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# CONVEXITY AND GENERALIZED BERNSTEIN POLYNOMIALS

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Dedicated to S. L. Lee

In a recent generalization of the Bernstein polynomials, the approximated function $f$ is evaluated at points spaced at intervals which are in geometric progression on [0, 1], instead of at equally spaced points. For each positive integer $n$, this replaces the single polynomial $B_nf$ by a one-parameter family of polynomials $B_n^qf$, where $0 < q \leq 1$. This paper summarizes briefly the previously known results concerning these generalized Bernstein polynomials and gives new results concerning $B_n^qf$ when $f$ is a monomial. The main results of the paper are obtained by using the concept of total positivity. It is shown that if $f$ is increasing then $B_n^qf$ is increasing, and if $f$ is convex then $B_n^qf$ is convex, generalizing well known results when $q = 1$. It is also shown that if $f$ is convex then, for any positive integer $n$, $B_n^rf \leq B_n^qf$ for $0 < q \leq r \leq 1$. This supplements the well known classical result that $f \leq B_nf$ when $f$ is convex.

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## 1. Introduction

In this paper we discuss further properties of the generalized Bernstein polynomials defined by

$$B_n(f;x) = \sum_{r=0}^{n} f_r \begin{bmatrix} n \\ r \end{bmatrix} x^r \prod_{s=0}^{n-r-1} (1 - q^s x), \tag{1.1}$$

where an empty product denotes 1 and $f_r = f([r]/[n])$. It is necessary to explain the notation. The function $f$ is evaluated at the ratios of the $q$-integers $[r]$ and $[n]$, where $q$ is a positive real number and

$$[r] = \begin{cases} (1-q^r)/(1-q), & q \neq 1, \\ r, & q = 1. \end{cases}$$

We then define the $q$-factorial $[r]!$ by

$$[r]! = \begin{cases} [r].[r-1]\ldots[1], & r = 1, 2, \ldots, \\ 1, & r = 0 \end{cases}$$

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and the $q$-binomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$ by

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{[n]!}{[r]![n-r]!}$$

for integers $n \geq r \geq 0$. These $q$-binomial coefficients satisfy the recurrence relations

$$\begin{bmatrix} n \\ r \end{bmatrix} = q^{n-r} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ r \end{bmatrix}$$

and

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} + q^{r} \begin{bmatrix} n-1 \\ r \end{bmatrix}.$$

We note from the above recurrence relations that $\begin{bmatrix} n \\ r \end{bmatrix}$ is positive for $n \geq r \geq 0$ and all $q \geq 0$. It is then clear from (1.1) that if $f$ is positive on $[0, 1]$ then, for all $q$ such that $0 < q \leq 1$, $B_n f$ is positive on $[0, 1]$. It is also easily verified that $B_n(f; 0) = f(0)$, $B_n(f; 1) = f(1)$ and $B_n(f; x) = f(x)$, $0 \leq x \leq 1$, when $f(x)$ is a polynomial of degree 1 or less.

In [4] there is a discussion of convergence and a Voronovskaya theorem on the rate of convergence, and a de Casteljau algorithm is given in [5] for computing $B_n(f; x)$ recursively. In [3] it is shown that, if $f$ is convex,

$$B_n(f; x) \leq B_{n-1}(f; x), \quad 0 \leq x \leq 1,$$

for $n > 1$ and $0 < q \leq 1$.

This paper is concerned with the behaviour of the generalized Bernstein polynomials as $q$ varies. When we need to emphasize the dependence on $q$ we will write $B_n^q(f; x)$ in place of $B_n(f; x)$. In Section 2 we discuss the Bernstein polynomials for the monomials, which have a particularly simple form. In Section 3 we quote some results on the theory of total positivity which are used in the following sections. In Section 4 we discuss a change of basis, in order to show later how $B_n(f; x)$ varies with the parameter $q$. Finally it is proved for all $n \geq 1$ and $0 < q \leq 1$ that if $f$ is increasing, $B_n^q f$ is increasing, and if $f$ is convex then $B_n^q f$ is convex. We also show that if $f$ is convex on $[0, 1]$ then $B_n^r f \leq B_n^q f$ for $0 < q \leq r \leq 1$.

## 2. The monomials

We require some preliminaries. For any real function $f$ we define $\Delta^0 f_i = f_i$ for $i = 0, 1, \ldots n$ and, recursively,

$$\Delta^{k+1}f_i = \Delta^k f_{i+1} - q^k \Delta^k f_i$$

for $k = 0, 1, \ldots, n-i-1$, where $f_i$ denotes $f([i]/[n])$. It is easily shown by induction on $k$ that $q$-differences satisfy the relation

$$\Delta^k f_i = \sum_{r=0}^{k} (-1)^r q^{r(r-1)/2} \begin{bmatrix} k \\ r \end{bmatrix} f_{i+k-r}, \tag{2.1}$$

see Schoenberg [6], Lee and Phillips [2]. The generalized Bernstein polynomial (1.1) may also be written in the $q$-difference form (see [4])

$$B_n(f; x) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} \Delta^j f_0 x^j. \tag{2.2}$$

We now express the $q$-binomial coefficients as

$$\begin{bmatrix} n \\ j \end{bmatrix} = \frac{[n]^j}{[j]! q^{j(j-1)/2}} \pi_j^n, \quad 0 \le j \le n, \tag{2.3}$$

where

$$\pi_j^n = \prod_{r=0}^{j-1} \left(1 - \frac{[r]}{[n]}\right)$$

and an empty product denotes 1. It follows from (2.2) that $B_n(x^i; x)$ is a polynomial of degree less or equal to $\min(i, n)$ and, using (2.2), (2.1) and (2.3), we obtain

$$B_n(x^i; x) = \sum_{j=0}^{i} \pi_j^n [n]^{j-i} S_q(i, j) x^j, \tag{2.4}$$

where

$$S_q(i, j) = \frac{1}{[j]! q^{j(j-1)/2}} \sum_{r=0}^{j} (-1)^r q^{r(r-1)/2} \begin{bmatrix} j \\ r \end{bmatrix} [j-r]^i. \tag{2.5}$$

We may verify by induction on $i$ that

$$S_q(i+1, j) = S_q(i, j-1) + [j] S_q(i, j) \tag{2.6}$$

for $i \ge 0$ and $j \ge 1$ with $S_q(0, 0) = 1$, $S_q(i, 0) = 0$ for $i > 0$ and we define $S_q(i, j) = 0$ for $j > i$. We call $S_q(i, j)$ the Stirling polynomials of the second kind since when $q = 1$ they are the Stirling numbers of the second kind. The recurrence relation (2.6) shows that, for $q > 0$, the Stirling polynomials are polynomials in $q$ with non-negative integer coefficients and so are positive monotonic increasing functions of $q$. Thus $B_n(x^i; x)$ and all its derivatives are non-negative on $[0, 1]$. In particular, $B_n(x^i; x)$ is convex. In Section 4, we will find that, more generally, $B_n(f; x)$ is convex when $f$ is convex.

## 3. Total positivity

In this section we will cite some results concerning totally positive matrices, which we require later to verify the shape-preserving properties of the generalized Bernstein polynomials.

**Definition 3.1.** For any real sequence $v$, finite or infinite, we denote by $S^-(v)$ the number of strict sign changes in $v$.

We use the same notation to denote sign changes in a function, as follows.

**Definition 3.2.** For a real-valued function $f$ on an interval $I$, we define $S^-(f)$ to be the number of sign changes of $f$, that is

$$S^-(f) = \sup S^-(f(x_0), \ldots, f(x_m))$$

where the supremum is taken over all increasing sequences $(x_0, \ldots, x_m)$ in $I$ for all $m$.

**Definition 3.3.** We say that a matrix $\mathbf{A} = (a_{ij})$ is $m$-banded if, for some $l$, $a_{ij} \neq 0$ implies $l \leq j - i \leq l + m$.

**Definition 3.4.** A matrix is said to be totally positive if all its minors are non-negative.

It is easily verified that, with $0 < x_0 < x_1 < \ldots < x_n$ the $(n+1) \times (n+1)$ Vandermonde matrix whose $(i, j)$th element is $x_i^j$, $0 \leq i, j \leq n$, is totally positive.

**Theorem 3.1.** *A finite matrix is totally positive if and only if it is a product of* 1-*banded matrices with non-negative elements.*

**Theorem 3.2** (Variation diminishing property). *If* $\mathbf{T}$ *is a totally positive matrix and* $\mathbf{v}$ *is any vector for which* $\mathbf{Tv}$ *is defined, then* $S^-(\mathbf{Tv}) \leq S^-(\mathbf{v})$.

**Definition 3.5.** We say that a sequence $(\phi_0, \ldots, \phi_n)$ of real-valued functions on an interval $I$ is totally positive if, for any points $x_0 < \ldots < x_n$ in $I$, the collocation matrix $(\phi_j(x_i))_{i,j=0}^n$ is totally positive.

**Theorem 3.3.** *If* $(\phi_0, \ldots, \phi_n)$ *is totally positive on* $I$ *then, for any numbers* $a_0, \ldots, a_n$,

$$S^-(a_0\phi_0 + \ldots + a_n\phi_n) \le S^-(a_0, \ldots, a_n).$$

For the proofs of these theorems see [1].

Thus, from the total positivity of the Vandermonde matrix, we see that $(1, x, \ldots, x^n)$ is totally positive in any subinterval of $[0, \infty)$. On making the change of variable $t = x/(1-x)$, noting that $t$ is increasing function of $x$, we see that

$$(1, x/(1-x), x^2/(1-x)^2, \ldots, x^n/(1-x)^n)$$

is totally positive on [0, 1] and thus

$$\left((1-x)^n, x(1-x)^{n-1}, x^2(1-x)^{n-2}, \ldots, x^n\right)$$

is totally positive on [0, 1]. For some $0 < q \le 1, n \ge 1, j = 0, \ldots, n$, let

$$P_j^{n,q}(x) = x^j \prod_{s=0}^{n-j-1} (1 - q^s x), \quad 0 \le x \le 1, \tag{3.1}$$

denote the functions which appear in the generalized Bernstein polynomials (1.1). We have seen above that

$$(P_0^{n,1}, P_1^{n,1}, \ldots, P_n^{n,1})$$

is totally positive on [0, 1] and we will see in Section 4 that the same is true of $(P_0^{n,q}, P_1^{n,q}, \ldots, P_n^{n,q})$ for any $q, 0 < q \le 1$.

## 4. Change of basis

In this section we present results which will be used to show how $B_n(f; x)$ varies with the value of the parameter $q$.

Since the functions defined in (3.1) are a basis for the subspace of the polynomials of degree at most $n$ then, for any $q, r, 0 < q, r \le 1$, there exists a non-singular matrix $\mathbf{T}^{n,q,r}$ such that

$$\begin{bmatrix} P_0^{n,q}(x) \\ \vdots \\ P_n^{n,q}(x) \end{bmatrix} = \mathbf{T}^{n,q,r} \begin{bmatrix} P_0^{n,r}(x) \\ \vdots \\ P_n^{n,r}(x) \end{bmatrix}.$$

**Theorem 4.1.** *For $0 < q \le r$ all elements of the matrix $\mathbf{T}^{n,q,r}$ are non-negative.*

**Proof.** We use induction on $n$. The result holds for $n = 1$ since $\mathbf{T}^{1,q,r}$ is the $2 \times 2$ identity matrix. Let us assume the result holds for some $n \ge 1$. Then, since

$$P_{j+1}^{n+1,q}(x) = xP_j^{n,q}(x), \quad 0 \le j \le n,$$

we have

$$\begin{bmatrix} P_1^{n+1,q}(x) \\ \vdots \\ P_{n+1}^{n+1,q}(x) \end{bmatrix} = \mathbf{T}^{n,q,r} \begin{bmatrix} P_1^{n+1,r}(x) \\ \vdots \\ P_{n+1}^{n+1,r}(x) \end{bmatrix}. \tag{4.1}$$

Also, we have

$$\begin{aligned} P_0^{n+1,q}(x) &= (1-x)\dots(1-q^{n-1}x)(1-q^n x) \\ &= (1-q^n x)\sum_{j=0}^{n} T_{0,j}^{n,q,r} P_j^{n,r}(x). \end{aligned}$$

On substituting

$$(1-q^n x)P_j^{n,r}(x) = P_j^{n+1,r}(x) + (r^{n-j} - q^n)P_{j+1}^{n+1,r}(x)$$

and simplifying, we obtain

$$\begin{aligned} P_0^{n+1,q}(x) =& T_{0,0}^{n,q,r} P_0^{n+1,r}(x) + (1-q^n) T_{0,n}^{n,q,r} P_{n+1}^{n+1,r}(x) \\ &+ \sum_{j=1}^{n} \left( (r^{n+1-j} - q^n) T_{0,j-1}^{n,q,r} + T_{0,j}^{n,q,r} \right) P_j^{n+1,r}(x). \end{aligned} \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\begin{bmatrix} P_0^{n+1,q}(x) \\ P_1^{n+1,q}(x) \\ \vdots \\ P_{n+1}^{n+1,q}(x) \end{bmatrix} = \begin{bmatrix} T_{0,0}^{n,q,r} & \mathbf{v}_{n+1}^T \\ & \\ \mathbf{0} & \mathbf{T}^{n,q,r} \end{bmatrix} \begin{bmatrix} P_0^{n+1,r}(x) \\ P_1^{n+1,r}(x) \\ \vdots \\ P_{n+1}^{n+1,r}(x) \end{bmatrix}, \tag{4.3}$$

where the elements of the row vector $\mathbf{v}_{n+1}^T$ are the coefficients of $P_1^{n+1,r}(x), \dots, P_{n+1}^{n+1,r}(x)$ given by (4.2). Thus $\mathbf{T}^{n+1,q,r}$ is the matrix in block form in (4.3) which, together with (4.2), shows that all elements of $\mathbf{T}^{n+1,q,r}$ are non-negative. This completes the proof. □

We now show that $\mathbf{T}^{n,q,r}$ can be factorized as a product of 1-banded matrices. First we require the following lemma.

**Lemma 4.1.** *For $m \ge 1$ and $r, a \in \mathbf{R}$, let $\mathrm{A}(m, a)$ denote the $m \times (m+1)$ matrix*

$$\begin{bmatrix} 1 & r^m - a & & & \\ & 1 & r^{m-1} - a & & \\ & & \ddots & \ddots & \\ & & & 1 & r - a \end{bmatrix}.$$

*Then*

$$\mathbf{A}(m, a)\mathbf{A}(m+1, b) = \mathbf{A}(m, b)\mathbf{A}(m+1, a). \tag{4.4}$$

**Proof.** For $i = 0, \ldots, m-1$ the $i$th row of each side of (4.4) is

$$[0, \ldots, 0, 1, r^{m+1-i} + r^{m-i} - a - b, (r^{m-i} - a)(r^{m-i} - b), 0, \ldots, 0]. \qquad \square$$

**Theorem 4.2.** *For $n \geq 2$ and any $q, r$ the matrix $\mathbf{T}^{n,q,r}$ is given by the product*

$$\begin{bmatrix} 1 & r - q^{n-1} & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 1 & r^2 - q^{n-2} & & & \\ & 1 & r - q^{n-2} & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \cdots$$

$$\begin{bmatrix} 1 & r^{n-1} - q & & & & & \\ & 1 & r^{n-2} - q & & & & \\ & & & \ddots & \ddots & & \\ & & & & 1 & r - q & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}.$$

**Proof.** We use induction on $n$. The result holds for $n = 2$. Denote the above product by $\mathbf{S}^{n,q,r}$ and assume that, for some $n \geq 2, \mathbf{T}^{n,q,r} = \mathbf{S}^{n,q,r}$. Then we can express $\mathbf{S}^{n+1,q,r}$ as the product, in block form,

$$\mathbf{S}^{n+1,q,r} = \begin{bmatrix} 1 & \mathbf{c}_0^T \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} 1 & \mathbf{c}_1^T \\ \mathbf{0} & \mathbf{B}_1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{c}_2^T \\ \mathbf{0} & \mathbf{B}_2 \end{bmatrix} \cdots \begin{bmatrix} 1 & \mathbf{c}_{n-1}^T \\ \mathbf{0} & \mathbf{B}_{n-1} \end{bmatrix}$$

where $\mathbf{c}_0^T, \ldots, \mathbf{c}_{n-1}^T$ are row vectors, $\mathbf{0}$ denotes the zero vector, $\mathbf{I}$ the unit matrix and

$$\mathbf{B}_1\mathbf{B}_2 \ldots \mathbf{B}_{n-1} = \mathbf{S}^{n,q,r} = \mathbf{T}^{n,q,r}.$$

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Also, the first column of $\mathbf{S}^{n+1,q,r}$ has 1 in the first row and zeros below. Thus it remains only to verify that the first rows of $\mathbf{T}^{n+1,q,r}$ and $\mathbf{S}^{n+1,q,r}$ are equal. We have

$$[S_{0,0}^{n+1,q,r}, \ldots, S_{0,n+1}^{n+1,q,r}] = [\mathbf{w}^T, 0],$$

where, in the notation defined in the above lemma,

$$\mathbf{w}^T = \mathbf{A}(1, q^n)\mathbf{A}(2, q^{n-1})\ldots\mathbf{A}(n-1, q^2)\mathbf{A}(n, q). \tag{4.5}$$

In view of the lemma, we may permute the quantities $q^n, q^{n-1}, \ldots, q$ in (4.5), leaving $\mathbf{w}^T$ unchanged. In particular, we may write

$$\mathbf{w}^T = \mathbf{A}(1, q^{n-1})\mathbf{A}(2, q^{n-2})\ldots\mathbf{A}(n-1, q)\mathbf{A}(n, q^n). \tag{4.6}$$

Now the product of the first $n-1$ matrices in (4.6) is simply the first row of $\mathbf{S}^{n,q,r}$ and thus

$$\mathbf{w}^T = [S_{0,0}^{n,q,r}, \ldots, S_{0,n-1}^{n,q,r}]\begin{bmatrix} 1 & r^n - q^n & & \\ & \ddots & \ddots & \\ & & 1 & r - q^n \end{bmatrix}$$
$$= [T_{0,0}^{n,q,r}, \ldots, T_{0,n-1}^{n,q,r}]\begin{bmatrix} 1 & r^n - q^n & & \\ & \ddots & \ddots & \\ & & 1 & r - q^n \end{bmatrix}.$$

This gives

$$S_{0,0}^{n+1,q,r} = T_{0,0}^{n,q,r}$$

and

$$S_{0,j}^{n+1,q,r} = (r^{n+1-j} - q^n)T_{0,j-1}^{n,q,r} + T_{0,j}^{n,q,r}, \quad j = 1, \ldots, n,$$

noting that $T_{0,n}^{n,q,r} = 0$. Then from (4.2)

$$S_{0,j}^{n+1,q,r} = T_{0,j}^{n+1,q,r}, \quad j = 0, \ldots, n,$$

and since $S_{0,n+1}^{n+1,q,r} = 0 = T_{0,n+1}^{n+1,q,r}$, the result is true for $n+1$ and the proof is complete. □

The following is a consequence of Theorem 4.2 and Theorem 3.1.

**Theorem 4.3.** *For $0 < q \leq r^{n-1}$ the matrix $\mathbf{T}^{n,q,r}$ is totally positive.*

We note that if $0 < q \leq r^{n-1}$ and

$$p = a_0^q P_0^{n,q} + \ldots + a_n^q P_n^{n,q} = a_0^r P_0^{n,r} + \ldots + a_n^r P_n^{n,r} \tag{4.7}$$

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then Theorem 3.2 shows that

$$S^-(a_0^r, \ldots, a_n^r) \le S^-(a_0^q, \ldots, a_n^q),$$

see [1, p. 166]. Since $(P_0^{n,1}, \ldots, P_n^{n,1})$ is totally positive it follows from Theorem 3.3 that, for $0 < q \le r^{n-1} \le 1$ and $p$ as in (4.7),

$$S^-(p) \le S^-(a_0^r, \ldots, a_n^r) \le S^-(a_0^q, \ldots, a_n^q). \tag{4.8}$$

## 5. Convexity

From (4.8) we see that, for $0 < q \le 1$, $S^-(B_n^q f) \le S^-(f)$. Since $B_n^q$ reproduces linear polynomials, this has the following consequence.

**Theorem 5.1.** *For any function f and any linear polynomial p,*

$$S^-(B_n^q f - p) = S^-(B_n^q(f - p)) \le S^-(f - p),$$

*for* $0 < q \le 1$.

This is illustrated by Figure 1. The function $f(x)$ is $\sin 2\pi x$ and the generalized Bernstein polynomials are of degree $n = 20$ with $q = 0.8$ and $q = 0.9$.

The next result follows from Theorem 5.1.

**Theorem 5.2.** *If f is increasing (decreasing) on* [0, 1], *then* $B_n^q f$ *is also increasing (decreasing) on* [0, 1], *for* $0 < q \le 1$.

**Proof.** Let $f$ be increasing on (0, 1). Then, for any constant $c$,

$$S^-(B_n^q f - c) \le S^-(f - c) \le 1$$

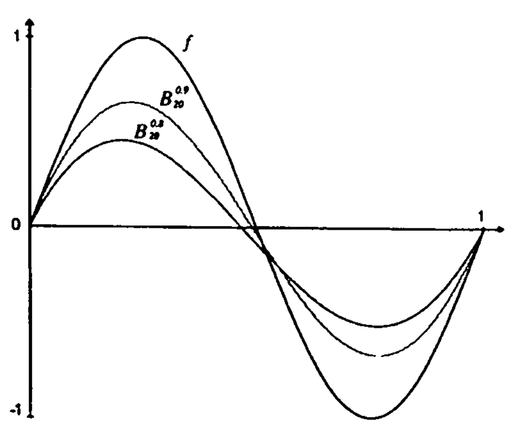


FIGURE 1: Sign changes of generalized Bernstein polynomials for $f(x) = \sin 2\pi x$. The polynomials are $B_{20}^{0.8} f$ and $B_{20}^{0.9} f$.

and thus $B_n^q f$ is monotonic. Since

$$B_n^q(f; 0) = f(0) \le f(1) = B_n^q(f; 1),$$

$B_n^q f$ is monotonic increasing. (If $f$ is decreasing we may replace $f$ by $-f$.) □

Next we recall the definition of a convex function.

**Definition 5.1.** A function $f$ is said to be convex on $[0, 1]$ if, for any $t_0, t_1$ such that $0 \le t_0 < t_1 \le 1$ and any $\lambda, 0 < \lambda < 1, f(\lambda t_0 + (1-\lambda)t_1) \le \lambda f(t_0) + (1-\lambda)f(t_1)$.

Geometrically, this definition states that no chord of $f$ lies below the graph of $f$. We now state a result on convexity.

**Theorem 5.3.** *If $f$ is convex on* $[0, 1]$, *then $B_n^q f$ is also convex on* $[0, 1]$, *for* $0 < q \le 1$.

**Proof.** Let $p$ denote any linear polynomial. Then if $f$ is convex we have

$$S^-(B_n^q f - p) = S^-(B_n^q(f - p)) \le S^-(f - p) \le 2.$$

Thus if $p(a) = B_n^q(f; a)$ and $p(b) = B_n^q(f; b)$ for $0 < a < b < 1$ then $B_n^q f - p$ cannot change sign in $(a, b)$. As we vary $a$ and $b$, a continuity argument shows that the sign of $B_n^q f - p$ on $(a, b)$ is the same for all $a$ and $b, 0 < a < b < 1$. From the convexity of $f$ we see that, when $a = 0$ and $b = 1, 0 \le p - f$, so that

$$0 \le B_n^q(p - f) = p - B_n^q f$$

for $0 < q \le 1$ and thus $B_n^q f$ is convex. □

We conclude this section by proving that, if $f$ is convex, the generalized Bernstein polynomials $B_n^q f$, for $n$ fixed, are monotonic in $q$.

**Theorem 5.4.** *For $0 < q \le r \le 1$ and for $f$ convex on* $[0, 1]$, *then*

$$B_n^r f \le B_n^q f.$$

**Proof.** Let us write $\zeta_j^{n,q} = \dfrac{[j]}{[n]}$ and $a_j^{n,q} = \begin{bmatrix} n \\ j \end{bmatrix}$. Then, for any function $g$ on $[0, 1]$,

$$B_n^q g = \sum_{j=0}^{n} g(\zeta_j^{n,q}) a_j^{n,q} P_j^{n,q} = \sum_{j=0}^{n} \sum_{k=0}^{n} g(\zeta_j^{n,q}) a_j^{n,q} T_{j,k}^{n,q,r} P_k^{n,r}$$

and thus

$$B_n^q g = \sum_{k=0}^{n} P_k^{n,r} \sum_{j=0}^{n} T_{j,k}^{n,q,r} g(\zeta_j^{n,q}) a_j^{n,q}. \tag{5.1}$$

With $g=1$, this gives

$$1=\sum_{j=0}^{n} a_j^{n,q} P_j^{n,q}=\sum_{k=0}^{n} P_k^{n,r} \sum_{j=0}^{n} T_{j,k}^{n,q,r} a_j^{n,q}$$

and hence

$$\sum_{j=0}^{n} T_{j,k}^{n,q,r} a_j^{n,q}=a_k^{n,r}, \quad k=0,\ldots,n. \tag{5.2}$$

On putting $g(x)=x$ in (5.1), we obtain

$$x=\sum_{j=0}^{n} \zeta_j^{n,q} a_j^{n,q} P_j^{n,q}=\sum_{k=0}^{n} P_k^{n,r} \sum_{j=0}^{n} T_{j,k}^{n,q,r} \zeta_j^{n,q} a_j^{n,q}.$$

Since

$$\sum_{j=0}^{n} \zeta_j^{n,r} a_j^{n,r} P_j^{n,r}=x$$

we have

$$\sum_{j=0}^{n} T_{j,k}^{n,q,r} \zeta_j^{n,q} a_j^{n,q}=\zeta_k^{n,r} a_k^{n,r}, \quad k=0,\ldots,n. \tag{5.3}$$

Now if $f$ is convex, it follows from (5.2) and (5.3) that

$$\begin{aligned} f(\zeta_k^{n,r}) &= f\left(\sum_{j=0}^{n} (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} \zeta_j^{n,q} a_j^{n,q}\right) \\ &\le \sum_{j=0}^{n} (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} a_j^{n,q} f(\zeta_j^{n,q}). \end{aligned}$$

Then (5.1) gives

$$\begin{aligned} B_n^q f &= \sum_{j=0}^{n} f(\zeta_j^{n,q}) a_j^{n,q} P_j^{n,q} \\ &= \sum_{k=0}^{n} a_k^{n,r} P_k^{n,r} \sum_{j=0}^{n} (a_k^{n,r})^{-1} T_{j,k}^{n,q,r} f(\zeta_j^{n,q}) a_j^{n,q} \\ &\ge \sum_{k=0}^{n} a_k^{n,r} P_k^{n,r} f(\zeta_k^{n,r}) = B_n^r f. \end{aligned}$$

□

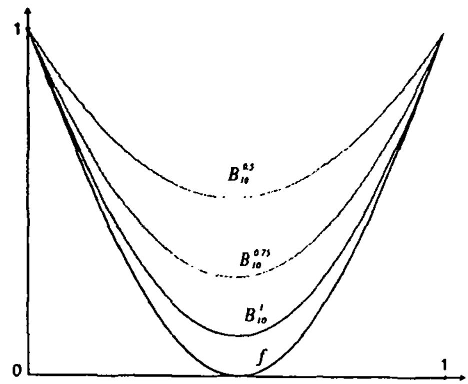

FIGURE 2: Monotonicity of generalized Bernstein polynomials in the parameter $q$, for $f(x) = 1 - \sin \pi x$. The polynomials are $B_{10}^{0.5}f$, $B_{10}^{0.75}f$ and $B_{10}^{1}f$.

Figure 2 illustrates the monotonicity in $q$ of the generalized Bernstein polynomials $B_n^q(f; x)$ for the convex function $f(x) = 1 - \sin \pi x$.

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