CONNECTIVITY IN MATROIDS

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1. Introduction. An edge of a 3-connected graph G is called *essential* if the 3-connection of G is destroyed both when the edge is deleted and when it is contracted to a single vertex. It is known **(1)** that the only 3-connected graphs in which every edge is essential are the "wheel-graphs." A *wheel-graph* of *order* n, where n is an integer $\geqslant 3$, is constructed from an n-gon called its "rim" by adding one new vertex, called the "hub," and n new edges, or "spokes" joining the new vertex to the n vertices of the rim; see Figure 4A.

A matroid can be regarded as a generalized graph. One way of developing the theory of matroids is therefore to generalize known theorems about graphs. In the present paper we do this with the theorem stated above. We state the relevant definitions and theorems of matroid theory in the next section. For proofs of the theorems reference may be made to **(2; 3;** and **4)**.

2. Matroids. Let E be a finite set. A class M of non-null subsets of E is called a *matroid* on E if it satisfies the following two axioms.

I. *No member of M is contained in another.*

II. *If X and Y are members of M and a and b are members of E such that $a \in X \cap Y$ and $b \in X - Y$, then there exists $Z \in M$ such that*

$$b \in Z \subseteq (X \cup Y) - \{a\}.$$

We refer to the members of E and M as the *cells* and *circuits* of M respectively. Circuits are called "atoms" in **(2)** and **(3)**.

As an example we may take E to be the set of edges of a graph G. We can then define a circuit of M to be the set of edges of a polygon of G. Each polygon of G is to give rise to a circuit of M in this way. The two axioms are readily verified. We denote the resulting matroid by $P(G)$ and call it the *polygon-matroid* of G.

A matroid which cannot be interpreted as the polygon-matroid of a graph can be constructed in the following way. We take E to be any finite set of four or more elements and define a circuit of M to be any set of three distinct elements of E.

The *rank* $r(M)$ of a matroid M on E is the least possible number of cells such that there is at least one in each circuit. It should be noted, however, that the "rank" used by Whitney in **(4)** is the difference between this and the number of members of E. For the polygon-matroid of a graph G the rank is

Received January 4, 1966. This research was supported by a grant from the National Research Council of Canada.

thus the least number of edges whose deletion destroys every polygon in the graph. This is the "cyclomatic number" or "Betti number of dimension 1." It is given by

(1) $$p_1(G) = \alpha_1(G) - \alpha_0(G) + p_0(G),$$

where $p_1(G)$ is the cyclomatic number and $\alpha_1(G)$, $\alpha_0(G)$, and $p_0(G)$ are the numbers of edges, vertices, and components of G, respectively.

We denote the number of members of a given finite set X by $|X|$.

Given M, we may consider the class L of all non-null subsets X of E such that $|X \cap Y| \neq 1$ whenever $Y \in M$. The class of all members of L not containing others can be shown to be a matroid M^* on E called the *dual* matroid of M; see (**2**, §§2.6 and 3.5). Dual matroids satisfy the following theorems:

(2) $$M^{**} = M,$$

(3) $$r(M) + r(M^*) = |E|.$$

Let S be any subset of E. We define $M \times S$ as the class of all circuits of M contained in S. Letting L_S denote the class of all non-null intersections with S of circuits of M, we write $M \cdot S$ for the class of all members of L that do not contain others. The classes $M \times S$ and $M \cdot S$ are matroids on S, by (**2**, §3.3). We refer to them as the *contraction* and *reduction* of M to S respectively. Contractions and reductions satisfy the following theorems:

(4) $$(M \times S)^* = M^* \cdot S \qquad (\mathbf{2}, 3.352),$$

(5) $$(M \cdot S)^* = M^* \times S \qquad (\mathbf{2}, 3.351),$$

(6) $$r(M \times S) + r(M \cdot (E - S)) = r(M) \qquad (\mathbf{2}, 3.54),$$

(7) $$r(M \times (S \cup T)) + r(M \times (S \cap T)) \geqslant r(M \times S) + r(M \times T) \qquad (\mathbf{2}, 3.56).$$

For complementary subsets S and T of E we write

(8) $$\xi(M; S, T) = r(M) - r(M \times S) - r(M \times T) + 1.$$

We then have

(9) $$\xi(M; S, T) \geqslant 1,$$

by (7). By (3), (5), and (6) we can rewrite (8) in the two following forms:

(10) $$\xi(M; S, T) = r(M \cdot S) - r(M \times S) + 1,$$

(11) $$\xi(M; S, T) = |S| - r(M \times S) - r(M^* \times S) + 1.$$

As a corollary of (11) we have

(12) $$\xi(M; S, T) = \xi(M^*; S, T).$$

We say that M is *k-separated*, where k is a positive integer, if there are complementary non-null subsets S and T of E such that

$$\xi(M; S, T) = k,$$

$$\operatorname{Min}(|S|, |T|) \geqslant k.$$

If there is a least positive integer k such that M is k-separated, we call it the *connectivity* of M and denote it by $\lambda(M)$. If there is no such integer, we write $\lambda(M) = \infty$. We say M is *n-connected*, where n is a positive integer, if $n \leqslant \lambda(M)$.

3. The connectivity of a polygon-matroid. We can give a somewhat analogous definition of connectivity for graphs. As usual we write $E(G)$ and $V(G)$ for the sets of edges and vertices, respectively, of a finite graph G. If $S \subseteq E(G)$, we write $G \cdot S$ for the subgraph of G determined by the edges of S and their incident vertices. It is evident that

$$P(G \cdot S) = P(G) \times S. \tag{13}$$

If S and T are complementary subsets of $E(G)$, we write $\eta(G; S, T)$ for the number of common vertices of $G \cdot S$ and $G \cdot T$. We say G is *k-separated*, where k is a positive integer, if G is connected and there are complementary subsets S and T of $E(G)$ such that

$$\eta(G; S, T) = k,$$

$$\operatorname{Min}(|S|, |T|) \geqslant k.$$

We say G is 0-*separated* if and only if it is not connected.

If there is a least non-negative integer k such that G is k-separated, we call it the *connectivity* of G and denote it by $\lambda(G)$. If there is no such integer, we write $\lambda(G) = \infty$. This happens, for example, when G is a polygon with not more than three edges. We say G is *n-connected*, where n is a non-negative integer, if $n \leqslant \lambda(G)$. Thus G is 1-connected if and only if it is connected.

The above definition of n-connection for graphs is equivalent to that given in **(1)**, though the preliminary definition of k-separation is more restrictive. In this section we shall relate the notions of n-connection for graphs and matroids by showing that $\lambda(P(G)) = \lambda(G)$ for any connected graph G.

3.1. *Let G be a connected graph and let S and T be complementary subsets of $E(G)$. Then*

$$\xi(P(G); S, T) = \eta(G; S, T) - p_0(G \cdot S) - p_0(G \cdot T) + 2.$$

Proof.

$$\xi(P(G); S, T) = p_1(G) - p_1(G \cdot S) - p_1(G \cdot T) + 1,$$

by (13),

$$= -\alpha_0(G) + \alpha_0(G \cdot S) + \alpha_0(G \cdot T) + 2 - p_0(G \cdot S) - p_0(G \cdot T),$$

by (1),

$$= \eta(G; S, T) - p_0(G \cdot S) - p_0(G \cdot T) + 2.$$

https://doi.org/10.4153/CJM-1966-129-2 Published online by Cambridge University Press

3.2. *If G is connected, then* $\lambda(P(G)) \leqslant \lambda(G)$.

Proof. If $\lambda(G) = \infty$, the theorem is trivial. In the remaining case, we have $\lambda(G) \geqslant 1$ since G is connected. We can then choose complementary subsets S and T of $E(G)$ so that

$$\eta(G; S, T) = \lambda(G), \tag{14}$$

$$\text{Min}\ (|S|, |T|) \geqslant \lambda(G). \tag{15}$$

But then

$$\xi(P(G); S, T) \leqslant \lambda(G),$$

by 3.1. Using (9) we deduce that $P(G)$ is k-separated for some positive integer $k \leqslant \lambda(G)$. The theorem follows.

3.3. *Let G be a connected graph. Let S and T be complementary subsets of* $E(G)$, *and let k be a positive integer such that*

$$\eta(G; S, T) \leqslant k + p_0(G\cdot S) + p_0(G\cdot T) - 2, \tag{16}$$

$$\text{Min}\ (|S|, |T|) \geqslant k. \tag{17}$$

Then $\lambda(G) \leqslant k$.

Proof. Assume $\lambda(G) > k$. We may suppose S and T chosen, consistently with (16) and (17), so that $\eta(G; S, T)$ is as small as possible.

We may further suppose that G has no loop A. For otherwise we would have $\eta(G; \{A\}, E(G) - \{A\}) = 1 \leqslant k$, which is contrary to assumption.

Since S and T are non-null, we have

$$\text{Min}\ (p_0(G\cdot S), p_0(G\cdot T)) \geqslant 1. \tag{18}$$

If $p_0(G\cdot S) = p_0(G\cdot T) = 1$, we have $\eta(G; S, T) \leqslant k$ by (16). But then $\lambda(G) \leqslant k$, contrary to assumption. We may therefore suppose that

$$p_0(G\cdot S) + p_0(G\cdot T) \geqslant 3. \tag{19}$$

Since G has no loop, it follows from (19) that

$$\alpha_0(G) \geqslant 4. \tag{20}$$

Consider any component H of $G\cdot S$ or $G\cdot T$. Let $x(H)$ denote the number of vertices of H in the intersection W of $V(G\cdot S)$ and $V(G\cdot T)$. Then

$$\alpha_1(H) \geqslant \alpha_0(H) - 1 \geqslant x(H) - 1. \tag{21}$$

We say H is of Type I if $\alpha_1(H) \geqslant x(H)$ and of Type II if $\alpha_1(H) = x(H) - 1$. In the latter case, H is a tree whose vertices are all in W. We say H is *transferable* if $|S - E(H)| \geqslant k$ and $|T - E(H)| \geqslant k$. We proceed to show that either $G\cdot S$ or $G\cdot T$ has a transferable component.

Case I. *One of the graphs* $G\cdot S$ *and* $G\cdot T$ *has more than k edges and all its components are of Type* II.

We may suppose that $|S| > k$ and that all the components of $G \cdot S$ are of Type II. Then we can find a monovalent vertex v of $G \cdot S$. Let A be the edge of $G \cdot S$ incident with v and let w be its other end.

If w is monovalent in $G \cdot S$, then $G \cdot \{A\}$ is a transferable component of $G \cdot S$.

If w is not monovalent in $G \cdot S$, we write $S' = S - \{A\}$, $T' = T \cup \{A\}$. We then have

$$\eta(G; S', T') = \eta(G; S, T) - 1,$$

$$p_0(G \cdot S') = p_0(G \cdot S),$$

$$p_0(G \cdot T') = p_0(G \cdot T) \quad \text{or} \quad p_0(G \cdot T) - 1.$$

Hence, by (16),

$$\eta(G; S', T') \leqslant k + p_0(G \cdot S') + p_0(G \cdot T') - 2.$$

Since Min $(|S'|, |T'|) \geqslant k$, the choice of S and T is contradicted.

Case II. *The components of* $G \cdot S$ *and* $G \cdot T$ *are all of Type* II. *Moreover* $|S| = |T| = k$.

In each of the graphs $G \cdot S$ and $G \cdot T$, the average valency is less than 2. Hence the average valency in G is less than 4.

Let w be a vertex of G of least valency γ. Then $\gamma \leqslant 3$. Let S_w be the set of the γ edges incident with w. Using (20) we find that $\alpha_1(G) \geqslant 2\gamma$. Hence

$$|E(G) - S_\omega| \geqslant |S_\omega| = \gamma.$$

But

$$\eta(G; S_w, E(G) - S_w) \leqslant |S_w| \leqslant 3.$$

It follows that $\lambda(G) \leqslant |S_\omega| \leqslant 3$. Hence $k = 1$ or 2. The first of these alternatives, however, is ruled out by (19).

By (18) and (19), we may now suppose that $G \cdot S$ has just two components, each with exactly one edge. Then

$$\alpha_0(G) = \alpha_0(G \cdot S) = \alpha_0(G \cdot T) = 4.$$

Since $|T| = k = 2$, the graph $G \cdot T$ also has exactly two components, each with a single edge. The connected graph G is therefore a quadrilateral. Hence $\lambda(G) = 2 = k$, which is contrary to assumption.

Case III. *Either* $G \cdot S$ *or* $G \cdot T$ *has two or more components, one of which is of Type* I.

We may suppose $G \cdot S$ to have a component H of Type I and at least one other component. We have

$$\alpha_1(H) \geqslant x(H) \geqslant 1,$$

$$|E(G) - E(H)| > |T| \geqslant k.$$

Moreover,

$$\eta(G; E(H), E(G) - E(H)) = x(H).$$

Now if $x(H) \leqslant k$, the above relations imply that $\lambda(G) \leqslant x(H) \leqslant k$, contrary to assumption. But if $x(H) > k$, the other components of $G \cdot S$ are transferable.

Case IV. *One of the graphs* $G \cdot S$ *and* $G \cdot T$ *is connected, and its only component is of Type* I.

We may suppose that $G \cdot S$ is connected. We may further suppose that the components of $G \cdot T$ are all of Type II and that $|T| = k$. For otherwise the conditions of Case I or Case III are satisfied. We then have

$$\begin{aligned}\eta(G; S, T) &= \alpha_0(G \cdot T) = |T| + p_0(G \cdot T) \\ &= k + p_0(G \cdot S) + p_0(G \cdot T) - 1,\end{aligned}$$

which is contrary to hypothesis.

The four cases discussed above exhaust all the possibilities. We deduce that either $G \cdot S$ or $G \cdot T$ has a transferable component.

We may suppose that $G \cdot S$ has a transferable component $G \cdot S_0$. Write

$$S' = S - S_0, \qquad T' = T \cup S_0.$$

Then

$$\begin{aligned}\eta(G; S', T') &= \eta(G; S, T) - x(G \cdot S_0), \\ p_0(G \cdot S') &= p_0(G \cdot S) - 1, \\ p_0(G \cdot T') &\geqslant p_0(G \cdot T) - x(G \cdot S_0) + 1.\end{aligned}$$

Hence, by (16),

$$\eta(G; S', T') \leqslant k + p_0(G \cdot S') + p_0(G \cdot T') - 2.$$

Moreover Min $(|S'|, |T'|) \geqslant k$, since $G \cdot S_0$ is transferable. But these results contradict the choice of S and T.

This contradiction establishes the theorem.

3.4. *If G is connected, then* $\lambda(G) \leqslant \lambda(P(G))$.

Proof. If $\lambda(P(G)) = \infty$, the theorem is trivial. In the remaining case, we can choose complementary subsets S and T of $E(G)$ such that

$$\xi(P(G); S, T) = \lambda(P(G)),$$

$$\text{Min}\,(|S|, |T|) \geqslant \lambda(P(G)).$$

We then have, by 3.1,

$$\eta(G; S, T) = \lambda(P(G)) + p_0(G \cdot S) + p_0(G \cdot T) - 2.$$

Hence $\lambda(G) \leqslant \lambda(P(G))$, by 3.3.

3.5. *If G is connected, then* $\lambda(P(G)) = \lambda(G)$ (by 3.2 and 3.4).

The preceding discussion is intended to justify the claim that the notion of connectivity for matroids is a generalization of that of connectivity for graphs. The results of the present section are not used in what follows.

4. Wheels and whirls. Let E be a set of $2n$ elements, where n is an integer $\geqslant 3$. We enumerate them as $A_1, A_2, \ldots, A_{2n}$ and adopt the convention that a suffix may be replaced by any other integer in its residue class (mod $2n$). We write Q for the set of all elements of E with even suffixes, and R for the set of all elements of E with odd ones.

Let j and k be integers such that $1 \leqslant j \leqslant n$ and $j < k < j + n$. Then we write

$$C_{j,k} = \{A_{2j}, A_{2j+1}, A_{2j+3}, \ldots, A_{2k-1}, A_{2k}\}.$$

The members of $C_{j,k}$, apart from A_{2j} and A_{2k}, have consecutive odd suffixes. We denote the class of all such subsets $C_{j,k}$ of E by C, and we write $W_n = C \cup \{R\}$.

For each integer j we write $R_j = R \cup \{A_{2j}\}$. We write Wr_n for the class of subsets of E obtained from C by adjoining the n subsets R_j $(1 \leqslant j \leqslant n)$.

We refer to W_n and Wr_n as the *wheel* and *whirl* respectively of *order* n. In the graphic context of **(1)** the term "wheel" is used for a wheel-graph. It can be verified that W_n is the polygon-matroid of a wheel-graph with the structure shown in Figure 4A. We shall not use this result in the arguments which follow, but we shall use the corresponding representation of E as the edge-set of a wheel-graph to illustrate some definitions. Thus in Figure 4B, representing the case $n = 7$, the thickened lines indicate $C_{1,2}$, and the broken ones $C_{3,7}$. The polygon made up of A_1, A_3, A_5, A_6, and A_{14} corresponds to $C_{7,3}$. The rim of the wheel-graph corresponds to R and the set of spokes to Q.

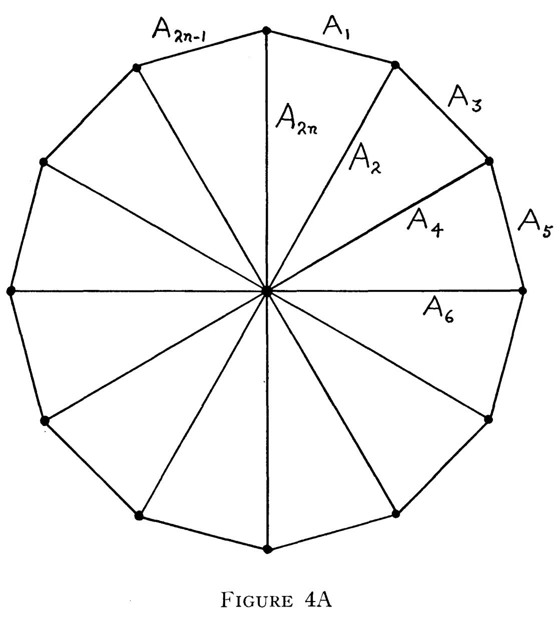

FIGURE 4A

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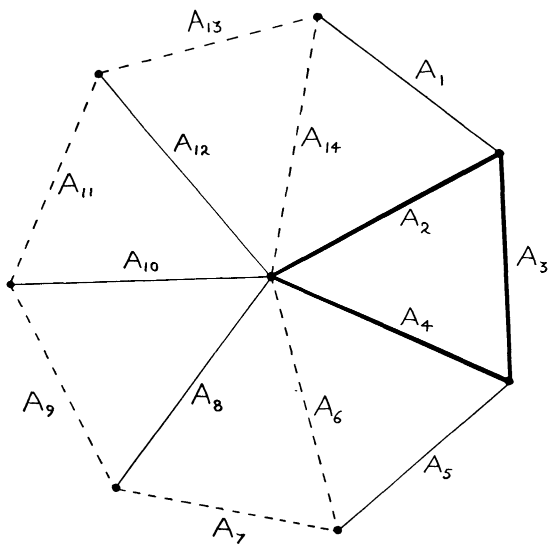

FIGURE 4B

4.1. *Let i be an integer and let U be a* (mod 2) *sum of members of W_n. Then $|U \cap \{A_{2i-1}, A_{2i}, A_{2i+1}\}|$ is even.*

Proof. It is readily verified that the theorem holds whenever U is a member of W_n. It must therefore hold for any (mod 2) sum of such members.

4.2. *Let U be a* (mod 2) *sum of members of W_n. Then if U is non-null, it is a union of one or more disjoint members of W_n.*

Proof. If $R \subseteq U$, it follows from 4.1 that $R = U$. The theorem then holds. If $R \cap U$ is null, it follows from 4.1 that U is null.

In the remaining case $R \cap U$ is a non-null proper subset of R. Let us define a *U-arc* of R as a maximal set of elements of $R \cap U$ having consecutive odd suffixes. Evidently the U-arcs of R are disjoint and their union is $R \cap U$.

Given a particular U-arc U_i of R we can write its elements as A_{2j+1}, $A_{2j+3}, \ldots, A_{2k-1}$, where $j < k < j + n$. Write $V_i = C_{j,k}$. Then the sets V_i corresponding to the U-arcs U_i are disjoint members of W_n. Let V be their union, which is also their (mod 2) sum. The (mod 2) sum of U and V has a null intersection with R. Hence $V = U$, by 4.1.

4.3. *W_n and Wr_n are matroids on E.*

Proof. It is clear that Axiom I holds for each of these classes. To verify Axiom II, let X and Y be any distinct members of the class $M = W_n$ or Wr_n under consideration, and let a and b be members of E such that $a \in X \cap Y$ and $b \in X - Y$. We must show that M has a member Z such that $b \in Z \subseteq (X \cup Y) - \{a\}$. If $M = W_n$, this result follows from 4.2, since b is in the (mod 2) sum of X and Y and this sum is a subset of $(X \cup Y) - \{a\}$.

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Suppose therefore that $M = Wr_n$. If X and Y are in C, there is a circuit Z' of W_n such that $b \in Z' \subseteq (X \cup Y) - \{a\}$. If $Z' \in Wr_n$, we may take it as Z. In the remaining case we have $Z' = R$. We can then write $X = C_{j,k}$ and $Y = C_{k,j+n}$. Accordingly Z is R_j or R_k, whichever does not include a.

If $X = R_j$ and $Y = R_k$, where $j < k < j + n$, we can put $Z = C_{j,k}$ or $C_{k,j+n}$, whichever does not include a.

In the remaining case, we may suppose one of X and Y to be of the form R_i and the other of the form $C_{j,k}$. If $A_{2i} = a$, so that $i \equiv j$ or k (mod $2n$), we write $Z = R_k$ or R_j respectively. If $A_{2i} = b$ we adjust the notation, by adding a multiple of n to j, so that $i < j < n + i$. We then put $Z = C_{i,j}$ or $C_{j,i+n}$, whichever does not include a. If A_{2i} is neither a nor b, there is a circuit T of W_n such that

$$b \in T \subseteq (R \cup C_{j,k}) - \{a\} \subseteq (X \cup Y) - \{a\}.$$

Evidently T is not R. It is therefore a member of Wr_n and we may take it as Z.

Let j and k be integers such that $1 \leqslant j \leqslant n$ and $j < k < j + n$. We write

$$D_{j,k} = \{A_{2j+1}, A_{2j+2}, A_{2j+4}, \ldots, A_{2k}, A_{2k+1}\}.$$

The members of $D_{j,k}$, apart from A_{2j+1} and A_{2k+1}, have consecutive even suffixes. We denote the class of all such subsets $D_{j,k}$ of E by D. In Figure 8C, representing the case $n = 7$, the thickened lines indicate $D_{4,5}$ and the broken ones $D_{6,3}$.

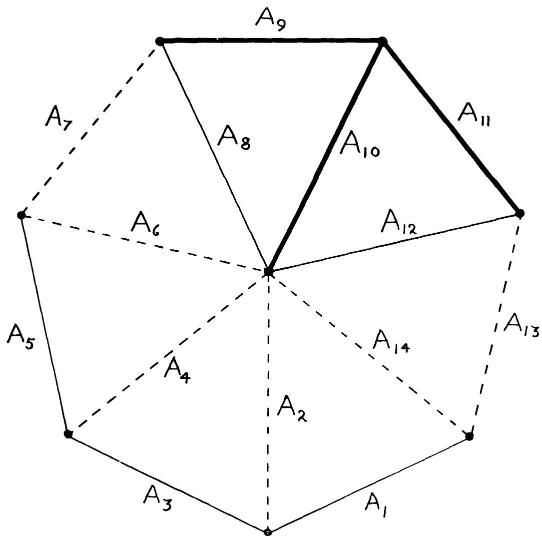

FIGURE 4C

For each integer j we write $Q_j = Q \cup \{A_{2j+1}\}$.

We define L as the class of all non-null subsets Y of E such that $|Y \cap X| \neq 1$ for each $X \in C$. Replacing C by W_n and Wr_n in this definition, we define sets L_1 and L_2 respectively in place of L.

4.4. *Let Y be any member of L. Then there exists $X \in D \cup \{Q\}$ such that $X \subseteq Y$.*

Proof. If $R \subseteq Y$, then $A_{2i} \in Y$ for some i since $|Y \cap C_{j,j+1}| \neq 1$ for each j. But then $D_{i-1,i} \subseteq Y$.

If $R \cap Y$ is null, consideration of the sets $C_{j,j+1}$ shows that $Q \subseteq Y$.

In the remaining case, $R \cap Y$ is a proper subset of R. Suppose first that we can find an integer j such that A_{2j+1} and A_{2j+2} belong to Y but A_{2j+3} does not. Let k be the first positive integer such that $A_{2j+2k+1}$ is in Y. Consideration of the sets $C_{j+1,j+2}$, $C_{j+2,j+3}$, etc. shows that $A_{2j+2}, A_{2j+4}, \ldots, A_{2j+2k}$ are all in Y. Hence if $k < n$, we have $D_{j,j+k} \subseteq Y$. The only alternative is $k = n$, which implies that $Q \subseteq Y$.

When $R \cap Y$ is a proper subset of R, we can always find an integer h such that $A_{2h+1} \in Y$ and $A_{2h+3} \notin Y$. If $A_{2h+2} \in Y$, we apply the preceding argument with $j = h$. If $A_{2h+2} \notin Y$, consideration of $C_{h,h+1}$ shows that $A_{2h} \in Y$. In this case, let k be the first positive integer such that $A_{2h-2k+1}$ is in Y. Evidently k exists and is at most n. Consideration of the sets $C_{h-1,h}$, $C_{h-2,h-1}$, etc. shows that $A_{2h}, A_{2h-2}, \ldots, A_{2h-2k+2}$ are all in Y. Accordingly either $D_{h-k,h} \subseteq Y$ or $Q \subseteq Y$.

4.5. $(W_n)^* = D \cup \{Q\}$.

Proof. It is easy to verify that each member of $D \cup \{Q\}$ is in L_1, and that no member of $D \cup \{Q\}$ contains another. But each member of L_1 belongs to L and so contains a member of $D \cup \{Q\}$, by 4.4. Hence $D \cup \{Q\}$ is the set of minimal members of L_1, that is $(W_n)^*$.

4.6. $(Wr_n)^* = D \cup \{Q_1, Q_2, \ldots, Q_n\}$.

Proof. Write $T = D \cup \{Q_1, Q_2, \ldots, Q_n\}$. It is easy to verify that each member of T is in L_2, and that no member of T contains another.

Suppose $Y \in L_2$. Then Y belongs to L and therefore contains either Q or a member of D, by 4.4. But if $Q \subseteq Y$, then $A_{2i+1} \in Y$ for some integer i, since $|Y \cap R_j| \neq 1$ for each integer j. Hence $Q_i \subseteq Y$.

We deduce that T is the set of minimal members of L_2, that is $T = (Wr_n)^*$.

The operation of increasing the suffix of each member A_h of E by 1 transforms C into D, R into Q, and R_i into Q_i, for each integer i. Hence, by 4.5 and 4.6, we have

4.7. *The dual of a wheel is a wheel, and the dual of a whirl is a whirl.*

5. Rank and connectivity in wheels and whirls. Let M be any matroid on a set E, and suppose $A \in S \subseteq E$. We define an integer $\theta(M; S, A)$ as follows: $\theta(M; S, A) = 1$ if there is a circuit X of M such that $A \in X \subseteq S$, and $\theta(M; S, A) = 0$ otherwise. Thus

(22) $$\theta(M; S, A) = r((M \times S) \cdot \{A\}),$$

(23) $$\theta(M; S, A) = r(M \times S) - r(M \times (S - \{A\})),$$

by (6).

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We make the following abbreviations:

$$M'_A = M \times (E - \{A\}), \tag{24}$$

$$M''_A = M \cdot (E - \{A\}). \tag{25}$$

Now let S and T be complementary subsets of $E - \{A\}$. Applying (23) to (8), we find that

$$\xi(M'_A; S, T) = \xi(M; S \cup \{A\}, T) - \theta(M; E, A) + \theta(M; S \cup \{A\}, A). \tag{26}$$

In a similar way we find

$$\xi(M''_A; S, T) = \xi(M; S \cup \{A\}, T) + \theta(M; \{A\}, A) - \theta(M; T \cup \{A\}, A). \tag{27}$$

However, in the proof of (27) we need the general theorem

$$(M \cdot U) \cdot V = M \cdot V, \tag{28}$$

where $V \subseteq U \subseteq E$, see (**2**, 3.332). This enables us to write, for example,

$$\begin{aligned} r(M'' \times S) &= r(M \cdot (E - \{A\})) - r(M \cdot T) \\ &= r(M \times (S \cup \{A\})) - r(M \times \{A\}), \end{aligned}$$

by (6). The details are given in (**3**, §3).

We proceed to apply these results to the theory of wheels and whirls. We use the notation of §4 for W_n and Wr_n.

5.1. *Write $M = W_n$ or Wr_n, and suppose $S \subseteq E$. Then the following propositions hold:*

(i) *If $S \subseteq R$ then $r(M \times S) = 0$, unless $S = R$ and $M = W_n$, in which case $r(M \times S) = 1$.*

(ii) *If $S \subseteq Q$, then $r(M \times S) = 0$.*

(iii) *If $R \subset S$, then $r(M \times S) = |S - R|$.*

(iv) *If $Q \subset S$, then $r(M \times S) = |S - Q|$.*

(v) *If $S \cap R$ is a proper non-null subset of R and S, then*

$$r(M \times S) \leqslant |S \cap Q| - 1.$$

Proof. Proposition (i) expresses the fact that S contains no circuit of M, except that S is itself a circuit when $S = R$ and $M = W_n$. Proposition (ii) follows from the fact that Q contains no circuit of M.

To prove (iii) we suppose first that $|S - R| = 1$. Then $M \times S$ has just one circuit, R if $M = W_n$ and one of the sets R_i if $M = Wr_n$. Hence

$$r(M \times S) = 1 = |S - R|.$$

If $|S - R| \geqslant 2$, we choose $A \in S - R$. There is a circuit X of M, belonging to C, such that $A \in X \subseteq S$. Hence $r(M \times (S - \{A\})) = r(M \times S) - 1$, by (23). Proposition (iii) now follows by induction on $|S - R|$.

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If $Q \subset S$, then each cell of $R \cap S$ belongs to a circuit X of M, of the form $C_{j,j+1}$, such that $X \subseteq S$. Thus for each $A \in R \cap S$ we have

$$r(M \times (S - \{A\})) = r(M \times S) - 1,$$

by (23). Repeated application of this result yields

$$r(M \times S) = |S - Q| + r(M \times Q) = |S - Q|, \qquad \text{by (ii)}.$$

To prove (v) we suppose first that $|S \cap Q| = 1$. It is then clear that S contains no circuit of M and so $r(M \times S) = 0 = |S \cap Q| - 1$. Hence we deduce, using $|S \cap Q| - 1$ applications of (23), that in the general case $r(M \times S) \leqslant |S \cap Q| - 1$.

5.2. *If* $M = W_n$ *or* Wr_n, *and* $A \in E$, *then* $r(M) = n$, $r(M'_A) = n - 1$, *and* $r(M''_A) = n$.

Proof. The first of these results follows from 5.1, (iii). The second then follows from (23). For the third we have

$$r(M''_A) = (2n - 1) - r((M^*)'_A),$$

by (3) and (5),

$$= (2n - 1) - (n - 1) = n,$$

by 4.7.

5.3. $\lambda(W_n) = \lambda(Wr_n) = 3$.

Proof. Write $M = W_n$ or Wr_n. Let S and T be complementary subsets of E. We note that $|E| = 2n \geqslant 6$.

Suppose $|S| \leqslant 2$. Since a wheel or whirl has no circuit of fewer than three cells, it follows from 4.7 that $r(M \times S) = r(M^* \times S) = 0$. Hence

$$\xi(M; S, T) > |S|,$$

by (11).

We may now suppose that $|S| \geqslant 3$ and $|T| \geqslant 3$. If $S \subseteq Q$, we have

$$\xi(M; S, T) = n - 0 - |T \cap Q| + 1, \qquad \text{by (8) and 5.1,} = |S| + 1 > 3.$$

If $S \subseteq R$, we have

$$\xi(M; S, T) \geqslant n - 1 - |T \cap R| + 1, \qquad \text{by (8) and 5.1,} = |S| \geqslant 3.$$

Similar results are obtained if $T \subseteq Q$ or $T \subseteq R$.

In the remaining case each of S and T meets both Q and R. Applying 5.1, (v), we have

$$\xi(M; S, T) \geqslant n - |Q \cap S| - |Q \cap T| + 3 = 3.$$

From the above results we see that $\lambda(M) \geqslant 3$.

Consider, however, the special case in which S is the circuit $C_{1,2}$ of M. Then $|S| = 3$, $|T| \geqslant 3$, and $r(M \times S) = 1$. Moreover, S contains no circuit of M^*, by 4.5 and 4.6, and therefore $r(M^* \times S) = 0$. It follows that $\xi(M; S, T) = 3$, by (11). We deduce that $\lambda(M) \leqslant 3$. The theorem follows.

5.4. *If* $M = W_n$ *or* Wr_n *and* $A \in E$, *then* $\lambda(M'_A) \leqslant 2$ *and* $\lambda(M''_A) \leqslant 2$.

Proof. We have $A \in D_{j,j+1}$ for some integer j. Write $S = D_{j,j+1}$ and $T = E - S$. Then $|S| = 3$ and $|T| \geqslant 3$. Evidently $r(M \times S) = 0$ and $r(M^* \times S) = 1$. Hence $\xi(M; S, T) = 3$, by (11).

We have $\theta(M; E, A) = 1$ and $\theta(M; S, A) = 0$. Hence

$$\xi(M'_A; S - \{A\}, T) = 2,$$

by (26). Since $|S - \{A\}| = 2$ and $|T| > 2$, it follows that $\lambda(M'_A) \leqslant 2$.

We have also $\lambda(M''_A) = \lambda((M^*)'_A)$, by (12) and (5), $\leqslant 2$, by 4.7 and the first part of the present proof.

If M is a 3-connected matroid and A is a cell of M such that neither M'_A nor M''_A is 3-connected, then we say that A is an *essential* cell of M. We may summarize the results of 5.3 and 5.4 as follows.

5.5. *The wheels and whirls are* 3-*connected matroids in which every cell is essential.*

6. Some theorems on connectivity. In what follows we are mainly concerned with proving that a 3-connected matroid in which every cell is essential must be either a wheel or a whirl. In the present section we set out some auxiliary theorems. We take M to be an arbitrary matroid on a set E.

From (7) and (8) we deduce that

$$(28) \quad \xi(M; S, T) + \xi(M; U, V) \geqslant \xi(M; S \cup U, T \cap V) + \xi(M; S \cap U, T \cup V),$$

where $\{S, T\}$ and $\{U, V\}$ are any two pairs of complementary subsets of E.

6.1. *Suppose* $A \in S \subseteq T \subseteq E$. *Then*

$$\theta(M; S, A) \leqslant \theta(M; T, A).$$

This theorem follows at once from the definition of θ.

6.2. *Let* S *and* T *be complementary subsets of* E *such that each circuit of* M *is contained in either* S *or* T. *Then* $\xi(M; S, T) = 1$.

Proof. By the definition of rank, we must have

$$r(M) = r(M \times S) + r(M \times T).$$

Hence $\xi(M; S, T) = 1$, by (8).

6.3. *If* M *is* 2-*connected and* $|E| \geqslant 2$, *then* $\theta(M; E, A) = 1$ *for each* $A \in E$.

Proof. If the theorem fails, we can find $A \in E$ such that each circuit of M is contained in $E - \{A\}$. Then $\xi(M; \{A\}, E - \{A\}) = 1$, by 6.2. Accordingly $\lambda(M) = 1$, which is contrary to hypothesis.

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6.4. *If M is 2-connected and $|E| \leqslant 3$, then* $\lambda(M) = \infty$.

Proof. Since $|E| \leqslant 3$ it is not possible for M to be k-separated for any integer $k \geqslant 2$.

The next proof can be regarded as a simple example of the kind of argument encountered in §7.

6.5. *If M is 2-connected and $A \in E$, then either M'_A or M''_A is 2-connected.*

Proof. Suppose the theorem fails. Then there are pairs $\{S, T\}$ and $\{U, V\}$ of complementary subsets of $E - \{A\}$ such that

$$\text{Min}\,(|S|, |T|, |U|, |V|) \geqslant 1, \tag{29}$$

$$\xi(M'_A; S, T) = 1, \tag{30}$$

$$\xi(M''_A; U, V) = 1. \tag{31}$$

Using (26) and the 2-connection of M, we deduce from (30) that

$$\xi(M; S \cup \{A\}, T) = \xi(M; S, T \cup \{A\}) = 2, \tag{32}$$

$$\theta(M; S \cup \{A\}, A) = \theta(M; T \cup \{A\}, A) = 0. \tag{33}$$

Similarly, by (27) and (31),

$$\xi(M; U \cup \{A\}, V) = \xi(M; U, V \cup \{A\}) = 2, \tag{34}$$

$$\theta(M; V \cup \{A\}, A) = \theta(M; U \cup \{A\}, A) = 1. \tag{35}$$

By (26), (34), and 6.3 we have

$$\xi(M'_A; U, V) \leqslant 2. \tag{36}$$

Hence, by (28) and (30),

$$3 \geqslant \xi(M'_A, S \cap U, T \cup V) + \xi(M'_A, S \cup U, T \cap V).$$

We can therefore adjust the notation, by interchanging S with T and U with V if necessary, so that

$$\xi(M'_A; S \cap U, T \cup V) = 1. \tag{37}$$

We note that $\theta(M; T \cup V \cup \{A\}, A) = 1$, by 6.1 and (35). Hence

$$\xi(M; S \cap U, T \cup V \cup \{A\}) = 1,$$

by (26), (37), and 6.3. It follows from (29) and the 2-connection of M that

$$|S \cap U| = 0.$$

But then $U \subseteq T$, whence $\theta(M, U \cup \{A\}, A) \leqslant \theta(M; T \cup \{A\}, A)$ by 6.1. This is contrary to (33) and (35).

6.6. *Suppose M is 3-connected and $|E| \geqslant 4$. Then $\lambda(M'_A) \geqslant 2$ and $\lambda(M''_A) \geqslant 2$ for each $A \in E$.*

Proof. Suppose the theorem fails for some $A \in E$. Then we can find complementary subsets S and T of $E - \{A\}$ such that

$$\text{Min}\,(|S|, |T|) \geqslant 1 \quad \text{and} \quad \xi(N; S, T) = 1,$$

where N is either M'_A or M''_A. Without loss of generality we may suppose that $1 \leqslant |S| \leqslant |T| \geqslant 2$. It follows from (26) and (27) that

$$\xi(M; S \cup \{A\}, T) \leqslant 2.$$

This implies that $\lambda(M) \leqslant 2$, contrary to hypothesis.

6.7. *If M is 3-connected and $|E| \geqslant 4$, then each circuit Y of M satisfies* $|Y| \geqslant 3$.

Proof. Suppose Y is a circuit of M such that $|Y| \leqslant 2$. Then $r(M \times Y) = 1$ and $r(M^* \times Y) \geqslant 0$. Hence

$$\xi(M; Y, E - Y) \leqslant |Y| - 1 - 0 + 1 = |Y|,$$

by (11). Since $1 \leqslant |Y| \leqslant E - Y|$, it follows that $\lambda(M) \leqslant |Y| \leqslant 2$, contrary to hypothesis.

6.8. *Suppose M is 3-connected and has an essential cell A. Then* $|E| \geqslant 5$.

Proof. Suppose $|E| \leqslant 4$. Then one of M'_A and M''_A is 2-connected by 6.5, and therefore 3-connected by 6.4. Hence A is not essential, contrary to hypothesis.

7. Triangles and triads.

In this section, M denotes a 3-connected matroid on a set E. We note that M^* is also 3-connected by (12).

A *triangle* of M is a circuit of M having just three cells. A *triad* of M is a triangle of M^*. We note that the triads of M^* are the triangles of M, by (2).

7.1. *Let A be an essential cell of M. Then A belongs to a triad or triangle of M.*

Proof. Since A is essential, it follows from 6.8 and 6.6 that there are pairs $\{S, T\}$ and $\{U, V\}$ of complementary subsets of $E - \{A\}$ such that

$$\text{Min}\,(|S|, |T|, |U|, |V|) \geqslant 2, \tag{38}$$

$$\xi(M'_A; S, T) = 2, \tag{39}$$

$$\xi(M''_A; U, V) = 2. \tag{40}$$

In analogy with the proof of 6.5 we find by (26) and (27) and the 3-connection of M that

$$\begin{aligned}\xi(M; S \cup \{A\}, T) &= \xi(M; S, T \cup \{A\}) \\ &= \xi(M\colon U \cup \{A\}, V) = \xi(M; U, V \cup \{A\}) = 3,\end{aligned} \tag{41}$$

$$\theta(M; S \cup \{A\}, A) = \theta(M; T \cup \{A\}, A) = 0, \tag{42}$$

$$\theta(M; U \cup \{A\}, A) = \theta(M; V \cup \{A\}, A) = 1. \tag{43}$$

From (26, (41), and (43) we have

(44) $$\xi(M'_A; U, V) = 3,$$

since $\theta(M; E, A) = 1$ by 6.3. Hence, by (28),

(45) $$\xi(M'_A; S \cup U, T \cap V) + \xi(M'_A; S \cap U, T \cup V) \leqslant 5.$$

We may therefore adjust the notation, interchanging S with T and U with V if necessary, so that

(46) $$\xi(M'_A; S \cap U, T \cup V) \leqslant 2.$$

By 6.1 and (43), we have $\theta(M; T \cup V \cup \{A\}, A) = 1$. Hence, by (26),

(47) $$\xi(M; S \cap U, T \cup V \cup \{A\}) \leqslant 2.$$

It follows by the 3-connection of M that $|S \cap U| \leqslant 1$. But if $|S \cap U| = 0$ we have $U \subseteq T$, $\theta(M; U \cup \{A\}, A) \leqslant \theta(M; T \cup \{A\}, A)$, by 6.1. This is contrary to (42) and (43). We deduce that $S \cap U$ consists of a single cell B.

We now repeat the first part of the preceding argument with U and V interchanged. From the analogue of (45) we deduce that either

$$\xi(M'_A; S \cup V, T \cap U) \leqslant 2$$

or

$$\xi(M'_A; S \cap V, T \cup U) \leqslant 2.$$

But

$$\theta(M; S \cup V \cup \{A\}, A) = \theta(M; T \cup U \cup \{A\}, A) = 1,$$

by (43) and 6.1. Hence, by (26), either

$$\xi(M; S \cup V \cup \{A\}, T \cap U) \leqslant 2$$

or

$$\xi(M; S \cap V, T \cup U \cup \{A\}) \leqslant 2.$$

It follows by the 3-connection of M that

(48) $$|S \cap V| = 1 \quad \text{or} \quad |T \cap U| = 1,$$

since the relations $V \subseteq T$ and $U \subseteq S$ are false by 6.1, (42), and (43).

If $S \cap V$ consists of a single cell C, we have $S = \{B, C\}$. Then

(49) $$r(M \times \{A, B, C\}) + r(M^* \times \{A, B, C\}) = 1,$$

by (11) and (41). If $T \cap U$ consists of a single cell C, then $U = \{B, C\}$, and (49) again follows from (11) and (41). By (49) and 6.7, the set $\{A, B, C\}$ is a circuit of either M or M^*, that is it is a triangle or triad of M.

7.2. *Suppose* $|E| \geqslant 4$. *Let* $\{A, B, C\}$ *be a triangle of* M *such that* $\lambda(M'_A) < 3$ *and* $\lambda(M'_B) < 3$. *Then there exists a triad of* M *which includes* A *and just one other cell of* $\{A, B, C\}$.

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Proof. By 6.6, there are pairs $\{S, T\}$ and $\{U, V\}$ of complementary subsets of $E - \{A\}$ and $E - \{B\}$, respectively, such that

(50) $$\text{Min}\,(|S|, |T|, |U|, |V|) \geqslant 2,$$

(51) $$\xi(M'_A; S, T) = 2,$$

(52) $$\xi(M'_B; U, V) = 2.$$

Using (26) and the 3-connection of M, we deduce that

(53) $$\xi(M; S \cup \{A\}, T) = \xi(M; S, T \cup \{A\}) = \xi(M; U \cup \{B\}, V) = \xi(M; U, V \cup \{B\}) = 3,$$

(54) $$\theta(M; S \cup \{A\}, A) = \theta(M; T \cup \{A\}, A) = \theta(M; U \cup \{B\}, B) = \theta(M; V\ U\,\{B\}, B) = 0.$$

If B and C are both in S or both in T, formula (54) is contradicted, since $\{A, B, C\}$ is a circuit of M. So without loss of generality we may write

(55) $$B \in S, \qquad C \in T.$$

Similarly we may put

(56) $$A \in U, \qquad C \in V.$$

Let us now make the assumption that the theorem fails.

Suppose $\theta(M; S, B) = 0$. Then, by (26) and (53),

(57) $$\xi(M'_B; S - \{B\}, T \cup \{A\}) = 2.$$

Now $\theta(M; T \cup \{A, B\}, B) = 1$ because of the triangle $\{A, B, C\}$. Hence, by (26) and (57),

(58) $$\xi(M; S - \{B\}, T \cup \{A, B\}) = 2.$$

Since M is 3-connected, it follows, by (50), that $|S - \{B\}| = 1$. Accordingly

(59) $$|S \cup \{A\}| = 3.$$

We now have

$$r(M \times (S \cup \{A\})) + r(M^* \times (S \cup \{A\}) = 1,$$

by (11) and (53). But $r(M \times (S \cup \{A\})) = 0$ by (54) and 6.7. Hence $S \cup \{A\}$ is a triad of M, by 6.7. Since $S \cup \{A\}$ contains A and B but not C, this result is contrary to assumption.

From this contradiction we deduce that

(60) $$\theta(M; S, B) = 1.$$

A similar argument, in which the roles of S and T, and also B and C, are interchanged, shows that

(61) $$\theta(M; T, C) = 1.$$

Similarly we have

$$(62) \qquad \theta(M;\, U, A) = 1,$$

$$(63) \qquad \theta(M;\, V, C) = 1.$$

By (26) and (53),

$$(64) \qquad \xi(M'_A;\, U - \{A\},\, V \cup \{B\}) = 3.$$

Applying (28) to (51) and (64), we obtain

$$(65) \quad \xi(M'_A; S \cap U, T \cup V \cup \{B\}) + \xi(M'_A; S \cup (U - \{A\}), T \cap V) \leqslant 5.$$

Hence either

$$\xi(M'_A; S \cap U, T \cup V \cup \{B\}) \leqslant 2$$

or

$$\xi(M'_A; S \cup (U - \{A\}), T \cap V) \leqslant 2.$$

Now $\theta(M; T \cup V \cup \{A, B\}, A) = 1$ because of the triangle $\{A, B, C\}$, and $\theta(M; S \cup U, A) = 1$ by (62) and 6.1. So, by (26), we have either

$$\xi(M; S \cap U, T \cup V \cup \{A, B\}) \leqslant 2$$

or

$$\xi(M; S \cup U, T \cap V) \leqslant 2.$$

Now $S \cap U$ is non-null, for otherwise we would have $U - \{A\} \subseteq T$, which implies that $\theta(M; U, A) \leqslant \theta(M; T \cup \{A\}, A)$ by 6.1, and the latter result is contrary to (54) and (62). Moreover, $T \cap V$ is not null since it contains C by (55) and (56). It follows by the 3-connection of M that either

$$(66) \qquad |S \cap U| = 1 \quad \text{or} \quad |T \cap V| = 1.$$

In a similar way we can deduce from (64) that either

$$\xi(M'_A; S \cap (V \cup \{B\}), T \cup (U - \{A\})) \leqslant 2$$

or

$$\xi(M'_A; T \cap U, S \cup V) \leqslant 2.$$

We have $\theta(M; T \cup U, A) = 1$ by (62) and 6.1, and

$$\theta(M; S \cup V \cup \{A\}, A) = 1$$

because of the triangle $\{A, B, C\}$. So, by (26), we have either

$$\xi(M; S \cap (V \cup \{B\}), T \cup U) \leqslant 2$$

or

$$\xi(M; T \cap U, S \cup V \cup \{A\}) \leqslant 2.$$

Now $T \cap U$ is not null, for otherwise we would have $U - \{A\} \subseteq S$, contrary to (54), (62), and 6.1. Moreover, $S \cap (V \cup \{B\})$ is not null, for it includes B, by (55). It follows by the 3-connection of M that either

$$(67) \qquad |S \cap (V \cup \{B\})| = 1 \quad \text{or} \quad |T \cap U| = 1.$$

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Suppose that the first alternative of (67) is true. Then $S \cap V$ is null, by (55). Hence $|V| = 1$ or $|S| = 2$, by (55), (56), and (66). The first of these alternatives is contrary to (50). The second is contrary to 6.7 since, by (60), there is a circuit of M contained in S. We deduce that $|T \cap U| = 1$.

If the second alternative of (66) holds, we now have $|T| = 2$, by (55). This contradicts 6.7 since T contains a circuit of M by (61). We conclude that

(68) $$|S \cap U| = |T \cap U| = 1.$$

We denote the members of $S \cap U$ and $T \cap U$ by J and K, respectively. Thus,

$$U = \{A, J, K\},$$

by (56). We note that U is a triangle of M, by (62) and 6.7.

J is distinct from A, B, and C since $A \notin S$, $B \notin U$, and $C \notin U$. Moreover, K is distinct from A, B, C, and J since $A \notin T$, $B \notin U$, $C \notin U$, and $J \notin T$.

By our assumption, the set $\{A, B, J\}$ is not a triad of M. Hence

$$r(M^* \times \{A, B, J\}) = 0, \qquad \text{by (12) and 6.7,}$$

$$r(M \cdot \{A, B, J\}) = 3, \qquad \text{by (2), (3) and (4),}$$

(69) $$r(M \times (E - \{A, B, J\})) = r(M) - 3,$$

by (6).

Suppose $\theta(M; E - \{A, B, J\}, K) = 1$. Then

$$r(M'_B \times (E - (U \cup \{B\}))) = r(M \times (E - (U \cup \{B\}))) = r(M) - 4,$$

by (23) and (69). We have also $r(M'_B) = r(M) - 1$, by (23) and 6.3. Moreover, $r(M'_B \times U) = r(M \times U) = 1$ since U is a circuit of M. Hence, by (8)

$$\xi(M'_B; U, V) = (r(M) - 1) - (r(M) - 4) - 1 + 1 = 3,$$

which is contrary to (52).

We deduce that $\theta(M; E - \{A, B, J\}, K) = 0$, whence $\theta(M; T, K) = 0$ by 6.1. Hence, by (26) and (53),

$$\xi(M'_K; S \cup \{A\}, T - \{K\}) = 2.$$

But $\theta(M; S \cup \{A, K\}, K) = 1$, by the triangle $\{A, J, K\}$. So by another application of (26) we have

$$\xi(M; S \cup \{A, K\}, T - \{K\}) = 2.$$

It follows by the 3-connection of M that $|T - \{K\}| \leqslant 1$, whence $|T| \leqslant 2$. But T contains a circuit of M, by (61). These results contradict 6.7. Our assumption is thus false, and the theorem follows.

7.3. *Suppose* $|E| \geqslant 4$. *Let* $\{A, B, C\}$ *be a triad of* M *such that* $\lambda(M''_A) < 3$ *and* $\lambda(M''_B) < 3$. *Then there exists a triangle of* M *which includes* A *and just one other cell of* $\{A, B, C\}$.

We obtain this result by applying 7.2 to M^*. For $\lambda(M^*) = \lambda(M)$, $\lambda(M''_A) = \lambda((M^*)'_A)$, and $\lambda(M''_B) = \lambda((M^*)'_B)$, by (5) and (12).

8. The main theorem.

8.1. *Let M be a 3-connected matroid, on a set E, in which every cell is essential. Then M^* is a 3-connected matroid having the same property*, (by (4), (5), and (12)).

8.2. *Let M be a 3-connected matroid, on a set E, in which every cell is essential. Then M is a wheel or a whirl.*

Proof. By 4.7 and 8.1 it does not matter whether we prove this theorem for M or for M^*.

We write a triad or triangle $\{A, B, C\}$ simply as ABC. We shall find it possible to make use of diagrams in which cells of M are represented by edges of a graph. We then represent a triangle of M by a graphic triangle, and a triad of M by three edges meeting at a vertex.

M has at least 5 cells, by 6.8. Let A_1 be one of them. It belongs to a triangle or triad of M, by 7.1. Replacing M by M^* if necessary, we may suppose that A_1 belongs to a triad $A_1 A_2 A_3$ of M. By 7.3, we can adjust the notation so that M has a triangle $A_2 A_3 A_4$, where $A_4 \neq A_1$. By 7.2, there is a triad $A_2 A_4 A_5$ or $A_3 A_4 A_5$ of M, where A_5 is not A_2 or A_3. We can evidently adjust the notation so that $A_3 A_4 A_5$ is a triad of M; cf. Figure 8A.

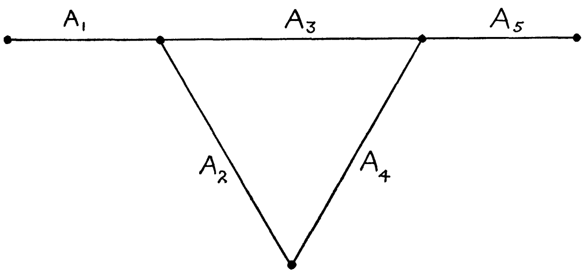

FIGURE 8A

In what follows, we make frequent use of the principle that a circuit of M and a circuit of M^* cannot have exactly one cell in common.

Assume $A_1 = A_5$. By 7.3 there is a triangle of M which includes $A_1 = A_5$ and just one other cell of $A_1 A_2 A_3$. By the principle just stated, it includes at least one cell of $A_3 A_4 A_5$ other than A_5. Thus either there is a triangle $A_1 A_2 A_4$ or $A_1 A_3 A_4$ of M, or there is a triangle $A_1 A_3 A_6$ of M, where A_6 is distinct from A_1, A_2, A_3, and A_4.

In the first alternative, we write $U = \{A_1, A_2, A_3, A_4\}$ and note that U contains two distinct triangles and two distinct triads of M. Using (23) we deduce that $r(M \times U) \geqslant 2$ and $r(M^* \times U) \geqslant 2$. Hence

$$\xi(M; U, E - U) = 1,$$

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by (9) and (11). It follows from the 3-connection of M that $U = E$. But this is impossible, by 6.8.

In the second alternative, we renumber A_6, A_1, A_3, A_2, and A_4 as A_1, A_2, A_3, A_4, and A_5 respectively, and interchange M and M^*.

We can thus always reduce to the case in which M has five distinct cells A_1, A_2, A_3, A_4, and A_5 such that $A_1 A_2 A_3$ and $A_3 A_4 A_5$ are triads and $A_2 A_3 A_4$ is a triangle.

Now, by 7.3, M has a triangle $A_3 A_5 A_6$ or $A_4 A_5 A_6$. In the first alternative, A_6 is A_1 or A_2 by the principle stated above. However, if $A_2 A_3 A_5$ is a triangle we find, using Axiom II and 6.7, that $A_3 A_4 A_5$ is a triangle. But this is impossible since $A_3 A_4 A_5$ has just one common cell with the triad $A_1 A_2 A_3$.

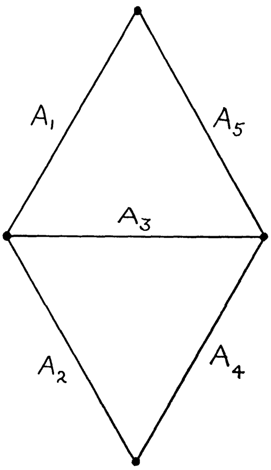

FIGURE 8B

In the first alternative, we therefore have triangles $A_2 A_3 A_4$ and $A_1 A_3 A_5$ and triads $A_1 A_2 A_3$ and $A_3 A_4 A_5$; see Figure 8B. Hence if

$$U = \{A_1, A_2, A_3, A_4, A_5\},$$

we have $r(M \times U) \geqslant 2$ and $r(M^* \times U) \geqslant 2$. Moreover, if the two cells A_3 and A_4 are removed, any remaining circuit of $M \times U$ must be a triangle of M having only one cell in common with the triad $A_3 A_4 A_5$, by 6.7. Since this is impossible, we have $r(M \times U) = 2$, by (23). Similarly all the circuits of $M^* \times U$ are destroyed by the deletion of the cells A_3 and A_4 of the triangle $A_2 A_3 A_4$, and $r(M^* \times U) = 2$. Hence

$$\xi(M; U, E - U) = 2,$$

by (11). We note that $E - U$ is not null, by (8). Hence, by the 3-connection of M, $E - U$ consists of a single cell, A_7 say.

By 7.1, 7.2, and 7.3 the cell A_7 must belong to a triangle of M. This triangle cannot include A_3, since it would then have to have another cell in each of the triads $A_1 A_2 A_3$ and $A_3 A_4 A_5$. Evidently the triangle is $A_1 A_2 A_7$ or $A_4 A_5 A_7$, and we can adjust the notation so that the latter possibility holds.

We deduce that the notation can be adjusted so that our second alternative holds, that is, there is a triangle $A_4 A_5 A_6$. Then A_6 is distinct from A_1, A_2, A_3, A_4, and A_5, since it cannot belong to the triad $A_1 A_2 A_3$; see Figure 8C.

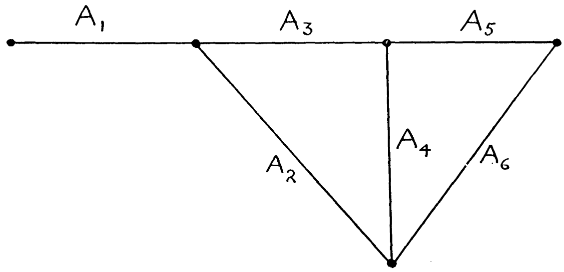

FIGURE 8C

We may now make the following assertion. There is an integer $n \geqslant 3$ such that M has $2n$ distinct cells $A_1, A_2, A_3, \ldots, A_{2n}$ with the following property: $A_j A_{j+1} A_{j+2}$ is a triangle of M if j is even and a triad of M if j is odd $(1 \leqslant j \leqslant 2n - 2)$; see Figures 8C and 8D. We may further suppose that these cells are chosen to correspond to the greatest possible value of n.

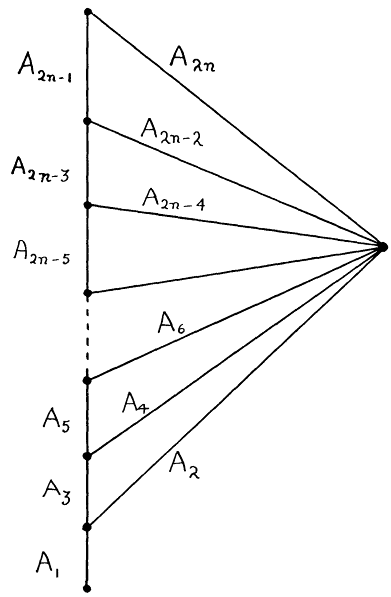

FIGURE 8D

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By 7.2 there is a triad T of M which contains A_{2n} and just one other member of the triangle $A_{2n-2} A_{2n-1} A_{2n}$. We denote the third member of T by A_{2n+1}.

Suppose $A_{2n-2} \in T$. Considering the triangle $A_{2n-4} A_{2n-3} A_{2n-2}$, we see that A_{2n+1} must be either A_{2n-4} or A_{2n-3}. Applying Axiom II to the circuits T and $A_{2n-3} A_{2n-2} A_{2n-1}$ of M^*, we find that there is a circuit Y of M^* contained in the set $U = \{A_{2n-4}, A_{2n-3}, A_{2n-1}, A_{2n}\}$. We have indeed $Y = U$, by 6.7, since Y cannot have only one cell in common with either of the triangles $A_{2n-4} A_{2n-3} A_{2n-2}$ and $A_{2n-2} A_{2n-1} A_{2n}$ of M. Applying Axiom II to the circuits Y and $A_{2n-5} A_{2n-4} A_{2n-3}$ of M^*, we find that there is a circuit Z of M^* contained in $\{A_{2n-5}, A_{2n-4}, A_{2n-1}, A_{2n}\}$. Then $Z = \{A_{2n-5}, A_{2n-1}, A_{2n}\}$ since Z cannot have only one cell in common with the triangle $A_{2n-4} A_{2n-3} A_{2n-2}$ of M, and by 6.7. If $n \geqslant 4$, then Z has just one cell in common with the triangle $A_{2n-6} A_{2n-5} A_{2n-4}$ of M. We deduce that $n = 3$, so that Z is the triad $A_5 A_6 A_1$ of M.

In the remaining case, A_{2n-1} is in T. Then A_{2n+1} cannot be one of the cells $A_2, A_3, A_4, \ldots, A_{2n-2}$, for otherwise it would be the only common cell of T and some particular triangle of M. We conclude that in each case M has a triad $A_{2n-1} A_{2n} A_{2n+1}$, where A_{2n+1} is either identical with A_1 or is distinct from each of the cells A_j $(1 \leqslant j \leqslant 2n)$.

We can now diminish each suffix by 1, interchange M and M^*, and repeat the preceding argument. The result, in terms of the original notation, is that M has a triangle $A_{2n} A_{2n+1} A_{2n+2}$, where A_{2n+2} is either identical with A_2 or distinct from each of the cells A_k $(1 < k \leqslant 2n + 1)$.

Consider the case in which A_{2n+1} is not A_1. Then A_{2n+2} is not A_1 or A_2, since the triangle $A_{2n} A_{2n+1} A_{2n+2}$ does not have only one cell in common with the triad $A_1 A_2 A_3$. But this result is contrary to our choice of n.

We may now suppose that A_{2n+1} is A_1. Considering the triad $A_1 A_2 A_3$, we see that A_{2n+2} must be A_2.

Write $U = \{A_1, A_2, \ldots, A_{2n}\}$. Deleting the cells of odd suffix, one by one, from $M \times U$ we find, using (23), that $r(M \times U) \geqslant n$. Similarly, deleting cells of even suffix from $M^* \times U$ we find that $r(M^* \times U) \geqslant n$. Hence

$$\xi(M; U, E - U) = 1,$$

by (9) and (11). It follows, by the 3-connection of M, that $U = E$.

Reverting to the notation of §4 we may say that the sets $C_{j,j+1}$ are circuits of M and that the sets $D_{j,j+1}$ are circuits of M^*. But suppose $C_{i,j}$ and $C_{j,k}$ are circuits of M, where $i < j < k < n + i$. Then, by Axiom II, there is a circuit Y of M contained in $C_{i,k}$. But then Y must be identical with $C_{i,k}$, since it cannot have only one cell in common with any of the triads $A_{2h+1} A_{2h+2} A_{2h+3}$, where $i - 1 \leqslant h \leqslant k - 1$. We deduce that $C \subseteq M$. Similarly $D \subseteq M^*$.

Applying Axiom II to the circuits $C_{j,j+1}$ and $C_{j+1,j+n}$ of M we find that there is a circuit Z_j of M contained in R_j. Consideration of the triads of M shows that Z_j must contain the whole of R. We deduce that either R is a circuit of M or R_j is a circuit of M for each j in the range $1 \leqslant j \leqslant n$. Similarly, either Q is a circuit of M^* or each Q_j is a circuit of M^*.

By 4.4 each circuit of M^* contains either Q or a member of D. Combining this with the results of the two preceding paragraphs, and using Axiom I, we find that $M^* = D \cup \{Q\}$ or $D \cup \{Q_1, Q_2, \ldots, Q_n\}$. Thus M^* is a wheel or a whirl, and the theorem is proved.

The main result of this paper is the combination of 5.5 and 8.2.

8.3. *A 3-connected matroid has all its cells essential if and only if it is a wheel or a whirl.*

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