u_0\|_2 + \|f\|_0\right), \tag{72}$$

where p' satisfies (16)*.*

Proof. We first observe that

(73)
$$\left|T_0\lambda_0 + \varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\lambda_0}{\partial x_j} - T_0\mu^h - \varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\mu^h}{\partial x_j}\right|_{1,h} \le \left|T_0\lambda_0 - T_0\mu^h\right|_{1,h} + \left|\varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\lambda_0}{\partial x_j} - \varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\mu^h}{\partial x_j}\right|_{1,h}$$

and

$$\left|\varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\lambda_0}{\partial x_j} - \varepsilon\,\chi^j_\varepsilon \frac{\partial T_0\mu^h}{\partial x_j}\right|_{1,h} \le \varepsilon\,\|\boldsymbol{\chi}\|_{0,\infty,Y}\|T_0\lambda_0 - T_0\mu^h\|_{2,h} + \|\boldsymbol{\chi}\|_{1,\infty,Y}\|T_0\lambda_0 - T_0\mu^h\|_{1,h}\,. \tag{74}$$

Noting that $T_0\lambda^h$ and $\hat{T}_0 f$ belong to $H^2(\mathcal{T}_h)$ and choosing $\mu^h \in \Lambda^h_*$ such that $\mu^h\,|_F = \frac{1}{|F|}\int_F \lambda_0$, we conclude from [33, Theorem 4.1] that

$$\|T_0\lambda_0 - T_0\mu^h\|_{1,h} \le c\,h\,\|u_0\|_2 \quad \text{and} \quad \|T_0\lambda_0 - T_0\mu^h\|_{2,h} \le c\,\|u_0\|_2\,. \tag{75}$$

Therefore, result (71) follows from Lemma 7. Estimate (72) results from (71), ellipticity condition (2) and observing that, for all $K \in \mathcal{T}_h$,

$$\nabla \cdot \left(\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_\varepsilon^h\right) = -\nabla \cdot \left(\mathcal{A}_\varepsilon \nabla T_\varepsilon(\lambda_\varepsilon - \lambda_\varepsilon^h)\right) = 0 \tag{76}$$

holds, since $\lambda_\varepsilon - \lambda_\varepsilon^h$ belongs to space (55). ◻

We next present a convergence result assuming minimal regularity from χ^j and u_0. As expected, we do not obtain sharp error estimates in terms of h.

Theorem 9. *Let u_ε be the solution of* (1) *and let u_ε^h be its numerical approximation defined by* (50). *Also, let $\boldsymbol{\sigma}_\varepsilon := -\mathcal{A}_\varepsilon \nabla u_\varepsilon$ be the post-processed exact dual variable and let $\boldsymbol{\sigma}_\varepsilon^h$ be its approximation given by* (53). *Assume $u_0 \in H^2(\Omega)$, $\chi^j \in W^{1,q}_{\mathrm{per}}(\Omega)$, $q > d$, $f \in L^2(\Omega)$, and* (61) *and* (65) *hold. Then,*

$$|u_\varepsilon - u_\varepsilon^h|_{1,h} \leq c_1(p') \left(\varepsilon^{1/2-1/p} + h^{1-\frac{d}{q}}\right) \|\boldsymbol{\chi}\|_{1,q,Y} (\|u_0\|_2 + \|f\|_0)\,, \tag{77}$$

$$\|\boldsymbol{\sigma}_\varepsilon - \boldsymbol{\sigma}_\varepsilon^h\|_{\mathrm{div}} \leq c_2(p') \left(\varepsilon^{1/2-1/p} + h^{1-\frac{d}{q}}\right) \|\boldsymbol{\chi}\|_{1,q,Y} (\|u_0\|_2 + \|f\|_0)\,, \tag{78}$$

where p' satisfies (16).

Proof. This result is obtained from Theorem 1 and an argument similar to the one used in the proof of Theorem 8. In particular, to estimate the second term on the right-hand side of (73) we set $1/p + 1/q = 1/2$ and use Hölder's inequality to obtain

$$\begin{aligned}
\left\| \varepsilon \frac{\partial \chi_\varepsilon^j}{\partial x_k} \frac{\partial (T_0\lambda_0 - T_0\mu^h)}{\partial x_j} \right\|_0 &\leq \|\boldsymbol{\chi}\|_{1,q,Y} \|T_0\lambda_0 - T_0\mu^h\|_{1,p,h} \\
&\leq c\, \|\boldsymbol{\chi}\|_{1,q,Y} \|T_0\lambda_0 - T_0\mu^h\|_{1+s,h} \\
&\leq c\, h^{1-s} \|\boldsymbol{\chi}\|_{1,q,Y} \|T_0\lambda_0 - T_0\mu^h\|_{2,h}\,,
\end{aligned}$$

with (see [1, Theorem 7.57])

$$\frac{2d}{d-2s} = p \text{ and hence } s = \frac{d}{q}\,.$$

The other terms are bounded following the proof of Theorem 1 with straightforward modifications. ◻

Remark 10. Under the assumptions that $T_0\, \mu^h \in W^{2,p}(\mathcal{T}_h)$ and $\hat{T}_0\, f \in W^{2,p}(\mathcal{T}_h)$, and $\chi^j \in W^{1,\infty}_{\mathrm{per}}(Y)$ and $u_0 \in W^{2,p}(\Omega)$, with $p > d$, it holds from Remark 6 that the approximation error in the broken H^1 norm is $O(h + \varepsilon^{1/2})$.

Our final result measures the error in the L^2 norm. Unlike classical approaches, we do not employ duality techniques and, therefore, no extra regularity is assumed.

Theorem 11. *Let u_ε be the solution of* (1) *and let u_ε^h be its numerical approximation defined by* (50). *Assume $u_0 \in H^2(\Omega)$, $\chi^j \in W^{1,\infty}_{\mathrm{per}}(Y)$, $f \in L^2(\Omega)$, and* (61) *and* (65) *hold. Then,*

$$\|u_\varepsilon - u_\varepsilon^h\|_0 \leq c_1(p')\, h \left(\varepsilon^{1/2-1/p} + h\right) \|\boldsymbol{\chi}\|_{1,\infty,Y} (\|u_0\|_2 + \|f\|_0)\,, \tag{79}$$

where p' satisfies (16).

Proof. From the definition of u_ε and u_ε^h we get

$$\begin{aligned}\|u_\varepsilon - u_\varepsilon^h\|_0 &\leq \|u_\varepsilon^0 + T_\varepsilon\lambda_\varepsilon - u_\varepsilon^{0,h} - T_\varepsilon\lambda_\varepsilon^h\|_0 \\ &\leq \|u_\varepsilon^0 - u_\varepsilon^{0,h}\|_0 + c\,h\,|T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h|_{1,h},\end{aligned} \tag{80}$$

where we used the triangle inequality, the Poincaré inequality, and the assumption on the regularity of the mesh.

Next, we estimate $\|u_\varepsilon^0 - u_\varepsilon^{0,h}\|_0$. Without losing generality, we assume that $u_\varepsilon^0 - u_\varepsilon^{0,h} \in V_0$ does not vanish in $K \in \mathcal{T}_h$. Let $\boldsymbol{\sigma}^\star$ be the vector-valued function belonging to the lowest order Raviart-Thomas space such that $\nabla\cdot\boldsymbol{\sigma}^\star = u_\varepsilon^0 - u_\varepsilon^{0,h}$. We recall that $\boldsymbol{\sigma}^\star\cdot\boldsymbol{n}_K\,|_{\partial K}$ is piecewise constant for all K in $\mathcal{T}_h$. Now, from (47), the fact that $T_\varepsilon\lambda_\varepsilon\,|_K$ and $T_\varepsilon\lambda_\varepsilon^h\,|_K$ belong to $L_0^2(K)$ for all $K\in\mathcal{T}_h$ and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\|u_\varepsilon^0 - u_\varepsilon^{0,h}\|_0 &= \frac{(\nabla\cdot\boldsymbol{\sigma}^\star, u_\varepsilon^0 - u_\varepsilon^{0,h})_{\mathcal{T}_h}}{\|\nabla\cdot\boldsymbol{\sigma}^\star\|_0} \\ &= \frac{\sum_{K\in\mathcal{T}_h}(\boldsymbol{\sigma}^\star\cdot\boldsymbol{n}_K, u_\varepsilon^0 - u_\varepsilon^{0,h})_{\partial K}}{\|\nabla\cdot\boldsymbol{\sigma}^\star\|_0} \\ &= -\frac{\sum_{K\in\mathcal{T}_h}(\boldsymbol{\sigma}^\star\cdot\boldsymbol{n}_K, T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h)_{\partial K}}{\|\nabla\cdot\boldsymbol{\sigma}^\star\|_0} \\ &= -\frac{(\boldsymbol{\sigma}^\star, \nabla(T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h))_{\mathcal{T}_h} + (\nabla\cdot\boldsymbol{\sigma}^\star, T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h)_{\mathcal{T}_h}}{\|\nabla\cdot\boldsymbol{\sigma}^\star\|_0} \\ &= -\frac{(\boldsymbol{\sigma}^\star, \nabla(T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h))_{\mathcal{T}_h}}{\|\nabla\cdot\boldsymbol{\sigma}^\star\|_0}\end{aligned}$$

and hence

$$\begin{aligned}\|u_\varepsilon^0 - u_\varepsilon^{0,h}\|_0 &\leq \frac{\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\sigma}^\star\|_{0,K}\|\nabla(T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h)\|_{0,K}}{\|\nabla\cdot\boldsymbol{\sigma}^\star\|_0} \\ &\leq c\,h\,|T_\varepsilon\lambda_\varepsilon - T_\varepsilon\lambda_\varepsilon^h|_{1,h}\,,\end{aligned}$$

where we used that $\|\boldsymbol{\sigma}^\star\|_{0,K} \leq c\,h_K\,\|\nabla\cdot\boldsymbol{\sigma}^\star\|_{0,K}$ from a scaling argument (cf. [11, page 111]) and the regularity of the mesh. Collecting the previous results, we get from (80) the existence of c such that

$$\|u_\varepsilon - u_\varepsilon^h\|_0 \,\leq\, c\,h\,|u_\varepsilon - u_\varepsilon^h|_{1,h}\,,$$

and the result follows from Theorem 8. ◻

Remark 12. If we further assume regularity $u_0 \in H^{k+1}(\Omega)$ in Theorems 8 and 11 and $\hat{T}_0\,f \in H^{k+1}(\mathcal{T}_h)$, with $k\geq 1$, then we can choose Λ^h such that high-order h-convergence is achieved. To this end, select Λ^h such that it embeds the space $\mathbb{P}^k(F)$ of piecewise polynomial functions of degree less than or equal to k on $F\in\mathcal{E}^h$, and take $\mu_h\in\mathbb{P}^k(F)$ as the L^2 projection of λ_0 on $\mathbb{P}^k(F)$. From [33, Theorem 4.1] it holds that $\|T_0\lambda_0 - T_0\mu^h\|_{1,h} \leq c\,h^k\,\|u_0\|_{k+1}$, and following closely the proof of Theorems 8 and 11, we get

$$\begin{aligned}|u_\varepsilon - u_\varepsilon^h|_{1,h} &\leq c_1(p')\,\left(\varepsilon^{1/2-1/p} + h^k\right)\|\boldsymbol{\chi}\|_{1,\infty,Y}(\|u_0\|_{k+1} + \|\hat{T}_0\,f\|_{k+1,h})\,, \\ \|u_\varepsilon - u_\varepsilon^h\|_0 &\leq c_2(p')\,h\,\left(\varepsilon^{1/2-1/p} + h^k\right)\|\boldsymbol{\chi}\|_{1,\infty,Y}(\|u_0\|_{k+1} + \|\hat{T}_0\,f\|_{k+1,h})\,,\end{aligned}$$

where p' satisfies (16).

4. Numerical validation

We assess the theoretical convergence results through a problem with a highly-oscillatory coefficient. The domain is a unit square with prescribed homogeneous Dirichlet boundary conditions, $f(\boldsymbol{x}) = \sin(x_1)\sin(x_2)$, and coefficient given by

$$\mathcal{A}_\varepsilon(\boldsymbol{x}) = \left[1 + 100\cos^2(\frac{\pi x_1}{\varepsilon})\sin^2(\frac{\pi x_2}{\varepsilon})\right] \mathcal{I}\,,$$

where ε is a small parameter defining the periodicity and $\mathcal{I}$ is the identity matrix. The reference solution is depicted in Figure 1 with $\varepsilon = \frac{1}{64}$. It is constructed using a mesh composed by $16,777,216$ quadrilateral bilinear elements. The MHM method is validated using quadrilateral elements with piecewise constant interpolation on edges to approximate the Lagrange multipliers. The multiscale basis functions η_i and $\hat{T}_\varepsilon f$ are approximated at the local level by the standard Galerkin method over the bilinear continuous polynomial space defined on structured submeshes. The submeshes are selected such that functions η_i and $\hat{T}_\varepsilon f$ are accurately approximated so as the underlying errors do not impact the MHM method. A typical basis function ($\varepsilon = \frac{1}{64}$) is shown in Figure 1.

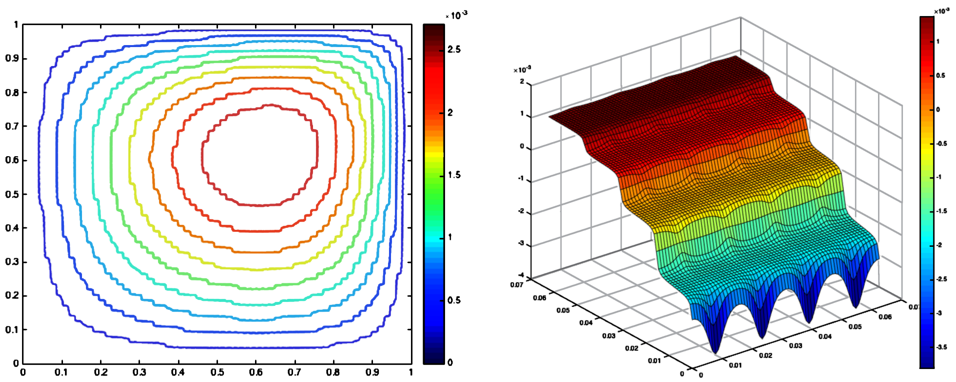

Figure 1. Isolines of the reference solution (left) and the elevation of a multiscale basis function (right). Here $\varepsilon = \frac{1}{64}$.

First, we verify that the theoretical errors with respect to ε in the H^1 and L^2 norms hold. This is shown in Figure 2. To avoid any pollution of the error estimates by the mesh parameter h, we decrease it proportionally to $\varepsilon^{1/2}$ as ε tends to zero. We observe that the numerics agree with the predicted results presented in Theorems 8 and 11.

The next test verifies that the MHM method produces resonance-free error estimates under the assumption (61). To this end, we investigate the error in the L^2 norm and broken H^1 semi-norm when ε and h tend to zero all together. Here we set $\frac{\varepsilon}{h} = \frac{1}{8}$. Figure 3 depicts the convergence results which are in agreement with the estimates presented in Theorems 8 and 11, with the upshot that super-convergence is found at the first points.

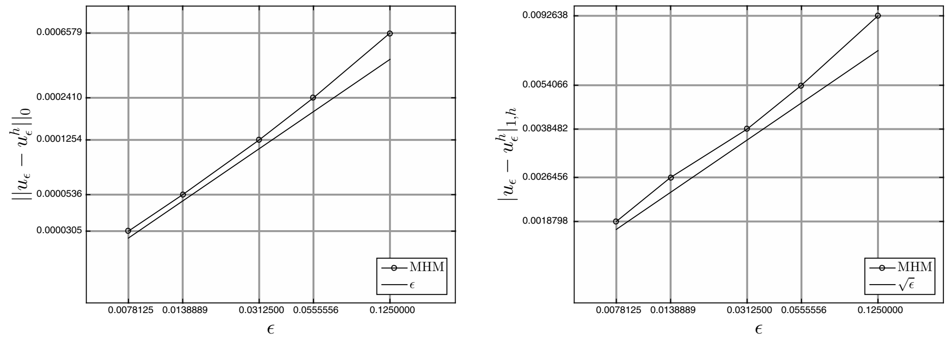

FIGURE 2. Convergence history with respect to ε agree with the theoretical estimates.

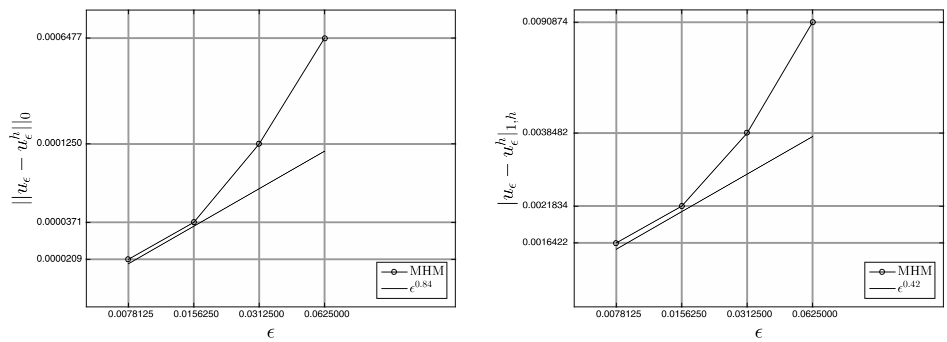

FIGURE 3. Convergence history with respect to $\frac{\varepsilon}{h}$ such that $\frac{\varepsilon}{h} = \frac{1}{8}$. We observe resonance-free errors as predicted by the theory.

Next, we investigate the convergence with respect to h for a fixed (small) value of ε. To this end, we set $\varepsilon = \frac{\pi}{150}$ to fit the benchmark proposed in [28] in which some of the most relevant and recent multiscale finite element methods are compared. In Figure 4, we present the relative error in the L^2 and broken H^1 norms. We find that the results from the MHM method are qualitatively equivalent to the ones in [28].

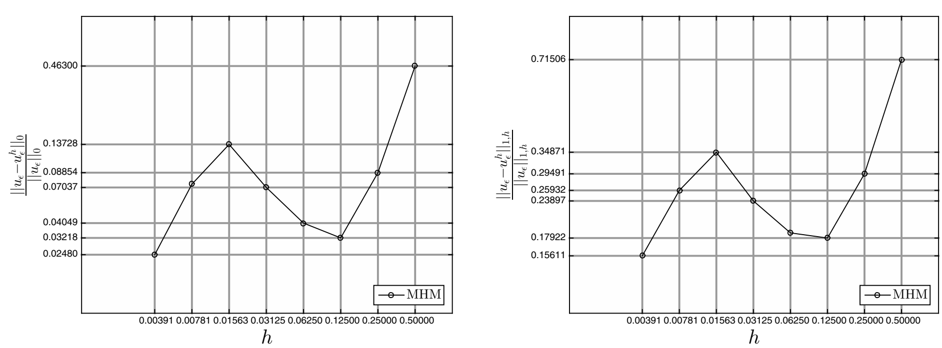

FIGURE 4. Convergence history with respect to h fixing $\varepsilon = \frac{\pi}{150}$.

As in [28], we identify three different regimes. First, we recover the expected theoretical convergence in the case $\frac{\varepsilon}{h} \leq \frac{1}{6}$. Such an upper-bound is comparable to the one found with the MsFEM method presented in [24]. It is worth recalling that the numerical results from the MHM method are obtained *without* using any oversampling techniques. A second regime is found in the interval $\frac{1}{6} < \frac{\varepsilon}{h} \leq 1$ in which the numerical results show an increasing error of order h^{-1}. Observe that this regime stays outside of the current theory. The convergence is recovered when $\frac{\varepsilon}{h} \geq 1$ as expected since the mesh is fine enough to capture the overall scales of the reference solution.

Despite being beyond the scope of the theory, the convergence in the intermediate regime ($\frac{1}{6} < \frac{\varepsilon}{h} \leq 1$) can be recovered by enhancing the space of approximation of the Lagrange multipliers. Specifically, we replace the interpolation space using one piecewise constant function per edge by the space spanned by multiple piecewise constant functions on each edge. See Figure 5 for an illustrative comparison between these different interpolation choices on triangles.

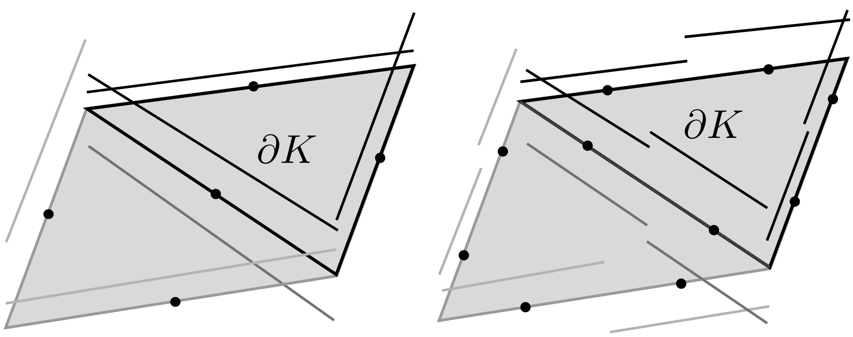

FIGURE 5. Illustration of different interpolation spaces on edges. The one constant (left) and the two constants (right) cases [21].

Specifically, we set a structured quadrilateral mesh with $h = \frac{1}{8}$ and progressively increase the number of degrees of freedom on each edge. We compute the relative error in the broken H^1 norm using this strategy (called space-based) and compare it to the one obtained from successive mesh refinements (called mesh-based). The results are depicted in Figure 6. On the left side of Figure 6, we analyze the result from the perspective of the diameter h, where h stands for the diameter of the edge partition in the space-based case (here the mesh is fixed with diameter $\frac{1}{8}$) and h recovers its usual meaning in the mesh-based case. On the right side of Figure 6, we perform the same analysis but now with respect to the number of degrees of freedom N_{DOF}.

We observe that the drawback shown in the mesh refinement strategy is completely overcome by the space-based enhancing approach. As a result the underlying error is drastically decreased with the upshot that the region of error divergence (of order h^{-1}) is no longer presented. In addition, considerably fewer degrees of freedom are necessary to achieve a given error threshold. We recall that such behavior is achieved without any oversampling technique and it is not predicted by the current theory. This indicates that such a very promising aspect of the MHM method deserves further investigations.

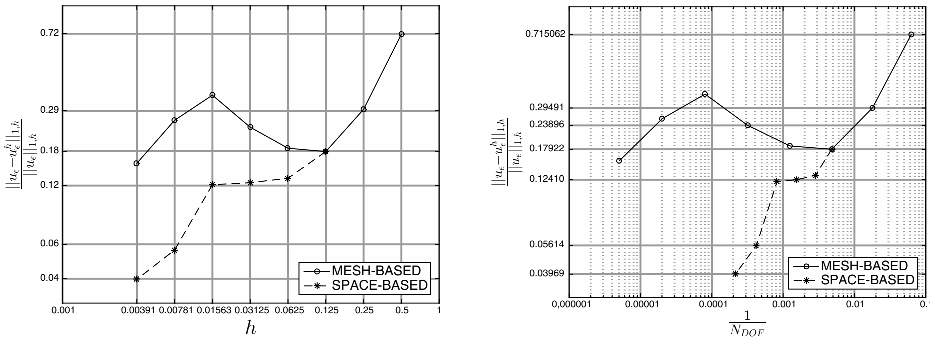

FIGURE 6. Convergence history. Comparison between the one piecewise constant interpolation per edge case with mesh refinement (mesh-based) and the multiple piecewise constant interpolation per edge case on a fixed mesh (space-based). The latter induces a tremendous improvement in the quality of the numerical results with fewer degrees of freedom.

5. Conclusion

We showed that the MHM method is robust with respect to the small parameter ε under mild regularity conditions. This was made possible by the association of the new asymptotic error estimates with the innovative form and properties of the MHM method. To our knowledge, such uniform bound estimates are the first to be established for a multiscale numerical algorithm within the standard Galerkin method. Also, the local boundary conditions are built from an entirely local strategy. It is worth mentioning that theoretical results are supported by the numerics presented in this work and in [4, 20].

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