Lightweight Description Logics with Meta-modelling

Paula Severi[1] and Edelweis Rohrer[2]

[1] Department of Computer Science, University of Leicester, England
[2] Instituto de Computación, Facultad de Ingeniería, Universidad de la República, Uruguay

Abstract. In order to incorporate meta-modelling in Description Logic, we add equations between individuals and concepts. We naturally require that the individual and the concept are semantically equal and the interpretation domain should be a well-founded set. In this paper, we consider the Description Logics behind the three OWL 2 profiles extended with meta-modelling and study the problems of checking consistency, classification and instance checking which turn out to be all tractable.

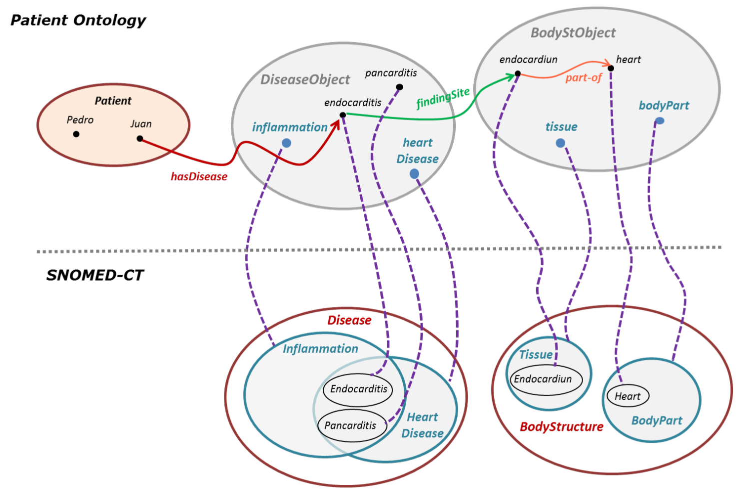
1 Introduction

We extend Description Logic with axioms $a =_{\mathsf{m}} A$ for equating an individual a to a concept A in a knowledge base [30,31,28]. Since the interpretations of a and A should be exactly the same, the domain of the interpretation cannot longer consist of only basic objects. We require that the domain should be a well-founded set. In particular, we do not allow sets that belong to themselves. Suppose we have a set $X = \{X\}$ that belongs to itself. Intuitively, X is the set $\{\{\{\ldots\}\}\}$. We exclude sets like X because they cannot represent any real object in the area of semantic information systems (in other areas or aspects of Computer Science, representing such objects is useful [4]).

This extension is motivated by real-world case studies where it is necessary to integrate and relate standard (or existing) ontologies where the same real object has been represented at different levels of granularity. In this case, we say that the ontologies are related through *meta-modelling* and the equation $a =_{\mathsf{m}} A$ is called a *meta-modelling axiom*. By adding meta-modelling axioms, concepts can become instances of another concept (called *meta-concept*) which itself can be an instance of yet another concept (called *meta meta-concept*) and so on.

The Web Ontology language OWL 2 profiles are lightweight Description Logics that have some modelling restrictions in order to improve reasoning efficiency and scalability, so that consistency, classification and instance checking can be performed in polynomial time [24,41]. Then, our main challenge is, taking into account the particular formal properties of $\mathcal{OWL}$-$\mathcal{EL}$, $\mathcal{OWL}$-$\mathcal{RL}$ and $\mathcal{OWL}$-$\mathcal{QL}$, to extend them with meta-modelling keeping their nice computational properties.

As an example of an ontology in the $\mathcal{OWL}$-$\mathcal{EL}$ profile, we consider SNOMED-CT which is a comprehensive clinical and medical ontology that covers a wide range of concepts in the health domain [38,18]. On one hand, suppose we have the ontology of SNOMED-CT with diseases represented as concepts, e.g. $Endocarditis$, and on the other hand, we have an ontology of clinical records linking patients with diseases represented as individuals, e.g. $Juan\ hasDisease\ endocarditis$ [7,5]. In order to connect both views, we equate each individual in one ontology with the corresponding concept in the other, e.g. $endocarditis =_{\mathsf{m}} Endocarditis$. This is illustrated in the next figure.



In order to exploit the SNOMED-CT hierarchy, it is useful to get inferences like "if Juan has endocarditis then Juan has a heart disease", since $Endocarditis$ is a subconcept of $HeartDisease$. To do this we introduce a new role characteristic

$$\mathsf{Propagate}(hasDisease)$$

in such a way that from the following explicit knowledge:

$$\begin{array}{ll} Juan\ hasDisease\ endocarditis & Endocarditis \sqsubseteq HeartDisease \\ endocarditis =_{\mathsf{m}} Endocarditis & heartDisease =_{\mathsf{m}} HeartDisease \end{array}$$

the reasoner will infer that $Juan\ hasDisease\ heartDisease$. The role characteristic $\mathsf{Propagate}(R)$ has the advantage of solving some common queries that happen often in practice such as "find out all the patients that suffer from heart disease" by just using the DL-query $\exists hasDisease.\{heartDisease\}$. Because of the ad-hoc evolution of SNOMED-CT, without ontological rigour, several studies have suggested rethinking the SNOMED-CT logical model. The need to represent some concepts as individuals and propagate some properties in the hierarchy has already been observed in earlier work but these problems have been tackled in different ways [40,36,39,35].

The kind of meta-modelling we consider in this paper can be expressed in OWL Full (OWL together with its metalanguage RDF) but it cannot be expressed in OWL DL. The fact that it is expressed in OWL Full is not very useful since the meta-modelling provided by OWL Full is so expressive that leads to undecidability [29]. OWL 2 DL has a very restricted form of meta-modelling called *punning* where the same identifier can be used as an individual and as a concept [11,15]. These identifiers are treated as different objects by the *reasoner* (engine that validates ontologies) and it is not possible to detect certain inconsistencies.

Glimm et al. do not define a set-theoretical semantics for meta-modelling. Instead, they *codify meta-modelling within OWL DL* [10]. This encoding is used to formalize the rules from the OntoClean methodology in OWL [14] but it has the limitation of having only two levels of meta-modelling (concepts and meta-concepts) and it is not enough to "fully" detect inconsistencies coming from meta-modelling [31]. A similar approach is followed by Rector et al. who consider codes to represent diseases as individuals and *bind* SNOMED with HL 7 [35,5].

There are several papers on meta-modelling in Description Logic that follow a *Hilog style semantics* [29,9,16,17,27,25,13]. In this style of semantics, the same object can have different interpretations depending on the position or role it plays in a sentence (subject or predicate). From the Description Logic point of view, this means that the same object can have different interpretations depending on the way it is represented. For example, the object endocarditis as an individual in $DiseaseObject(endocarditis)$ does not have the same interpretation as the same object endocarditis represented as a class in $Endocarditis \sqsubseteq Disease$.

We use a direct semantics (Henkin's style semantics) and give the object endocarditis the same semantics independent from the way it was represented. Our semantics detects inconsistencies that the Hilog semantics cannot. For example, if the user wrongly states that $Pancarditis \equiv Endocardits$ and $pancarditis \neq endocarditis$ then Hilog semantics cannot detect the inconsistency while we can.

Hilog semantics also ignores the delicate issue on *well-founded sets*. For example, suppose the user adds the axiom $DiseaseObject \sqsubseteq HeartDisease$ to the ontology. Then, we would infer that $HeartDisease(heartdisease)$ which is non-sense because the notion of "heart disease" can not itself be a heart disease. Our approach reports this ontology as inconsistent because we require that the interpretation domain should be a well-founded set.

In our approach, the well-foundness of the interpretation domain is not ensured by means of fixing layers beforehand as in [33,19,16,17] but it is our reasoner which checks for circularities. The *fixed layer approach* forces the user to explicitly write the information of the layer in the syntax and this means that standard ontologies have to be modified by adding this extra information in all its classes and individuals, e.g. $Endocarditis^1$ in SNOMED CT and $Juan^1$ and $endocarditis^1$ in the ontology of patients records. Another drawback of the fixed layer approach is that we cannot mix levels. In particular, we would not be able to re-write the concept hierarchy of SNOMED because inclusions of concepts at different levels are not allowed, e.g. $DiseaseObject^2 \equiv Disease^1$.

Using our approach, the layers are *silent in the syntax*. The user does not have to write or know the layer of the concept because the reasoner will infer it for him. The user has the flexibility of changing the status of an object at any point without having to make any substantial change to the ontology. Our approach does not restrict the level j of B to be the successor of the level i of a when $B(a)$ holds. This has the advantage that individuals with meta-modelling can co-exist with individuals without meta-modelling in the same concept. In a real scenario of evolving ontologies, that need to be integrated, not all objects of a given class need to have meta-modelling and hence, they do not have to belong to the same level.

Contributions and Outline. This is the first paper to study the lightweight ontology languages extended with equations between individuals and concepts based on a direct semantics of well-founded sets. The role characteristic Propagate has not been considered before and it is a useful and simple device to solve some common queries involving meta-modelling.

Section 2 recalls the Description Logics behind the three OWL2 profiles. Section 3 gives the syntax and semantics of the extension of an arbitrary Description Logic with meta-modelling. This section also studies general conditions for building a model of a knowledge base with meta-modelling. Section 4 gives a general algorithm for computing semantic consequences for certain logics called *consequential*. Section 5 outlines our future work.

2 Preliminaries on the Lightweight Ontology Languages

In this section, we recall the syntax for the three OWL profiles $\mathcal{OWL}$-$\mathcal{EL}$, $\mathcal{OWL}$-$\mathcal{RL}$ and a variant of $\mathcal{OWL}$-$\mathcal{QL}$. The semantics for the constructors is shown below.

Name	Syntax	Semantics
top	$\top$	$\Delta^{\mathcal{I}}$
bottom	$\bot$	$\emptyset$
nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
complement	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
disjunction	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
existential restriction	$\exists R.C$	$\{x \in \Delta^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}$
universal restriction	$\forall R.C$	$\{x \in \Delta^{\mathcal{I}} \mid \text{for all } y \in \Delta^{\mathcal{I}} \text{ if } (x,y) \in R^{\mathcal{I}} \text{ then } y \in C^{\mathcal{I}}\}$
cardinality restriction	$\leqslant 1R.C$	$\{x \in \Delta^{\mathcal{I}} \mid \sharp\{y \in \Delta^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \leq 1\}$
range	$\mathsf{range}(R)$	$\{y \in \Delta^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}}\}$
self restriction	$\exists R.Self$	$\{x \in \Delta^{\mathcal{I}} \mid (x,x) \in R^{\mathcal{I}}\}$
empty role chain	ϵ	$\{x \in \Delta^{\mathcal{I}} \mid (x,x) \in \Delta^{\mathcal{I}}\}$
inverse role	R^-	$(R^-)^{\mathcal{I}} = \{(x,y) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (y,x) \in R^{\mathcal{I}}\}$
GCI	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
key restriction	$C \text{ key } R_1 \ldots R_n$	if $a_1^{\mathcal{I}} \in C^{\mathcal{I}}$, $a_2^{\mathcal{I}} \in C^{\mathcal{I}}$, $(a_1^{\mathcal{I}}, b_i^{\mathcal{I}}) \in {R_i}^{\mathcal{I}}$ and $(a_2^{\mathcal{I}}, b_i^{\mathcal{I}}) \in {R_i}^{\mathcal{I}}$ for $1 \leq i \leq n$ then $a_1^{\mathcal{I}} = a_2^{\mathcal{I}}$
RI	$R_1 \circ \cdots \circ R_k \sqsubseteq R$	$R_1^{\mathcal{I}} \circ \cdots \circ R_k^{\mathcal{I}} \subseteq R^{\mathcal{I}}$
disjoint roles	$\mathsf{Dis}(R,S)$	$R^{\mathcal{I}} \cap S^{\mathcal{I}} = \emptyset$
irreflexive role	$\mathsf{Irr}(R)$	$(x,x) \notin R^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}}$
concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
role assertion	$R(a,b)$	$(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$
same individuals	$a = b$	$a^{\mathcal{I}} = b^{\mathcal{I}}$
different individuals	$a \neq b$	$a^{\mathcal{I}} \neq b^{\mathcal{I}}$

The Description Logic $\mathcal{OWL}$-$\mathcal{EL}$ is essentially $\mathcal{EL}^{++}$ extended with reflexive roles, range restrictions and keys [2,3,34,41]. Concepts and axioms are defined as follows.

$$\begin{array}{ll}\mathsf{Axiom} := & \mathsf{C} \sqsubseteq \mathsf{C} \mid \mathsf{C}\ \mathsf{key}\ R_1\ \ldots\ R_n \mid R_1 \circ \cdots \circ R_k \sqsubseteq R \mid \mathsf{range}(R) \sqsubseteq \mathsf{C} \\ \mathsf{C} := & A \mid \top \mid \bot \mid \mathsf{C} \sqcap \mathsf{C} \mid \exists R.\mathsf{C} \mid \{a\} \mid \exists R.Self\end{array}$$

When $k = 0$, we assume that the role inclusion is $\epsilon \sqsubseteq R$ which allows us to declare that R is reflexive. Roles can have range declarations provided they satisfy the restriction that if $R_1 \circ R_2 \circ \ldots \circ R_k \sqsubseteq S \in \mathcal{K}$ with $k \geq 1$ and $\mathcal{K} \models \mathsf{range}(S) \sqsubseteq C$ then $\mathcal{K} \models \mathsf{range}(R_k) \sqsubseteq C$ [3].

Recall that the axioms of an Abox can be expressed using nominals and inclusions as follows: $C(a)$ is equivalent to $\{a\} \sqsubseteq C$, $R(a,b)$ is equivalent to $\{a\} \sqsubseteq \exists R.\{b\}$, $a = b$ is equivalent to $\{a\} \equiv \{b\}$ and $a \neq b$ is equivalent to $\{a\} \sqcap \{b\} = \bot$. Moreover, the $\mathcal{OWL}$-$\mathcal{EL}$ profile allows class disjointness, transitive object properties, domain restrictions and negative object property assertions, which can be expressed as $C \sqcap D \sqsubseteq \bot$, $R \circ R \sqsubseteq R$, $\exists R.\top \sqsubseteq C$ and $\{a\} \sqcap \exists R.\{b\} \sqsubseteq \bot$.

For simplicity, we do not include concrete domains. Since meta-modelling makes sense only between individuals and concepts, adding concrete domains seems like an orthogonal problem and it should not be difficult to add them.

We now recall the Description Logic $\mathcal{OWL}$-$\mathcal{RL}$ (without ObjectHasValue in the superclasses) as described in [23]. Concepts, roles and axioms are defined as follows.

$$
\begin{array}{ll}
\mathsf{Axiom} := & \mathsf{CL} \sqsubseteq \mathsf{CR} \mid \mathsf{CR}(a) \mid R(a,b) \mid a = b \mid a \neq b \mid R_1 \circ \cdots \circ R_k \sqsubseteq R, k > 0 \mid \\
 & \mathsf{CL}\ \mathsf{key}\ R_1\ \ldots\ R_n \mid \mathsf{Dis}(R,S) \mid \mathsf{Irr}(R) \\
\mathsf{CL} := & \bot \mid \top \mid A \mid \bot \mid \mathsf{CL} \sqcap \mathsf{CL} \mid \exists R.\mathsf{CL} \mid \mathsf{CL} \sqcup \mathsf{CL} \mid \{a\} \\
\mathsf{CR} := & \bot \mid \top \mid A \mid \neg\mathsf{CL} \mid \mathsf{CR} \sqcap \mathsf{CR} \mid \forall R.\mathsf{CR} \mid \leqslant 1 R.\mathsf{CL} \\
\mathsf{R} := & \mathsf{R} \mid \mathsf{R}^{-}
\end{array}
$$

$\mathcal{OWL}$-$\mathcal{RL}$ includes range restrictions, symmetric and asymmetric rol characteristics which can be expressed as $\top \sqsubseteq \forall R.C$, $R \sqsubseteq R^-$ and $\mathsf{Dis}(R, R^-)$ respectively.

We also consider $\mathcal{OWL}$-$\mathcal{QL}^*$, a variant of $\mathcal{OWL}$-$\mathcal{QL}$ with axioms $a = b$ and without role disjointness, irreflexive and asymmetric roles [1]. In the presence of meta-modelling where individuals can represent concepts, we cannot assume the unique name assumption. An implementation could replace a by b but the user should be able to see the inference $a = b$ when $A \equiv B$, $a =_{\mathsf{m}} A$, $b =_{\mathsf{m}} B$ hold.

$$
\begin{array}{ll}
\mathsf{Axiom} := & \mathsf{CL} \sqsubseteq \mathsf{CR} \mid \mathsf{CR}(a) \mid R(a,b) \mid a = b \mid a \neq b \mid R \sqsubseteq S \mid \epsilon \sqsubseteq R \\
\mathsf{CL} := & A \mid \top \mid \bot \mid \exists R.\top \\
\mathsf{CR} := & A \mid \top \mid \bot \mid \neg\mathsf{CL} \mid \mathsf{CR} \sqcap \mathsf{CR} \mid \exists R.\mathsf{CR} \\
\mathsf{R} := & \mathsf{R} \mid \mathsf{R}^{-}
\end{array}
$$

$\mathcal{OWL}$-$\mathcal{QL}^*$ includes class disjointness, range restrictions and symmetric rol characteristics which can be expressed as $C \sqsubseteq \neg D$, $\exists R^-.\top \sqsubseteq C$ and $R \sqsubseteq R^-$ respectively.

An *isomorphism between two interpretations* $\mathcal{I}$ and $\mathcal{I}'$ of a knowledge base $\mathcal{K}$ is a bijective function $f : \Delta^{\mathcal{I}} \to \Delta^{\mathcal{I}'}$ such that

- $f(a^{\mathcal{I}}) = a^{\mathcal{I}'}$
- $x \in A^{\mathcal{I}}$ if and only if $f(x) \in A^{\mathcal{I}'}$
- $(x,y) \in R^{\mathcal{I}}$ if and only if $(f(x), f(y)) \in R^{\mathcal{I}'}$.

The description logics $\mathcal{OWL}$-$\mathcal{EL}$, $\mathcal{OWL}$-$\mathcal{RL}$ or $\mathcal{OWL}$-$\mathcal{QL}^*$ all satisfy the following lemma which holds for first order logic:

Lemma 1 (Isomorphism). *Let $\mathcal{I}$ and $\mathcal{I}'$ be two isomorphic interpretations of a knowledge base $\mathcal{K}$ in $\mathcal{L}$ where $\mathcal{L}$ is either $\mathcal{OWL}$-$\mathcal{EL}$, $\mathcal{OWL}$-$\mathcal{RL}$ or $\mathcal{OWL}$-$\mathcal{QL}^*$. Then, $\mathcal{I}$ is a model of $\mathcal{K}$ if and only if $\mathcal{I}'$ is a model of $\mathcal{K}$.*

From now on, we assume that when we refer to a generic description logic $\mathcal{L}$, that logic also satisfies the above lemma.

3 Extending a Description Logic with Meta-modelling

The logic $\mathcal{LM}$ extends an arbitrary description logic $\mathcal{L}$ by adding metamodelling axioms and propagations. A metamodelling axiom is an equation $a =_{\mathsf{m}} A$ between an individual a and atomic concept A [30,31]. Propagations are role characteristics of the form $\mathsf{Propagate}(R)$ where R is a role (not a chain of roles). The precise semantics of these new constructors is described in the table below.

Name	Syntax	Semantics
Metamodelling Axiom	$a =_{\mathsf{m}} A$	$a^{\mathcal{I}} = A^{\mathcal{I}}$
Propagation	$\mathsf{Propagate}(R)$	if $(c^{\mathcal{I}}, a^{\mathcal{I}}) \in R^{\mathcal{I}}$ and $a^{\mathcal{I}} \subseteq b^{\mathcal{I}}$ then $(c^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$ for all $c \in (\mathcal{K}, \mathcal{M})$, $a =_{\mathsf{m}} A, b =_{\mathsf{m}} B$ in $\mathcal{M}$

We recall that a set X is *well-founded* if X does not have infinite $\ni$-decreasing sequences, i.e. there is no $\{x_i \mid i \in \mathbb{N}\} \subseteq X$ such that $x_i \ni x_{i+1}$ for all $i \in \mathbb{N}$.

Let $\mathcal{K}$ be a knowledge base in the logic $\mathcal{L}$ without meta-modelling. A knowledge base in $\mathcal{LM}$ is $\mathcal{KM} = (\mathcal{K}, \mathcal{M})$ where $\mathcal{M}$ is a set of metamodelling axioms and propagations, called *Mbox*.

We will show how to build a model for a knowledge base $(\mathcal{K}, \mathcal{M})$ with meta-modelling from a model for the knowledge base $\mathcal{K}$ without meta-modelling. For this, we start introducing some examples.

Suppose we have an ontology $(\mathcal{K}, \mathcal{M})$ with four individuals a, b, c and d with axioms $B(a)$, $A(c)$, $A(d)$ and the meta-modelling axioms given by $a =_{\mathsf{m}} A$ and $b =_{\mathsf{m}} B$. We consider the model $\mathcal{I}$ of $\mathcal{K}$ defined as follows.

$$\Delta^{\mathcal{I}} = \{a, b, c, d\} \quad A^{\mathcal{I}} = \{c, d\} \quad B^{\mathcal{I}} = \{a\}$$

In order to get a model $\mathcal{I}'$ of the knowledge base with meta-modelling, we need to impose the equations $a^{\mathcal{I}'} = A^{\mathcal{I}'} = \{c, d\}$ and $b^{\mathcal{I}'} = B^{\mathcal{I}'} = \{a^{\mathcal{I}'}\}$. We can see that $a^{\mathcal{I}'}$ and $b^{\mathcal{I}'}$ are not longer basic objects. We can build a model $\mathcal{I}'$ of $(\mathcal{K}, \mathcal{M})$ from the model $\mathcal{I}$ of $\mathcal{K}$ as follows.

$$\Delta^{\mathcal{I}'} = \{\{c, d\}, \{\{c, d\}\}, c, d\} \quad (A)^{\mathcal{I}'} = \{c, d\} \quad (B)^{\mathcal{I}'} = \{\{c, d\}\}$$

Suppose now we have an ontology $(\mathcal{K}, \mathcal{M})$ with two individuals a and b, the individual assertions $B(a)$ and $A(b)$, and the meta-modelling axioms $a =_{\mathsf{m}} A$ and $b =_{\mathsf{m}} B$. We consider the model $\mathcal{I}$ of the ontology without meta-modelling given by $\Delta^{\mathcal{I}} = \{a, b\}$,

$A^{\mathcal{I}} = \{b\}$ and $B^{\mathcal{I}} = \{a\}$. In order to take into account the meta-modelling, we impose the equations $a^{\mathcal{I}'} = A^{\mathcal{I}'} = \{b^{\mathcal{I}'}\}$ and $b^{\mathcal{I}'} = B^{\mathcal{I}'} = \{a^{\mathcal{I}'}\}$. This interpretation cannot be made into a model of $(\mathcal{K}, \mathcal{M})$ because $a^{\mathcal{I}'} \in b^{\mathcal{I}'} \in a^{\mathcal{I}'}$.

Suppose now we have a knowledge base $(\mathcal{K}, \mathcal{M})$ with three individuals a, b and c, the individual assertions: $A_2(c)$, $A_1(b)$ and the meta-modelling axioms: $a =_{\mathsf{m}} A_1$ and $a =_{\mathsf{m}} A_2$. We consider the model $\mathcal{I}$ of $\mathcal{K}$ given by

$$\Delta^{\mathcal{I}} = \{a, b, c\} \quad A_1^{\mathcal{I}} = \{b\} \quad A_2^{\mathcal{I}} = \{c\}$$

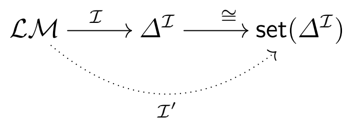
In order to build a model of $(\mathcal{K}, \mathcal{M})$ we need to force the equations $a^{\mathcal{I}'} = \{b\}$ and $a^{\mathcal{I}'} = \{c\}$. In this case, we cannot build a model of $(\mathcal{K}, \mathcal{M})$ because the interpretation of a is not uniquely determined when b and c are different.

A relation $\prec$ is *well-founded on* X if X has no infinite $\prec$-decreasing sequences, i.e. there are no $\{x_i \mid i \in \mathbb{N}\} \subseteq X$ such that $x_{i+1} \prec x_i$ for all $i \in \mathbb{N}$. Note that the relation $\prec$ does not have to be transitive [42].

Definition 1 (Coherent Model). *We say that a model $\mathcal{I}$ of $\mathcal{K}$ is coherent w.r.t. $\mathcal{M}$ if it satisfies the following conditions:*

1. $a^{\mathcal{I}} = b^{\mathcal{I}}$ *if and only if* $A^{\mathcal{I}} = B^{\mathcal{I}}$ *for all* $a =_{\mathsf{m}} A$ *and* $b =_{\mathsf{m}} B$ *in* $\mathcal{M}$.
2. $\prec$ *is a well-founded relation where* $\prec$ *as the smallest relation on* $\Delta^{\mathcal{I}}$ *such that* $y \prec x$ *if and only if* $a^{\mathcal{I}} = x$ *and* $a =_{\mathsf{m}} A$ *and* $y \in A^{\mathcal{I}}$.
3. *If* $(c^{\mathcal{I}}, a^{\mathcal{I}}) \in R^{\mathcal{I}}$ *and* $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ *then* $(c^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$ *for all* $a =_{\mathsf{m}} A$*,* $b =_{\mathsf{m}} B$ *and* R *such that* $\mathsf{Propagate}(R)$.

The model of a knowledge base $(\mathcal{K}, \mathcal{M})$ is built as the composition of two functions: a coherent model of $\mathcal{K}$ and a function set that computes the set associated to an individual with meta-modelling recursively.



More precisely, we define $\mathcal{I}'$ by

$$\begin{aligned} \Delta^{\mathcal{I}'} &= \{\mathsf{set}(x) \mid x \in \Delta^{\mathcal{I}}\} & a^{\mathcal{I}'} &= \mathsf{set}(a^{\mathcal{I}}) \\ A^{\mathcal{I}'} &= \{\mathsf{set}(x) \mid x \in A^{\mathcal{I}}\} & R^{\mathcal{I}'} &= \{(\mathsf{set}(x), \mathsf{set}(y)) \mid (x, y) \in R^{\mathcal{I}}\} \end{aligned}$$

where the function set is defined by recursion on $\prec$ as follows.

$$\begin{aligned} \mathsf{set}(x) &= \{\mathsf{set}(y) \mid y \in A^{\mathcal{I}}\} & &\text{if } a^{\mathcal{I}} = x \text{ and } a =_{\mathsf{m}} A \\ \mathsf{set}(x) &= x & &\text{otherwise} \end{aligned}$$

In the case when x is an individual a with meta-modelling, the recursive step applies the function set to y where $y \prec x$. The idea of the function set is that for an individual a with meta-modelling, $\mathsf{set}(a)$ gives the set of objects that a represents.

Theorem 1 (Model Construction). *Let $\mathcal{K}$ be a knowledge base in $\mathcal{L}$. If $\mathcal{K}$ has a coherent model $\mathcal{I}$ w.r.t. $\mathcal{M}$ then, the interpretation $\mathcal{I}'$ defined above is a model of* $(\mathcal{K}, \mathcal{M})$.

Proof. Suppose $\mathcal{I}$ is a coherent model of $\mathcal{K}$. We first prove that set is indeed a function and then prove that set is injective.

To prove that set is a function we consider $x = a^{\mathcal{I}}$ with $a =_{\mathsf{m}} A$ and $x = b^{\mathcal{I}}$ with $b =_{\mathsf{m}} B$. As $a^{\mathcal{I}} = b^{\mathcal{I}}$, by Item 1 of Definition 1 we have that $A^{\mathcal{I}} = B^{\mathcal{I}}$. Hence, $\mathsf{set}(x)$ is uniquely determined.

To prove that set is injective, we assume that $\mathsf{set}(x) = \mathsf{set}(y)$ and prove that $x = y$ by induction on $(\Delta^{\mathcal{I}}, \prec)$. We can do this induction because $\prec$ is well-founded on $\Delta^{\mathcal{I}}$ by Item 2 of Definition 1. Suppose $\mathsf{set}(x) = a$ and x has no meta-modelling. Hence, $x = \mathsf{set}(x) = \mathsf{set}(y) = y$. Suppose now that $x = a^{\mathcal{I}}$, and $a =_{\mathsf{m}} A$. Then, $y = b^{\mathcal{I}}$ and $b =_{\mathsf{m}} B$. Moreover,

$$\mathsf{set}(x) = \{\mathsf{set}(x') \mid x' \in A^{\mathcal{I}}\}$$
$$\mathsf{set}(y) = \{\mathsf{set}(y') \mid y' \in B^{\mathcal{I}}\}$$

Since $\mathsf{set}(x) = \mathsf{set}(y)$, we have that for all $x' \in A^{\mathcal{I}}$, there exists $y' \in B^{\mathcal{I}}$ such that $\mathsf{set}(x') = \mathsf{set}(y')$. Since $x' \prec x$, it follows from induction hypothesis that $x' = y'$. Hence, $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. Similarly, we can conclude that $B^{\mathcal{I}} \subseteq A^{\mathcal{I}}$. Hence, $A^{\mathcal{I}} = B^{\mathcal{I}}$. By Item 1 of Definition 1 we have that $a^{\mathcal{I}} = b^{\mathcal{I}}$ which means that $x = y$.

One can prove that $\Delta^{\mathcal{I}'}$ is well-founded using the fact that $\mathsf{set}(x) \in \mathsf{set}(y)$ if and only if $x \prec y$. Since set is a bijection from $\Delta^{\mathcal{I}}$ into $\Delta^{\mathcal{I}'}$, the interpretation $\mathcal{I}'$ is isomorphic to $\mathcal{I}$. By Isomorphism Lemma, it is also a model of $\mathcal{K}$. It only remains to prove that $\mathcal{I}'$ is a model of $\mathcal{M}$. Suppose that $a =_{\mathsf{m}} A$ in $\mathcal{M}$. It follows from the definitions of $\mathcal{I}'$ and set that

$$a^{\mathcal{I}'} = \mathsf{set}(a^{\mathcal{I}}) = \{\mathsf{set}(y) \mid y \in A^{\mathcal{I}}\} = A^{\mathcal{I}'} \quad (1)$$

Suppose $a =_{\mathsf{m}} A$, $b =_{\mathsf{m}} B$ and $\mathsf{Propagate}(R)$ are in $\mathcal{M}$, $(c^{\mathcal{I}'}, a^{\mathcal{I}'}) \in R^{\mathcal{I}'}$ and $a^{\mathcal{I}'} \subseteq b^{\mathcal{I}'}$. By Eq. (1), $a^{\mathcal{I}'} = A^{\mathcal{I}'}$ and $b^{\mathcal{I}'} = B^{\mathcal{I}'}$, so $A^{\mathcal{I}'} \subseteq B^{\mathcal{I}'}$ and by the isomorphism of $\mathcal{I}$ and $\mathcal{I}'$, we have that $(c^{\mathcal{I}}, a^{\mathcal{I}}) \in R^{\mathcal{I}}$ and $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$. It follows from the fact that $\mathcal{I}$ is coherent that $(c^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$. By the isomorphism, $(c^{\mathcal{I}'}, b^{\mathcal{I}'}) \in R^{\mathcal{I}'}$. □

4 A General Subsumption Algorithm for Consequential Logics

Since there are several algorithms for computing semantic consequences in a logic $\mathcal{L}$ (without meta-modelling), we wonder whether it is possible to define a corresponding algorithm for a knowledge base $(\mathcal{K}, \mathcal{M})$ in a logic $\mathcal{LM}$ (with meta-modelling) reusing any of the existing algorithms for $\mathcal{L}$. For this, we study under what conditions we can give a general polynomial reduction of the subsumption problem from $\mathcal{LM}$ to $\mathcal{L}$. The search of these conditions lead us to identify some logics which, for any knowledge base $\mathcal{K}$ in that logic, have the particularity of admitting models which capture the semantic consequences of $\mathcal{K}$. We call these kinds of models "consequential models".

Definition 2 (Consequential Model). *We say that a model $\mathcal{I}$ of $\mathcal{K}$ is consequential if the domain $\Delta^{\mathcal{I}}$ consists of only basic objects and for all α, if $\mathcal{I} \models \alpha$ then $\mathcal{K} \models \alpha$ where α is either $A \sqsubseteq B$, $A(a)$, $R(a, b)$ or $a = b$.*

Note in the above definition we used the converse of what holds for all model $\mathcal{I}$ of $\mathcal{K}$.

Definition 3 (Consequential Description Logic). *We say that a description logic $\mathcal{L}$ is consequential if all consistent knowledge bases in $\mathcal{L}$ have a consequential model.*

The consequential models in the following proof are variants of the counter-models used to prove completeness of the subsumption algorithms for $\mathcal{OWL}$-$\mathcal{EL}$ and $\mathcal{OWL}$-$\mathcal{RL}$ [8,23,22]. We prefer to give an explicit and direct definition of consequential model because, besides being more neat, it is independent of any particular algorithm.

Theorem 2. *The description logics $\mathcal{OWL}$-$\mathcal{EL}$, $\mathcal{OWL}$-$\mathcal{RL}$ and $\mathcal{OWL}$-$\mathcal{QL}^*$ are all consequential.*

Proof. Given a consistent knowledge base $\mathcal{K}$ of $\mathcal{OWL}$-$\mathcal{EL}$, we define $\mathcal{I}$ as follows.

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{[C] \mid \mathcal{K} \not\models C \sqsubseteq \bot\} \text{ where } [C] = \{C' \mid \mathcal{K} \models C \equiv C'\}\\
A^{\mathcal{I}} &= \{[C] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq A\}\\
(\exists R.Self)^{\mathcal{I}} &= \{[C] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq \exists R.Self\}\\
a^{\mathcal{I}} &= [\{a\}]\\
R^{\mathcal{I}} &= \{([C], [D]) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq \exists R.D \text{ and } \mathcal{K} \models D \sqsubseteq \mathsf{range}(R)\}
\end{aligned}$$

Since $\mathcal{K}$ is consistent, we have that $\Delta^{\mathcal{I}} \neq \emptyset$. The fact that $\mathcal{I}$ is consequential follows immediately from its definition. In order to prove that $\mathcal{I}$ is a model of $\mathcal{K}$, it is necessary to prove the following property which follows by induction on C.

$$C^{\mathcal{I}} = \{[D] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models D \sqsubseteq C\} \tag{2}$$

Eq. (2) holds for $C = \bot$ because we excluded $[D]$ such that $\mathcal{K} \models D \sqsubseteq \bot$ from $\Delta^{\mathcal{I}}$. For convenience in the proof, even though the knowledge base $\mathcal{K}$ should satisfy the required restrictions on the syntax for $\mathcal{OWL}$-$\mathcal{EL}$, we consider semantic consequences with more permissive syntax, i.e. $\mathcal{K} \models D \sqsubseteq \mathsf{range}(R)$ and also allow $[D \sqcap \mathsf{range}(R)]$ as an element of $D^{\mathcal{I}}$.

Given a consistent knowledge base $\mathcal{K}$ of $\mathcal{OWL}$-$\mathcal{RL}$, we define $\mathcal{I}$ as follows.

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{[C] \mid \mathcal{K} \not\models C \sqsubseteq \bot\} \cup \{[a] \mid a \text{ is an individual}\}\\
a^{\mathcal{I}} &= [a] \text{ where } [a] = \{b \mid \mathcal{K} \models a = b\}\\
A^{\mathcal{I}} &= \{[C] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq A\} \cup \{[a] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models A(a)\}\\
R^{\mathcal{I}} &= \{([a], [b]) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{K} \models R(a, b)\}
\end{aligned}$$

The fact that $\mathcal{I}$ is consequential follows immediately from its definition. One can prove that $\mathcal{I}$ is a model of $\mathcal{K}$ using the following two claims. The first claim is proved by induction on $C \in \mathsf{CL}$ and the second one by induction on $C \in \mathsf{CR}$.

Claim 2.1: Let $C \in \mathsf{CL}$.

1. If $[D] \in C^{\mathcal{I}}$ then $\mathcal{K} \models D \sqsubseteq C$.
2. If $[a] \in C^{\mathcal{I}}$ then $\mathcal{K} \models C(a)$.

Claim 2.2: Let $C \in \mathsf{CR}$.

1. If $\mathcal{K} \models D \sqsubseteq C$ and $\mathcal{K} \not\models D \sqsubseteq \bot$ then $[D] \in C^{\mathcal{I}}$.

2. If $\mathcal{K} \models C(a)$ then $[a] \in C^{\mathcal{I}}$.

Given a consistent knowledge base $\mathcal{K}$ of $\mathcal{OWL}$-$\mathcal{QL}^*$, we define $\mathcal{I}$ as follows.

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{[C] \mid \mathcal{K} \not\models C \sqsubseteq \bot \text{ for } C \in \mathsf{CR}\} \cup \{[a] \mid a \text{ is an individual}\} \\
a^{\mathcal{I}} &= [a] \text{ where } [a] = \{b \mid \mathcal{K} \models a = b\} \\
A^{\mathcal{I}} &= \{[C] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq A\} \cup \{[a] \in \Delta^{\mathcal{I}} \mid \mathcal{K} \models A(a)\} \\
R^{\mathcal{I}} &= \{([C], [D]) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq \exists R.D \text{ and } \mathcal{K} \models D \sqsubseteq \exists R^{-}.C\} \cup \\
&\quad \{([a], [b]) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{K} \models R(a, b)\} \cup \\
&\quad \{([a], [C]) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{K} \models \exists R.C(a) \text{ and } \mathcal{K} \models C \sqsubseteq \exists R^{-}.\top\} \cup \\
&\quad \{([C], [a]) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid \mathcal{K} \models C \sqsubseteq \exists R.\top \text{ and } \mathcal{K} \models \exists R^{-}.C(a)\}
\end{aligned}$$

Again, the fact that $\mathcal{I}$ is consequential follows immediately from its definition. One needs to prove two claims similar to the above ones. □

In the above proof, the interpretation domains are infinite. The size of $\Delta^{\mathcal{I}}$ is irrelevant since this result is independent of the algorithm. We have not included irreflexive roles in $\mathcal{OWL}$-$\mathcal{QL}^*$ because $([C], [C])$ could belong to $R^{\mathcal{I}}$ even if $\mathsf{Irr}(R)$ is in $\mathcal{K}$ (similar reason for not including disjoint roles).

Definition 4. *We say that $\mathcal{K}$ is saturated w.r.t. $\mathcal{M}$ if for all $a =_{\mathsf{m}} A$ and $b =_{\mathsf{m}} B$ in $\mathcal{M}$, the following conditions hold:*

1. *$\mathcal{K} \models A \equiv B$ if and only if $\mathcal{K} \models a = b$.*
2. *If $\mathcal{K} \models R(c, a)$, $\mathcal{K} \models A \sqsubseteq B$ then $\mathcal{K} \models R(c, b)$ for all c in $\mathcal{K} \cup \mathcal{M}$ and for all R such that $a =_{\mathsf{m}} A, b =_{\mathsf{m}} B, \mathsf{Propagate}(R) \in \mathcal{M}$.*

Definition 5 (Circularity). *Let $\mathcal{K}$ be a knowledge base in $\mathcal{L}$. We say that $\mathcal{K}$ has a circularity w.r.t. $\mathcal{M}$ if there is a sequence of meta-modelling axioms $a_1 =_{\mathsf{m}} A_1$, $a_2 =_{\mathsf{m}} A_2, \ldots, a_n =_{\mathsf{m}} A_n$ all in $\mathcal{M}$ such that $\mathcal{K} \models A_2(a_1)$, $\mathcal{K} \models A_3(a_2), \ldots, \mathcal{K} \models A_n(a_{n-1})$, $\mathcal{K} \models A_1(a_n)$.*

Lemma 2. *Let $\mathcal{K}$ be saturated and without circularities w.r.t. $\mathcal{M}$. If $\mathcal{I}$ is a consequential model of $\mathcal{K}$ then $\mathcal{I}$ is a coherent model w.r.t. $\mathcal{M}$.*

Proof. It is easy to show that $\mathcal{K} \models a = b$ iff $\mathcal{K} \models A \equiv B$ using the fact that $\mathcal{K}$ is saturated and $\mathcal{I}$ is a consequential model of $\mathcal{K}$. We now show that $\prec$ is well-founded. Suppose by contradiction that $\prec$ is not a well-founded relation. Hence, there exists an infinite $\prec$-decreasing sequence of the form $\ldots \prec x_i \prec \ldots \prec x_1$. By definition of $\prec$, we have that $x_i \in \Delta^{\mathcal{I}}$ for all $i \in \mathbb{N}$ and there exists an individual a_i in the Mbox such that $a_i^{\mathcal{I}} = x_i$. Since the Mbox is finite, there exists an element in the above sequence that should occur at least twice. Then, there exist $x_1 \ldots x_n \in \Delta^{\mathcal{I}}$ such that $x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \prec x_1$. So, we have that ${a_i}^{\mathcal{I}} = x_i \in {A_{i+1}}^{\mathcal{I}}$ for $1 \leq i \leq n-1$ and ${a_n}^{\mathcal{I}} = x_n \in {A_1}^{\mathcal{I}}$. Since $\mathcal{I}$ is a consequential model, $\mathcal{K} \models A_2(a_1)$, $\mathcal{K} \models A_3(a_2), \ldots, \mathcal{K} \models A_n(a_{n-1})$ and $\mathcal{K} \models A_1(a_n)$. This contradicts the fact that $\mathcal{K}$ has no circularities w.r.t. $\mathcal{M}$.

We now show Item 3 of Definition 1. Suppose $(c^{\mathcal{I}}, a^{\mathcal{I}}) \in R^{\mathcal{I}}$ and $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ with $a =_{\mathsf{m}} A$, $b =_{\mathsf{m}} B$ and $\mathsf{Propagate}(R)$ in $\mathcal{M}$. Since $\mathcal{I}$ is a consequential model, we have that $\mathcal{K} \models R(c, a)$ and $\mathcal{K} \models A \sqsubseteq B$. By Item 2 of Definition 4, $\mathcal{K} \models R(c, b)$. Since $\mathcal{I}$ is a model of $\mathcal{K}$, $(c^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$. □

Below, we show a general procedure for calculating a saturated knowledge base w.r.t. $\mathcal{M}$. We iteratively transfer equalities at the level of individuals to equalities at the level of concepts and conversely. This procedure could be seen as an algorithm parametric on the way we compute the semantic consequences of the form $\mathcal{K} \models \alpha$ where α is either $A \sqsubseteq B$, $a = b$ or $R(a, b)$.

Saturation Procedure
Input: knowledge base $\mathcal{K}$ and Mbox $\mathcal{M}$
Output: a saturated knowledge base w.r.t. $\mathcal{M}$

---

1. Initialization is $\mathcal{K}_{old} := \mathcal{K}$.
2. We calculate $\mathcal{K}_{new}$ as follows.

$$\begin{aligned}\mathcal{K}_{new} := \mathcal{K}_{old} \;\cup & \\ & \{A \equiv B \mid \mathcal{K}_{old} \models a \equiv b \in \mathcal{M} \text{ and } a =_{\mathsf{m}} A \text{ and } b =_{\mathsf{m}} B \in \mathcal{M}\} \;\cup \\ & \{a \equiv b \mid \mathcal{K}_{old} \models A \equiv B \text{ and } a =_{\mathsf{m}} A \text{ and } b =_{\mathsf{m}} B \in \mathcal{M}\} \;\cup \\ & \{R(c, b) \mid \mathsf{Propagate}(R) \text{ and } a =_{\mathsf{m}} A \text{ and } b =_{\mathsf{m}} B \text{ and} \\ & \mathcal{K}_{old} \models A \sqsubseteq B \text{ and } \mathcal{K}_{old} \models R(c, a)\}\end{aligned}$$

3. If $\mathcal{K}_{new} = \mathcal{K}_{old}$ then go to 4) else $\mathcal{K}_{old} := \mathcal{K}_{new}$ and go to 2).
4. $\mathsf{saturate}(\mathcal{K}, \mathcal{M}) := \mathcal{K}_{new}$.

Lemma 3 (Termination). *Let $T(m)$ be the complexity of computing the entailments of a knowledge base $\mathcal{K}$ of size m. The saturation procedure terminates and has complexity $T(n^3) \times n^3$ where n is the size of the input knowledge base $\mathcal{K} \cup \mathcal{M}$.*

Proof. The saturation procedure terminates because each step augments $\mathcal{K}_{new}$ which is bounded by $\mathcal{K} \cup \{A \equiv B, a = b \mid a =_{\mathsf{m}} A, b =_{\mathsf{m}} B \in \mathcal{M}\} \cup \{R(a, b) \mid a, b \in (\mathcal{K}, \mathcal{M})\}$. The size of $\mathcal{K}_{new}$ is at most n^3 and the number of times we compute the entailments of $\mathcal{K}_{new}$ is at most n^3. Hence, the whole procedure is at most $T(n^3) \times n^3$. □

Theorem 3. *Let $\mathcal{L}$ be a consequential description logic. Then, the following statements are equivalent:*

1. *either* $\mathsf{saturate}(\mathcal{K}, \mathcal{M})$ *has circularities w.r.t.* $\mathcal{M}$ *or* $\mathsf{saturate}(\mathcal{K}, \mathcal{M}) \models \alpha$
2. $(\mathcal{K}, \mathcal{M}) \models \alpha$

where α is either $\top \sqsubseteq \bot$, $A \sqsubseteq B$ or $A(a)$ or $R(a, b)$ or $a = b$.

Proof. Since the saturation procedure terminates, there exists an $n \in \mathbb{N}$ and an increasing sequence $\mathcal{K} = \mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \mathcal{K}_2 \subseteq \ldots \subseteq \mathcal{K}_n$ which satisfies:

Claim 3.1: If $(\mathcal{K}_i, \mathcal{M}) \models \alpha$ then $(\mathcal{K}, \mathcal{M}) \models \alpha$ for all $1 \leq i \leq n$.

Claim 3.2: $\mathcal{K}_n$ is saturated w.r.t. $\mathcal{M}$.

(1 ⇒ 2). We have that $\mathcal{K}_n = \mathsf{saturate}(\mathcal{K}, \mathcal{M})$.

1. Suppose $\mathcal{K}_n$ has a circularity w.r.t. $\mathcal{M}$, it is easy to see that $(\mathcal{K}_n, \mathcal{M})$ is inconsistent. By Claim 3.1, $(\mathcal{K}, \mathcal{M})$ is inconsistent too.
2. Suppose $\mathcal{K}_n \models \alpha$. It is easy to see that $(\mathcal{K}_n, \mathcal{M}) \models \alpha$. By Claim 3.1, $(\mathcal{K}, \mathcal{M}) \models \alpha$.

(2 ⇐ 1). Suppose $\mathcal{K}_n$ does not have a circularity w.r.t. $\mathcal{M}$ and $\mathcal{K}_n \not\models \alpha$. Since the logic $\mathcal{L}$ is consequential and $\mathcal{K}_n$ is consistent, there exists a consequential model $\mathcal{I}$ of $\mathcal{K}_n$. If α is $\top \sqsubseteq \bot$ then obviously, $\mathcal{I} \not\models \top \sqsubseteq \bot$ since $\mathcal{I}$ is a model. Otherwise, it follows from Definition 2 that $\mathcal{I} \not\models \alpha$. Since $\mathcal{K}_n$ is saturated and has no circularities w.r.t. $\mathcal{M}$, it follows from Lemma 2 that $\mathcal{I}$ is a coherent model. By Theorem 1, there exists an isomorphic interpretation $\mathcal{I}'$ of $\mathcal{I}$ such that $\mathcal{I}'$ is a model of $(\mathcal{K}, \mathcal{M})$. Since $\mathcal{I}$ and $\mathcal{I}'$ are isomorphic, we have that $\mathcal{I}' \not\models \alpha$. □

It is easy to see that Theorem 3 provides us with an algorithm for computing the semantic consequences of the form $(\mathcal{K}, \mathcal{M}) \models \alpha$ provided we have a way of computing $\mathcal{K} \models \alpha$ where α is either $\top \sqsubseteq \bot$, $A \sqsubseteq B$, $a = b$ or $R(a, b)$ for any consequential logic $\mathcal{L}$. In particular, we know that $(\mathcal{K}, \mathcal{M})$ is consistent if and only if $\mathsf{saturate}(\mathcal{K}, \mathcal{M})$ has no circularities w.r.t. $\mathcal{M}$ and $\mathcal{K}$ is consistent.

Checking that $\mathsf{saturate}(\mathcal{K}, \mathcal{M})$ has circularities w.r.t. $\mathcal{M}$ is equivalent to checking that a graph is acyclic. The nodes of the graph are the individuals with metamodelling and there is an edge from a to b if $a \prec b$. Determining whether a directed graph contains a cycle can be done in polinomial time [37,12].

Using Theorem 2 and Lemma 3, we can conclude that checking consistency, classifying and instance checking in $\mathcal{OWL}$-$\mathcal{ELM}$, $\mathcal{OWL}$-$\mathcal{RLM}$ or $\mathcal{OWL}$-$\mathcal{QLM}^*$ can all be done in polinomial time on the size of $\mathcal{K} \cup \mathcal{M}$ [2,3,22,21]. For that purpose, we can invoke any algorithm that computes the entailments of the form $\mathcal{K} \models \alpha$ [1,2,3,22,21,23].

In the presence of meta-modelling, it still makes sense to distinguish between combined and data complexity when the size of the Mbox (and the Tbox) is negligible compared with the size of the Abox.

It is easy to see that consistency, classification and instance checking in $\mathcal{OWL}$-$\mathcal{ELM}$ and $\mathcal{OWL}$-$\mathcal{RLM}$ are P-complete for combined and data complexity [2,3,23].

Even though these three problems are NLOGSPACE for $\mathcal{OWL}$-$\mathcal{QL}^*$ [1], having to check for circularities increases the combined complexity making them P-complete. To prove this, we use the fact that checking cycles in a graph is P-complete [12,26] and define the knowledge base $(\mathcal{K}_\mathcal{G}, \mathcal{M}_\mathcal{G})$ associated to a graph $\mathcal{G}$ by

$$\mathcal{K}_\mathcal{G} = \{A_v(a_w) \mid \text{there is an edge from } w \text{ to } v\} \quad \mathcal{M}_\mathcal{G} = \{a_v =_\mathsf{m} A_v \mid v \in \mathcal{G}\}$$

where a_v, A_v are names created for each vertex v of $\mathcal{G}$. It is easy to see that $\mathcal{G}$ has a cycle if and only if $\mathcal{K}_\mathcal{G}$ has a circularity w.r.t. $\mathcal{M}_\mathcal{G}$.

The procedure just described does not work for logics beyond $\mathcal{ALCM}$ as shown by the following two examples.

Consider the knowledge base $(\mathcal{K}, \mathcal{M})$ where $\mathcal{K}$ has the axioms $A \sqcup B(b)$ and $A \sqcup B(a)$ and $\mathcal{M}$ has the axiomas $a =_\mathsf{m} A$ and $b =_\mathsf{m} B$. It is not difficult to see that $(\mathcal{K}, \mathcal{M}) \models \top \sqsubseteq \bot$ but $\mathcal{K}$ is saturated, it has no circularities w.r.t. $\mathcal{M}$ and $\mathcal{K} \not\models \top \sqsubseteq \bot$.

In the previous example, we used the fact that the domain of an interpretation of $\mathcal{LM}$ has to be well-founded. We now show an example where the algorithm does not work even if we did not have this requirement. The idea is to build a knowledge base in $\mathcal{ALCOM}$ that expresses the following facts:

1. If a and b are equal then A and B are disjoint.
2. If a and b are different then A and B are equal.

$$\mathcal{K} \quad \{c\} \sqsubseteq A \quad \top \sqsubseteq (\mathsf{Eq}(\{a\},\{b\}) \sqcup \mathsf{Eq}(A,B)) \quad \top \sqsubseteq (\mathsf{Disj}(A,B) \sqcup \mathsf{Disj}(\{a\},\{b\}))$$
$$\mathcal{M} \quad a =_{\mathsf{m}} A \quad b =_{\mathsf{m}} B$$

where we use $\mathsf{Eq}(C,D)$ as an abbreviation for $(\neg C \sqcup D) \sqcap (\neg D \sqcup C)$ and $\mathsf{Disj}(C,D)$ for $\neg C \sqcap D \sqcup C \sqcap \neg D$. We have that $(\mathcal{K},\mathcal{M})$ is inconsistent. However, $\mathcal{K}$ is consistent, saturated and does not have circularities w.r.t. $\mathcal{M}$.

The procedure for checking consistency in the fixed level approach uses a saturation procedure similar to ours and in their case it works for logics beyond $\mathcal{ALC}$ [33]. The reason for this is that the fixed level approach is less expressive than ours and cannot express the above two examples. The first one because an individual at level i has to belong to a concept at level $i+1$. The second one is because we are mixing levels by having the disjunction of a meta-concept with a concept.

5 Future Work

We have studied the problems of consistency, classification and instance checking in the three OWL 2 profiles extended with meta-modelling. Though, it still remains to see how we can include irreflexive and disjoint roles to the third profile. We also leave the study of data complexity for the third profile as future research. There is still work to be done by combining meta-modelling with other variants of DLs. It should also be possible to generalize Theorem 2 to Horn-Description Logics [20,32].

We could also consider a more liberal semantics of $\mathsf{Propagate}(R)$ defined by: if $(x, a^{\mathcal{I}}) \in R^{\mathcal{I}}$ and $a^{\mathcal{I}} \subseteq b^{\mathcal{I}}$ then $(x, b^{\mathcal{I}}) \in R^{\mathcal{I}}$ for all $a =_{\mathsf{m}} A, b =_{\mathsf{m}} B$ in $\mathcal{M}$. It will also be interesting to study ways of incorporating the rules of Ontoclean using role characteristics similar to $\mathsf{Propagate}(R)$ [14].

The next important step will be to study ontology-mediated query answering for Description Logics extended with our approach to meta-modelling and to solve queries that cannot be covered by $\mathsf{Propagate}(R)$ [6].

As a final remark, we would like to add that we are extending the $\mathcal{OWL}$-$\mathcal{EL}$ reasoner ELK with metamodelling [22].

6 Acknowledgements

The third author would like to acknowledge a Daphne Jackson fellowship sponsored by EPSRC and the University of Leicester. We would like to thank Frank Wolter, Ian Horrocks, Giuseppe de Giacomo, Yevgeny Kazakov, Mónica Martinez and Regina Motz for discussions.

References

1. A. Artale, D. Calvanese, R. Kontchakov, and M. Zakharyaschev. The DL-Lite Family and Relations. *J. Artif. Intell. Res. (JAIR)*, 36:1–69, 2009.
2. F. Baader, S. Brandt, and C. Lutz. Pushing the EL envelope. In *IJCAI-05, Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence*, pages 364–369, 2005.
3. F. Baader, S. Brandt, and C. Lutz. Pushing the el envelope further. In K. Clark and P. F. Patel-Schneider, editors, *In Proceedings of the OWLED 2008 DC Workshop on OWL: Experiences and Directions*, 2008.
4. K. Barwise and L. Moss. *Vicious Circles. On the Mathematics of Non-Wellfounded Phenomena*. Cambridge University Press, 1996.
5. T. Benson and G. Grieve. *Principles of Health Interoperability. SNOMED CT, HL7 and FHIR*. Springer-Verlag, 2016.
6. M. Bienvenu and M. Ortiz. Ontology-mediated query answering with data-tractable description logics. In *Reasoning Web. Web Logic Rules - 11th International Summer School 2015, Berlin, Germany, July 31 - August 4, 2015, Tutorial Lectures*, pages 218–307, 2015.
7. S. El-Sappagh, M. E. Mogy, and A. M. Riad. A standard fragment of EHR relational data model for diabetes mellitus diagnosis. In *Informatics and Systems (INFOS), 2014 9th International Conference on*, pages DEKM–1–DEKM–9, 2014.
8. F. Baader, S. Brandt, and C. Lutz. Pushing the el envelope. LTCS-Report LTCS-05-01, Chair for Automata Theory, Institute for Theoretical Computer Science, Dresden University of Technology, Germany, 2005. See http://lat.inf.tu-dresden.de/research/reports.html.
9. G. D. Giacomo, M. Lenzerini, and R. Rosati. Higher-order description logics for domain metamodeling. In *Proceedings of the Twenty-Fifth AAAI Conference on Artificial Intelligence, AAAI 2011*. AAAI Press, 2011.
10. B. Glimm, S. Rudolph, and J. Völker. Integrated metamodeling and diagnosis in OWL 2. In *International Semantic Web Conference, ISWC 2010*, pages 257–272, 2010.
11. B. C. Grau, I. Horrocks, B. Motik, B. Parsia, P. F. Patel-Schneider, and U. Sattler. OWL 2: The next step for OWL. *J. Web Sem.*, 6:309–322, 2008.
12. R. Greenlaw, H. J. Hoover, and W. L. Ruzzo. A compendium of problems complete for p, 1991.
13. Z. Gu. Meta-modeling extension of horn-SROIQ and query answering. In *Proceedings of DL 2016*, 2016.
14. N. Guarino and C. A. Welty. An overview of ontoclean. In *Handbook on Ontologies*, pages 201–220. 2009.
15. P. Hitzler, M. Krötzsch, and S. Rudolph. *Foundations of Semantic Web Technologies*. Chapman & Hall/CRC, 2009.
16. M. Homola, J. Kluka, V. Svátek, and M. Vacura. Towards typed higher-order description logics. In *Proceedings of the 26th International Workshop on Description Logic*, pages 221–233, 2013.
17. M. Homola, J. Kluka, V. Svátek, and M. Vacura. Typed higher-order variant of SROIQ - why not? In *Informal Proceedings of the 27th International Workshop on Description Logics*, pages 567–578, 2014.
18. IHTSDO. SNOMED CT Starter Guide, 2014.
19. N. Jekjantuk, G. Gröner, and J. Z. Pan. Modelling and reasoning in metamodelling enabled ontologies. *International Journal Software and Informatics*, 4(3):277–290, 2010.
20. Y. Kazakov. Consequence-driven reasoning for horn SHIQ ontologies. In *Proceedings of IJCAI*, pages 2040–2045, 2009.
21. Y. Kazakov, M. Krötzsch, and F. Simancik. Practical reasoning with nominals in the EL family of description logics. In *Principles of Knowledge Representation and Reasoning: Proceedings of the Thirteenth International Conference, KR 2012*, 2012.

22. Y. Kazakov, M. Krötzsch, and F. Simancik. The incredible ELK - from polynomial procedures to efficient reasoning with EL ontologies. *J. Autom. Reasoning*, 53(1):1–61, 2014.
23. M. Krötzsch. The not-so-easy task of computing class subsumptions in OWL RL. In *The Semantic Web - ISWC 2012 - 11th International Semantic Web Conference, Proceedings, Part I*, pages 279–294, 2012.
24. M. Krötzsch. OWL 2 profiles: An introduction to lightweight ontology languages. In *Reasoning Web. Semantic Technologies for Advanced Query Answering - 8th International Summer School 2012, Proceedings*, pages 112–183, 2012.
25. P. Kubincová, J. Kluka, and M. Homola. Expressive description logic with instantiation metamodelling. In *Proceedings of KR 2016*, pages 569–572, 2016.
26. T. Lengauer and K. W. Wagner. *The correlation between the complexities of the non-hierarchical and hierarchical versions of graph problems*, pages 100–113. Springer Berlin Heidelberg, Berlin, Heidelberg, 1987.
27. M. Lenzerini, L. Lepore, and A. Poggi. A higher-order semantics for metaquerying in OWL 2 QL. In *Proceedings of KR 2016*, pages 577–580, 2016.
28. M. Martinez, E. Rohrer, and P. Severi. Complexity of the description logic ALCM. In *Principles of Knowledge Representation and Reasoning: Proceedings of the Fifteenth International Conference, KR 2016*, pages 585–588, 2016.
29. B. Motik. On the properties of metamodeling in OWL. *Journal of Logic and Computation*, 17(4):617–637, 2007.
30. R. Motz, E. Rohrer, and P. Severi. Reasoning for *ALCQ* extended with a flexible meta-modelling hierarchy. In *4th Joint International Semantic Technology Conference, JIST 2014*, volume 8943 of *Lecture Notes in Computer Science*, pages 47–62, 2014.
31. R. Motz, E. Rohrer, and P. Severi. The description logic *SHIQ* with a flexible meta-modelling hierarchy. *Journal of Web Semantics: Science, Services and Agents on the World Wide Web*, 2015.
32. M. Ortiz, S. Rudolph, and M. Simkus. Worst-case optimal reasoning for the Horn-DL fragments of OWL 1 and 2. In *Proceedings of KR 2010*, 2010.
33. J. Z. Pan, I. Horrocks, and G. Schreiber. OWL FA: A metamodeling extension of OWL DL. In *OWLED*, volume 188 of *CEUR Workshop Proceedings*, 2005.
34. B. Parsia, U. Sattler, and T. Schneider. Easy keys for OWL. In *Proceedings of the Fifth OWLED Workshop on OWL*, 2008.
35. A. L. Rector, R. Qamar, and T. Marley. Binding ontologies and coding systems to electronic health records and messages. *Applied Ontology*, 4:51–69, 2009.
36. S. Schulz, B. Suntisrivaraporn, F. Baader, and M. Boeker. SNOMED reaching its adolescence: Ontologists' and logicians' health check. *I. J. Medical Informatics*, 78:S86–S94, 2009.
37. R. Sedgewick and K. Wayne. *Algorithms, 4th Edition*. Addison-Wesley, 2011.
38. M. Q. Stearns, C. Price, K. A. Spackman, and A. Y. Wang. SNOMED clinical terms: overview of the development process and project status. *Proceedings of the AMIA Symposium*, pages 662–666, 2001.
39. B. Suntisrivaraporn. *Polynomial time reasoning support for design and maintenance of large-scale biomedical ontologies*. PhD thesis, Dresden University of Technology, 2009.
40. B. Suntisrivaraporn, F. Baader, S. Schulz, and K. A. Spackman. Replacing sep-triplets in SNOMED CT using tractable description logic operators. In *Artificial Intelligence in Medicine, 11th Conference on Artificial Intelligence in Medicine, AIME 2007, Amsterdam, The Netherlands, July 7-11, 2007, Proceedings*, pages 287–291, 2007.
41. W3C. OWL 2 Web Ontology Language Profiles (Second Edition), Last date accessed May 2016.

42. G. Winskel. Notes and Exercises for Set Theory for Computer Science (pdf) an MPhil course in Advanced Computer Science at the University of Cambridge. (Last data accessed January 2015).

A Model Construction

We show more examples of models of $\mathcal{K}$ that cannot be transformed into a model of $(\mathcal{K}, \mathcal{M})$.

Example 1. Consider the knowledge base $(\mathcal{K}, \mathcal{M})$ with individuals a, b, c, d, e and axioms: $A(c)$, $B(c)$, $R(a, d)$ and $R(b, e)$. We also have that R is functional. The metamodelling is given by the axioms: $a =_{\mathsf{m}} A$ and $b =_{\mathsf{m}} B$. We consider the model $\mathcal{I}$ of $\mathcal{K}$ given by:

$$\begin{array}{l} \Delta^{\mathcal{I}} = \{a, b, c, d, e\} \text{ where } a, b, c, d \text{ are pairwise different elements,} \\ A^{\mathcal{I}} = \{c\} \\ B^{\mathcal{I}} = \{c\} \\ R^{\mathcal{I}} = \{(a, d), (b, e)\} \end{array}$$

In order to get a model of the knowledge base with metamodelling, we need to force the equations $a = \{c\}$ and $b = \{c\}$. Now, we obtain an interpretation $\mathcal{I}'$ whose domain is:

$$\Delta^{\mathcal{I}'} = \{\{c\}, c, d, e\}$$

Since $A^{\mathcal{I}'} = B^{\mathcal{I}'} = \{c\}$, we see that $a = a^{\mathcal{I}'} = b^{\mathcal{I}'} = b$. However, $\mathcal{I}'$ is not a model of $(\mathcal{K}, \mathcal{M})$ because in order to be a model we would need besides that $a = b$ also that $d = e$ to make R functional.

Note that if we add the axiom $d \neq e$ to the original ontology then this ontology is inconsistent.

Example 2. We consider the following knowledge base:

$\mathcal{K}$	$A(c)$ $B(c)$ $C(a)$ $D(b)$ $C \sqcap D \equiv \bot$
$\mathcal{M}$	$a =_{\mathsf{m}} A$ $b =_{\mathsf{m}} B$

Suppose we have an interpretation that satisfies the following:

$$\begin{array}{l} A^{\mathcal{I}} = B^{\mathcal{I}} = \{c^{\mathcal{I}}\} \\ C^{\mathcal{I}} = \{a^{\mathcal{I}}\} \\ D^{\mathcal{I}} = \{b^{\mathcal{I}}\} \end{array}$$

Then, $\mathcal{I}$ is a model of $\mathcal{K}$ if and only if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. But $A^{\mathcal{I}} = B^{\mathcal{I}}$ and $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. It is not possible to transform this interpretation into a model of $(\mathcal{K}, \mathcal{M})$.

B Consequential Logics

We prove that that $[D] \in C^{\mathcal{I}}$ iff $\mathcal{K} \models D \sqsubseteq C$ and $\mathcal{K} \not\models D \sqsubseteq \bot$ by induction on C.

- Case $C = A$ follows by definition of $A^{\mathcal{I}}$.
- Case $C = \exists R.Self$ follows by definition of $(\exists R.Self)^{\mathcal{I}}$.
- Case C is $\exists R.E$. Suppose $[D] \in (\exists R.E)^{\mathcal{I}}$. Hence, there is F such that $([D], [F]) \in R^{\mathcal{I}}$ and $[F] \in E^{\mathcal{I}}$. It follows from the definition of $R^{\mathcal{I}}$ that $\mathcal{K} \models D \sqsubseteq \exists R.F$ and from induction hypothesis that $\mathcal{K} \models F \sqsubseteq E$. It is easy to show that $\mathcal{K} \models D \sqsubseteq \exists R.E$. We now prove the converse. Suppose $\mathcal{K} \models D \sqsubseteq \exists R.E$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. Hence, $\mathcal{K} \models D \sqsubseteq \exists R.(E \sqcap \mathsf{range}(R))$. By definition of $R^{\mathcal{I}}$, $([D], [E \sqcap \mathsf{range}(R)]) \in R^{\mathcal{I}}$. By induction hypothesis, $[E \sqcap \mathsf{range}(R)] \in E^{\mathcal{I}}$. Hence, $[D] \in (\exists R.E)^{\mathcal{I}}$.
- Case C is $E \sqcap F$. Suppose $[D] \in (E \sqcap F)^{\mathcal{I}}$. Then $[D] \in E^{\mathcal{I}}$ and $[D] \in F^{\mathcal{I}}$. By induction hypothesis, $\mathcal{K} \models D \sqsubseteq E$ and $\mathcal{K} \models D \sqsubseteq F$. It is easy to see that $\mathcal{K} \models D \sqsubseteq E \sqcap F$. We now prove the converse. Suppose $\mathcal{K} \models D \sqsubseteq E \sqcap F$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. It is easy to show that $\mathcal{K} \models D \sqsubseteq E$ and $\mathcal{K} \models D \sqsubseteq F$. By induction hypothesis, $[D] \in E^{\mathcal{I}}$ and $[D] \in F^{\mathcal{I}}$. So, $[D] \in (E \sqcap F)^{\mathcal{I}}$.
- Case C is a nominal $\{a\}$. Suppose $[D] \in \{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$. Hence, $[D] = [\{a\}]$. Obviously, $\mathcal{K} \models D \sqsubseteq \{a\}$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. We now prove the converse. Suppose $\mathcal{K} \models D \sqsubseteq \{a\}$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. It is not difficult to show that $\mathcal{K} \models D \equiv \{a\}$.
- Case C is $\bot$. It is trivially true since $C^{\mathcal{I}} = \emptyset$.
- Case C is $\top$. By definition of $\Delta^{\mathcal{I}}$, $[D] \in \top^{\mathcal{I}} = \Delta^{\mathcal{I}}$ if and only if $\mathcal{K} \models D \sqsubseteq \top$ and $\mathcal{K} \not\models D \sqsubseteq \bot$.

Proof that the interpretation $\mathcal{I}$ for $\mathcal{OWL}$-$\mathcal{EL}$ is a model of $\mathcal{K}$.

- Suppose $C \sqsubseteq D$ is in $\mathcal{K}$ and $[E] \in C^{\mathcal{I}}$. It follows from Eq. (2) that $\mathcal{K} \models E \sqsubseteq C$. Hence, $\mathcal{K} \models E \sqsubseteq D$. By applying Eq. (2) again, we have that $[E] \in D^{\mathcal{I}}$. This proves that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- Suppose that $R_1 \circ \cdots \circ R_k \sqsubseteq R$ is in $\mathcal{K}$.
  - Case $k = 0$. It is not difficult to see that $\mathcal{K} \models C \sqsubseteq \exists R.C$ if $\epsilon \sqsubseteq R$ is in $\mathcal{K}$. Hence, $([C], [C]) \in R^{\mathcal{I}}$ for all $[C] \in \Delta^{\mathcal{I}}$.
  - Case $k \geq 1$. Suppose $([C_1], [C_2]) \in R_1^{\mathcal{I}}$, ..., $([C_k], [C_{k+1}]) \in R_k^{\mathcal{I}}$. Hence, $\mathcal{K} \models C_1 \sqsubseteq \exists R_1.C_2, \ldots, \mathcal{K} \models C_k \sqsubseteq \exists R_k.C_{k+1}$ and $\mathcal{K} \models C_{k+1} \sqsubseteq \mathsf{range}(R_k)$. It is not difficult to show that $\mathcal{K} \models C_1 \sqsubseteq \exists R.C_{k+1}$. According to the restriction imposed on chains, if $\mathcal{K} \models \mathsf{range}(R) \sqsubseteq C$ then $\mathcal{K} \models \mathsf{range}(R_k) \sqsubseteq C$. In particular, we have that $\mathcal{K} \models \mathsf{range}(R_k) \sqsubseteq \mathsf{range}(R)$. Hence, $\mathcal{K} \models C_{k+1} \sqsubseteq \mathsf{range}(R)$ and $([C_1], [C_{k+1}]) \in R^{\mathcal{I}}$.
- Suppose $\mathsf{range}(R) \sqsubseteq E$ is in $\mathcal{K}$ and $([C], [D]) \in R^{\mathcal{I}}$. It follows from definition of $R^{\mathcal{I}}$ that $\mathcal{K} \models D \sqsubseteq E$. Using Eq. (2), we conclude that $[D] \in E^{\mathcal{I}}$.
- Suppose C key $R_1, \ldots R_n$ is in $\mathcal{K}$ and consider individuals a_1, a_2 and $b_1, \ldots, b_n$ such that $a_i^{\mathcal{I}} \in C^{\mathcal{I}}$ and $(a_i^{\mathcal{I}}, b_j^{\mathcal{I}}) \in R_j^{\mathcal{I}}$ for all $1 \leq i \leq 2$ and $1 \leq j \leq n$. By the definition of $\mathcal{I}$ on individuals and Eq. (2), $\mathcal{K} \models \{a_i\} \sqsubseteq C$ and $\mathcal{K} \models \{a_i\} \sqsubseteq \exists R_j.\{b_j\}$. Then, $\mathcal{K} \models \{a_1\} \equiv \{a_2\}$. Using again the definition of $\mathcal{I}$ on individuals, we conclude that $a_1^{\mathcal{I}} = a_2^{\mathcal{I}}$.

Proof of Claim 2.1. We prove both parts by induction on C.

- Case $C = A$. Both parts follow from the definition of $A^{\mathcal{I}}$.
- Case $C = \top$. Both parts are trivial since $\mathcal{K} \models D \sqsubseteq \top$ and $\mathcal{K} \models \top(a)$ for all D and a.
- Case $C = \bot$. Both parts are trivial since $C^{\mathcal{I}} = \emptyset$.
- Case C is $E \sqcap F$. We prove the first part. Suppose $[D] \in (E \sqcap F)^{\mathcal{I}}$. Then $[D] \in E^{\mathcal{I}}$ and $[D] \in F^{\mathcal{I}}$. By induction hypothesis, $\mathcal{K} \models D \sqsubseteq E$ and $\mathcal{K} \models D \sqsubseteq F$. It is easy to see that $\mathcal{K} \models D \sqsubseteq E \sqcap F$. We now prove the second part. Suppose $[a] \in (E \sqcap F)^{\mathcal{I}}$. Then $[a] \in E^{\mathcal{I}}$ and $[a] \in F^{\mathcal{I}}$. By induction hypothesis, $\mathcal{K} \models E(a)$ and $\mathcal{K} \models F(a)$. It is easy to see that $\mathcal{K} \models (E \sqcap F)(a)$.
- Case C is $\exists R.E$. The first part is trivially true since $[D] \notin (\exists R.E)^{\mathcal{I}}$. We prove the second part. Suppose $[a] \in (\exists R.E)^{\mathcal{I}}$. Hence, there is and individual b such that $([a], [b]) \in R^{\mathcal{I}}$ and $[b] \in E^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, we have that $\mathcal{K} \models R(a, b)$ and by induction hypothesis that $\mathcal{K} \models E(b)$. It is easy to show that $\mathcal{K} \models (\exists R.E)(a)$.
- Case C is $E \sqcup F$. We prove the first part. Suppose $[D] \in (E \sqcup F)^{\mathcal{I}}$. Then $[D] \in E^{\mathcal{I}}$ or $[D] \in F^{\mathcal{I}}$. By induction hypothesis, $\mathcal{K} \models D \sqsubseteq E$ or $\mathcal{K} \models D \sqsubseteq F$. It is easy to see that $\mathcal{K} \models D \sqsubseteq E \sqcup F$. We prove the second part. Suppose $[a] \in (E \sqcup F)^{\mathcal{I}}$. Then $[a] \in E^{\mathcal{I}}$ or $[a] \in F^{\mathcal{I}}$. By induction hypothesis, $\mathcal{K} \models E(a)$ or $\mathcal{K} \models F(a)$. It is easy to see that $\mathcal{K} \models (E \sqcup F)(a)$.
- Case C is a nominal $\{a\}$. The first part holds trivially since $[D] \notin \{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$. We prove the second part. Suppose $[a] \in \{b\}^{\mathcal{I}} = \{b^{\mathcal{I}}\}$. By definition of $b^{\mathcal{I}}$, $[a] = [b]$. Hence, $\mathcal{K} \models \{b\}(a)$.

Proof of Claim 2.2.

- Case $C = A$. Both parts follow from the definition of $A^{\mathcal{I}}$.
- Case $C = \top$ follows because $\Delta^{\mathcal{I}}$ contains all concepts and all individuals.
- Case $C = \bot$. The first part follows because $\mathcal{K} \models D \sqsubseteq \bot$ and $\mathcal{K} \not\models D \sqsubseteq \bot$ is not possible. The second part follows because $\bot(a)$ is not possible for $\mathcal{K}$ consistent.
- Case C is $\neg E$. We prove the first part. Suppose $\mathcal{K} \models D \sqsubseteq \neg E$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. Since $\mathcal{K}$ is consistent, $\mathcal{K} \not\models D \sqsubseteq E$. By Claim 2.1 part 1, $[D] \notin E^{\mathcal{I}}$. Hence, $[D] \in (\neg E)^{\mathcal{I}}$.
  We now prove the second part. Suppose $\mathcal{K} \models (\neg E)(a)$. Since $\mathcal{K}$ is consistent, $\mathcal{K} \not\models E(a)$. By Claim 2.1 part 2, $[a] \notin E^{\mathcal{I}}$, so $[a] \in (\neg E)^{\mathcal{I}}$.
- Case C is $E \sqcap F$. We prove the first part. Suppose $\mathcal{K} \models D \sqsubseteq E \sqcap F$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. It is easy to show that $\mathcal{K} \models D \sqsubseteq E$ and $\mathcal{K} \models D \sqsubseteq F$. By induction hypothesis, $[D] \in E^{\mathcal{I}}$ and $[D] \in F^{\mathcal{I}}$. So $[D] \in (E \sqcap F)^{\mathcal{I}}$.
  We now prove the second part. Suppose $\mathcal{K} \models (E \sqcap F)(a)$. It is easy to show that $\mathcal{K} \models E(a)$ and $\mathcal{K} \models F(a)$. By induction hypothesis, $[a] \in E^{\mathcal{I}}$ and $[a] \in F^{\mathcal{I}}$. So $[a] \in (E \sqcap F)^{\mathcal{I}}$.
- Case C is $\forall R.E$. The first part is trivially true because there is no F such that $([D], [F]) \in R^{\mathcal{I}}$. We now prove the second part. Suppose $\mathcal{K} \models (\forall R.E)(a)$. Suppose $([a], [b]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models R(a, b)$. It is easy to show that $\mathcal{K} \models E(b)$. By induction hypothesis $[b] \in E^{\mathcal{I}}$. Hence, $[a] \in (\forall R.E)^{\mathcal{I}}$.
- Case C is $\leqslant 1R.E$. The first part is trivially true because there is no F such that $([D], [F]) \in R^{\mathcal{I}}$. We now prove the second part. Suppose $\mathcal{K} \models (\leqslant 1R.E)(a)$. Suppose also $([a], [b_1]), ([a], [b_2]) \in R^{\mathcal{I}}$ such that $[b_1], [b_2] \in E^{\mathcal{I}}$.

By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models R(a, b_1)$ and $\mathcal{K} \models R(a, b_2)$. By Claim 2.1 part 2, $\mathcal{K} \models E(b_1)$ and $\mathcal{K} \models E(b_2)$. It is easy to show that $\mathcal{K} \models b_1 = b_2$. Hence, $[a] \in (\leqslant 1R.E)^{\mathcal{I}}$.

It is easy to see that $R^{\mathcal{I}}$ is the inverse of $(R^-)^{\mathcal{I}}$. Suppose $([a], [b]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models R(a, b)$. From this, it is easy to show that $\mathcal{K} \models R^-(b, a)$. By definition of $R^{\mathcal{I}}$, $([b], [a]) \in (R^-)^{\mathcal{I}}$.

Proof that the interpretation $\mathcal{I}$ for $\mathcal{OWL}$-$\mathcal{RL}$ is a model of $\mathcal{K}$.

- Suppose $C \sqsubseteq D$ is in $\mathcal{K}$.
  1. Suppose $[E] \in C^{\mathcal{I}}$. It follows from Claim 2.1 part 1, that $\mathcal{K} \models E \sqsubseteq C$. Hence, $\mathcal{K} \models E \sqsubseteq D$. It follows from Claim 2.2 part 1, that $[E] \in D^{\mathcal{I}}$.
  2. Suppose $[a] \in C^{\mathcal{I}}$. It follows from Claim 2.1 part 2, that $\mathcal{K} \models C(a)$. Hence, $\mathcal{K} \models D(a)$. It follows from Claim 2.2 part 2, $[a] \in D^{\mathcal{I}}$.

  This proves that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.
- Suppose that $R_1 \circ \ldots R_k \sqsubseteq R$ is in $\mathcal{K}$ for $k > 0$. Suppose $([a_1], [a_2]) \in R_1^{\mathcal{I}}$, …, $([a_k], [a_{k+1}]) \in R_k^{\mathcal{I}}$. Hence, $\mathcal{K} \models R(a_1, a_2)$, …, $\mathcal{K} \models R(a_k, a_{k+1})$. It is not difficult to show that $\mathcal{K} \models R(a_1, a_{k+1})$. Hence, $([a_1], [a_{k+1}]) \in R^{\mathcal{I}}$.
- Suppose that $\mathsf{Irr}(R)$ in $\mathcal{K}$. Hence, $\mathcal{K} \not\models R(a, a)$ for all individuals a. Hence, $([a], [a]) \notin R^{\mathcal{I}}$.
- Suppose that $\mathsf{Dis}(R, S)$ in $\mathcal{K}$. Suppose towards a contradiction that $([a], [b]) \in R^{\mathcal{I}}$ and $([a], [b]) \in S^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models R(a, b)$ and $\mathcal{K} \models S(a, b)$.
- Suppose that $C(a)$ in $\mathcal{K}$ (only allowed for C in CR). By Claim 2.2 part 2, we have that $[a] \in C^{\mathcal{I}}$.
- Suppose that $R(a, b)$ in $\mathcal{K}$. By definition of $R^{\mathcal{I}}$, we have that $([a], [b]) \in R^{\mathcal{I}}$.
- The cases $a = b$ or $a \neq b$ in $\mathcal{K}$ follow immediately from the definition of the interpretation.

Proof that the interpretation $\mathcal{I}$ for $\mathcal{OWL}$-$\mathcal{QL}^*$ is a model of $\mathcal{K}$.
We show first that $R^{\mathcal{I}}$ is the inverse of $(R^-)^{\mathcal{I}}$. We have three cases:

1. Suppose $([C], [D]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models C \sqsubseteq \exists R.D$ and $\mathcal{K} \models D \sqsubseteq \exists R^-.C$. Now we apply the definition of the interpretation of roles on R^- and it is trivial to see that $([D], [C]) \in (R^-)^{\mathcal{I}}$.
2. Suppose $([a], [b]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models R(a, b)$. From this, it is easy to show that $\mathcal{K} \models R^-(b, a)$. By definition of $R^{\mathcal{I}}$, $([b], [a]) \in (R^-)^{\mathcal{I}}$.
3. It is also easy to show that $([a], [C]) \in R^{\mathcal{I}}$ if and only if $([C], [a]) \in (R^-)^{\mathcal{I}}$.

Claim 3.3: Let $C \in \mathsf{CL}$.

1. If $[D] \in C^{\mathcal{I}}$ then $\mathcal{K} \models D \sqsubseteq C$.
2. If $[a] \in C^{\mathcal{I}}$ then $\mathcal{K} \models C(a)$.

The above claim is proved by induction on $C \in \mathsf{CL}$.

- Case $C = A$. Both parts follow from the definition of $A^{\mathcal{I}}$.
- Case $C = \top$. Both parts are trivial since $\mathcal{K} \models D \sqsubseteq \top$ and $\mathcal{K} \models \top(a)$.

- Case $C = \bot$. Both parts are trivial since $C^{\mathcal{I}} = \emptyset$.
- Case C is $\exists R.\top$. We prove the first part. Suppose $[D] \in (\exists R.\top)^{\mathcal{I}}$. Hence, there is F such that $([D], [F]) \in R^{\mathcal{I}}$ or there is a such that $([D], [a]) \in R^{\mathcal{I}}$ In both cases, we have that $\mathcal{K} \models D \sqsubseteq \exists R.\top$ by definition of $R^{\mathcal{I}}$. We now prove the second part. Suppose $[a] \in (\exists R.\top)^{\mathcal{I}}$. Hence, there is an individual b such that $([a], [b]) \in R^{\mathcal{I}}$ or there is a concept E such that $([a], [E]) \in R^{\mathcal{I}}$. In both cases, we have that $\mathcal{K} \models (\exists R.\top)(a)$ by definition of $R^{\mathcal{I}}$.

Claim 3.4: Let $C \in \mathsf{CR}$.

1. If $\mathcal{K} \models D \sqsubseteq C$ and $\mathcal{K} \not\models D \sqsubseteq \bot$ then $[D] \in C^{\mathcal{I}}$.
2. If $\mathcal{K} \models C(a)$ then $[a] \in C^{\mathcal{I}}$.

The above claim is proved by induction on $C \in \mathsf{CR}$.

- Case $C = A$ from the definition of $A^{\mathcal{I}}$.
- Case $C = \top$ follows because $\Delta^{\mathcal{I}}$ contains all concepts and all individuals.
- Case $C = \bot$. The first part follows because $\mathcal{K} \models D \sqsubseteq \bot$ and $\mathcal{K} \not\models D \sqsubseteq \bot$ is a contradiction. The second part follows because $\bot(a)$ is not possible for $\mathcal{K}$ consistent.
- Case C is $\neg E$. We prove the first part. Suppose $\mathcal{K} \models D \sqsubseteq \neg E$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. Since $\mathcal{K}$ is consistent, $\mathcal{K} \not\models D \sqsubseteq E$. By Claim 3.3 part 1, $[D] \notin E^{\mathcal{I}}$. Hence, $[D] \in (\neg E)^{\mathcal{I}}$.
  We now prove the second part. Suppose $\mathcal{K} \models (\neg E)(a)$. Since $\mathcal{K}$ is consistent, $\mathcal{K} \not\models E(a)$. By Claim 3.3 part 2, $[a] \notin E^{\mathcal{I}}$, so $[a] \in (\neg E)^{\mathcal{I}}$.
- Case C is $E \sqcap F$. We prove the first part. Suppose $\mathcal{K} \models D \sqsubseteq E \sqcap F$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. It is easy to show that $\mathcal{K} \models D \sqsubseteq E$ and $\mathcal{K} \models D \sqsubseteq F$. By induction hypothesis, $[D] \in E^{\mathcal{I}}$ and $[D] \in F^{\mathcal{I}}$. So $[D] \in (E \sqcap F)^{\mathcal{I}}$.
  We now prove the second part. Suppose $\mathcal{K} \models (E \sqcap F)(a)$. It is easy to show that $\mathcal{K} \models E(a)$ and $\mathcal{K} \models F(a)$. By induction hypothesis, $[a] \in E^{\mathcal{I}}$ and $[a] \in F^{\mathcal{I}}$. So $[a] \in (E \sqcap F)^{\mathcal{I}}$.
- Case C is $\exists R.E$. We prove the first part. Suppose $\mathcal{K} \models D \sqsubseteq \exists R.E$ and $\mathcal{K} \not\models D \sqsubseteq \bot$. It is easy to show that $\mathcal{K} \models D \sqsubseteq \exists R.(E \sqcap \exists R^{-}.D)$. By definition of $R^{\mathcal{I}}$, we have that $([D], [E \sqcap \exists R^{-}.D]) \in R^{\mathcal{I}}$. By induction hypothesis, $[E \sqcap \exists R^{-}.D]) \in E^{\mathcal{I}}$. Hence, $[D] \in (\exists R.E)^{\mathcal{I}}$.
  We now prove the second part. Suppose $\mathcal{K} \models (\exists R.E)(a)$. Hence, $\mathcal{K} \models (\exists R.(E \sqcap \exists R^{-}.\top))(a)$. Hence, $([a], [E \sqcap \exists R^{-}.\top]) \in R^{\mathcal{I}}$ and so $[a] \in (\exists R.E)^{\mathcal{I}}$.

We now prove that $\mathcal{I}$ is a model of $\mathcal{K}$.

- Suppose $C \sqsubseteq D$ is in $\mathcal{K}$.
  1. Suppose $[E] \in C^{\mathcal{I}}$. It follows from Claim 3.3 part 1, that $\mathcal{K} \models E \sqsubseteq C$. Hence, $\mathcal{K} \models E \sqsubseteq D$. It follows from Claim 3.4 part 1, that $[E] \in D^{\mathcal{I}}$.
  2. Suppose $[a] \in C^{\mathcal{I}}$. It follows from Claim 3.3 part 2, that $\mathcal{K} \models C(a)$. Hence, $\mathcal{K} \models D(a)$. It follows from Claim 3.4 part 2, that $[a] \in D^{\mathcal{I}}$.

  This proves that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

– Suppose $R \sqsubseteq S$ is in $\mathcal{K}$.

1. Suppose $([E], [F]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models E \sqsubseteq \exists R.F$ and $\mathcal{K} \models F \sqsubseteq \exists R^{-}.E$. Hence, $\mathcal{K} \models E \sqsubseteq \exists S.F$ and $\mathcal{K} \models F \sqsubseteq \exists S^{-}.E$. So, by definition of $S^{\mathcal{I}}$, $([E], [F]) \in S^{\mathcal{I}}$.
2. Suppose $([a], [b]) \in R^{\mathcal{I}}$. By the definition of $R^{\mathcal{I}}$, $\mathcal{K} \models R(a, b)$. Hence, $\mathcal{K} \models S(a, b)$. So, by definition of $S^{\mathcal{I}}$, $([a], [b]) \in S^{\mathcal{I}}$.
3. Suppose $([a], [F]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models \exists R.F(a)$ and $\mathcal{K} \models F \sqsubseteq \exists R^{-}.\top$. Hence, $\mathcal{K} \models \exists S.F(a)$ and $\mathcal{K} \models F \sqsubseteq \exists S^{-}.\top$. So, by definition of $S^{\mathcal{I}}$, $([a], [F]) \in S^{\mathcal{I}}$.
4. Suppose $([E], [b]) \in R^{\mathcal{I}}$. By definition of $R^{\mathcal{I}}$, $\mathcal{K} \models E \sqsubseteq \exists R.\top$ and $\mathcal{K} \models \exists R^{-}.E(b)$. Hence, $\mathcal{K} \models E \sqsubseteq \exists S.\top$ and $\mathcal{K} \models \exists S^{-}.E(b)$. So, by definition of $S^{\mathcal{I}}$, $([E], [b]) \in S^{\mathcal{I}}$.

This proves that $R^{\mathcal{I}} \subseteq S^{\mathcal{I}}$.

– Suppose $\mathsf{Ref}(R)$ is in $\mathcal{K}$.

1. Since $\mathcal{K} \models E \sqsubseteq \exists R.E$, we have that $([E], [E]) \in R^{\mathcal{I}}$.
2. Since $\mathcal{K} \models R(a, a)$, we have that $([a], [a]) \in R^{\mathcal{I}}$.

– Suppose that $C(a)$ in $\mathcal{K}$ (only allowed for C in CR). By Claim 3.4 part 2, we have that $[a] \in C^{\mathcal{I}}$.
– Suppose that $R(a, b)$ in $\mathcal{K}$. By definition of $R^{\mathcal{I}}$, we have that $([a], [b]) \in R^{\mathcal{I}}$.
– The cases $a = b$ or $a \neq b$ in $\mathcal{K}$ follow from the definition of $\mathcal{I}$.

C Non Consequential Logics

In this section, we give more examples that show that the algorithm of Section 4 does not work for logics beyond $\mathcal{ALCM}$. We use $\mathsf{Eq}(C, D)$ as an abbreviation for $(\neg C \sqcup D) \sqcap (\neg D \sqcup C)$ and $\mathsf{Disj}(C, D)$ for $\neg C \sqcap D \sqcup C \sqcap \neg D$.

Example 3. We show another example in $\mathcal{ALCM}$.

$$\mathcal{M} \quad a =_{\mathsf{m}} A \quad b =_{\mathsf{m}} B \quad c =_{\mathsf{m}} C$$

$$\mathcal{K} \quad A(b) \quad B(c)$$

$$\top \sqsubseteq (\mathsf{Eq}(A, B) \sqcup \mathsf{Eq}(A, C))$$

Example 4. We show an example in $\mathcal{ALCQM}$.

$$\mathcal{M} \quad a =_{\mathsf{m}} A \quad b =_{\mathsf{m}} B$$

$$\mathcal{K} \quad A(b) \quad R(c, a) \quad R(c, b)$$

$$\top \sqsubseteq (\leqslant 1 R.\top) \sqcup \mathsf{Eq}(A, B)$$

Example 5. We consider the following knowledge base in $\mathcal{ALCQM}$.

$$\mathcal{M} \quad a =_{\mathsf{m}} A \quad b =_{\mathsf{m}} B$$

$$\mathcal{K} \quad A(e) \;\; R(c,a) \;\; R(c,b) \;\; C(a) \;\; D(b)$$

$$\top \sqsubseteq \mathsf{Disj}(C,D) \sqcup \mathsf{Disj}(A,B)$$

$$\top \sqsubseteq (\leqslant 1 R.\top) \sqcup \mathsf{Eq}(A,B)$$

None of these examples can be written in the logic of [33]