Dirichlet Processes and Nonparametric Bayesian Modelling

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Motivation

- Infinite models have recently gained a lot of attention in Bayesian machine learning
- They offer great flexibility and, in many applications, allow a more truthful representation, if compared to a parametric approach
- The most prominent representatives are Gaussian processes and Dirichlet processes

Gaussian Processes: Modeling Functions

- Gaussian processes define a prior over functions
- A sample of a Gaussian process is a function

$$f(\cdot) \sim \mathrm{GP}(\cdot|\mu(\cdot), k(\cdot,\cdot))$$

  where $\mu(\cdot)$ is the mean function and $k(\cdot,\cdot)$ is the covariance kernel function
- Gaussian processes are infinite-dimensional generalizations of finite-dimensional Gaussian distributions
- In a typical problem we have samples of the underlying true function and we want to calculate the posterior distribution of the function and make predictions at a new input (Gaussian process smoothing)
- In a related setting, we can only obtain noisy measurements of the true function (Gaussian process regression)

Dirichlet Processes: Modeling Probability Measures

- Dirichlet processes define a prior over probability measures
- A sample of a Dirichlet process is a probability measure

$$G \sim \mathrm{DP}(\cdot|G_0, \alpha_0)$$

  G_0 is the base distribution and α_0 is the concentration parameter

- Infinite-dimensional Dirichlet processes are generalizations to finite Dirichlet distributions
- In a typical problem we have samples of the underlying true probability measure and we want to calculate the posterior probability measure or the predictive distribution for a new sample; (note, that we do not have a measurement of the function, as in the GP case but a sample of the true probability measure; this is the main difference between GP and DP)
- In a related setting, we can only obtain noisy measurements of a sample; this is then a *Dirichlet process mixture model*

Outline

- I: Introduction to Bayesian Modeling
- II: Multinomial Sampling with a Dirichlet Prior
- III: Hierarchical Bayesian Modeling
- IV: Dirichlet Processes
- V: Applications and More on Nonparametric Bayesian Modeling

I: Introduction to Bayesian Modeling

Statistical Approaches to Learning and Statistics

- Probability theory is a branch of mathematics
- Statistics and (statistical) machine learning are attempts to applying probability theory to solving problems in the real world: effectiveness of a medication, text classification, medical expert systems, ...
- There are different approaches to applying probability theory to problems in the real world in a useful way: frequentist statistics, Bayesian statistics, statistical learning theory, ...
- All of them are useful in their own right
- In this tutorial, we take a Bayesian point of view

Review of Some Laws of Probability

Multivariate Distribution

- We start with two (random) variables X and Y. A multivariate (probability) distribution is defined as

$$P(x, y) := P(X = x, Y = y) = P(X = x \wedge Y = y)$$

Conditional Distribution

- *Definition* of a **conditional distribution**

$$P(Y = y|X = x) := \frac{P(X = x, Y = y)}{P(X = x)} \text{ where } P(X = x) > 0$$

Product Decomposition and Chain Rule

From the definition of a conditional distribution we obtain:

- **Product Decomposition**

$$P(x, y) = P(x|y)P(y) = P(y|x)P(x)$$

- and the **chain rule**

$$P(x_1, \ldots, x_M) = P(x_1)P(x_2|x_1)P(x_3|x_1, x_2) \ldots P(x_M|x_1, \ldots, x_{M-1})$$

Bayes' Rule

- **Bayes' Rule** follows immediately

$$P(x|y) = \frac{P(x,y)}{P(y)} = \frac{P(y|x)P(x)}{P(y)} \quad P(y) > 0$$

Marginal Distribution

- To calculate a **marginal distribution** from a joint distribution, one uses:

$$P(X = x) = \sum_{y} P(X = x, Y = y)$$

Bayesian Reasoning and Bayesian Statistics

Bayesian Reasoning

- Bayesian reasoning is the straight-forward application of the rules of probability to real world problems involving uncertain reasoning
- $P(H = 1)$: assumption about the truth of hypothesis H (a priori probability)
- $P(D|H = 1)$: Probability of observing (Data) D, if hypothesis H is true (likelihood); $P(D|H = 0)$: Probability of observing (Data) D, if hypothesis H is not true (likelihood)
- Bayes' rule:

$$P(H = 1|D) = \frac{P(D|H = 1)P(H = 1)}{P(D)}$$

- Evidence: $P(D) = P(D|H = 1)P(H = 1) + P(D|H = 0)P(H = 0)$

Bayesian Reasoning: Example

- A friend has a new car
- A priori assumption:

$$P(Car = SportsCar) = 0.5$$

- I learn that the car has exactly two doors; likelihood:

$$P(2Doors|Car = SportsCar) = 1 \quad P(2Doors|Car = \neg SportsCar) = 0.5$$

- Using Bayes' theorem:

$$P(Car = SportsCar|2Doors) = \frac{1 \times 0.5}{(1 \times 0.5 + 0.5 \times 0.5)} = 0.66$$

Bayesian Reasoning: Debate

- There is no disagreement that one can define an appropriate likelihood term $P(D|H)$
- There is disagreement, if one should be allowed to define and exploit the prior probability of a hypothesis $P(H)$, since in most cases, this can only present someone's prior belief
- Non-Bayesians often criticize the necessity to model someone's prior belief: this appears to be subjective and non-scientific
- To people sympathetic to Bayesian reasoning: the prior distribution can be used to incorporate valuable prior knowledge and constraints (e.g., medical expert system); it is a necessity for obtaining a complete statistical model
- Comment: Since in parametric modeling the assumption about the likelihood function is much more critical than assumptions concerning the prior distribution, the discussion might not be quite to the point

Bayesian Reasoning: Subjective Probabilities

- If one is willing to assign numbers to beliefs then under few assumptions of consistency and if 1 means that one is certain that an event will occur and if 0 means that one is certain that an event will not occur, then these numbers exactly behave as probabilities. Theorem: Any measure of belief is isomorphic to a probability measure (Cox, 1946)

- "One common criticism of the Bayesian definition of probability is that probabilities seem arbitrary. Why should degrees of belief satisfy the rules of probability? On what scale should probabilities be measured? In particular, it makes sense to assign a probability of one (zero) to an event that will (not) occur, but what probabilities do we assign to beliefs that are not at the extremes? Not surprisingly, these questions have been studied intensely. With regards to the first question, many researchers have suggested different sets of properties that should be satisfied by degrees of belief (e.g., Ramsey 1931, Cox 1946, Good 1950, Savage 1954, DeFinetti 1970). It turns out that each set of properties leads to the same rules: the rules of probability. Although each set of properties is in itself compelling, the fact that different sets all lead to the rules of probability provides a particularly strong argument for using probability to measure beliefs." Heckerman: A Tutorial on Learning With Bayesian Networks

Technicalities in Bayesian Statistics

Basic Approach in Statistical Bayesian Modeling

- Despite the fact that in Bayesian modeling any uncertain quantity of interest is treated as a random variable, one typically distinguishes between parameters and variables; variables might assume different states in each data point (e.g., object, measurement) whereas parameters are used to describe regularities in the domain. A typical assumption is that data points are exchangeable given the parameters. For Bayesian modeling, exchangeability is a more useful property than the independent identical distribution (i.i.d.) assumption often used in frequentist statistics (see for example: Jordan, 2005)

- In a typical setting might have observed data D, unknown parameters θ and a quantity to be predicted X. Furthermore, we might have latent variables H_D and H in the training data and in the test point, respectively

- One first builds a joint model, using the product rule (example)

$$P(\theta, H_D, D, H, X) = P(\theta)P(D, H_D|\theta)P(X, H|\theta)$$

  $P(\theta)$ is the prior distribution, $P(D, H_D|\theta)$ is the complete data likelihood; we might be interested in $P(X|D)$

- First we marginalize the latent variables $P(D|\theta) = \int P(D, H_D|\theta)\ dH_D$ and obtain the likelihood w.r.t the observed data $P(D|\theta)$

Basic Approach in Statistical Bayesian Modeling (2)

- Then we obtain the posterior parameters distribution using Bayes rule

$$P(\theta|D) = \frac{P(D|\theta)P(\theta)}{P(D)}$$

- Then we marginalize the parameters and obtain

$$P(X, H|D) = \int P(X, H|\theta)P(\theta|D)\ d\theta$$

- Finally, one marginalizes the latent variable in the test point

$$P(X|D) = \sum_{H} P(X, H|D)$$

- The demanding operations are the integrals, resp. sums; thus one might say with some justification: *the frequentist optimizes (e.g., in the maximum likleihood approach), and the Bayesian integrates*

Approximating the Integrals in Bayesian Modeling

- The integrals are often over high-dimensional quantities; typical approaches to solving or approximating the integrals
  - Closed-from solutions (exist for some special cases)
  - Laplace approximation (leads to an optimization problem)
  - Markov Chain Monte Carlo Sampling (e.g., Gibbs sampling, ...) (integration via Monte Carlo)
  - Variational approximations (e.g., mean field) (leads to an optimization problem)
  - Expectation Propagation

Conclusion on Bayesian Modeling

- Bayesian modeling is the straightforward application of the laws of probability to problems in the real world
- The Bayesian program is quite simple: build a model, get data, perform inference

II: Multinomial Sampling with a Dirichlet Prior

Likelihood, Prior, Posterior, and Predictive Distribution

Multinomial Sampling with a Dirichlet Prior

- Before we introduce the Dirichlet process, we need to get a good understanding of the finite-dimensional case: Multinomial sampling with a Dirichlet prior
- Learning and inference in the finite case find their equivalences in the infinite-dimensional case of Dirichlet processes
- Highly recommended: David Heckerman's tutorial: A Tutorial on Learning With Bayesian Networks (http://research.microsoft.com/research/pubs/view.aspx?msr_tr_id=MSR-TR-95-06)

Example: Tossing a Loaded Dice

- Running example: the repeated tossing of a loaded dice
- Let's assume that we toss a loaded dice; by $\Theta = \theta^k$ we indicate the fact that the toss resulted in showing θ^k
- Let's assume that we observe in N tosses N_k times θ^k
- A reasonable estimate is then that

$$\widehat{P}(\Theta = \theta^k) = \frac{N_k}{N}$$

Multinomial Likelihood

- In a formal model we would assume multinomial sampling; the observed variable Θ is discrete, having r possible states $\theta^1, \ldots, \theta^r$. The likelihood function is given by

$$P(\Theta = \theta^k | \mathbf{g}) = g_k, \quad k = 1, \ldots, r$$

  where $\mathbf{g} = \{g_2, \ldots, g_r, \}$ are the parameters and $g_1 = 1 - \sum_{k=2}^{r} g_k$, $g_k \geq 0, \forall k$

- Here, the parameters correspond to the physical probabilities

- The sufficient statistics for a data set $D = \{\Theta_1 = \theta_1, \ldots, \Theta_N = \theta_N\}$ are $\{N_1, \ldots, N_r\}$, where N_k is the number of times that $\Theta = \theta^k$ in D. (In the following, D will in general stand for the observed data)

Multinomial Likelihood for a Data Set

- The likelihood for the complete data set (here and in the following, C denotes normalization constants irrelevant for the discussion)

$$P(D|\mathbf{g}) = \text{Multinomial}(\cdot|\mathbf{g}) = \frac{1}{C}\prod_{k=1}^{r} g_k^{N_k}$$

- The maximum likelihood estimate is (exercise)

$$g_k^{ML} = \frac{N_k}{N}$$

Thus we obtain the very intuitive result that the parameter estimates are the empirical counts. If some or many counts are very small (e.g., when $N < r$) many probabilities might be (incorrectly) estimated to be zero; thus, a Bayesian treatment might be more appropriate

Dirichlet Prior

- In a Bayesian framework, one defines an a priori distribution for $\mathbf{g}$. A convenient choice is a conjugate prior, in this case a Dirichlet distribution

$$P(\mathbf{g}|\boldsymbol{\alpha}^*) = \mathrm{Dir}(\cdot|\alpha_1^*, \ldots, \alpha_r^*) \equiv \frac{1}{C} \prod_{k=1}^{r} g_k^{\alpha_k^* - 1}$$

- $\boldsymbol{\alpha}^* = \{\alpha_1^*, \ldots, \alpha_r^*\}$, $\alpha_k^* > 0$
- It is also convenient to re-parameterize

$$\alpha_0 = \sum_{k=1}^{r} \alpha_k^* \quad \alpha_k = \frac{\alpha_k^*}{\alpha_0} \quad k = 1, \ldots, r$$

  and $\boldsymbol{\alpha} = \{\alpha_1, \ldots, \alpha_r\}$ such that $\mathrm{Dir}(\cdot|\alpha_1^*, \ldots, \alpha_r^*) \equiv \frac{1}{C} \prod_{k=1}^{r} g_k^{\alpha_0 \alpha_k - 1}$

- The meaning of $\boldsymbol{\alpha}$ becomes apparent when we note that

$$P(\Theta = \theta^k|\boldsymbol{\alpha}^*) = \int P(\Theta = \theta^k|\mathbf{g}) P(\mathbf{g}|\boldsymbol{\alpha}^*)\, d\mathbf{g} = \int g_k \mathrm{Dir}(\mathbf{g}|\boldsymbol{\alpha}^*) d\mathbf{g} = \alpha_k$$

Posterior Distribution

- The posterior distribution is again a Dirichlet with

$$P(\mathbf{g}|D, \boldsymbol{\alpha}^*) = \mathrm{Dir}(\cdot|\alpha_1^* + N_1, \ldots, \alpha_r^* + N_r)$$

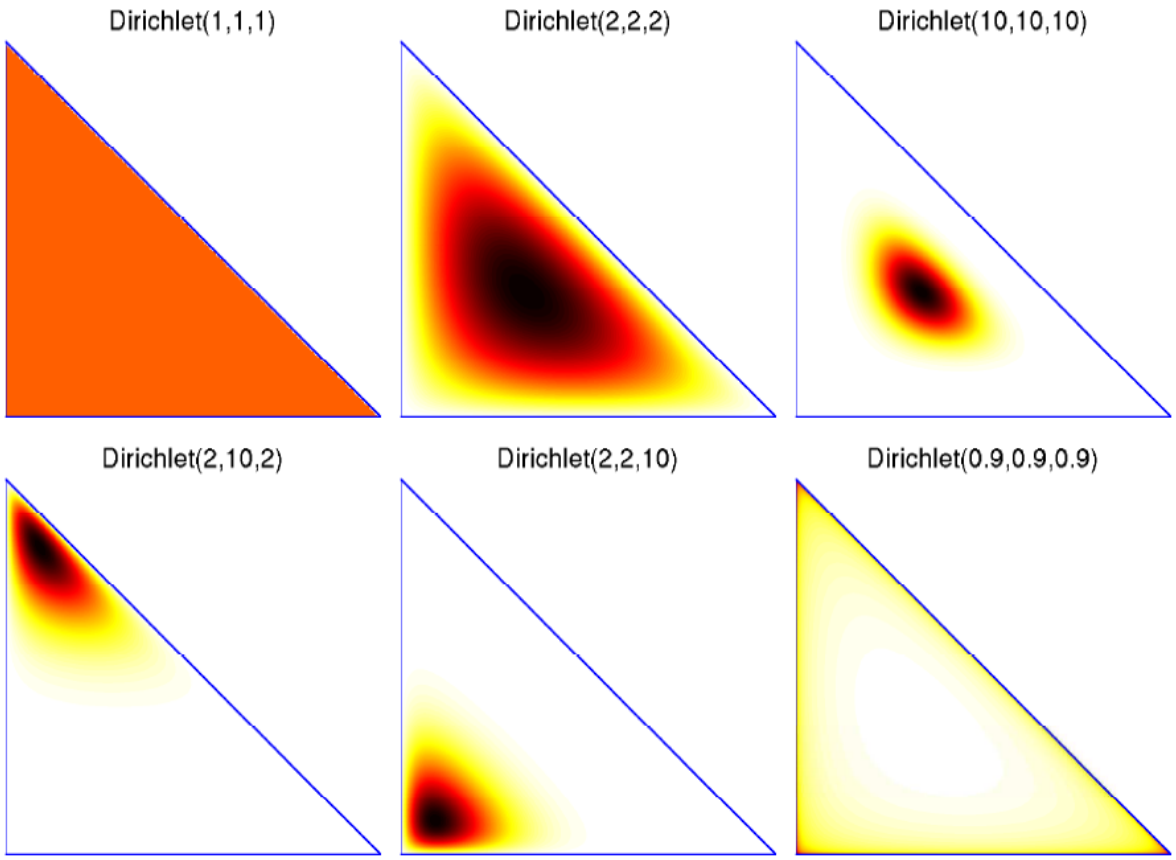
(Incidentally, this is an inherent property of a conjugate prior: the posterior comes from the same family of distributions as the prior)

- The probability for the next data point (after observing D)

$$P(\Theta_{N+1} = \theta^k|D, \boldsymbol{\alpha}^*) = \int g_k \mathrm{Dir}(\mathbf{g}|\alpha_1^*+N_1, \ldots, \alpha_r^*+N_r)d\mathbf{g} = \frac{\alpha_0\alpha_k + N_k}{\alpha_0 + N}$$

- We see that with increasing N_k we obtain the same result as with the maximum likelihood approach and the prior becomes negligible

Dirichlet Distributions for $\mathrm{Dir}(\cdot|\alpha_1^*, \alpha_2^*, \alpha_3^*)$



$$\mathrm{Dir}(\cdot|\alpha_1^*, \ldots, \alpha_r^*)$$

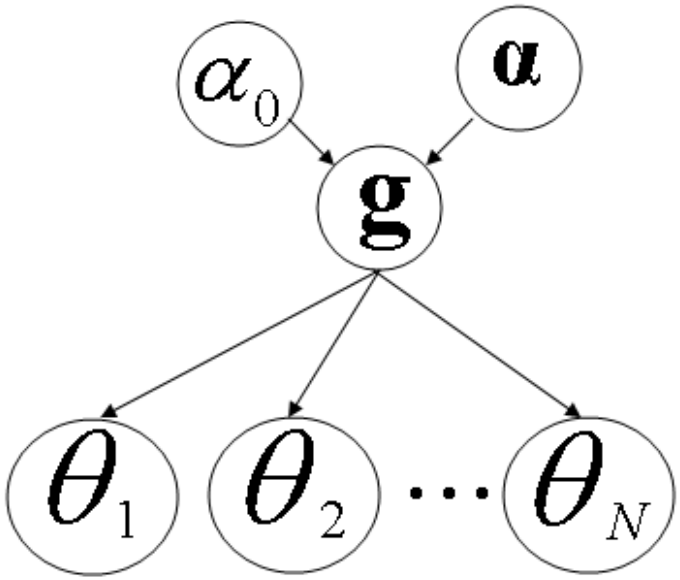
$$\equiv \frac{1}{C} \prod_{k=1}^{r} g_k^{\alpha_k^* - 1}$$

(From Ghahramani, 2005)

Generating Samples from g and θ

Generative Model

- Our goal is now to use the multinomial likelihood model with a Dirichlet prior as a generative model
- This means that want to "generate" loaded dices according to our Dirichlet prior and "generate" virtual tosses from those virtual dices
- The next slide shows a graphical representation



Graphical Model:
multinomial sampling
with a Dirichlet prior

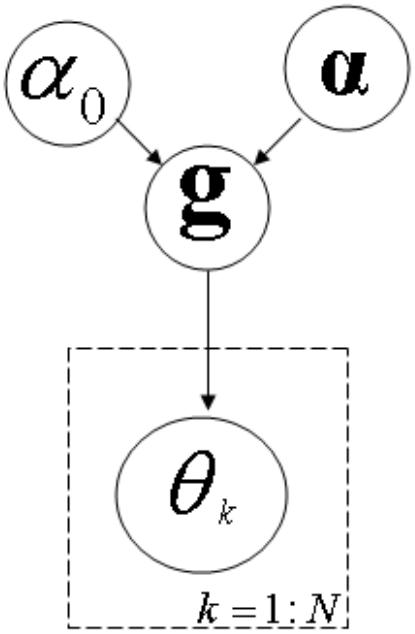


Plate representation

First Approach: Sampling from g

- We first generate a sample $\mathbf{g}$ from the Dirichlet prior
- This is not straightforward but algorithms for doing that exist; (one version involves sampling from independent gamma distributions using shape parameters $\alpha_1^*, \ldots, \alpha_r^*$ and normalizing those samples) (later in the DP case, this sample can be generate using the stick breaking presentation)
- Given a sample $\mathbf{g}$ it is trivial to generate independent samples for the tosses with

$$P(\Theta = \theta^k | \mathbf{g}) = g_k$$

Second Approach: Sampling from Θ directly

- We can also take the other route and sample from Θ directly
- Recall the probability for the next data point (after observing D)

$$P(\Theta_{N+1} = \theta^k | D) = \frac{\alpha_0 \alpha_k + N_k}{\alpha_0 + N}$$

We can use the same formula, only that now D are previously *generated* samples; *this simple equation is of central importance and will reappear in several guises repeatedly in the tutorial*

- Thus there is no need to generate an explicit sample from $\mathbf{g}$ first
- Note, that with probability proportional to N, we will sample from the empirical distribution with $P(\Theta = \theta^k) = N_k/N$ and with probability proportional to α_0 we will generate a sample according to $P(\Theta = \theta^k) = \alpha_k$

Second Approach: Sampling from $\ominus$ directly (2)

- Thus a previously generated sample increases the probability that the same sample is generated at a later stage; in the DP model this behavior will reappear in the Pólya urn representation and the Chinese restaurant process

$P(\Theta_{N+1} = \theta^k | D) = \frac{\alpha_0 \alpha_k + N_k}{\alpha_0 + N}$ with $\alpha_0 \to 0$: A Paradox?

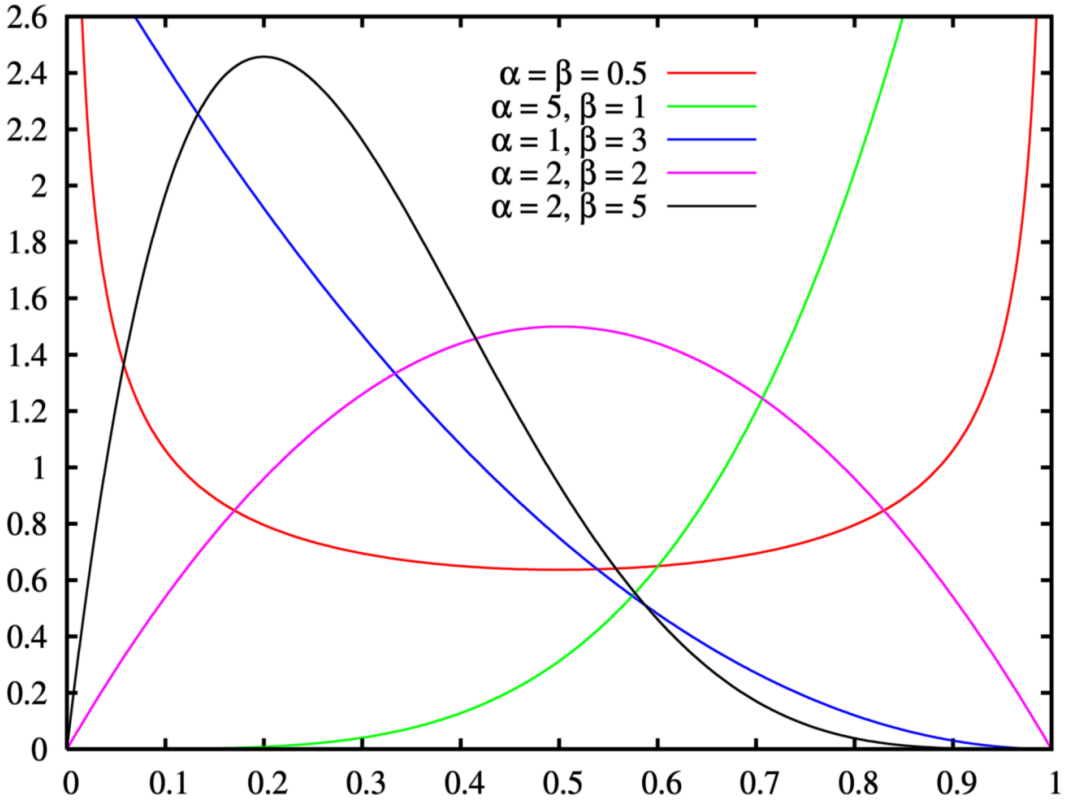
- If we let $\alpha_0 \to 0$, the first generated sample will dominate all samples generated thereafter: they will all be identical to the first sample; but note that independent of α_0 we have $P(\Theta = \theta^k) = \alpha_k$
- Note also that

$$\lim_{\alpha_0 \to 0} P(\mathbf{g}|\boldsymbol{\alpha}^*) \propto \prod_{k=1}^{r} \frac{1}{g_k}$$

  such that distributions with many zero-entries are heavily favored
- Here is the paradox: the generative model will almost never produce a fair dice but if actual data would indicate a fair dice, the prior is immediately and completely ignored
- Resolution: The Dirichlet prior with a small α_0 favors extreme solutions, but this prior belief is very weak and is easily overwritten by data
- This effect will reoccur with the DP: if α_0 is chosen to be small, sampling heavily favors clustered solutions

Beta-Distribution

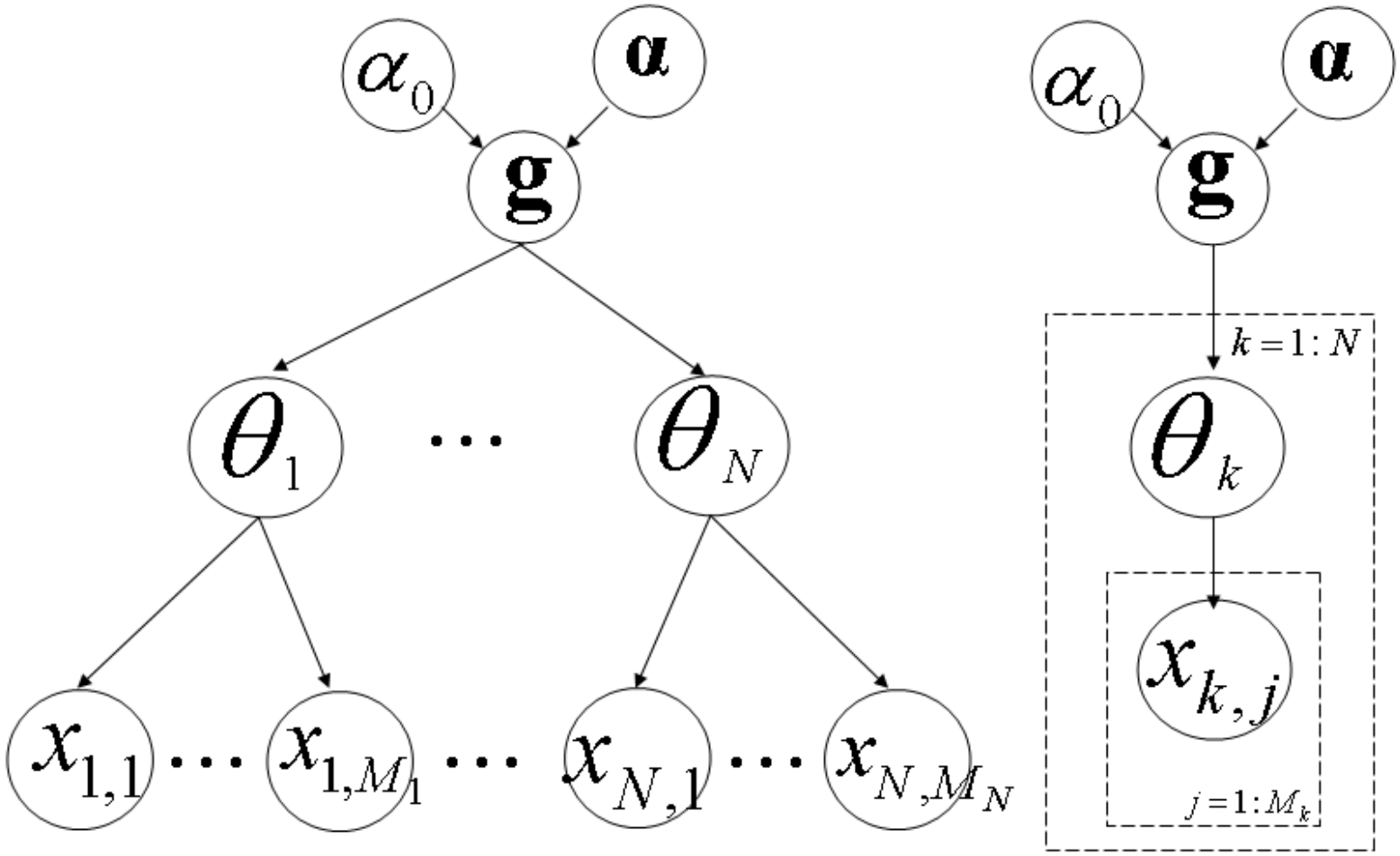
- The Beta-distribution is a two-dimensional Dirichlet with two parameters α and β; for small parameter values, we see that extreme solutions are favored



Including an Observation Model

Observation Model

- Now we want to make the model slightly more complex; we assume that we cannot observe the results of the tosses Θ directly but only (several) derived quantities (e.g., noisy measurements) X with some $P(X|\Theta)$. Let $D_k = \{x_{k,j}\}_{j=1}^{M_k}$ be the observed measurements of the $i-$th toss and let $P(x_{k,j}|\theta_k)$ be the probability distribution (several unreliable persons inform you about the results of the tosses)
- Again we might be interested in inferring the property of the dice by calculating $P(\mathbf{g}|D)$ (the probabilities of the properties of the dice) or in the probability of the actual tosses $P(\Theta_1, \ldots, \Theta_N|D)$
- This is now a problem with missing data (the Θ are missing); since it is relevant also for DP, we will we will only discuss approaches based on Gibbs sampling but we want to mention that the popular EM algorithm might also be used to obtain a point estimate of $\mathbf{g}$
- The next slide shows a graphical representation



Graphical Model: multinomial sampling with a Dirichlet prior and noisy observations

Inference based on Markov Chain Monte Carlo Sampling

- What we have learned about the model based on the data is incorporated in the predictive distribution

$$P(\Theta_{N+1}|D) = \sum_{\theta_1,\ldots,\theta_N} P(\Theta_1,\ldots,\Theta_N|D)P(\Theta_{N+1}|\Theta_1,\ldots,\Theta_N)$$

$$\approx \frac{1}{S}\sum_{s=1}^{S} P(\Theta_{N+1}|\theta_1^s,\ldots,\theta_N^s)$$

where (*Monte Carlo* approximation) $\theta_1^s,\ldots,\theta_N^s \sim P(\Theta_1,\ldots,\Theta_N|D)$

- In contrast to before, we now need to generate samples from the posterior distribution; ideally, one would generate samples independently, which is often infeasible
- In Markov chain Monte Carlo (MCMC), the next generated sample is only dependent on the previously generated sample (in the following we drop the s label in θ^s)

Gibbs Sampling

- Gibbs sampling is a specific form of an MCMC process
- In Gibbs sampling we initialize all variables in some appropriate way, and replace a value $\Theta_k = \theta_k$ by a sample of $P(\Theta_k | \{\Theta_i = \theta_i\}_{i \neq k}, D)$. One continuous to do this repeatedly for all k. Note, that Θ_k is dependent on its data $D_k = \{x_{k,j}\}_j$ but is independent of the remaining data given the samples of the other Θ
- The generated samples are from the correct distribution (after a burn in phase); a problem is that subsequent samples are not independent, which would be a desired property; if subsequent samples are highly correlated it is said that *the chain does not mix well*
- Note that we can integrate out $\mathbf{g}$ so we never have to sample from $\mathbf{g}$; this form of sampling is called *collapsed Gibbs sampling*

Gibbs Sampling (2)

- We obtain (note, that N_l are the counts without considering Θ_k)

$$P(\Theta_k = \theta^l | \{\Theta_i = \theta_i\}_{i \neq k}, D) = P(\Theta_k = \theta^l | \{\Theta_i = \theta_i\}_{i \neq k}, D_k)$$

$$= \frac{1}{C} P(\Theta_k = \theta^l | \{\Theta_i = \theta_i\}_{i \neq k}) P(D_k | \Theta_k = \theta^l)$$

$$= \frac{1}{C} (\alpha_0 \alpha_l + N_l) P(D_k | \Theta_k = \theta^l)$$

$(C = \sum_l (\alpha_0 \alpha_l + N_l) P(D_k | \Theta_k = \theta_l))$

Auxiliary Variables, Blocked Sampling and the Standard Mixture Model

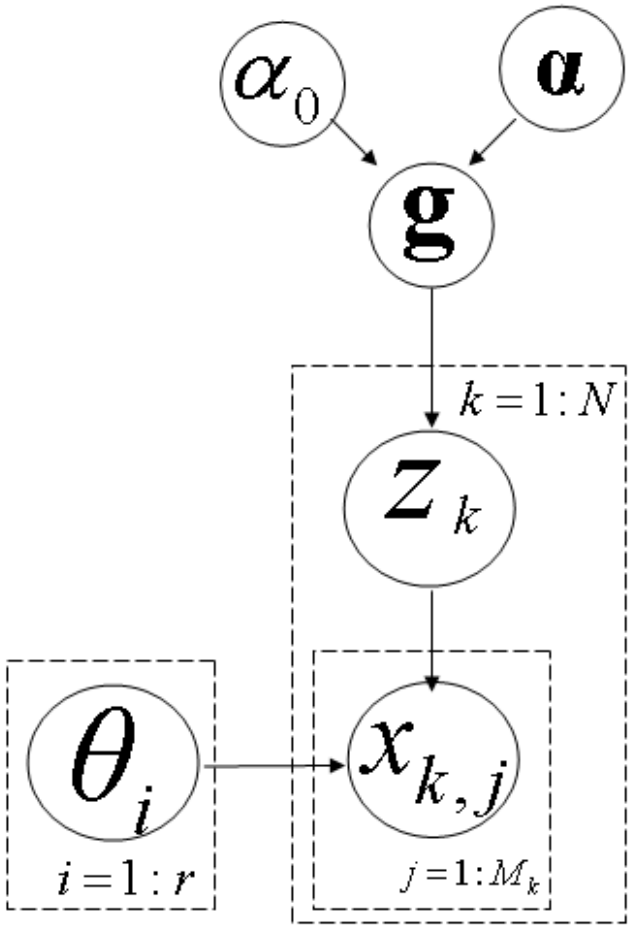
Introducing an Auxiliary Variable Z

- The figure shows a slightly modified model; here the auxiliary variables Z have been introduced with states $z^1, \ldots, z^r$
- We have

$$P(Z = z^k | \mathbf{g}) = g_k, \quad k = 1, \ldots, r$$

$$P(\Theta = \theta^j | Z = z^k) = \delta_{j,k}, \quad k = 1, \ldots, r$$

- If the $\boldsymbol{\theta}$ are fixed, this leads to the same probabilities as in the previous model and we can again use Gibbs sampling



Graphical Model: multinomial sampling with a Dirichlet prior and noisy observations including an auxiliary variable Z

Collapsing and Blocking

- So far we had used a *collapsed* Gibbs sampler, which means that we never explicitly sampled from $\mathbf{g}$
- This is very elegant but has the problem that the Gibbs sampler does not mix very well
- One often obtains better sampling by using a non-collapsed Gibbs sample, i.e., by sampling explicitly from $\mathbf{g}$
- The advantage is that given $\mathbf{g}$, one can independently sample from the auxiliary variables in a block (thus the term *blocked* Gibbs sampler)

The Blocked Gibbs Sampler

One iterates

- We generate samples from $Z_k|\mathbf{g}, D_k$ for $k = 1 \ldots, N$
- We generate a sample from

$$\mathbf{g}|Z_1, \ldots, Z_N \sim \mathrm{Dir}(\alpha_1^* + N_1, \ldots, \alpha_1^* + N_r)$$

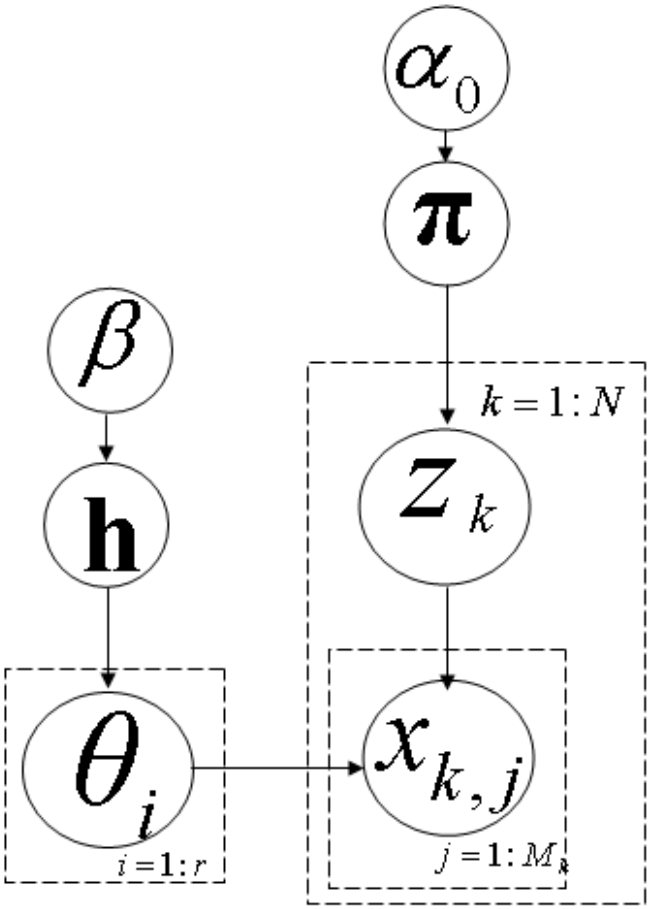
where N_k is the number of times that $Z_k = z^k$ in the current sample

Relationship to a Standard Mixture Model: Learning θ

- We can now relate our model to a standard mixture model; note, that the standard mixture model is related but not identical to the model considered so far (in the case of infinite models, although, we can indeed define an infinite mixture model which exactly corresponds to the infinite version of the previously defined model!)
- The main difference is that now we treat the the θ_k as random variables; this corresponds to the situation where Z would tell us which side of the dice is up and θ_k would correspond to a value associated with the $k-$th face
- We now need to put a prior on θ_k with hyperparameters $\mathbf{h}$ and learn θ_k from data (see figure)!
- A reasonable prior for the probabilities might be $P(\boldsymbol{\pi}|\alpha_0) = \mathrm{Dir}(\cdot|\alpha_0/r, \ldots, \alpha_0/r)$
- As a special case: when $M_k = 1$, and typically $r << N$, this corresponds to a typical mixture model; a mixture model is a probabilistic version of (soft) clustering
- Example: if the $P(X|\Theta)$ is a Gaussians distribution with parameters Θ, we obtain a Gaussian mixture model

Relationship to a Standard Mixture Model: Learning θ (2)

- Gibbs sampling as before can be used but needs to be extended to also generate sample for $\ominus$



A finite mixture model

Conclusions for the Multinomial Model with a Dirichlet Prior

- We applied the Bayesian program to a model with a multinomial likelihood and Dirichlet prior
- We discussed a number of variations on inference, in particular variations on Gibbs sampling
- But one might argue that we are still quite restrictive in the sense that if one is not interested in loaded dices or gambling in general this might all be not so relevant
- In the next section we show that by a process called Dirichlet enhancement, the Dirichlet model is the basis for nonparametric modeling in a very general class of hierarchical Bayesian models

III: Hierarchical Bayesian Modeling and Dirichlet Enhancement

Hierarchical Bayesian Modeling

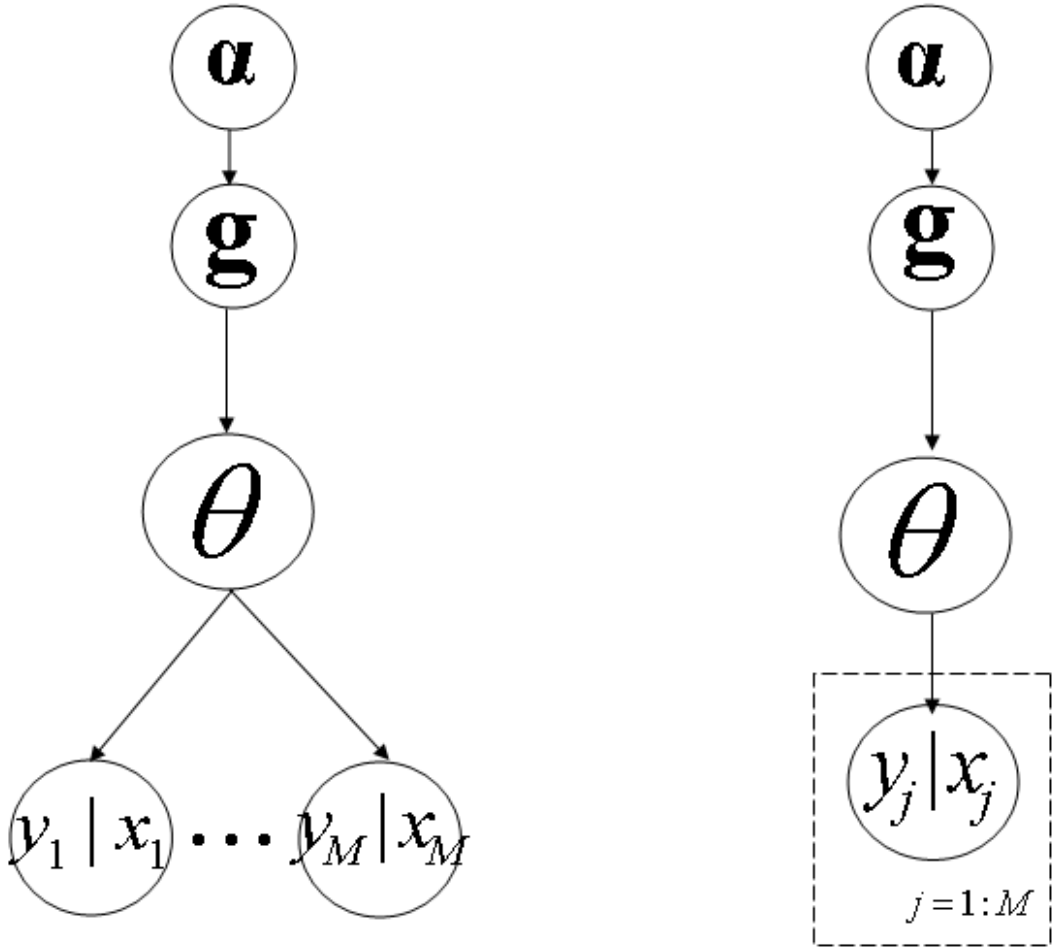
Hierarchical Bayesian Modelling

- In hierarchical Bayesian modeling both parameters and variables are treated equally as random variables (as we have done in the multinomial model)
- In the simplest case we would assume that there are random variables that might take on specific values in each instance. Example: diagnosis and length of stay of a person in a given hospital typically differs in different patients
- Then we would assume that there are variables, which we would model as being constant (but unknown) in a domain. These would typically be called parameters. Example: average length of stay given the diagnosis in a given hospital

The Standard Hierarchy

- The figure shows the standard Bayesian model for supervised learning; as a concrete example let's assume the goal is to predict the preference for an object y given object features x and given parameters $\boldsymbol{\theta}$. The parameters have a prior distribution with parameters $\mathbf{g}$, which itself originates from a distribution with parameters α
- The hierarchical probability model is (assuming that $Y_1, Y_2 \ldots$ are exchangeable given the inputs and given $\boldsymbol{\theta}$)

$$P(\boldsymbol{\alpha})P(\mathbf{g}|\boldsymbol{\alpha})P(\boldsymbol{\theta}|\mathbf{g})\prod_{j=1}^{M}P(\mathbf{y}_j|\mathbf{x}_j,\boldsymbol{\theta})$$



A standard hierarchical model

The Standard Hierarchy (2)

- The hyperparameters can be integrated out and one obtains

$$P(\boldsymbol{\theta}, D) = P(\boldsymbol{\theta})P(D|\boldsymbol{\theta}) = P(\boldsymbol{\theta}) \prod_{j=1}^{M} P(\mathbf{y}_j|\mathbf{x}_j, \boldsymbol{\theta})$$

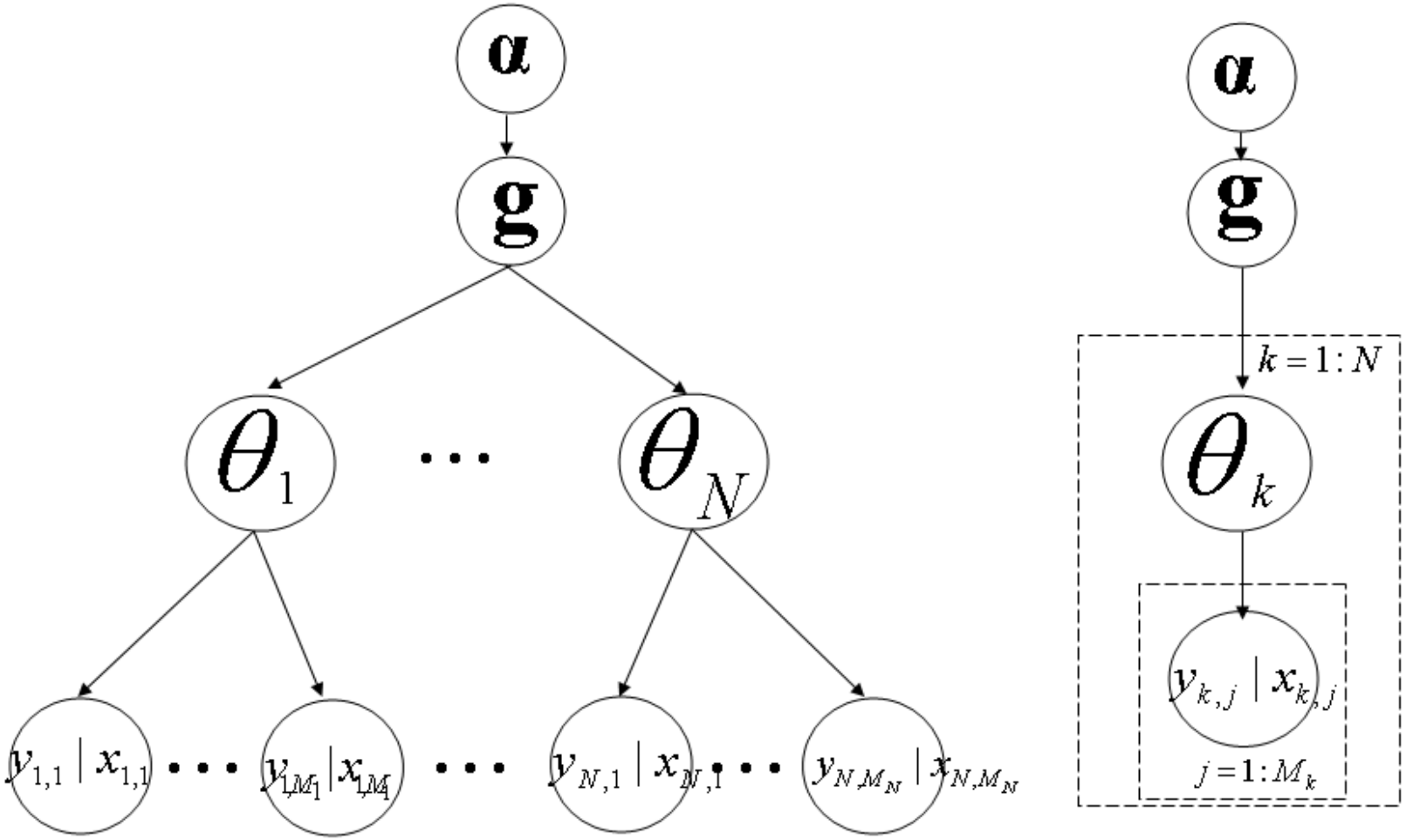
  with

$$P(\boldsymbol{\theta}) = \int P(\boldsymbol{\alpha})P(\mathbf{g}|\boldsymbol{\alpha})P(\boldsymbol{\theta}|\mathbf{g})d\boldsymbol{\alpha}d\mathbf{g}$$

- The effect of the prior vanishes when sufficient data are available: The posterior probability gets increasingly dominated by the likelihood function; thus the critical term to specify by the user is the *functional form of the likelihood*! One then needs to do an *a posterior* analysis and check if the assumptions about the likelihood were reasonable

Extended (Object Oriented) Hierarchical Bayesian

- Consider the situation of learning a model for predicting the outcome for patients with a particular disease based on patient information. Due to differences in patient mix and hospital characteristics such as staff experiences *the models are different for different hospitals but also will share some common effects.* This can be modeled by assuming that the model parameters originate from a particular distribution of parameters that can be learned from data from a sufficiently large number of hospitals. If applied to a new hospital, this learned distribution assumes the role of a learned prior
- A preference model for items (movies, books); the preference model is individual for each person
- The probability of a word is document specific; the word probabilities come out off a cluster of similar word documents
- The figure shows a graphical representation



An object oriented hierarchical model

Discussion of the Extended Hierarchical Model

- Inference and learning is more difficult but in principle nothing new (Gibbs sampling might be applied)
- Note, that $\theta_1, \theta_2, \ldots$ are exchangeable given $\mathbf{g}$ and that $Y_{i,1}, Y_{i,2} \ldots$ are exchangeable given the inputs and given $\boldsymbol{\theta}_i$
- As before, $P(\boldsymbol{\theta}_k|D_k)$ will converge to a point mass with $M_k \to \infty$
- With increasing numbers of situations and data for the situations, also $\mathbf{g}$ will converge to a point mass at some $\widehat{g}$
- This means that for a new object $N+1$, we can inherit the learned prior distribution

$$P(\boldsymbol{\theta}_{N+1}|D_1, \ldots, D_N) \approx P(\boldsymbol{\theta}_{N+1}|\widehat{g})$$

Towards a Nonparametric Approach: Dirichlet Enhancement

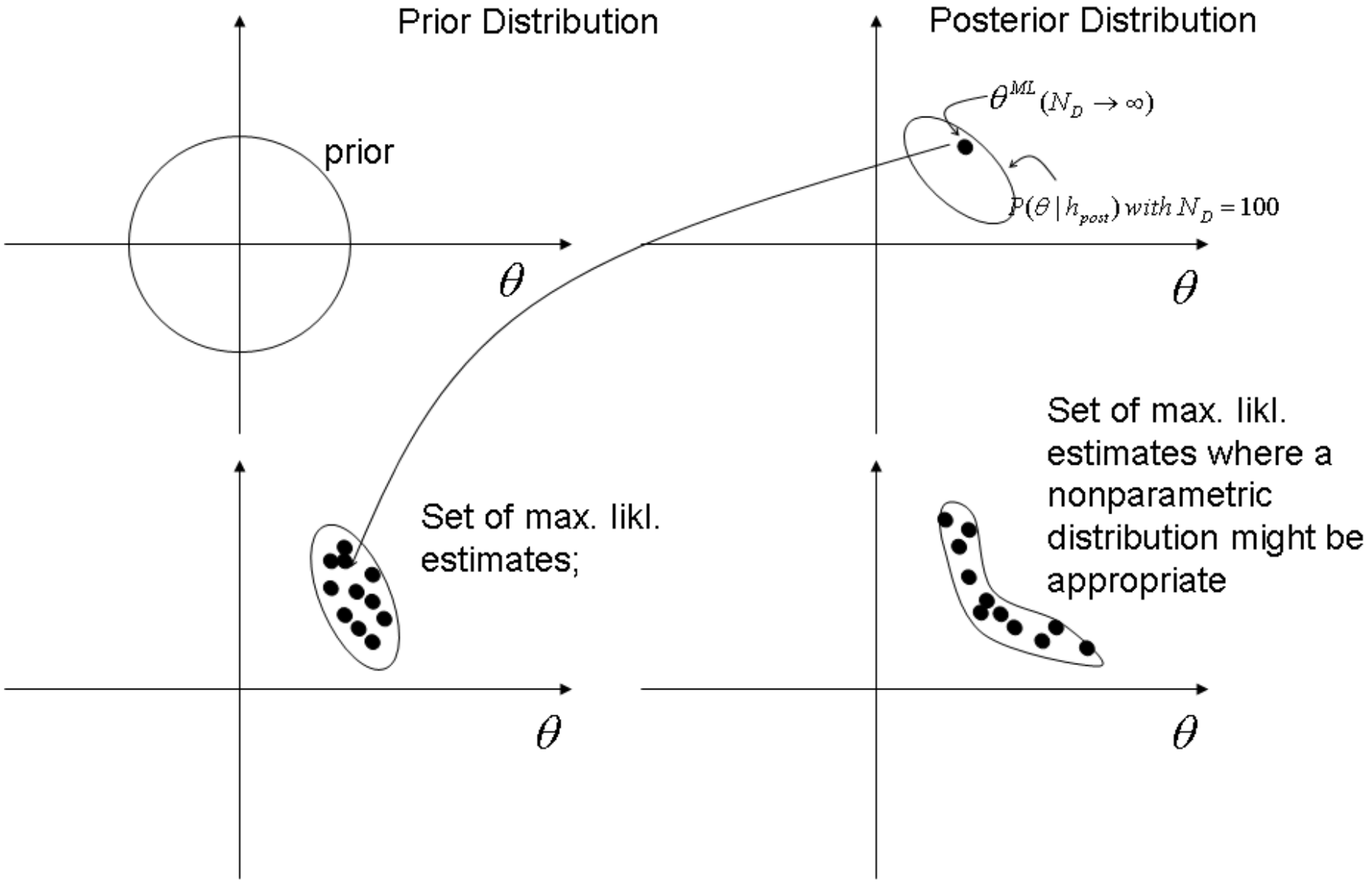
Model Check in Hierarchical Bayesian Modelling

- In the standard model, the likelihood was critical and should be checked to be correct
- In a hierarchical Bayesian model, in addition, the learned prior

$$P(\boldsymbol{\theta}_{N+1}|D_1,\ldots,D_N) \approx P(\boldsymbol{\theta}_{N+1}|\widehat{g})$$

should be checked; this distribution is critical for the sharing strength effect and the assumed functional form of the prior becomes much more important! Also note that $\boldsymbol{\theta}$ is often high dimensional (whereas the likelihood often reduces to evaluating scalar probabilities, e.g., in the case of additive independent noise)

- A simple parametric prior is typically too inflexible to represent the true distribution
- Thus one needs nonparametric distributions as priors such as derived from the the Dirichlet Process; the figure illustrates the point


Prior Distribution
Posterior Distribution
prior
$\theta^{ML}(N_D \to \infty)$
$P(\theta \mid h_{post})\ with\ N_D = 100$
θ
θ
Set of max. likl. estimates;
Set of max. likl. estimates where a nonparametric distribution might be appropriate
θ
θ

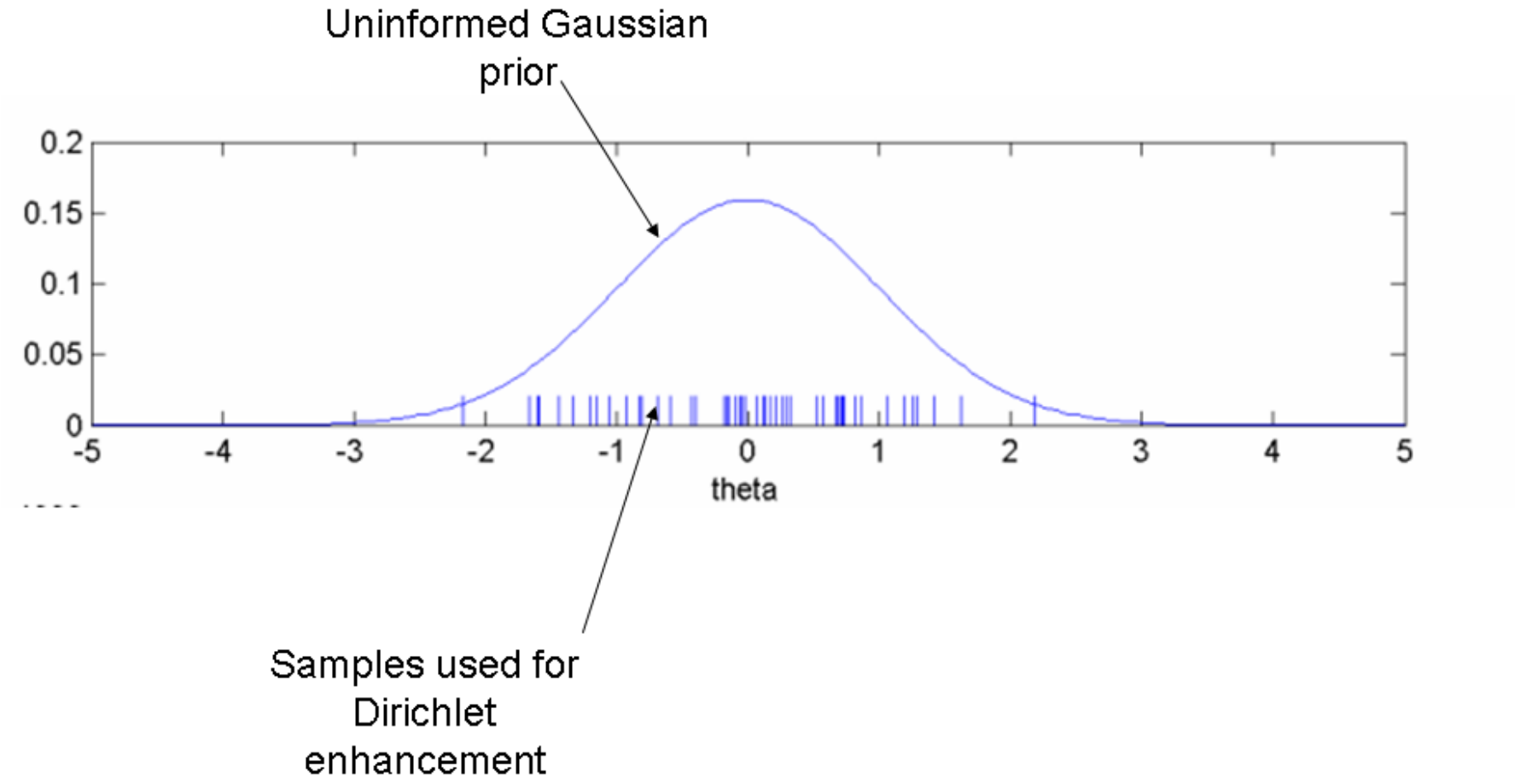
Dirichlet Enhancement: The Key Idea

- Let's assume that we consider only discrete $\theta \in \{\theta^1, \ldots, \theta^r\}$ with a very large r
- *Now we can re-parameterize the prior distribution in terms of a multinomial model with a Dirichlet prior*

$$P(\Theta = \theta^k | \mathbf{g}) = g_k, \quad k = 1, \ldots, r$$

$$P(\mathbf{g}|\boldsymbol{\alpha}^*) = \mathrm{Dir}(\cdot|\alpha_1^*, \ldots, \alpha_r^*) \equiv \frac{1}{C} \prod_{k=1}^{r} g_k^{\alpha_k^* - 1}$$

- We might implement our noninformative prior belief in various forms; for example, one might sample θ_i from $P(\theta_i)$ and set $\alpha_i^* = \alpha_0/r, \forall i$


Uninformed Gaussian prior
0.2
0.15
0.1
0.05
0
-5
-4
-3
-2
-1
0
1
2
3
4
5
theta
Samples used for Dirichlet enhancement

Dirichlet Enhancement (2)

- Thus we have obtained a model that technically is equivalent the the *multinomial likelihood model with a Dirichlet prior and noisy measurements* as discussed in the last section
- The process of replacing the original prior by a prior using the Dirichlet Process is sometimes referred to a *Dirichlet enhancement*
- For inference in the model we can immediately apply Gibbs sampling

Towards Dirichlet Processes

- Naturally there are computational problems if we let $r \longrightarrow \infty$
- Technically, we have two options:
  - We introduce an auxiliary variables Z as before and use a *standard mixture model* where a reasonable small r might be used; this might not be appropriate if the distribution is not really clustered
  - We let $r \longrightarrow \infty$, which leads us to nonparametric Bayesian modeling and the Dirichlet process
- In the latter vase we obtain a Dirichlet process prior and the corresponding model is called a Dirichlet process mixture (DPM)

IV: Dirichlet Processes

Basic Properties

Dirichlet Process

- We have studied the multinomial model with a Dirichlet prior and extended the model to the case of noisy measurements
- We have studied the hierarchical Bayesian model and found that in the case of repeated trials it makes sense to employ Dirichlet enhancement
- We have concluded that one can pursue two paths
  - Either one assumes a finite mixture model and one permits the adaptation of the parameters
  - Or uses an infinite model and makes the transition from a Dirichlet distribution to a Dirichlet process (DP)
- In this section we study the transition to the DP
- The Dirichlet Process is a generalization of the Dirichlet distribution; whereas a Dirichlet distribution is a distribution over probabilities, a DP is a measure on measures

Basic Properties

- Let's compare the finite case and the infinite case
- In the finite case we wrote $\mathbf{g} \sim \mathrm{Dir}(.|\alpha^*)$, in the infinite case we write

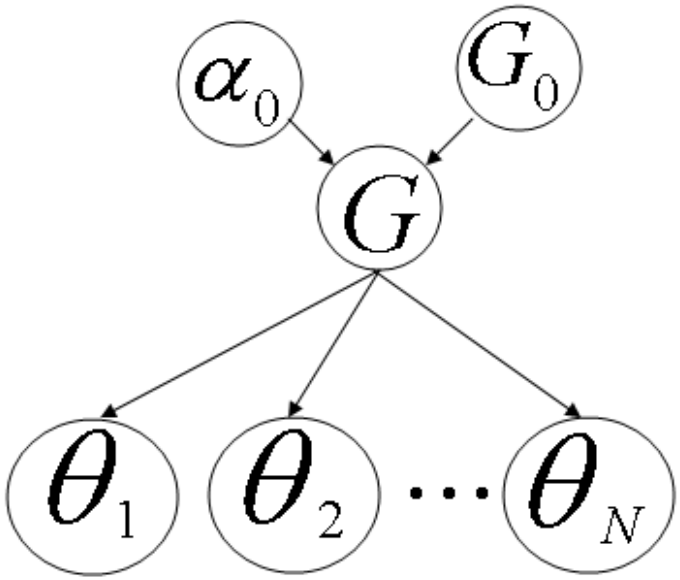
$$G \sim \mathrm{DP}(.|G_0, \alpha_0)$$

  where G is a measure (Ferguson, 1973)

- Furthermore, in the finite case we wrote $P(\Theta = \theta^k|\mathbf{g}) = g_k$; in the infinite case we write

$$\theta \sim G(\cdot)$$

- G_0 is the base distribution (corresponds to the $\boldsymbol{\alpha}$) and might be describes as a probability density, e.g., as a Gaussian $G_0 \sim N(.|0, I)$
- α_0 again is a concentration parameter; the graphical structure is shown in the figure



Graphical Model of a
Dirichlet Process

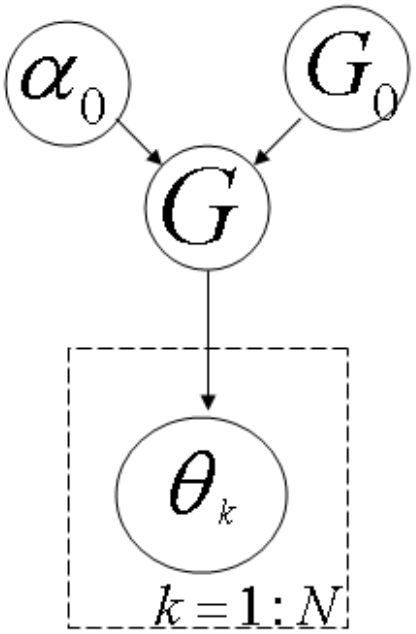


Plate representation

Basic Properties: Posteriors

- In analogy to the finite case, the posterior is again a DP with

$$G|\theta_1 \dots \theta_N \sim \mathrm{DP}\left(\frac{1}{\alpha_0 + N}\left[\alpha_0 G_0 + \sum_{k=1}^{N} \delta_{\theta_k}\right], \alpha_0 + N\right)$$

- δ_{θ_k} is a discrete measure concentrated at θ_k

  compare to the finite case

$$\mathbf{g}|\theta_1 \dots \theta_N = \mathrm{Dir}(\cdot|\alpha_1^* + N_1, \dots, \alpha_r^* + N_r)$$

Generating Samples from G and θ

Sampling from θ: Urn Representation

- Consider that N samples $\theta_1, \ldots, \theta_N$ have been generated
- In the Dirichlet distribution, we used $P(\Theta_{N+1} = \theta^k | D) = \frac{\alpha_0 \alpha_k + N_k}{\alpha_0 + N}$
- This generalizes in an obvious way in the Dirichlet process to (Blackwell and MacQueen, 1973)

$$\theta_{N+1} | \theta_1, \ldots, \theta_N \sim \frac{1}{\alpha_0 + N} \left(\alpha_0 G_0(\cdot) + \sum_{k=1}^{N} \delta_{\theta_k} \right)$$

- This is associated with the Pólya urn representation: one draws balls with different colors out of a urn (with G_0); If a ball is drawn, one puts the ball back *plus an additional ball with the same color* (δ_{θ_k}); thus in subsequent draws balls with a color already encountered become more likely to we drawn again
- Note, that there is no need to sample from G

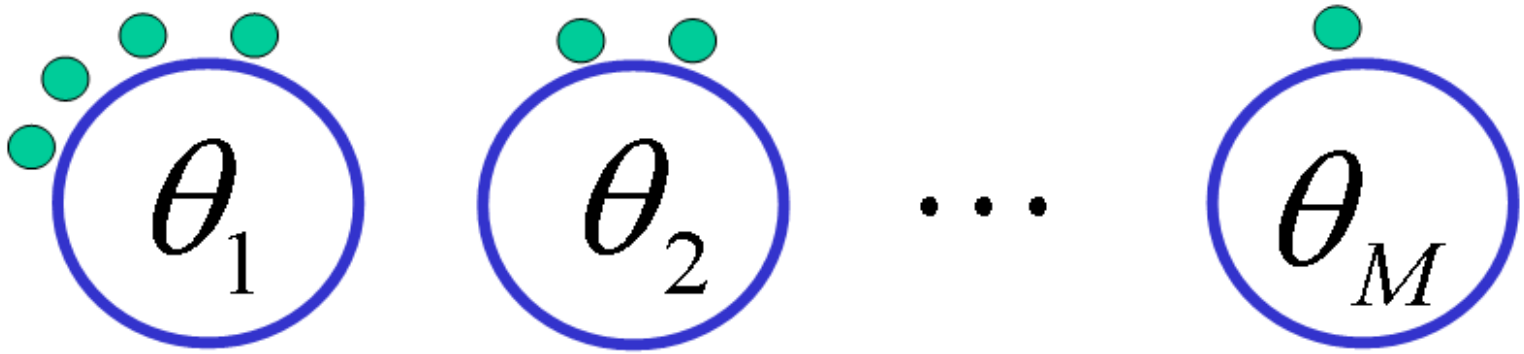
Sampling from θ (2)

- Note that the last equation can be interpreted as a mixture of distributions:
  - With prob. $\alpha_0/(\alpha_0 + N)$ a sample is generated from distribution G_0
  - With prob. $N/(\alpha_0 + N)$ a sample is generated uniformly from $\{\theta_1, \ldots, \theta_N\}$ (which are not necessarily distinct)
- Note, that in the urn process it is likely that identical parameters are repeatedly sampled

Chinese Restaurant Process (CRP)

- This is formalized as the Chinese restaurant process (Aldous, 1985); in the Chinese restaurant process it is assumed that customers sit down in a Chinese restaurant with an infinite number of tables; $Z_k = j$ means that customer k sits at table j. Associated with each table j is a parameter θ_j
- The first customer sits at the first table 1, $Z_1 = 1$; we generate a sample $\theta_1 \sim G_0$
- With probability $1/(1 + \alpha_0)$, the second customer also sits at the first table 1, $Z_2 = 1$, and inherits θ_1; with probability $\alpha_0/(1 + \alpha_0)$ the customer sits at table 2, $Z_2 = 2$, and a new sample is generated $\theta_2 \sim G_0$
- The figure shows the situation after N customers have entered the restaurant

Chinese Restaurant Process



N customers occupy M tables

N_j customers sit at table j

$1 \leq j \leq M$

Chinese Restaurant Process (CRP)(2)

- Customer $N+1$ enters the restaurant
- Customer $N+1$ sits with probability

$$\frac{N_j}{N+\alpha_0}$$

at a previously occupied table j and inherits θ_j. Thus: $Z_{N+1} = j$, $N_j \leftarrow N_j + 1$

- With probability

$$\frac{\alpha_0}{N+\alpha_0}$$

the customer sits at a new table $M+1$. Thus: $Z_{N+1} = M+1$, $N_{M+1} = 1$

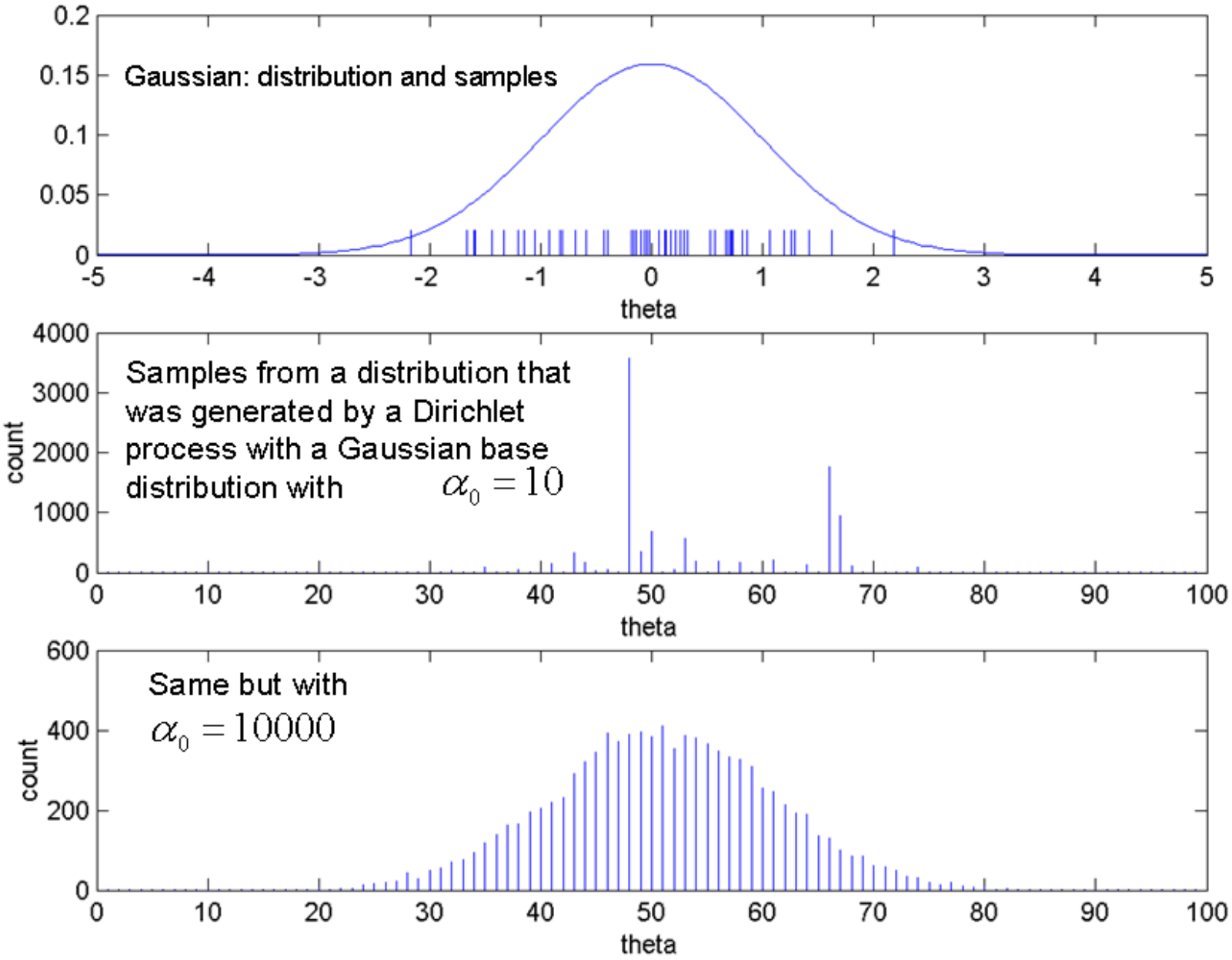
- For the new table a new parameter $\theta_{M+1} \sim G_0(\cdot)$ is generated. $M \leftarrow M+1$

Chinese Restaurant Process (CRP)(3)

- Obviously, the generated samples exactly correspond to the ones generated in the urn representation

Discussion

- So really not much new if compared to the finite case
- In particular we observe the same clustering if α_0 is chosen to be small
- The CRP makes the tendency to generate clusters even more apparent (see figure); again the tendency towards forming clusters can be controlled by α_0


Gaussian: distribution and samples
theta
Samples from a distribution that was generated by a Dirichlet process with a Gaussian base distribution with $\alpha_0 = 10$
count
theta
Same but with $\alpha_0 = 10000$
count
theta

Sampling from G: Stick Breaking Representation

- After an infinite number of samples are generated the underlying $G(\cdot)$ can be recovered
- Not surprisingly, the underlying measure can be written as (Sethuraman, 1994)

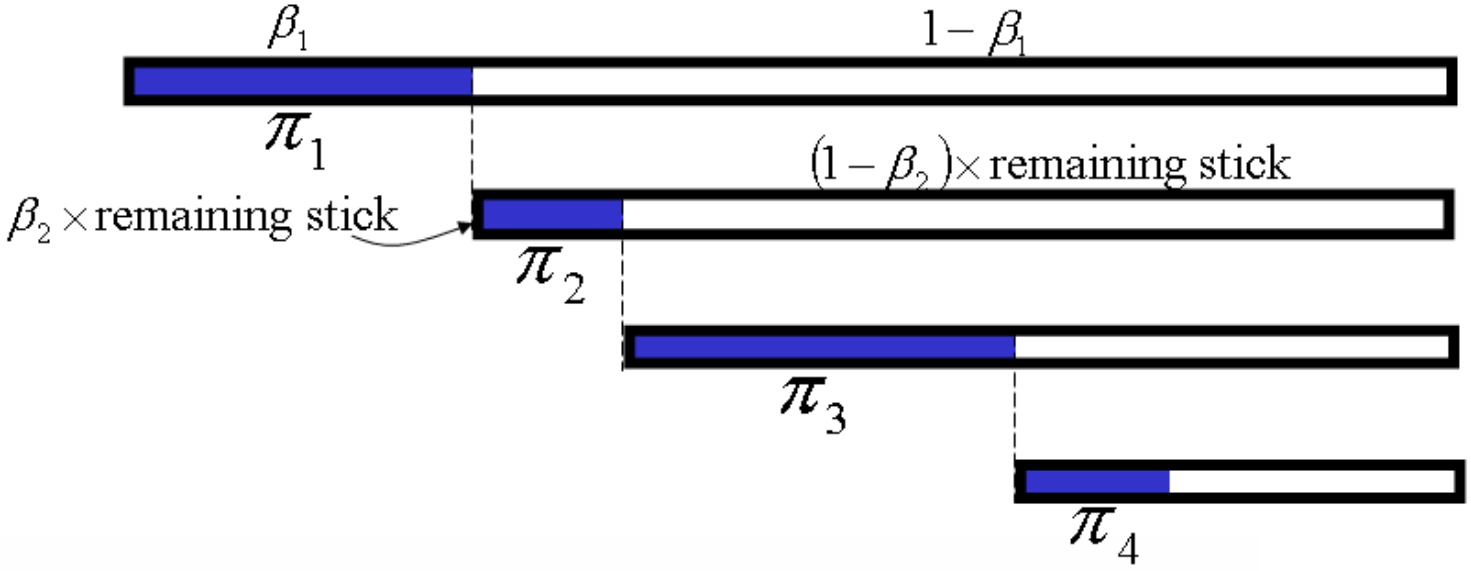
$$G(\cdot) \sim \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k}(.)$$

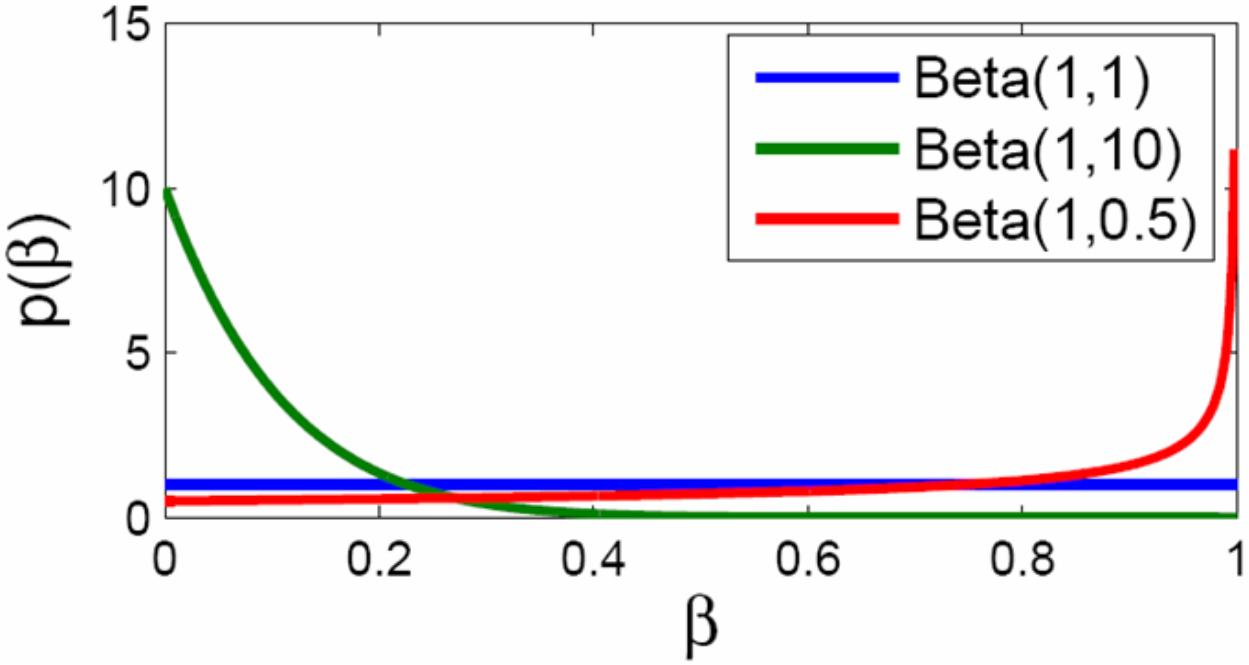
$\pi_k \geq 0$, $\sum_{k=1}^{\infty} \pi_k = 1 \quad \theta_k \sim G_0(\cdot)$

- Furthermore, the π_k can be generated recursively with $\pi_k = \beta_1$ and

$$\pi_k = \beta_k \prod_{j=1}^{k-1} (1 - \beta_j) \quad k \geq 2$$

- $\beta_1, \beta_2, \ldots$ are independent $\mathrm{Be}(1, \alpha_0)$ random variables
- One writes $\boldsymbol{\pi} \sim \mathrm{Stick}(\alpha_0)$





Stick Breaking

Introducing an Auxiliary Variable

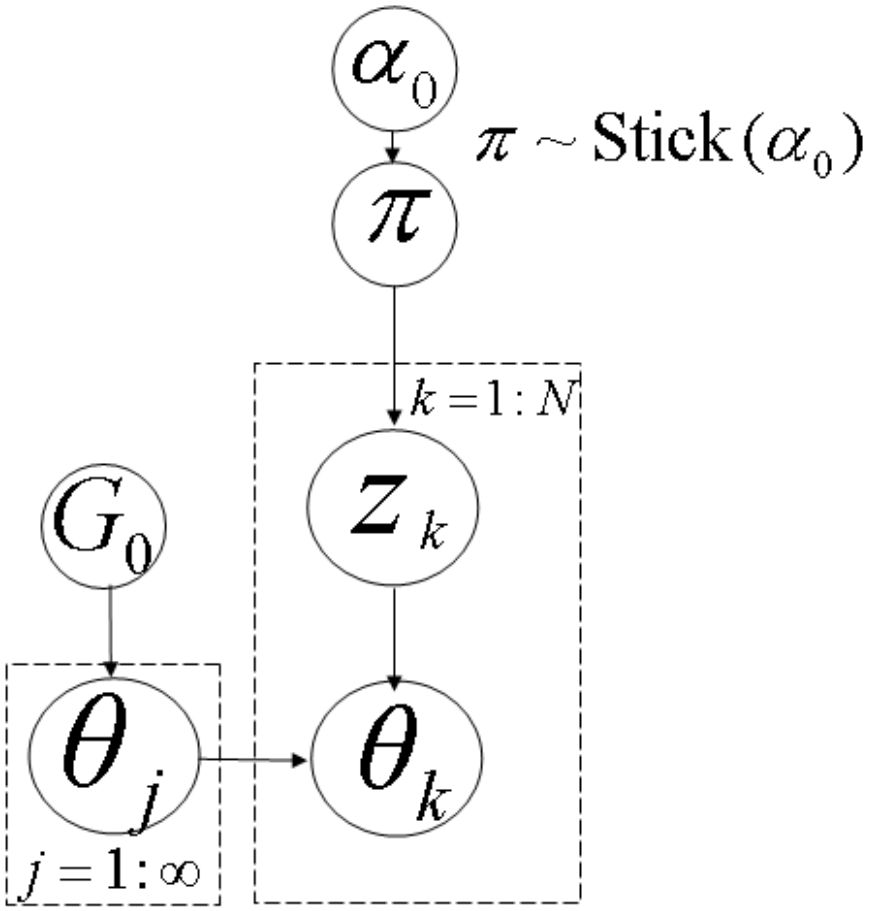
- Considering the particular form of the stick breaking prior, we can implement the DP model using an auxiliary variable Z with an infinite number of states $z^1, z^2, \ldots$
- With the stick breaking probability,

$$\boldsymbol{\pi} \sim \text{Stick}(\alpha_0)$$

  is generated
- Then, one generates independently for $k = 1, 2, \ldots$

$$Z_k \sim \boldsymbol{\pi} \qquad \theta_k \sim G_0$$

- The CRP produces samples of Z and θ in this model (integrating out $\boldsymbol{\pi}$); compare the graphical model in the figure

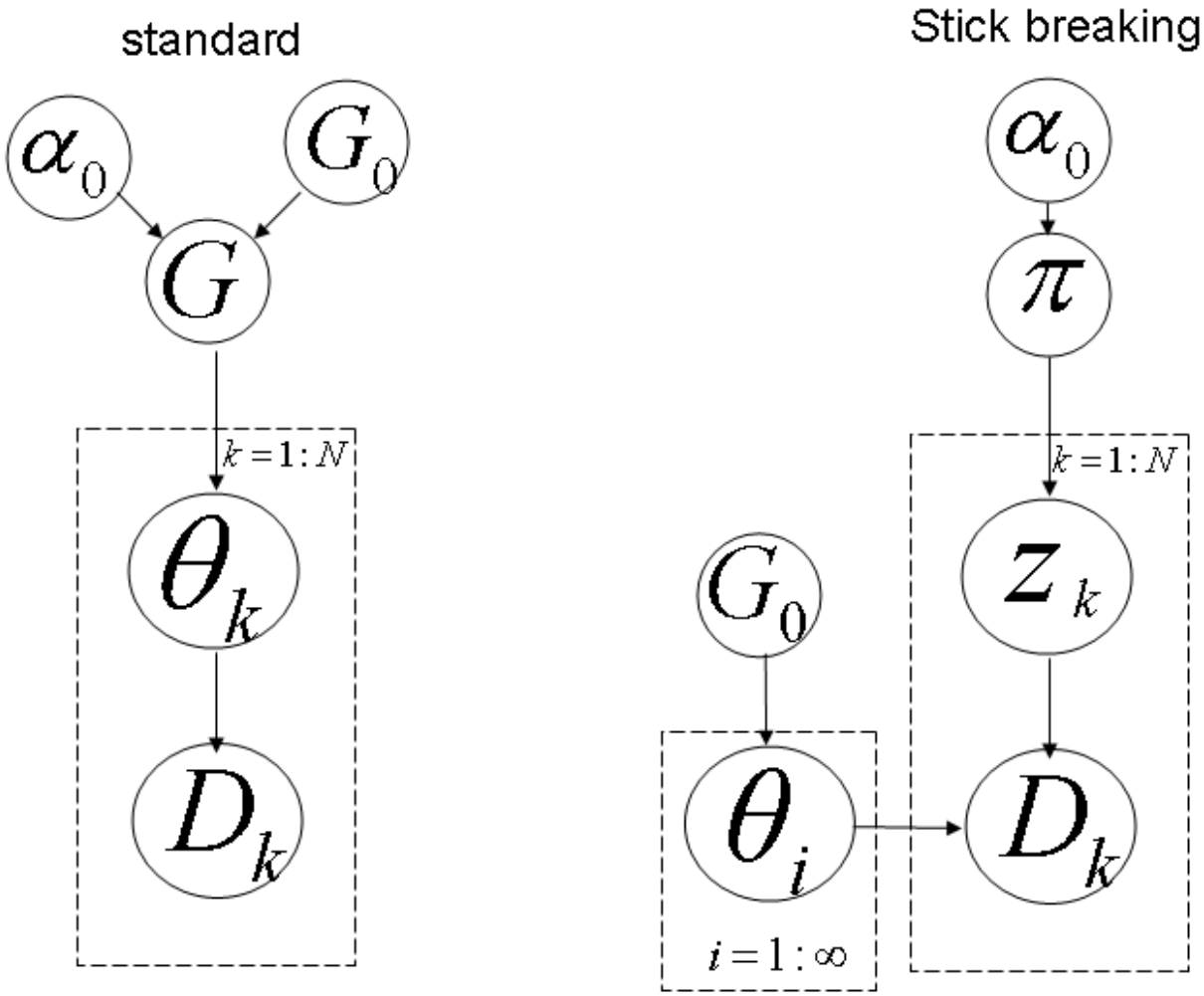


A DP model derived from the stick breaking representation

Including an Observation Model - The Dirichlet Process Mixture

The Dirichlet Process Mixture (DPM)

- Now we consider that the realizations of θ are unknown; furthermore we assume that derived quantities (e.g., noisy measurements) X with some $P(X|\theta)$ are available. Let $D_k = \{x_{k,j}\}_j$ be the data available for θ_k and let $P(x_{k,j}|\theta_k)$ be the probability distribution
- Note, that this also includes the case that the observer model is conditioned on some input $in_{k,j}$: $P(x_{k,j}|in_{k,j}, \theta_k)$
- Recall, that this is exactly the situation encountered in the Dirichlet enhanced hierarchical Bayesian model
- The Dirichlet process mixture is also called: Bayesian nonparametric hierarchical model (Ishwaran), and, not quite accurately, mixture of Dirichlet proceses



Dirichlet Process Mixture Models

- Gibbs sampling on the collapsed left model correpsonds to urn-type sampling
- Gibbs sampling on the collapsed right model correpsonds to CRP-type sampling

Gibbs Sampling from the DPM using the Urn Representation

- In analogy to the finite case, the crucial distribution is now

$$\theta_k|\{\theta_i\}_{i\neq k}, D \sim \frac{1}{C}\left(\alpha_0 G_0(\cdot) + \sum_{l:l\neq k}\delta_{\theta_l}\right) P(D_k|\theta_k)$$

- This can be re-written as

$$\theta_k|\{\theta_i\}_{i\neq k}, D \sim \frac{1}{C}\left(\alpha_0 P(D_k) P(\theta_k|D_k) + \sum_{l:l\neq k} P(D_k|\theta_l)\delta_{\theta_l}\right)$$

$C = \alpha_0 P(D_k) + \sum_{l:l\neq k} P(D_k|\theta_l)\delta_{\theta_l}$

Sampling from the DPM using the Urn Representation (2)

- Here,

$$P(D_k) = \int P(D_k|\theta) dG_0(\theta)$$

$$P(\theta_k|D_k) = \frac{P(D_k|\theta_k)G_0(\theta_k)}{P(D_k)}$$

- Both terms can be calculated in closed form, if $G_0(\cdot)$ and the likelihood are conjugate. In this case, sampling from $P(\theta_k|D_k)$ might also be simple

Sampling from the DPM using the CRP Representation

- We can again use the CRP model for sampling from the DPM
- Folding in the likelihood, we obtain
  - We randomly select customer k; the customer sat at table $Z_k = i$; we remove her/him from her/his table; thus $N_i \leftarrow N_i - 1$; $N \leftarrow N - 1$; if the table i is now unoccupied it is removed; assume, M tables are occupied
  - Customer i now sits with probability proportional to

    $$N_j P(D_k|\theta_j)$$

    at an already occupied table j and inherits θ_j. $Z_k = j$, $N_j \leftarrow N_j + 1$
  - With probability proportional to

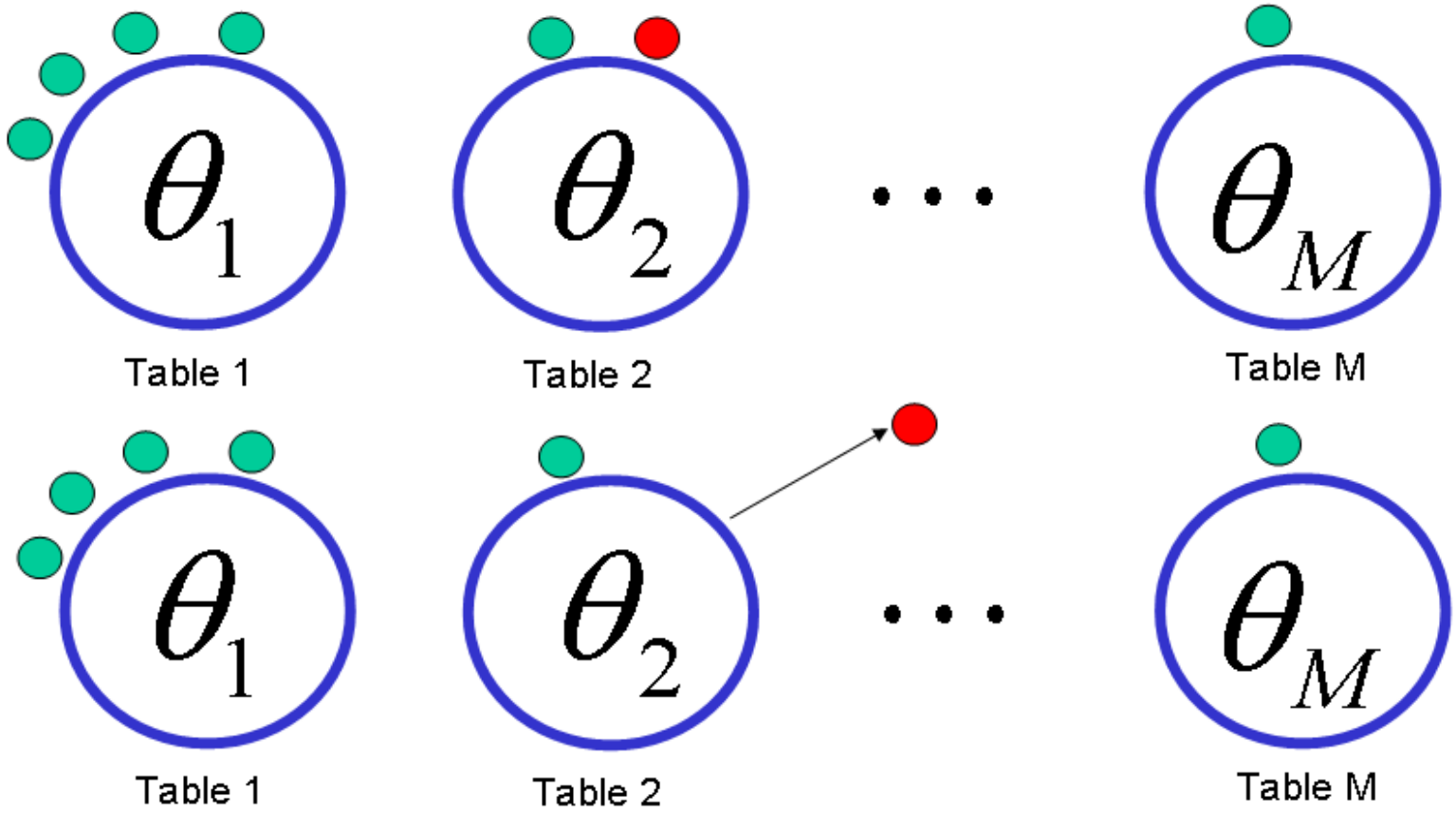
    $$\alpha_0 P(D_k)$$

    the customer sits at a new table $M + 1$. $Z_k = M + 1$, $N_{M+1} = 1$. For the new table a new parameter $\theta_{M+1} \sim P(\theta|D_k)$ is generated

Sampling from the DPM using the CRP Representation (2)

- In the CRP representation, $\theta_k, k = 1, \ldots$ *are re-sampled occasionally from the posterior parameter distribution given all data assigned to table* k
- Due to this re-estimation of all parameters assigned to the same table in one step, the Gibbs sampler mixes better than the sampler based on the urn represenation

Chinese Restaurant Process Gibbs Sampler



The customer is removed from the table and then is placed at a table that explains the customer's data well, giving a preference to highly occupied tables

- The customer is also allowed to open a new table
- Occasionally the parameters of the tables are re-estimated

Integrating out the Parameters

- This is a CRP sampler, where one never needs to generate samples from the θ_k
- $$N_j P(D_k|\theta_j)$$

  is replaced by

  $$N_j P(D_k|D^j) = N_j \int P(D_k|\theta) P(\theta|G_0, D^j) d\theta$$

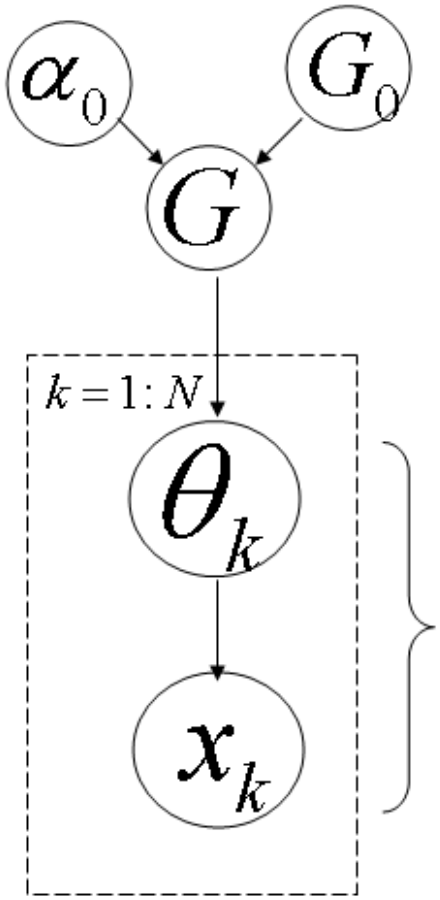
  where D^j are all data previously associated with $Z = j$.
- If G_0 is conjugate to the likelihood, a closed-form for the integral is available

Comments on the Base Distribution

- Note, that independent of α_0, parameters are generated from the posterior distribution using the base distribution as a prior
- So independent of α_0, the base distribution keeps playing an important role!

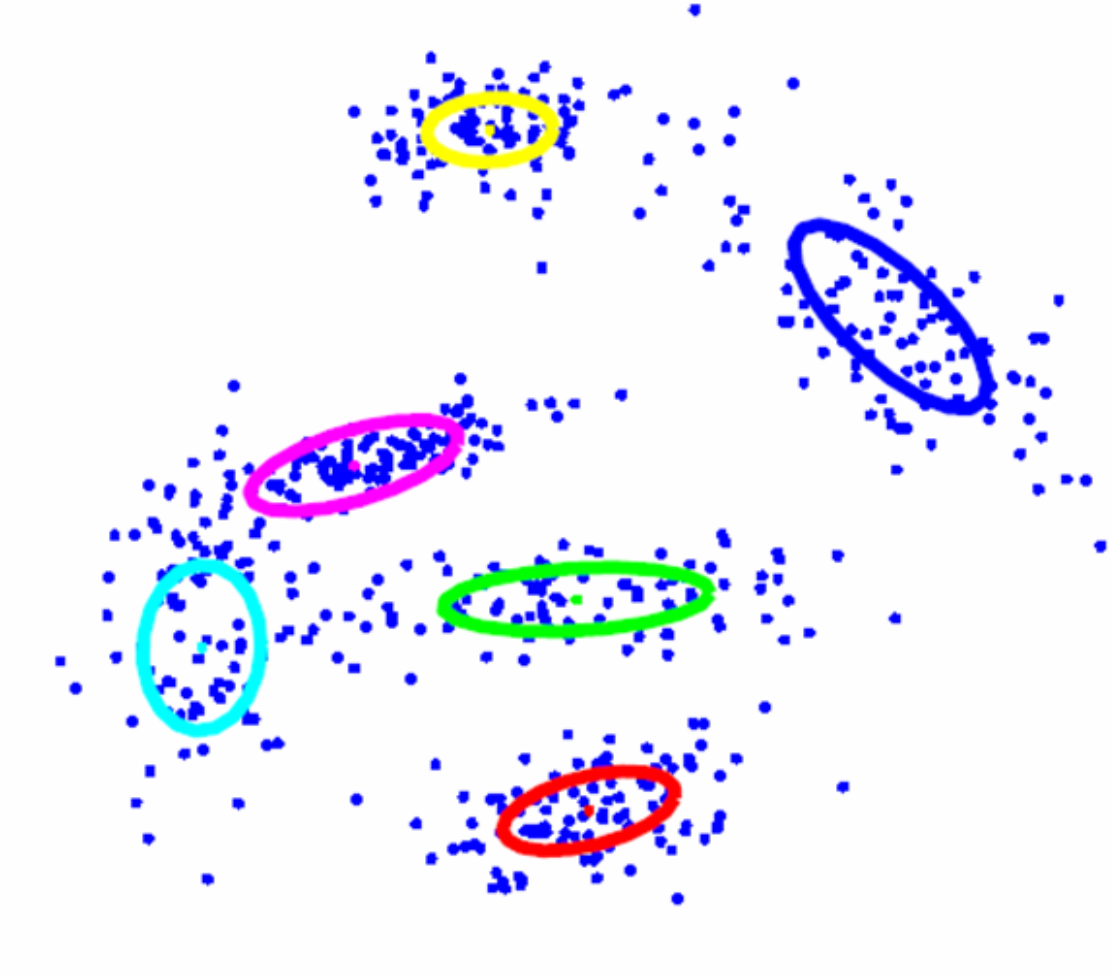
Example (what is all this good for (2))

- Let's assume that $P(x_k|\theta_k)$ is a Gaussian distribution, i.e., θ_k corresponds to the center and the covariance of a Gaussian distribution
- During CRP-sampling, all data points assigned to the same table k inherit identical parameters, thus can be though of to be generated from the same Gaussian
- Thus, the number of occupied tables gives us an estimate of the true number of clusters in the data
- Thus in contrast to a finite mixture model, we do not have to specify the number of clusters we are looking for in advance!
- α_0 is a tuning parameter, tuning the tendency to generate a large number (α_0: large) or a small number (α_0: small) of clusters



DPM with a Gaussian model corresponding to an infinite mixture of Gaussians

Gaussian distribution



Snapshot of the sampling of a DPM representing an infinite mixture of Gaussians

- The data points are the customers
- A customer is assigned uniquely to a cluster(i.e., table)
- 6 tables are occupied
- The parameters of the Gaussians are sampled from the posterior distribution of the parameters given the data assigned to the particular table

Relationship to the Standard Mixture Model and Blocked Sampling

Sampling from the DPM using the CRP Representation (2)

- The DPM model with an auxiliary variable can be derived from the standard mixture model (see last section) if the Dirichlet prior for the mixing proportion is

$$\boldsymbol{\pi} \sim \mathrm{Dir}(\cdot|\alpha_0/r, \ldots, \alpha_0/r)$$

  and with

$$\theta \sim G_0(\cdot)$$

  when we let $r \longrightarrow \infty$

- *Here, it is even mode clear that the DPM can be interpreted as a mixture model with an infinite number of components, where the prior distribution for the parameters is given by the base distribution*

Blocked Sampling Derived from a Finite Stick Representation

- As discussed in the part on the finite models, the sampler mixes better if one could generate samples from G, since then, the parameters can be sampled independently; thus it allows blocked updates
- The stick breaking process allows us to sample from G but the representation is infinite; a finite representation with K terms derived from the stick breaking representation would be the obvious solution

$$G(\cdot) \sim \sum_{k=1}^{K} \pi_k \delta_{\theta_k}(.)$$

Truncated Stick Breaking and Dirichlet-multinomial Allocation

- In the *truncated* approach one simply terminates the stick breaking procedure at K terms; one sets $\beta_K = 1$ so that the probabilities sum to one
- In the *Dirichlet-multinomial allocation* one sets

$$\boldsymbol{\pi} \sim \mathrm{Dir}(\cdot|\alpha_0/r, \ldots, \alpha_0/r)$$

- The latter case is identical to a finite mixture model with a large number of components r

Gibbs sampling for the Dirichlet mixture model

	Original Model	With auxiliary variable
Blocked Gibbs sampling		Finite approximation: •truncated stick breaking •Dirichlet multin. alloc.
Collapsed Gibbs sampling	Urn based	CRP based

Formal Definition of a DP

Let Θ be a measurable space, G_0 be a probability measure on Θ, and α_0 a positive real number

For every partition $B_1, B_2, ..., B_k$ of Θ

$$G \sim \mathrm{DP}(\cdot|G_0, \alpha_0)$$

means that

$$(G(B_1), G(B_2), \ldots, G(B_k)) \sim \mathrm{Dir}(\alpha_0 G_0(B_1), \alpha_0 G_0(B_2), \ldots, \alpha_0 G_0(B_k))$$

(Ferguson, 1973, Ghahramani, 2005)

Even More Formal Definition of a DP

The theorem asserts the existence of a Dirichlet process and also serves as a definition. Let $(\mathbb{R}, \mathcal{B})$ be the real line with the Borel σ-algebra $\mathcal{B}$ and let $M(\mathbb{R})$ bet the set of probability measures on $\mathbb{R}$, equipped with the σ-algebra $\mathcal{B}_M$

Theorem 1 *Let α be a finite measure on $(\mathbb{R}, \mathcal{B})$. Then there exists a unique probability measure D_α on $M(\mathbb{R})$ called the Dirichlet process with parameters α satisfying:*

For every partition $B_1, B_2, ..., B_k$ of $\mathbb{R}$ by Borel sets
$(P(B_1), P(B_2), \ldots, P(B_k)) \sim Dir(\alpha(B_1), \alpha((B_2), \ldots, \alpha((B_k))$

V: Applications and Extensions

Sharing Statistical Strength: A Recommendation System

Recommendation Systems

- Let's assume that the task is to build recommendation systems for different users based on the features of the items
- For a particular user there might not be sufficient data for obtaining a reasonable model
- Thus it is sensible to build a nonparametric hierarchical model to share statistical strength

book 1: like

book 2: like

book 3: ?

active user

feature 1: fashion:y/n

feature 2: love stories:y/n

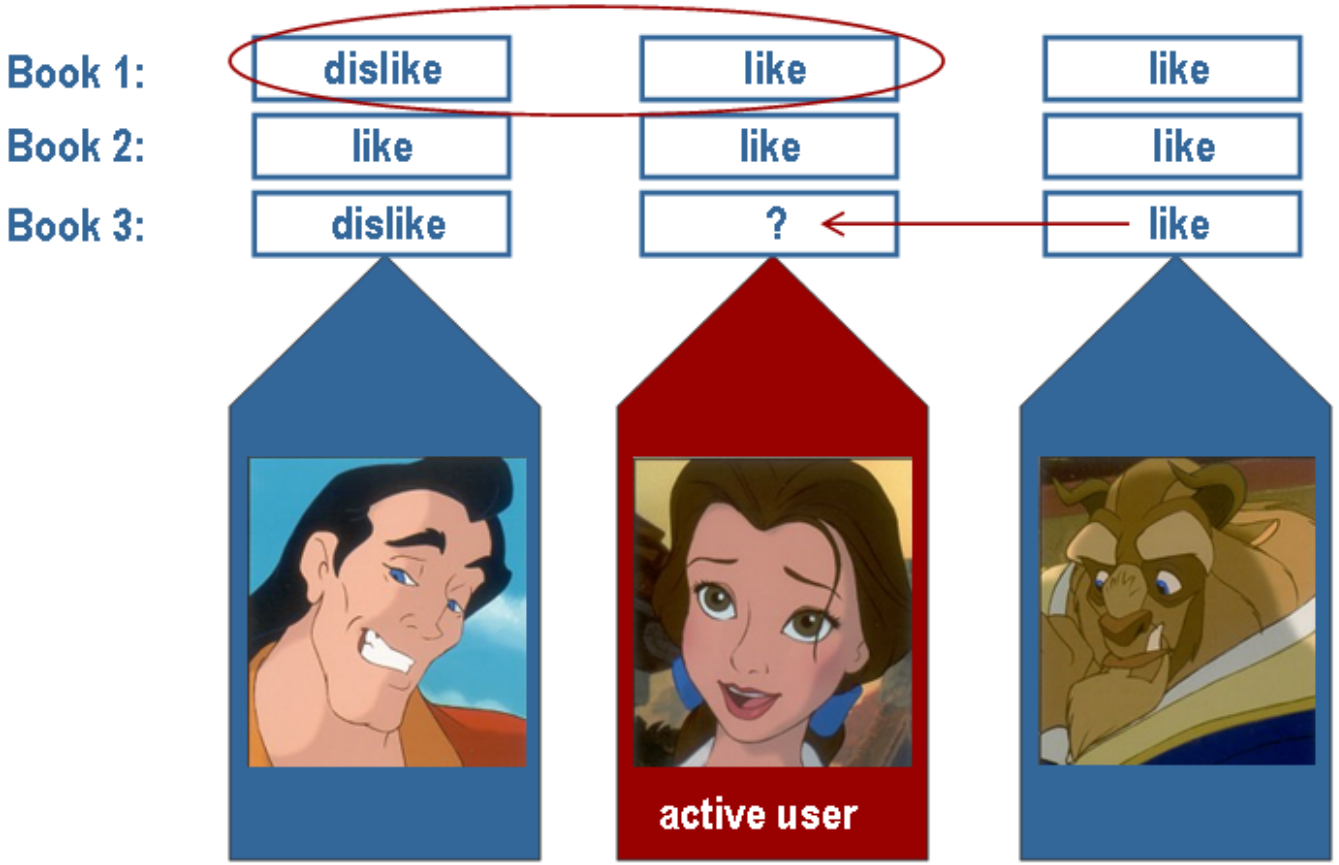
feature 3: action:y/n

Does Belle like book 3?

Predictive Models:
SVM, NN, GLM, …,

$$P(y \mid x, \theta)$$

$$P(y \mid x, D)$$

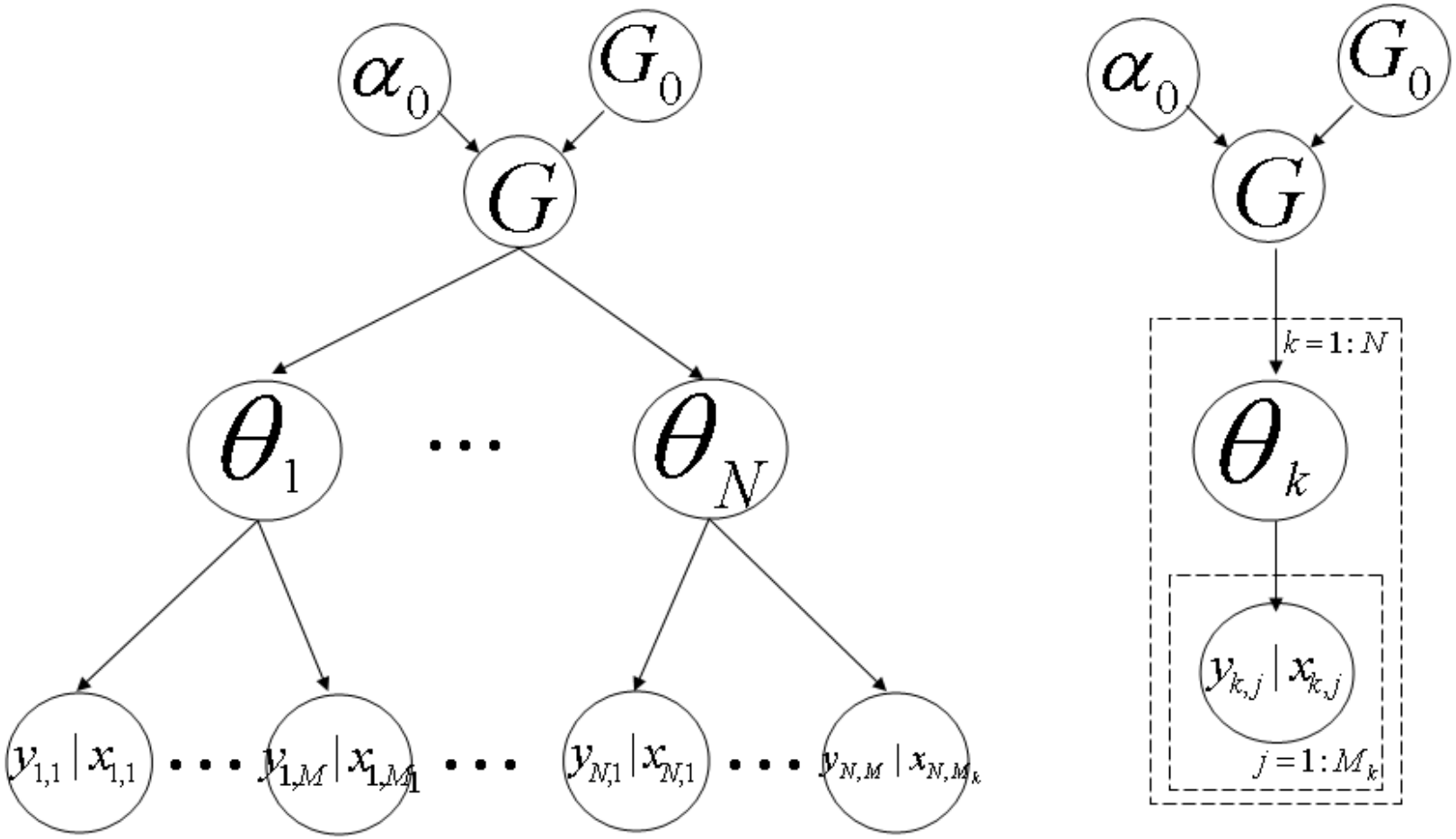


It seems that Belle agrees more with Beast!

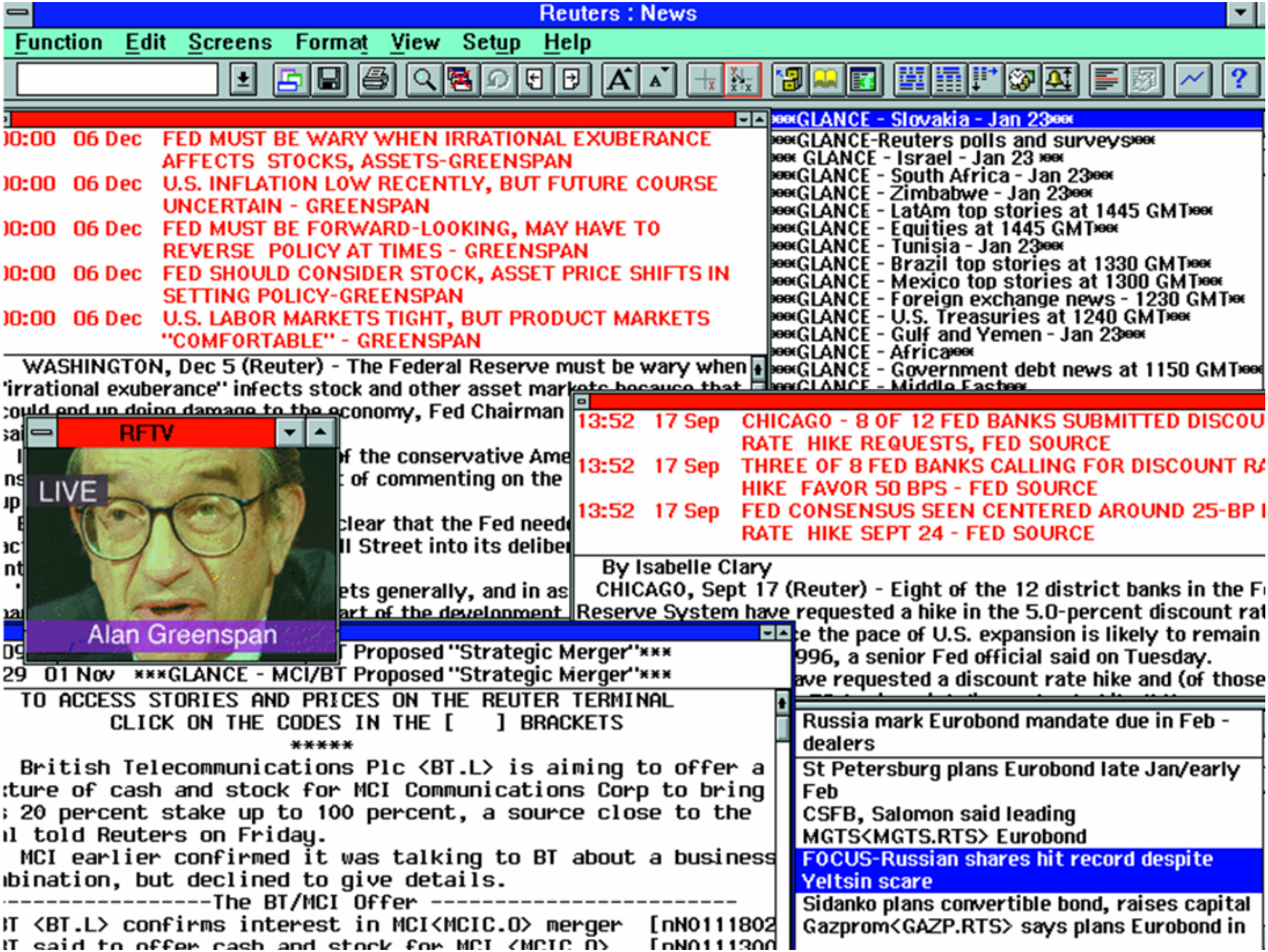
Different Modeling Assumptions

Different assumptions lead to different models:

- All users are the same: use one model trained on all data
- Each user is a complete individual: train a separate model for each user
- Learn from one another: each user has her/his own model generated from a common prior distribution, which is learned from data and shared between user models

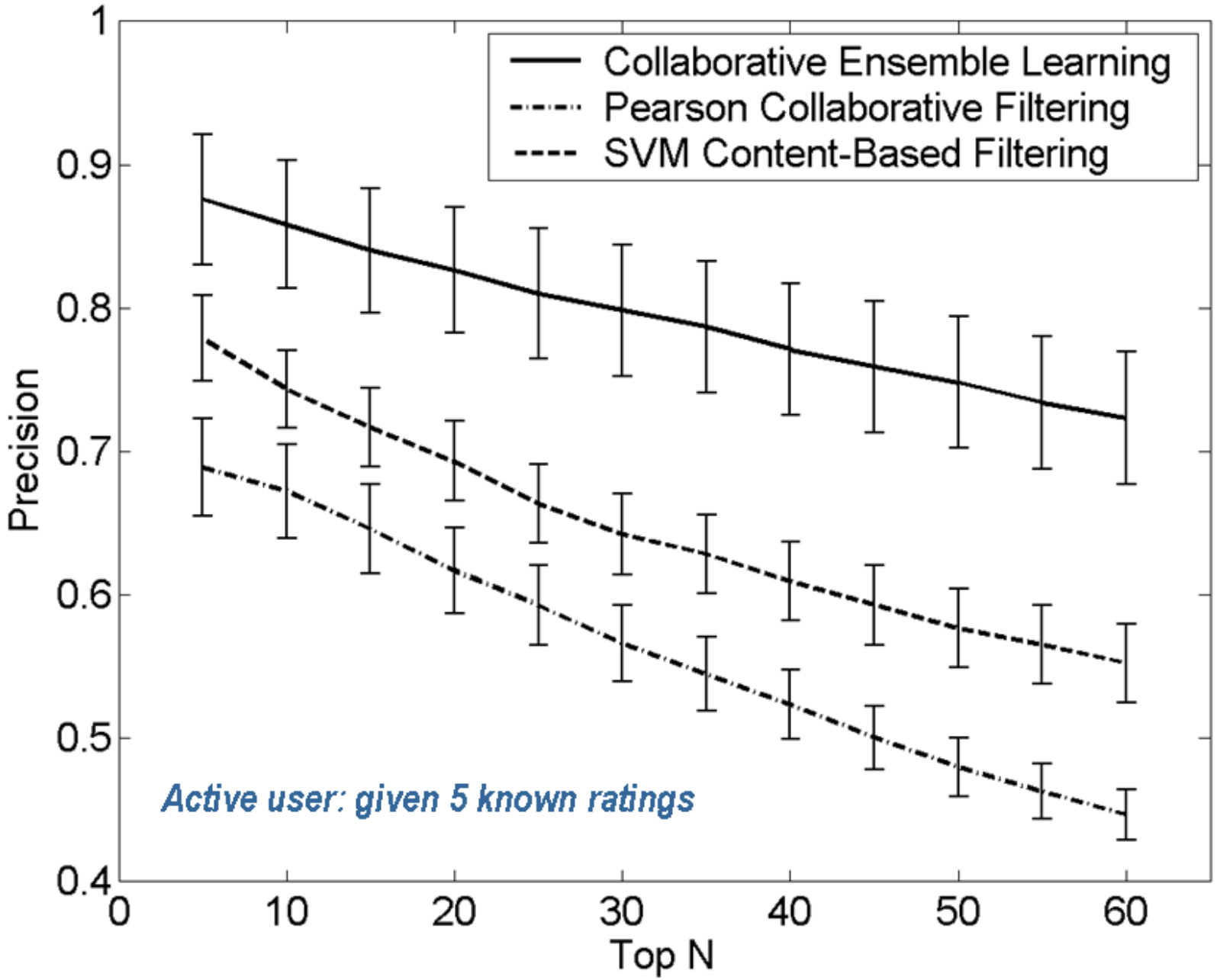


A DPM Recommendation System


Reuters : News
Function Edit Screens Format View Setup Help
)0:00 06 Dec FED MUST BE WARY WHEN IRRATIONAL EXUBERANCE AFFECTS STOCKS, ASSETS-GREENSPAN
)0:00 06 Dec U.S. INFLATION LOW RECENTLY, BUT FUTURE COURSE UNCERTAIN - GREENSPAN
)0:00 06 Dec FED MUST BE FORWARD-LOOKING, MAY HAVE TO REVERSE POLICY AT TIMES - GREENSPAN
)0:00 06 Dec FED SHOULD CONSIDER STOCK, ASSET PRICE SHIFTS IN SETTING POLICY-GREENSPAN
)0:00 06 Dec U.S. LABOR MARKETS TIGHT, BUT PRODUCT MARKETS "COMFORTABLE" - GREENSPAN
WASHINGTON, Dec 5 (Reuter) - The Federal Reserve must be wary when "irrational exuberance" infects stock and other asset markets because that could end up doing damage to the economy, Fed Chairman
GLANCE - Slovakia - Jan 23
GLANCE-Reuters polls and surveys
GLANCE - Israel - Jan 23
GLANCE - South Africa - Jan 23
GLANCE - Zimbabwe - Jan 23
GLANCE - LatAm top stories at 1445 GMT
GLANCE - Equities at 1445 GMT
GLANCE - Tunisia - Jan 23
GLANCE - Brazil top stories at 1330 GMT
GLANCE - Mexico top stories at 1300 GMT
GLANCE - Foreign exchange news - 1230 GMT
GLANCE - U.S. Treasuries at 1240 GMT
GLANCE - Gulf and Yemen - Jan 23
GLANCE - Africa
GLANCE - Government debt news at 1150 GMT
RFTV
LIVE
Alan Greenspan
13:52 17 Sep CHICAGO - 8 OF 12 FED BANKS SUBMITTED DISCOU RATE HIKE REQUESTS, FED SOURCE
13:52 17 Sep THREE OF 8 FED BANKS CALLING FOR DISCOUNT RA HIKE FAVOR 50 BPS - FED SOURCE
13:52 17 Sep FED CONSENSUS SEEN CENTERED AROUND 25-BP RATE HIKE SEPT 24 - FED SOURCE
By Isabelle Clary
CHICAGO, Sept 17 (Reuter) - Eight of the 12 district banks in the F Reserve System have requested a hike in the 5.0-percent discount rat
29 01 Nov ***GLANCE - MCI/BT Proposed "Strategic Merger"***
TO ACCESS STORIES AND PRICES ON THE REUTER TERMINAL
CLICK ON THE CODES IN THE [ ] BRACKETS
*****
British Telecommunications Plc <BT.L> is aiming to offer a
20 percent stake up to 100 percent, a source close to the
told Reuters on Friday.
MCI earlier confirmed it was talking to BT about a business
----------The BT/MCI Offer ----------
<BT.L> confirms interest in MCI<MCIC.O> merger [nN0111802
Russia mark Eurobond mandate due in Feb - dealers
St Petersburg plans Eurobond late Jan/early Feb
CSFB, Salomon said leading MGTS<MGTS.RTS> Eurobond
FOCUS-Russian shares hit record despite Yeltsin scare
Sidanko plans convertible bond, raises capital
Gazprom<GAZP.RTS> says plans Eurobond in

News

- We used 36 categories covering totally 10,034 news articles from the Reuters text data set (1,152 articles belong to more than one category)
- For the experiments, we assume that each user is interested in exactly one of these categories
- We generate example data for 360 (artificial) users by choosing at random a set of 30 (positive and negative) example items
- The goal is to predict the probability that a user likes an article
- Rank all unseen articles and select top N ranked articles;
- Plot: how many of the top N articles are truly positive


Collaborative Ensemble Learning
Pearson Collaborative Filtering
SVM Content-Based Filtering
Precision
Top N
1
0.9
0.8
0.7
0.6
0.5
0.4
0
10
20
30
40
50
60
Active user: given 5 known ratings

Welcome to Our Painting Survey Page!

Please click the Save button after rating complete!

Save Ratings and Exit

33 Like Dislike Not Sure

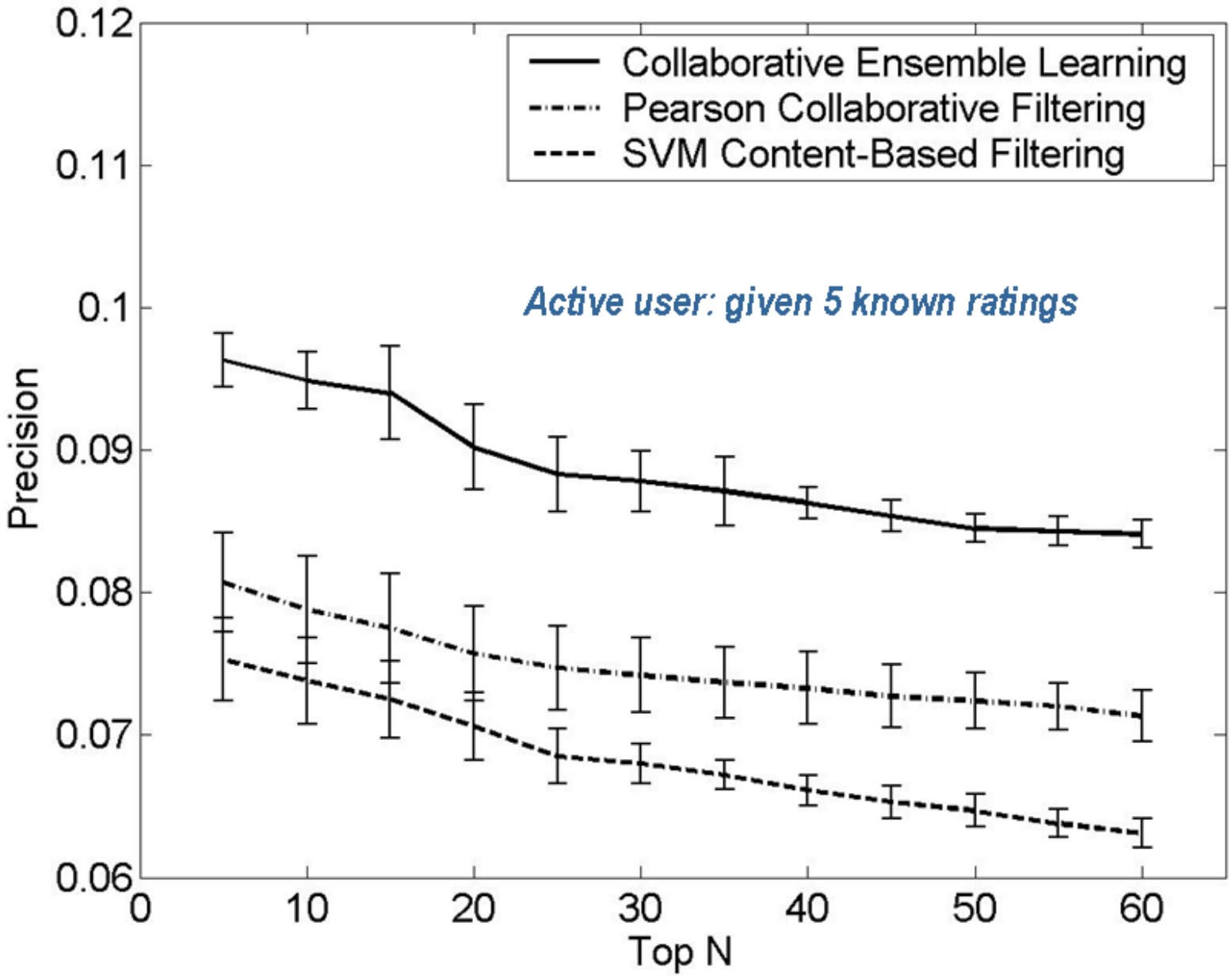
34 Like Dislike Not Sure

35 Like Dislike Not Sure

36 Like Dislike Not Sure

Paintings

- Task: Predict image rating (642 images)
- 190 users with 89 rated images on average
- Image features: 256 correlogram features (colour/texture) 10 features based on wavelet texture 9 features on color moments giving a 275-dimensional feature vector for each image; these are weak indicators of high-level information about an image
- The model predicts N highest ranked pictures; among those, how many were rated positively (in comparison to unrated or negatively rated)
- Classifier: probabilistic SVM with Gaussian kernel http://honolulu.dbs.informatik.uni-muenchen.de:8080/paintings/index.jsp


Collaborative Ensemble Learning
Pearson Collaborative Filtering
SVM Content-Based Filtering
Active user: given 5 known ratings
Precision
Top N
0.12
0.11
0.1
0.09
0.08
0.07
0.06
0
10
20
30
40
50
60

Implementation Details

- We used a deterministic variational EM approximation which leads to an approximation (Yu, Schwaighofer, Tresp, Ma, and Zhang, 2003)

$$G|D \propto \alpha_0 G_0(\cdot) + \sum_{k=1}^{N} \xi_k \delta_{\theta_k^{ML}}$$

- Here, θ_k^{ML} is the ML (or MAP) -estimate of each user model trained on its own data
- ξ_k is optimized in the variational EM approximation
- After convergence, the prediction of an active model $a \in 1, \ldots, N$ becomes

$$P(Y_a = y|x, \{D_k\}_{k=1}^N) \approx$$

$$\frac{1}{C}\left(\alpha_0 P(D_a) P(Y_a = y|x, D_a) + \sum_{k=1}^{N} \xi_k P(D_a|\theta_k^{ML}) P(Y_a = y|x, \theta_k^{ML})\right)$$

Implementation Details (2)

- Let's take a look at the second term
- Essentially it says that any user should make a prediction and that this prediction is weighted by the likelihood that a user can explain the past data of the active user
- Thus, initially, we obtain a simple average of all user's predictions
- If the active user has seen some data, those users get higher weight, who agree well with the past ratings of the active user
- Eventually, with many data for the active user, only the active user's own model will contribute to the prediction

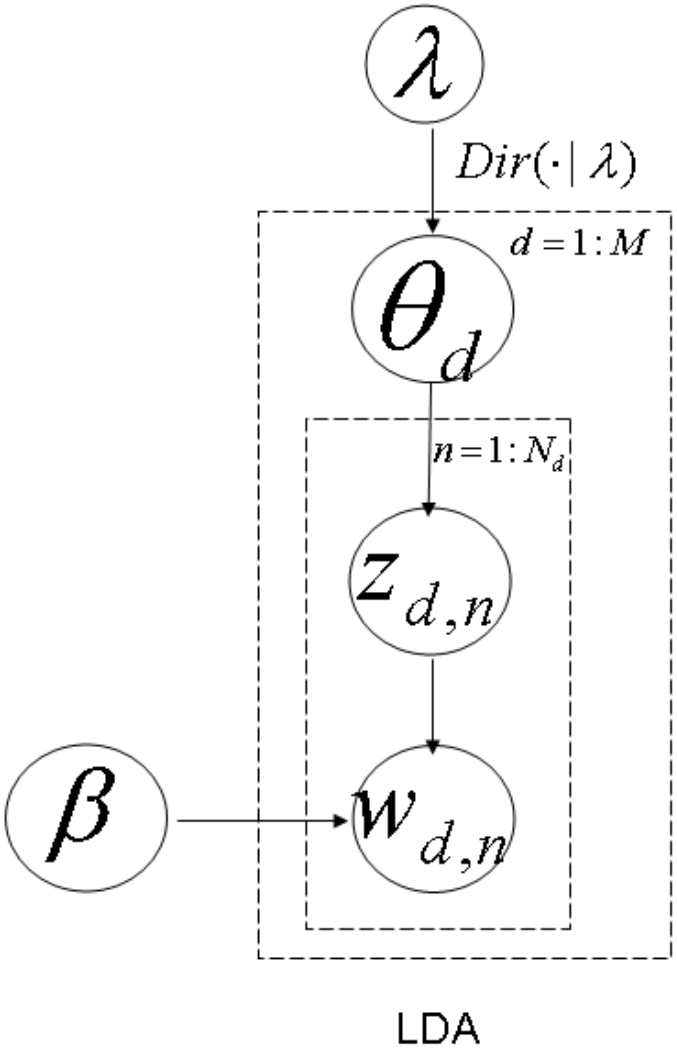
Cluster Analysis Using Textual Data: Dirichlet Enhanced Latent Semantic Analysis

The Goal

- In latent semantic analysis (LSA), we aim at modeling a large corpus of *high-dimensional discrete data* from a *probabilistic* perspective
- **The assumption:** one data point can be modeled by *latent factors*, which account for the co-occurrence of items within the data
- We are also interested in the *clustering* structure of the data, which may benefit from the latent factors of the items
- For example:
  - In document modeling, the data are document-word pairs
    Latent factors: topics for words
    Data clustering: categories of documents
  - In collaborative filtering, the data are user ratings (for, e.g., movies)
    Latent factors: categories or structures of movies
    Data clustering: user interest groups

Latent Dirichlet Allocation (LDA)

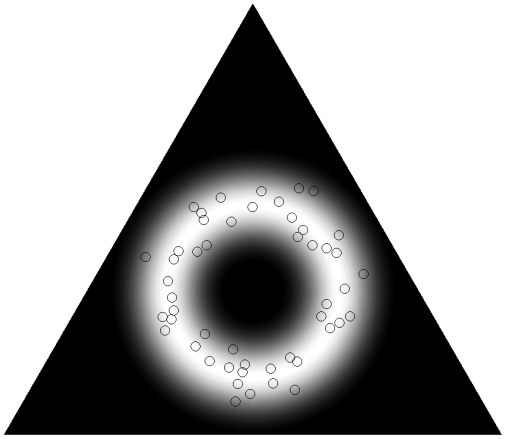
- **Latent Dirichlet Allocation (LDA)** Assign a discrete latent model to words and let each document maintain a random variable θ, indicating its probabilities of belonging to each topic



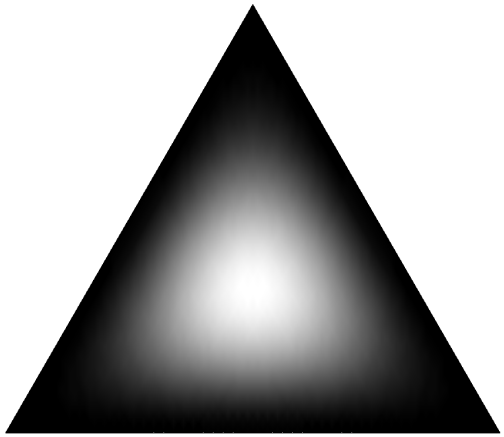
LDA

A Potential Problem with LDA

- Assumption: The prior for θ is a Dirichlet distribution (which is learned from data)
- Limitation: A single Dirichlet distribution is not flexible enough to represent interesting dependencies such as a clustering structure in documents



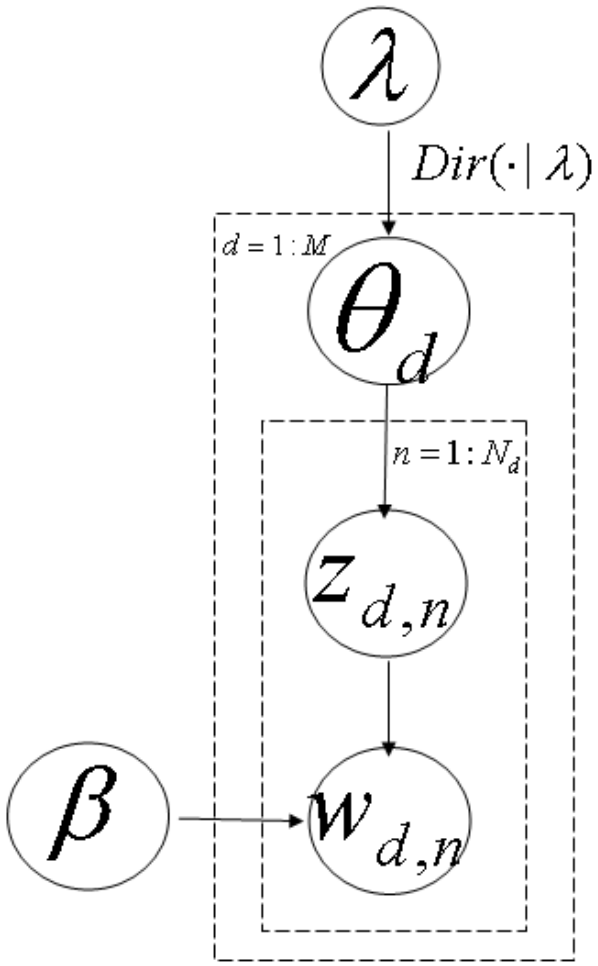
The true distribution of θ in a toy problem



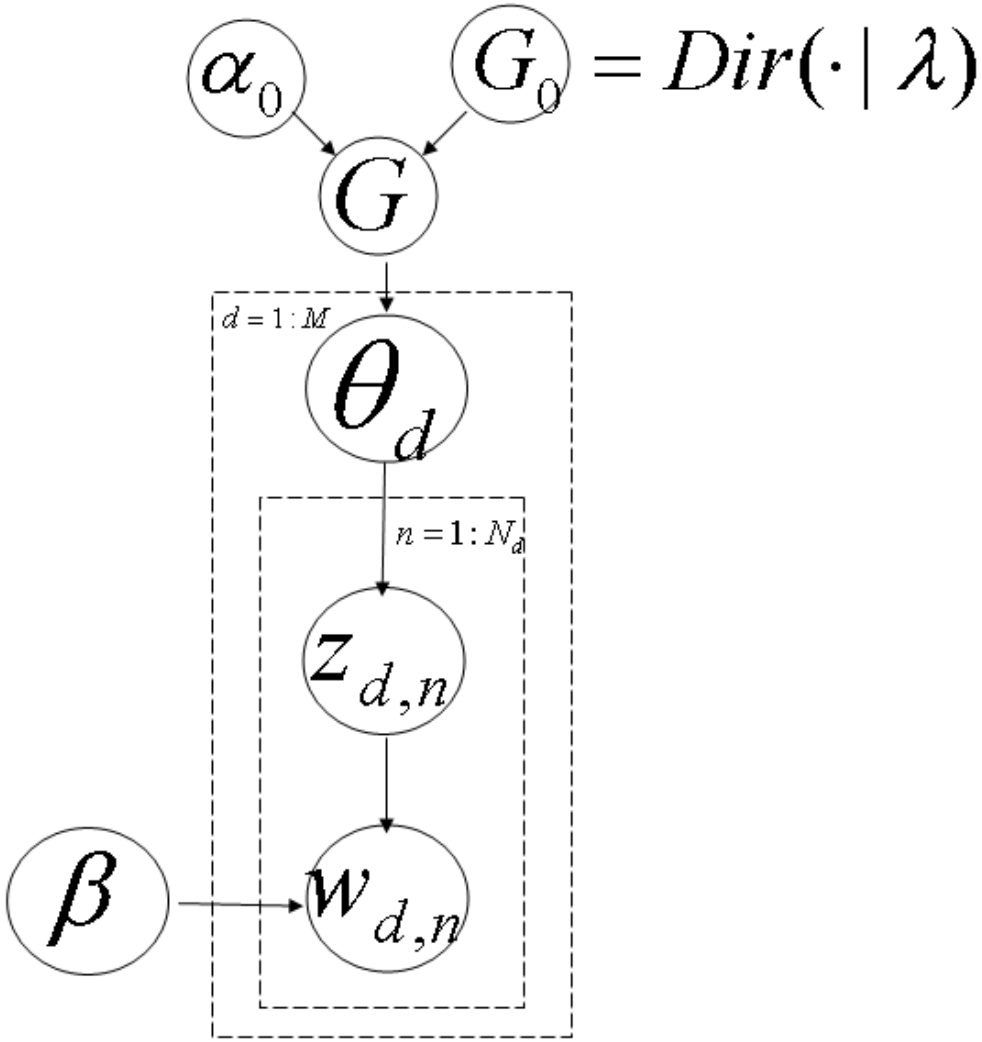
The learned Dirichlet distribution in LDA

Dirichlet Enhanced Latent Semantic Analysis

- The *key point* of the DELSA model (Yu, K., Yu, S., Tresp, V., 2005) is to replace the single Dirichlet distribution in LDA with a *nonparametric Dirichlet process prior*
- We employ the *Dirichlet-multinomial allocation (DMA)* denoted as DP_N as a finite approximation to DP
- Inference is based on a mean field approximation


λ
$Dir(\cdot \mid \lambda)$
$d = 1:M$
θ_d
$n = 1:N_d$
$z_{d,n}$
β
$w_{d,n}$

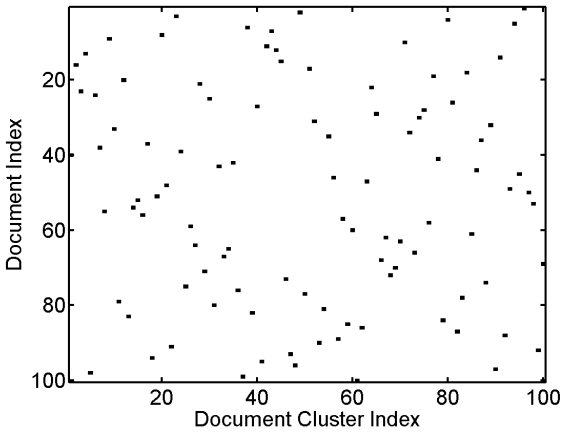
LDA


α_0
G_0
$= Dir(\cdot \mid \lambda)$
G
$d = 1:M$
θ_d
$n = 1:N_d$
$z_{d,n}$
β
$w_{d,n}$

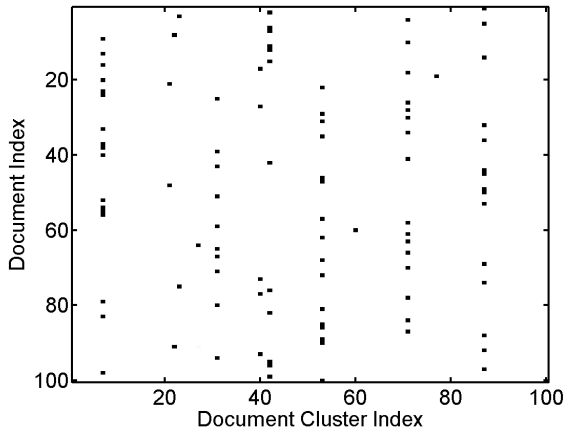
DELSA

Evaluation on Toy Data

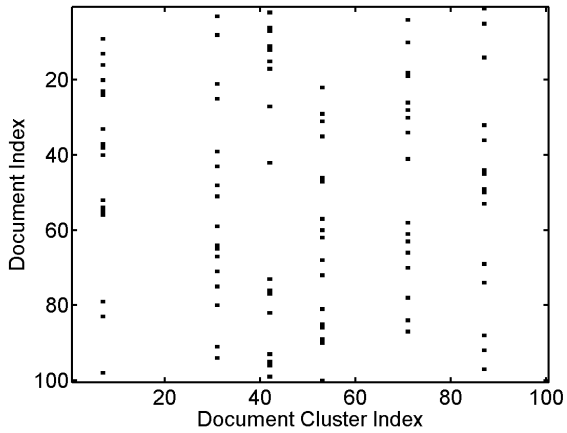
A dictionary of 200 words are associated with 5 latent topics. 100 documents are generated with *6* document clusters. $N = 100$ before learning



Random initialization



After 1 EM step

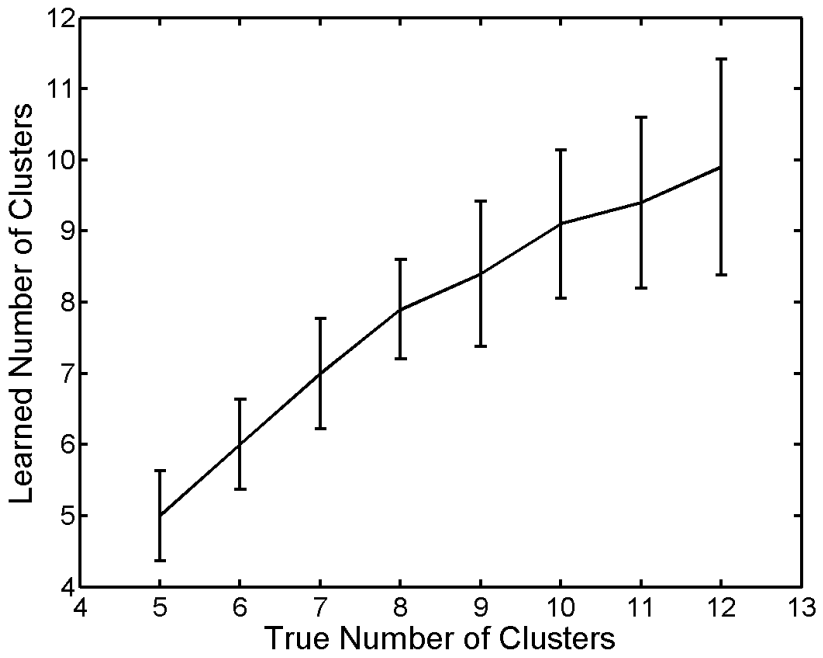


After 5 EM step (final)

Evaluation on Toy Data: Clustering

We then vary the number of clusters from 5 to 12 and randomize the data for 20 trials. We record the detected number of clusters

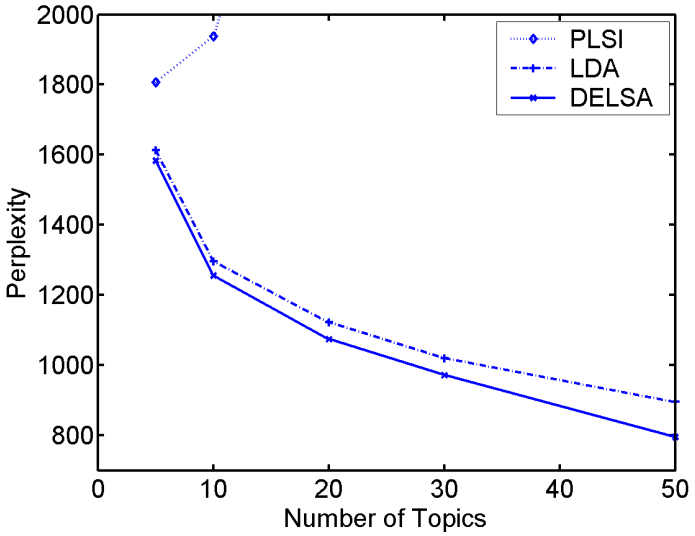
- We can correctly detect number of clusters
- The calculation is fast without overfitting
- The recovered parameter β is very good



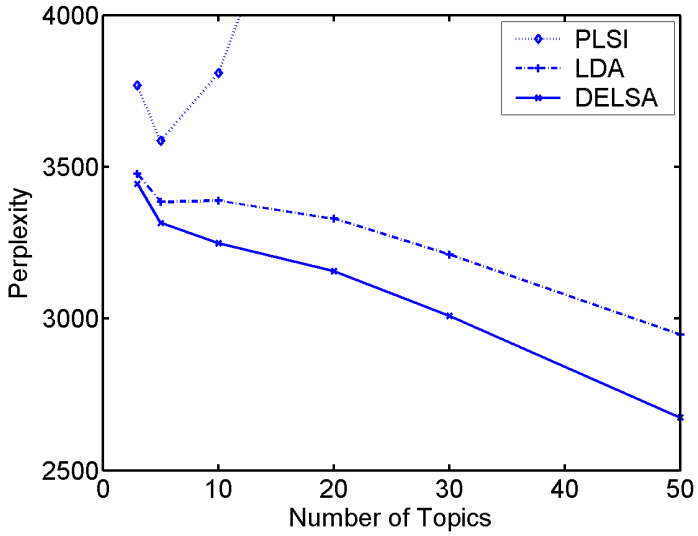
Document Modeling

We compare DELSA with PLSI and LDA on Reuters-21578 and 20-Newsgroup in terms of *perplexity*: $\text{Perp}(\mathcal{D}_t) = \exp\left(-\ln p(\mathcal{D}_t)/\sum_d |\mathbf{w}_d|\right)$

- DELSA is consistently better than PLSI and LDA without overfitting
- Better for data set with strong clustering structure (like 20-Newsgroup)



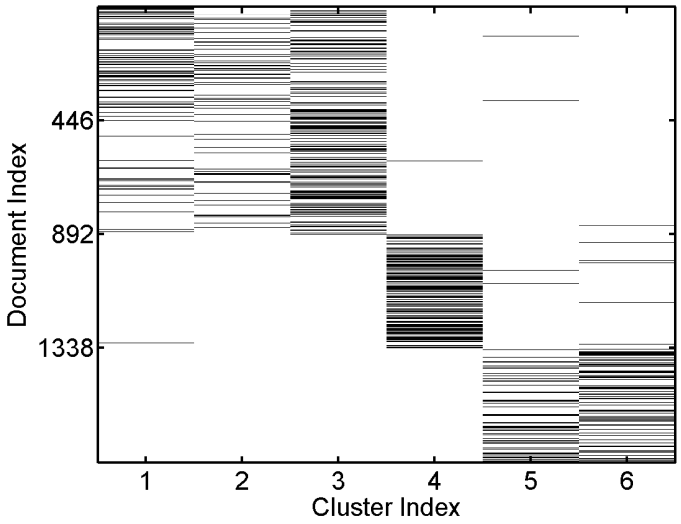
Perplexity for Reuters



Perplexity for Newsgroup

Document Clustering

- We test DELSA on 20-Newsgroup data with 4 categories *autos*, *motorcycles*, *baseball* and *hockey*, each taking 446 documents. 6 clusters are found
- Documents in one category show similar behavior
- Note: A very similar model to DELSEA was developed based on a mean field approximation using the truncated stick breaking approach (Blei and Jordan, 2005)



More Models

Automatically Determining the Right Number of Clusters

The following papers employ DPs to form infinite mixture models

The true number of mixture components is determined by the clustering effect in the Gibbs sampler

- Infinite Mixtures of Gaussians (Rasmussen, 2000)
- Infinite Mixtures of Gaussian Processes (Rasmussen and Ghahramani, 2002)
- Infinite Hidden Markov Networks (Beal, Ghahramani, and Rasmussen, 2002, Teh et al. 2004)

Relational Modeling Using DPs

- Probabilistic relational models form truthful statistical representations of relational data, e.g., data stored in a relational data base
- Effective learning can be realized using hierarchical Bayesian modeling
- DPs have been applied to relational modeling to obtain a sharing strengths effect and for clustering, exploiting the relational information (Kemp et al., 2004, Xu et al., 2005, Xu et al., 2006)

Conclusions

- Nonparametric Bayesian models allow for much flexibility in hierarchical modeling and other applications such as clustering
- The most important setting is the Dirichlet mixture model
- There is a large literature on nonparametric Bayesian modeling in the statistics fields and more needs to be explored
- As it is obvious by the peaky structure of G, this is not a very nice probability density; to achieve a probability density, one might introduce another hierarchical smoothing level (Tomlinson and Escobar, 2005)
- More processes to be explored, Hierarchical DP, Dirichlet Diffusion trees, Indian buffet processes, ...
- For more on DP, look at the related tutorials by Ghahramani (2005) and Jordan (2005) and the introductory paper by Tresp and Yu (2004)

Acknowledgements

- This is joint work with Kai Yu and Shipeng Yu

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