Learning in Pessiland via Inductive Inference

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Online Complexity Seminar

Backgrounds

Our Results

Proof Techniques

Conclusion

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Our Results

Proof Techniques

Conclusion

Pessiland

NP is Hard on Avg

Too Great Heuristics for SAT ✗

No One-Way Functions

(Classic) Crypto ✗

Suppose Our World is Unfortunately Pessiland.

What Can We Do?

Average-Case Inverter for Poly-Time Functions

↓

Algorithms in Pessiland

Hardness

Learning in Pessiland

$\nexists$OWF $\Longrightarrow$ Learning

"BigBang"

1990 [Impagliazzo-Levin] No Better Ways to Generate Hard NP Instances than Picking Uniformly at Random

- High-Level Ideas of "Universal Extrapolation"
- No Formal Statements

Which kind of learning tasks can be done ?

93 [Blum-Frust-Kearns-Lipton]

Average-Case PAC Learning

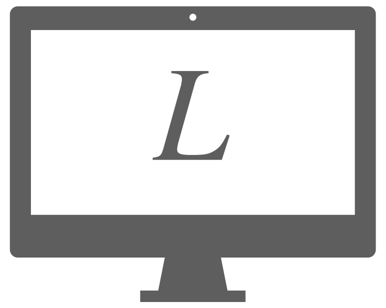
PAC Learning in Pessiland

[BFKL93] $\nexists$OWF$\Longrightarrow$ Avg PAC learning

a concept class $\mathscr{C} = \{\mathscr{C}_n\}_{n\in\mathbb{N}}$ $\mathscr{C}_n \subseteq \{f\colon \{0,1\}^n \to \{0,1\}\}$

an example distribution $\mathscr{E} = \{\mathscr{E}_n\}_{n\in\mathbb{N}}$ $\mathscr{E}_n$ is over $\{0,1\}^n$

a distribution over functions $\mathscr{F} = \{\mathscr{F}_n\}_{n\in\mathbb{N}}$ $\mathscr{F}_n$ is over $\mathscr{C}_n$



learner

in time poly(n, ϵ^{-1})

$\longleftarrow$ $(x^1, f(x^1)), \ldots, (x^m, f(x^m))$

$f \sim \mathscr{F}_n$

$x^1, \ldots, x^m \sim \mathscr{E}_n$

$\longrightarrow$ w.h.p. hypothesis h s.t. $\Pr_{x\sim\mathscr{E}}[h(x) \neq f(x)] \leq \epsilon$

[BFKL93] $\mathscr{C}$: efficiently evaluatable, $\mathscr{E}, \mathscr{F}$: samplable

Learning in Pessiland

$\nexists$OWF $\Longrightarrow$ Learning

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[Blum-Frust-Kearns-Lipton]

Average-Case PAC Learning

2006

[Naor-Rothblum]

Learning Adaptively Changing Distributions (ACDs)

Distributional Learning in Pessiland

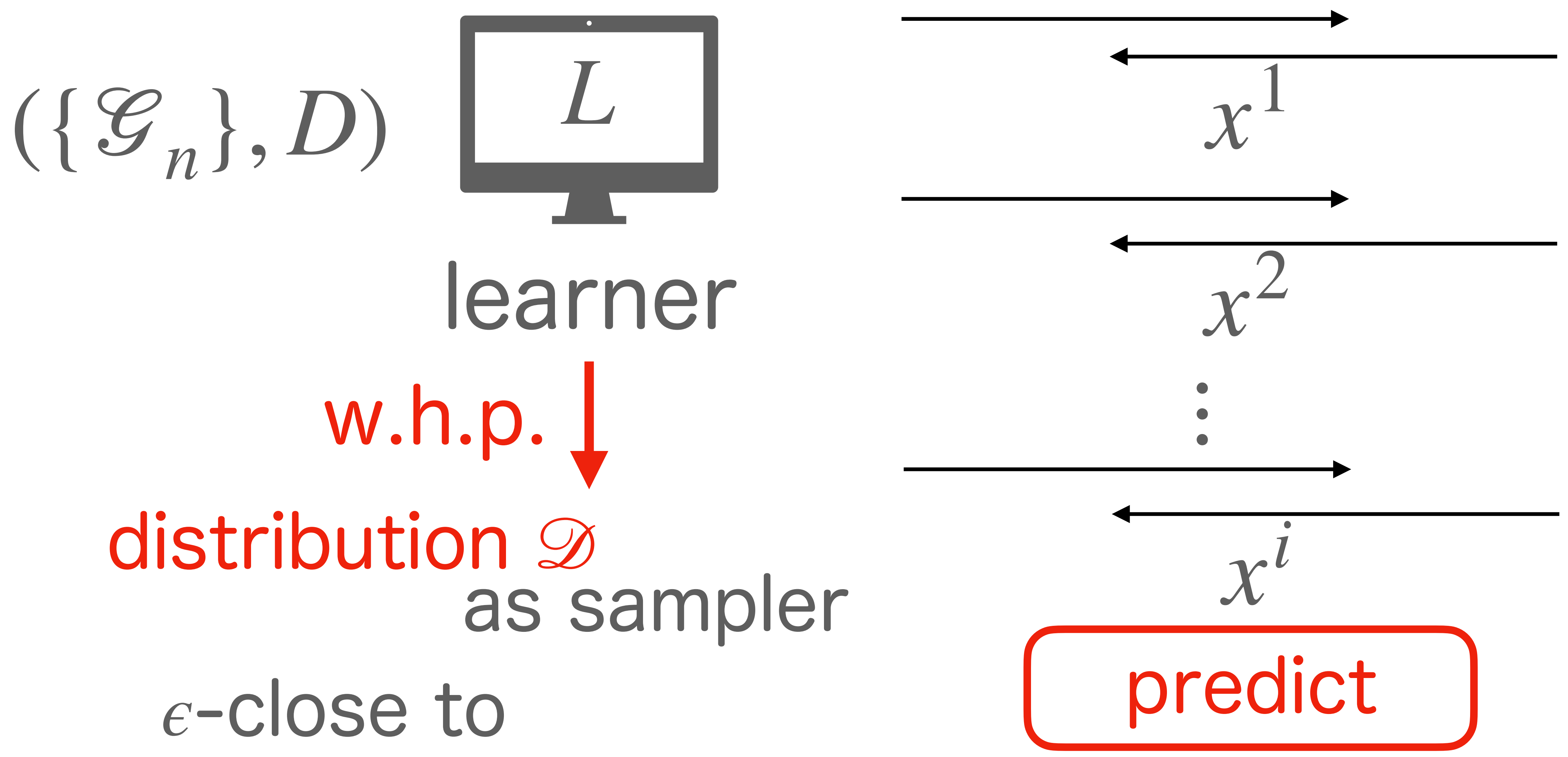
[NR06] $\nexists$OWF $\Longrightarrow$ Learning **known** ACDs

ACD $(\{\mathscr{G}_n\}, D)$

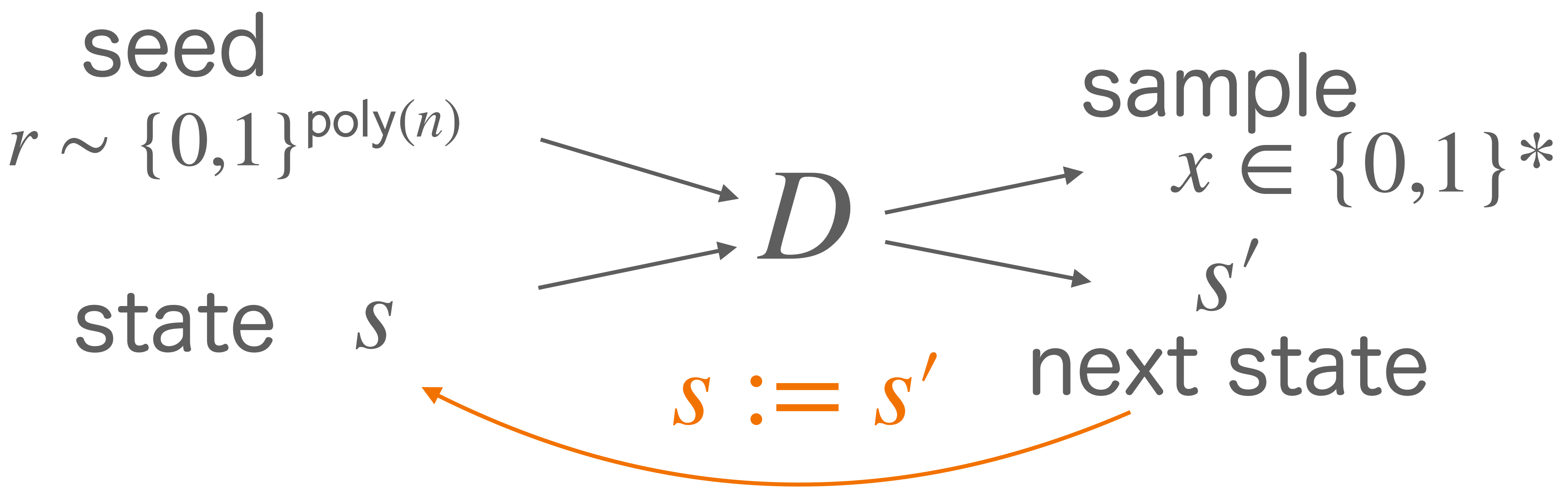
$\mathscr{G}$: samplable D : poly-time sampler

internal state s $= s_0$

Initialization: $s_0 \sim \mathscr{G}_n$
initial state **target**



ϵ-close to
conditional dist of x^{i+1}
given $x^1, \cdots, x^i$ & initial state s_0



[NR06] L uses the knowledge of $\{\mathscr{G}_n\}$ and D

Why is the knowledge of distributions important?

Critical Cases

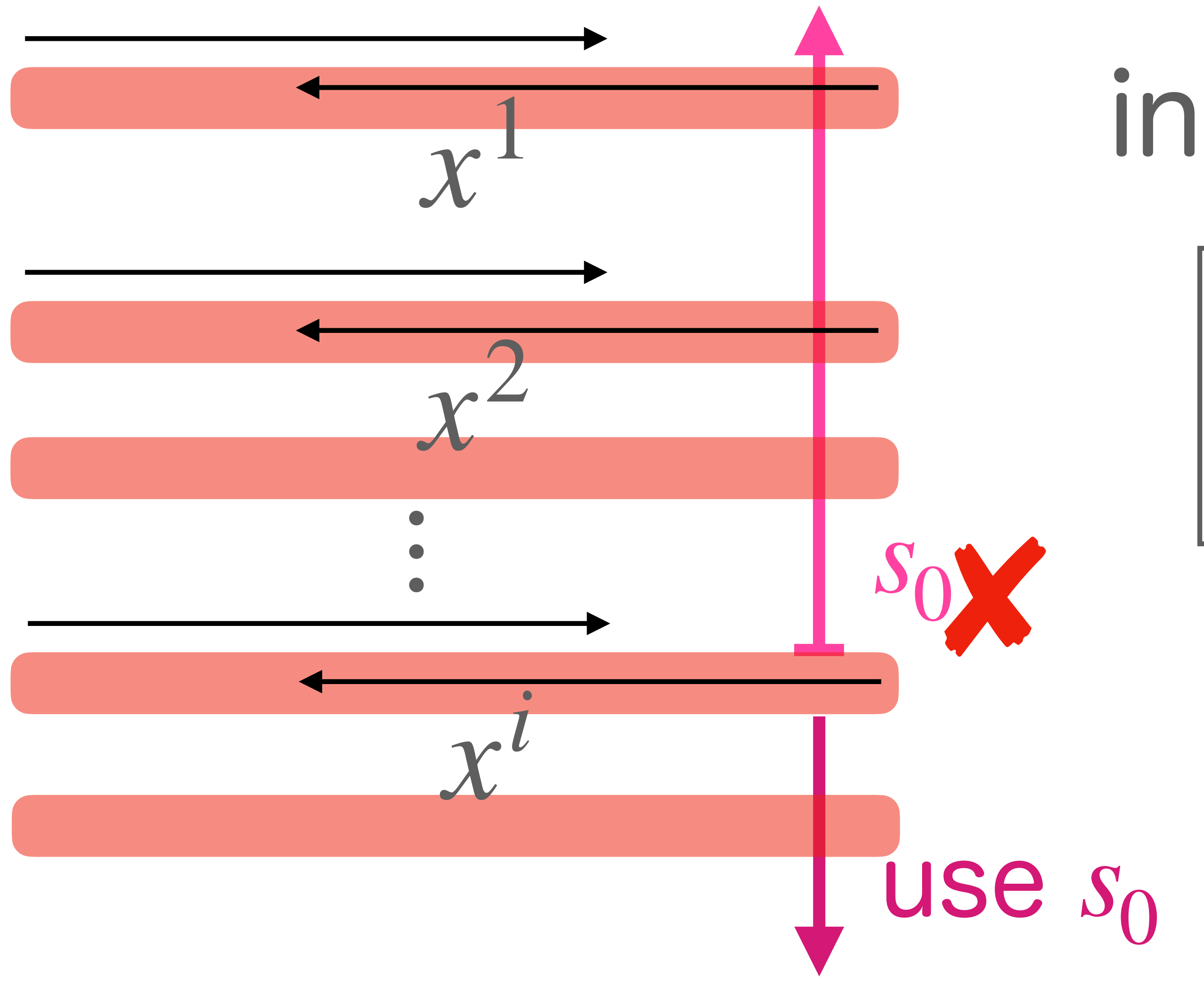
$(\{\mathscr{G}_n\}, D)$ ✗

learner L

w.h.p.

distribution $\mathscr{D}$ as sampler

ϵ-close to conditional dist of x^{i+1} given $x^1, \cdots, x^i$ & initial state s_0



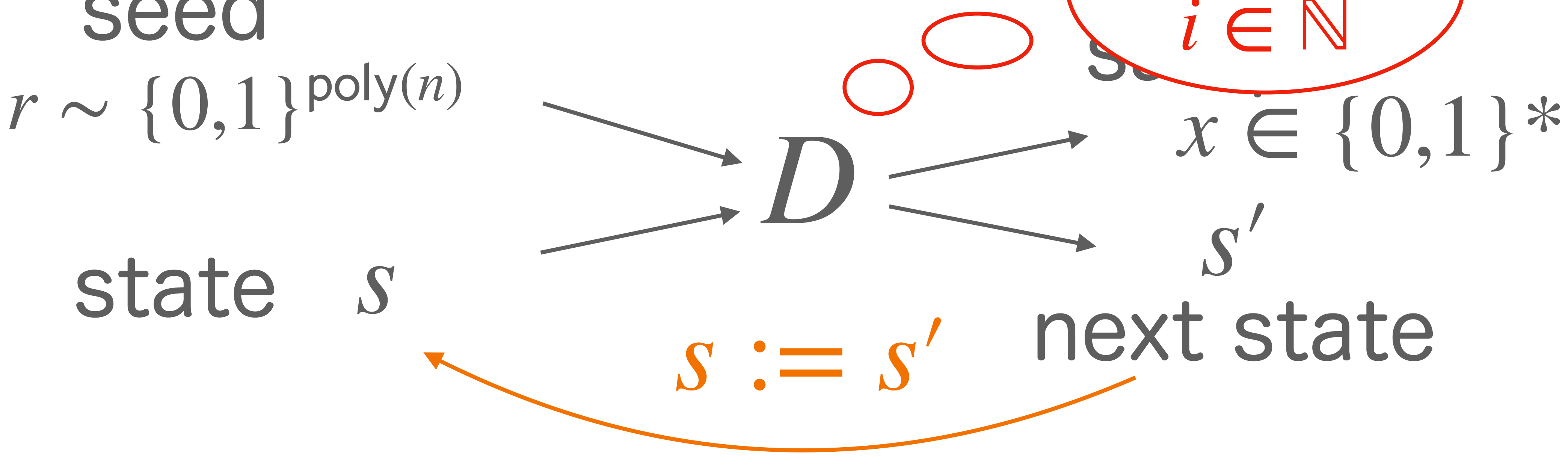
ACD $(\{\mathscr{G}_n\}, D)$

$\mathscr{G}$: samplable D : poly-time sampler

internal state s $= s_0$

Initialization: $s_0 \sim \mathscr{G}_n$
initial state **target**

specific $i \in \mathbb{N}$



[NR06] L uses the knowledge of $\{\mathscr{G}_n\}$ and D

to test whether the next sample is predictable

Improved Learning in Pessiland ?

[BFKL] examples and target function are separately selected

→ Q. Average-Case PAC Learning on Joint Distribution?

$\mathscr{D} \sim \mathscr{G}$ $(x, b) \sim \mathscr{D}$

↑
description of joint distribution

	Separated (BFKL)	Joint
	proper learning	→ DistNP-hard [Pitt-Valiant]

Q. Agnostic Learning ? $\Pr_{(x,b)}[h(x) \neq b] \leq \min_{f \in \mathscr{C}} \Pr_{(x,b)}[f(x) \neq b] + \epsilon$

[NR] learning known ACDs

→ Q. Learning unknown ACDs ?

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Our Contribution

Resolve Q. Joint Distribution? Q. Learning unknown ACDs ?
Q. Agnostic Learning ?

1990

Impagliazzo-Levin, No Better Ways to Generate Hard NP Instances than Picking Uniformly at Random

- High-Level Ideas of “Universal Extrapolation” revisit
- No Formal Statements

Unified & Simplified & Improved

93

Blum-Frust-Kearns-Lipton

Average-Case PAC Learning

2006

Naor-Rothblum

Learning Adaptively Changing Distributions (ACDs)

In this talk...

1. $\nexists$ Infinitely Often OWF

"Adversaries can invert functions for all sufficiently large parameters"

$\nexists$ (standard) OWF $\longrightarrow$ infinitely many size n

accuracy, confidence $\leq 1/\text{poly}(n)$

fixed as poly-time functions in n

2. No details for the choices of parameters

I do not discuss confidence parameters so much

minimize $\mathrm{K}^{\text{your\_time}}(\text{the paper} \mid \text{this talk})$

Learning ACDs in Pessiland

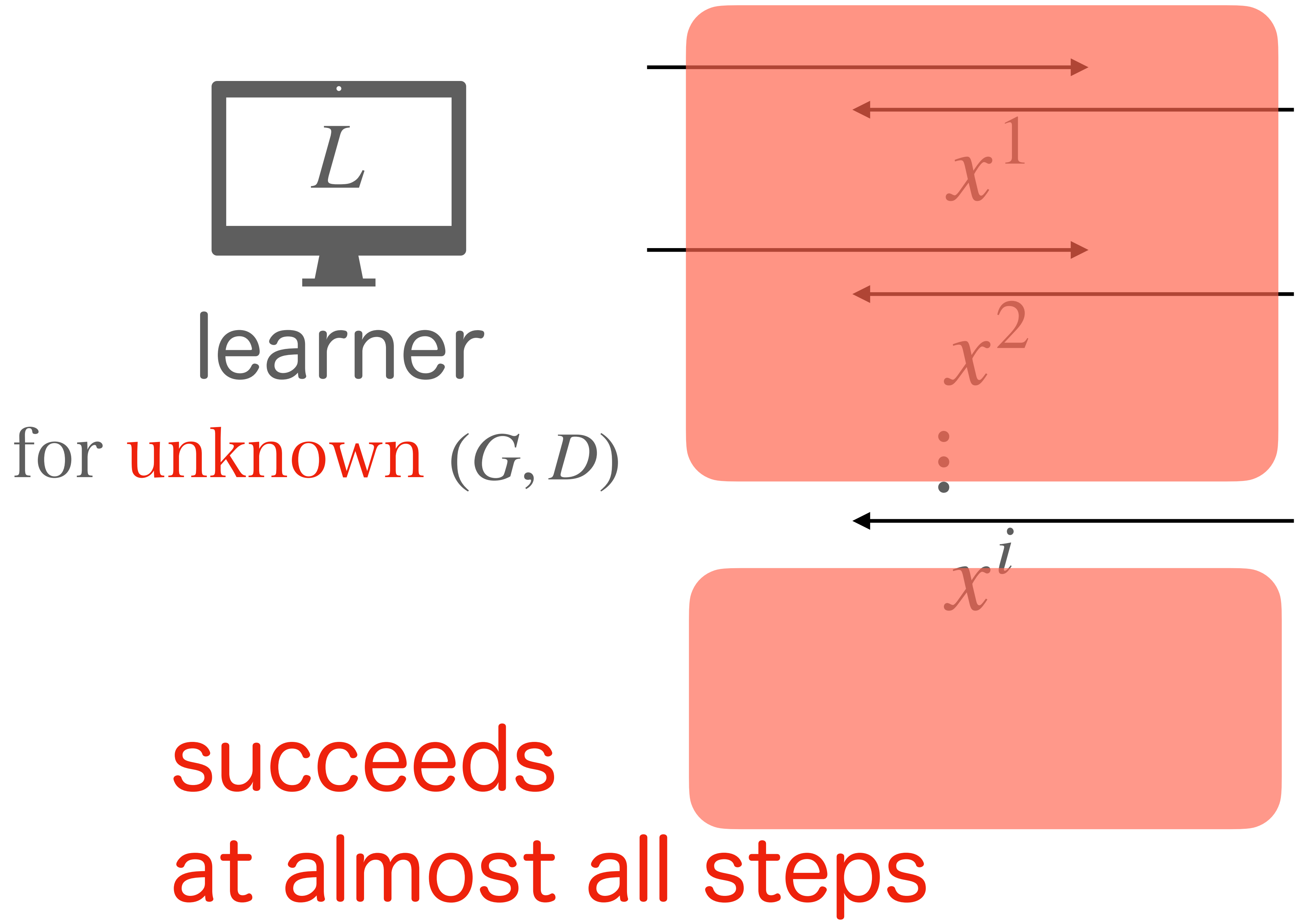
Q. Learning unknown ACDs ?

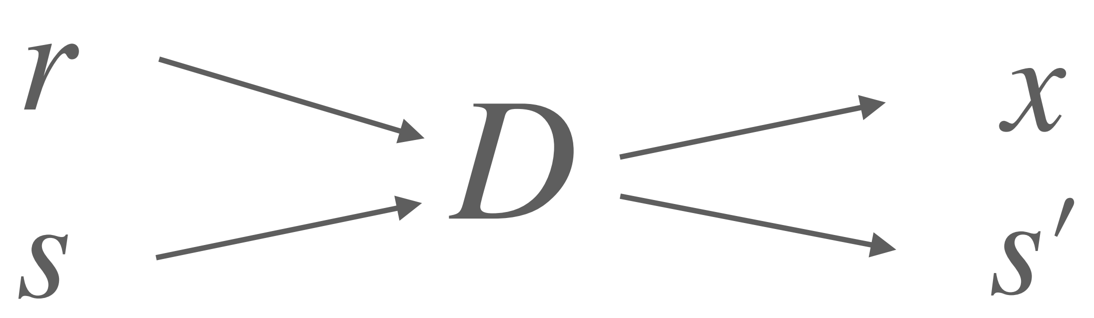
Thm. 1

∄OWF iff ∃ poly-time learner for all (unknown) ACDs $(\mathscr{G}, D)$
with sample complexity $O(s\epsilon^{-2})$ for s-bit initial states and accuracy ϵ

improved from $O(s\epsilon^{-4})$ [NR06]

How can we avoid the critical cases?





choose a prediction stage
uniformly at random

Agnostic Learning in Pessiland

Q. Joint Distribution?　　Q. Agnostic Learning ?

Thm. 2

$\nexists$OWF iff $\exists$ poly-time agnostic learner for $\mathscr{F} = \{f\colon \{0,1\}^n \to \{0,1\}^{\mathrm{poly}(n)}\}$ (with 0-1 loss)

(= learning by information theoretically optimal hypothesis)

on avg under a joint dist $\mathscr{D}$ on samples, where $\mathscr{D} \sim \mathscr{G}$, $\mathscr{G}$ is samplable

with sample complexity $O(s\epsilon^{-2})$ (for $s = |\mathscr{D}|$, accuracy ϵ)

optimal in general

[Previous]		[Ours]
PAC	▶	Agnostic
Separated Distributions		Joint Distributions
Binary Labels		Multi Labels

Improper learning
(General Hypothesis)

Improved Learning in Pessiland

$\nexists$OWF $\iff$ Worst-Case Learning

exp-time in Computational Depth of Secrets

U: universal TM

$$\mathrm{K}(x) = \min\{p \in \mathbb{N} : \exists \Pi \in \{0,1\}^p \text{ s.t. } U(\Pi) = x\}$$

Q^t := dist. of $U(w)$ executed in t steps for $w \sim \{0,1\}^t$

$$\mathrm{q}^t(x) := -\log \Pr[x \sim \mathrm{Q}^t]$$

$\mathrm{q}^t(x) \approx \mathrm{pK}^{t'}(x)$ introduced in [GKLO22]

$\mathrm{pK}^{\mathrm{poly}(t)}(x) \lesssim \mathrm{q}^t(x)$ Optimal coding [LOZ22]

$\mathrm{pK}^t(x) \gtrsim \mathrm{q}^{\mathrm{poly}(t)}(x)$ Domination

$$\mathrm{cd}^t(x) := \mathrm{q}^t(x) - \mathrm{K}(x)$$

$$\approx \mathrm{pK}^{t'}(x) - \mathrm{pK}^{\infty}(x)$$

for any samplable distribution $\mathscr{D} = \{\mathscr{D}_n\}$

$\mathrm{cd}^{\mathsf{poly}}(x) = O(\log n)$ w.h.p. $x \sim \mathscr{D}_n$

Slow Growth Law

$x \longrightarrow$ M:PPT $\longrightarrow x'$

$\mathrm{cd}^{O(t+time_M)}(x') \lesssim \mathrm{cd}^t(x)$ w.h.p.

Improved Learning in Pessiland

$\nexists$OWF $\iff$ Worst-Case Learning

exp-time in Computational Depth of Secrets

Thm. 3

The following are equivalent:

1. $\nexists$ OWF
2. learning unknown ACDs in time $\mathrm{poly}(n, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(s_0)})$, where s_0 is initial state (worst case on initial states)
3. agnostic learning on unknown joint dists $\mathscr{D}$ over samples in time $\mathrm{poly}(n, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(|\mathscr{D}|)})$ (worst case on joint distributions over samples)

Note Thm1 & 2 are implied by Thm3

Our Approach

Step I State “Universal Extrapolation” formally

Step II Translate “Universal Extrapolation” into Learning

Our Approach

Step I State "Universal Extrapolation" formally

Step II Translate "Universal Extrapolation" into Learning

Formulating Universal Extrapolation

Our Proposal Universal Extrapolation = Extrapolation under Q^t

(Time-Bounded Universal Distribution)

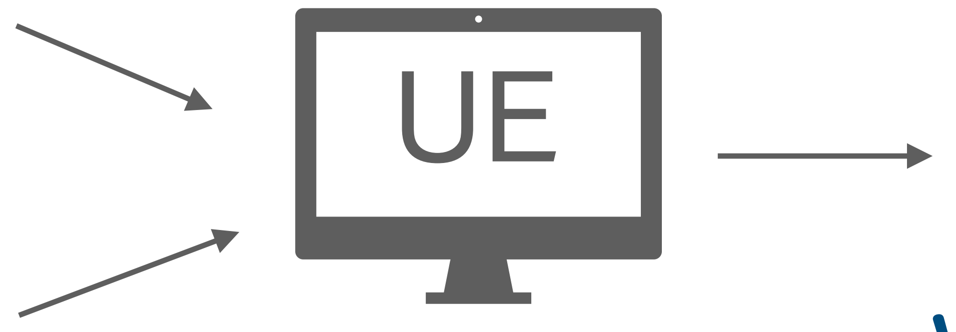
Notation distribution $\mathscr{D}$ prefix $x \in \{0,1\}^*$ $k \in \mathbb{N}$

$\mathsf{Next}_k(x; \mathscr{D})$ = distribution of k bits following x w.r.t. $\mathscr{D}$

prefix $x \in \{0,1\}^*$

paramaters

$k \in \mathbb{N}$ $t \in \mathbb{N}$ $\epsilon \in (0,1)$



$y \in \{0,1\}^{\leq k} \approx \mathsf{Next}_k(x; \mathrm{Q}^t)$

within statistical distance ϵ

Lemma (informal) $\nexists$ (io)OWF $\implies$ $\exists$UE that works in worst case on x

in time $\mathrm{poly}(|x|, k, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(x)})$

(UE is given 2^α (in unary) and works for every x with $\mathrm{cd}^t(x) < \alpha$)

Lemma (informal) $\nexists$ (io)OWF $\Longrightarrow$ $\exists$UE that works in worst case on x in time $\mathrm{poly}(|x|, k, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(x)})$

Distributional Inverting

Inverting

$f(x)$ $(x \sim \{0,1\}^*)$ $\longrightarrow$ Inv $\longrightarrow$ some x' $f(x') = f(x)$

Distributional Inverting

$f(x)$ $(x \sim \{0,1\}^*)$ $\longrightarrow$ $DInv$ $\longrightarrow$ some x' $f(x') = f(x)$

simulate unif sampling from $\{x' : f(x') = f(x)\}$

Thm [IL89]

$\nexists$ (io)OWF $\Longrightarrow$ for every poly-time function $f = \{f_n\}$

$\exists DInv$: PPT s.t. $\forall n, \epsilon^{-1}, \delta^{-1} \in \mathbb{N}$

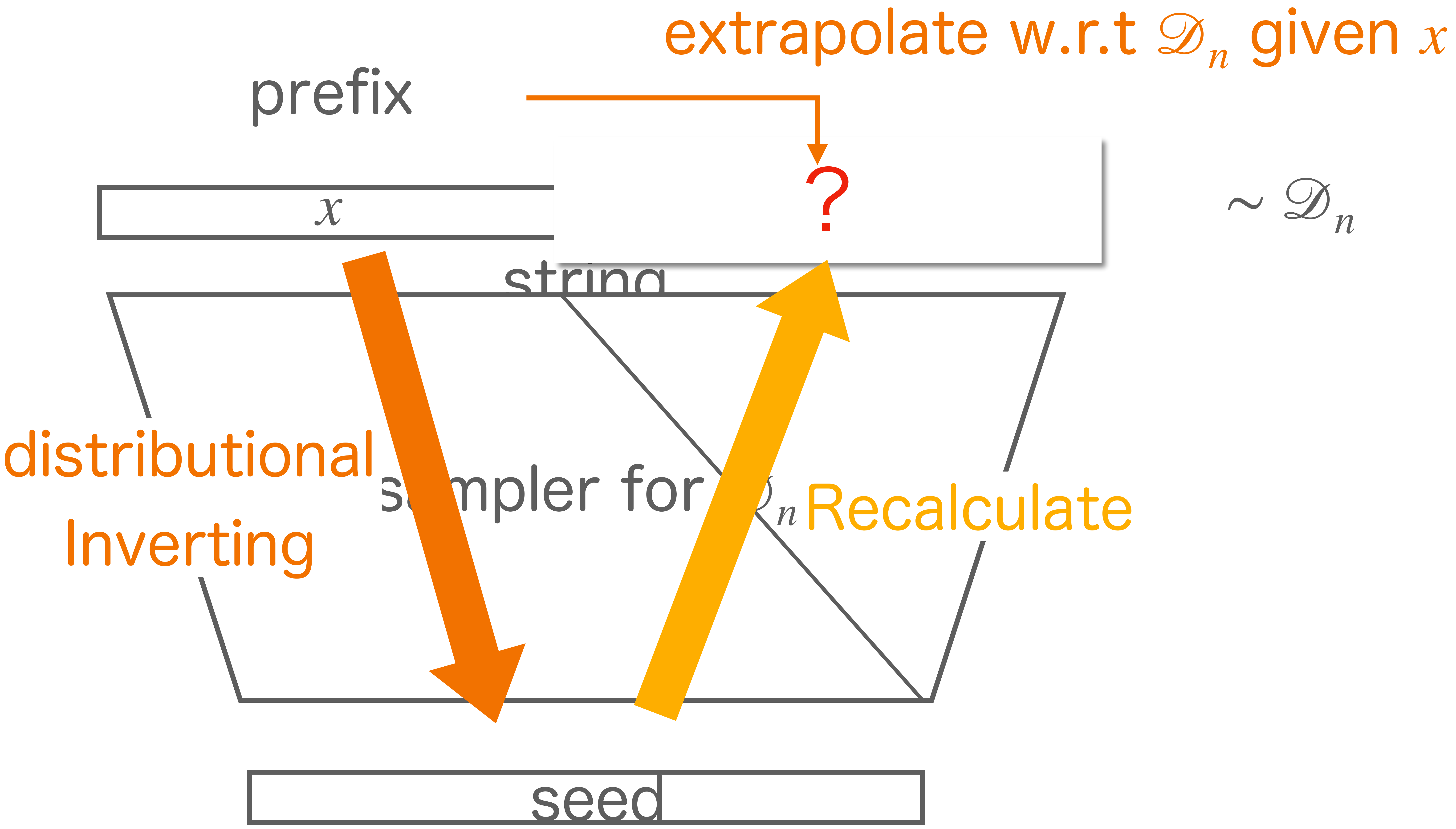
$$\Delta_{TV}\left(DInv(f_n(x); 1^n, 1^{\epsilon^{-1}}, 1^{\delta^{-1}}), \text{Unif over } f_n^{-1}(f_n(x))\right) \leq \epsilon$$

w.p. $\geq 1 - \delta$ over $x \sim \{0,1\}^{\mathrm{poly}(n)}$

Lemma (informal) $\nexists$ (io)OWF $\Longrightarrow$ $\exists$UE that works in worst case on x in time $\mathrm{poly}(|x|, k, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(x)})$

Distributional Inverting → Distribution-Specific Extrapolation [Ost91, OW93, NR06, . . .]

$\mathscr{D} = \{\mathscr{D}_n\}$ samplable distribution



Lemma (informal) $\nexists$ (io)OWF $\Longrightarrow$ $\exists$UE that works in worst case on x in time $\mathrm{poly}(|x|, k, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(x)})$

Distribution-Specific Extrapolation for Q^t

$\exists\mathrm{UE}'$: PPT s.t. $\forall t, \ell, k, \epsilon^{-1}, \delta^{-1} \in \mathbb{N}$

$$\Delta_{TV}\left(\mathrm{UE}'(x), \mathrm{Next}_k(x; \mathrm{Q}^t)\right) \le \epsilon \quad \text{w.p. } \ge 1-\delta \text{ over } x \sim \mathrm{Q}^t_{\le \ell}$$

the first ℓ bits of Q^t

Based on [AF09] $E := E_{t,\ell,k,\epsilon,\delta} = \left\{ x : \Delta_{TV}\left(\mathrm{UE}'(x), \mathrm{Next}_k(x; \mathrm{Q}^t)\right) > \epsilon \right\}$ error set

$\delta := 2^{-\alpha}$ Goal: $x \in E \Longrightarrow \mathrm{cd}^t(x) \ge \alpha - O(\log t\ell k \epsilon^{-1} \alpha)$

$$2^{-\alpha} \ge \sum_{x \in E} \Pr[x \sim \mathrm{Q}^t_{\le \ell}] \ge \sum_{x \in E} \Pr[x \sim \mathrm{Q}^t] \qquad \sum_{x \in E} 2^{\alpha} \Pr[x \sim \mathrm{Q}^t] \le 1$$

computable given UE$'$ & parameters

inefficiently computable distribution $\{\mathscr{E}_{t,\ell,k,\epsilon,\delta}\}$ $\forall x \in E$ $\Pr[x \sim \mathscr{E}] = 2^{\alpha} \Pr[x \sim \mathrm{Q}^t]$

$\forall x \in E$ $\mathrm{K}(x) \le -\log \Pr[x \sim \mathscr{E}] + O(\log t\ell k \epsilon \alpha)$ optimal coding

$$= -\alpha + \underbrace{(-\log \Pr[x \sim \mathrm{Q}^t])}_{\mathrm{q}^t(x)} + O(\log t\ell k \epsilon \alpha)$$

Step 1: Summary

Our Proposal Universal Extrapolation = Extrapolation under Q^t

(Time-Bounded Universal Distribution)

prefix $x \in \{0,1\}^*$

paramaters

$k \in \mathbb{N}$ $t \in \mathbb{N}$ $\epsilon \in (0,1)$

→ UE → $y \in \{0,1\}^k$ $\approx$ $\mathrm{Next}_k(x; \mathrm{Q}^t)$

with statistical distance within ϵ

Lemma (informal) $\nexists$ (io)OWF $\Longrightarrow$ $\exists$UE that works in worst case on x

in time $\mathrm{poly}(|x|, k, t, \epsilon^{-1}, 2^{\mathrm{cd}^t(x)})$

(UE is given 2^α (in unary) and works for every x with $\mathrm{cd}^t(x) < \alpha$)

Proof Use Distributional Inverter for Q^t

Q. How can we obtain learners (e.g., agnostic learners)?

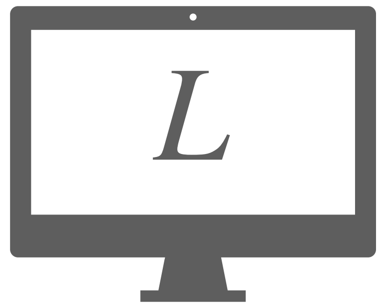
Step2 UE → Learning

offline stream

efficiently & randomly

$x^1 \ y^1 \ x^2 \ y^2 \ \cdots \ x^m \ y^m$ ← secret info. (sampler) $\Pi \in \{0,1\}^*$

possibly correlated



$x^1 \ y^1 \ x^2 \ y^2 \ \cdots \ x^{i-1} \ y^{i-1} \ x^i$ y^i **?**

predict

$x^1, y^1, x^2, y^2 \ldots \in \{0,1\}^*$

x : advice information

y : target

EX. learning ACDs

$\Pi \approx$ initial state x : empty y : sample

simulate the distribution of y^i

agnostic learning (with 0-1 loss)

$\Pi \approx$ joint distribution over samples

x : example y : label

answer the best possible y^i

Cheating Learner

offline stream

efficiently & randomly

$x^1 \ y^1 \ x^2 \ y^2 \ \cdots \ x^m \ y^m$ ← secret info. (sampler) $\Pi \in \{0,1\}^*$

possibly correlated

$x^1 \ y^1 \ x^2 \ y^2 \ \cdots \ x^{i-1} \ y^{i-1} \ x^i$ y^i **?**

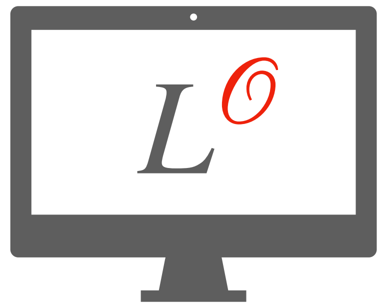
$x^1, y^1, x^2, y^2 \ldots \in \{0,1\}^*$

x : advice information

y : target

condition

$h^{\mathcal{O}}$



$y^i \ y^i \ y^i$ ← $\mathcal{O}$

w.r.t. Π

EX. learning ACDs

$\Pi \approx$ initial state x : empty y : sample

simulate the distribution of y^i

✅ by outputting $y^i \sim \mathcal{O}$

agnostic learning (with 0-1 loss)

$\Pi \approx$ joint distribution over samples

x : example y : label

answer the best possible y^i

✅ by collecting $y^i, y^{i'}, \ldots \sim \mathcal{O}$

UE $\rightarrow$ Learning

offline stream

efficiently & randomly

secret info. (sampler) $\Pi \in \{0,1\}^*$

$x^1 \; y^1 \; x^2 \; y^2 \; \cdots \; x^m \; y^m$

possibly correlated

$x^1 \; y^1 \; x^2 \; y^2 \; \cdots \; x^{i-1} \; y^{i-1} \; x^i$ $\quad y^i$?

$x^1, y^1, x^2, y^2 \ldots \in \{0,1\}^*$

x : advice information

y : target

condition

h L

$y^i \; y^i \; y^i$ $\leftarrow$ $\mathcal{O}$

w.r.t. Π

$\approx$

UE

w.h.p. when $i \sim [m]$

for statistical distance within ϵ

$m = O(|\Pi|\epsilon^{-2})$

Q. Why ?

learning ACDs ✓ agnostic learning (with 0-1 loss) ✓

sample complexity: linear in $|\Pi|$ time complexity: exp in $|\Pi|$

Solomonoff's Inductive Inference [Sol64, LV19]

When extrapolating symbols, attach a higher probability for a more precise hypothesis, particularly, with an exponential rate on the description size

$q^{poly}(\cdot\,|\,\cdot)$ or $pK^{poly}(\cdot\,|\,\cdot)$

$(\because 2^{-q^{poly}} = Q^{poly})$

by conditional coding [HILNO23]

Domination + Chain Rule for KL divergence

efficiently & randomly

$$x^1 \quad y^1 \quad x^2 \quad y^2 \quad \cdots \quad x^m \quad y^m \longleftarrow \boxed{\Pi \in \{0,1\}^*}$$

When $t \gg \text{time}(\Pi)$

$$\Pr\left[x^1y^1\ldots x^my^m \sim Q^t\right] \geq 2^{-O(|\Pi|)}\Pr\left[x^1y^1\ldots x^my^m \sim \Pi\right] \qquad \textbf{(Domination)}$$

$$\log \frac{\Pr\left[x^1y^1\ldots x^my^m \sim \Pi\right]}{\Pr\left[x^1y^1\ldots x^my^m \sim Q^t\right]} \leq O(|\Pi|)$$

Taking the expectation over $x^1, y^1, \ldots, x^m, y^m \sim \Pi$

$$\mathrm{KL}\left(\Pi \,\|\, Q^t\right) \leq O(|\Pi|)$$

Solomonoff's Inductive Inference [Sol64, LV19]

$\mathrm{KL}\left(\Pi \,\middle\|\, Q^t\right) \le O(|\Pi|)$

$\mathcal{O}$ 

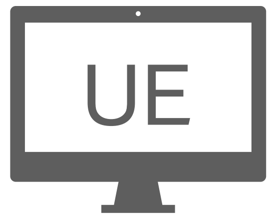
true distribution

$x^1\ y^1\ x^2\ y^2\ \cdots\ y^{i-1}\ x^i\ \ y^i\ \cdots\ x^m\ y^m \longleftarrow \boxed{\Pi \in \{0,1\}^*}$

$X^1\ Y^1 \quad X^m\ Y^m$

$\tilde{X}^1\ \tilde{Y}^1 \quad \tilde{X}^m\ \tilde{Y}^m$

$x^1\ y^1\ x^2\ y^2\ \cdots\ y^{i-1}\ x^i\ \ y^i\ \cdots\ x^m\ y^m \longleftarrow Q^t$



ignore the statistical error

$$O(|\Pi|) \ge \mathrm{KL}\left(X^1Y^1\cdots X^mY^m \,\middle\|\, \tilde{X}^1\tilde{Y}^1\cdots\tilde{X}^m\tilde{Y}^m\right)$$

$$= \sum_{i=1}^{m} \mathrm{KL}\left((Y^i \mid X^1Y^1\cdots Y^{i-1}X^i) \,\middle\|\, (\tilde{Y}^i \mid \tilde{X}^1\tilde{Y}^1\cdots\tilde{Y}^{i-1}\tilde{X}^i)\right)$$

(Chain Rule)

$$+ \sum_{i=1}^{m} \mathrm{KL}\left((X^i \mid X^1Y^1\cdots X^{i-1}Y^{i-1}) \,\middle\|\, (\tilde{X}^i \mid \tilde{X}^1\tilde{Y}^1\cdots\tilde{X}^{i-1}\tilde{Y}^{i-1})\right)$$

$$\ge \sum_{i=1}^{m} \mathrm{KL}\left((Y^i \mid X^1Y^1\cdots Y^{i-1}X^i) \,\middle\|\, (\tilde{Y}^i \mid \tilde{X}^1\tilde{Y}^1\cdots\tilde{Y}^{i-1}\tilde{X}^i)\right)$$

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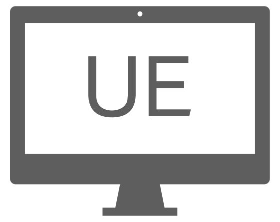

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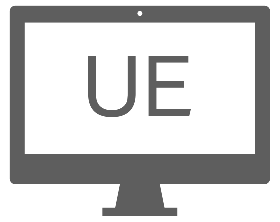
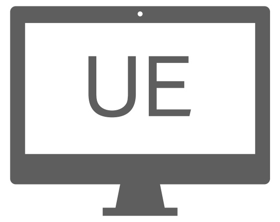
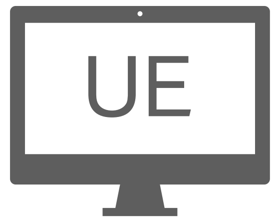
Solomonoff's Inductive Inference [Sol64, LV19]

$O(|\Pi|) \geq \sum_{i=1}^{m} \mathrm{KL}($  $_i \,\|\,$ 

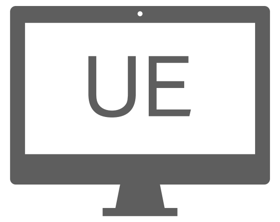
$_i\ )$

$\dfrac{O(|\Pi|)}{m} \geq \mathrm{E}_{i\sim[m]}[\mathrm{KL}($  $_i \,\|\,$ 

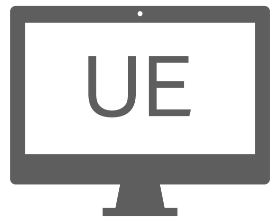
$_i\ )]$

$m \gg \dfrac{|\Pi|}{\epsilon^2} \implies \mathrm{E}_{i\sim[m]}[\mathrm{KL}($  $_i \,\|\,$ 

$_i\ )] \ll \epsilon^2$

(Markov) $\mathrm{KL}($  $_i \,\|\,$ 

$_i\ ) \ll \epsilon^2$ w.h.p. over $i \sim [m]$

(Pinsker) $\Delta_{TV}($  $_i\ ,$ 

$_i\ ) \leq \epsilon$ w.h.p. over $i \sim [m]$

Remarks

$x^1\ y^1\ x^2\ y^2\ \cdots\ x^{i-1}\ y^{i-1}\ x^i$ y^i **?**

condition

h ✗ L ✗ $y^i\ y^i\ y^i$ ← $\mathcal{O}$ ✗

$\approx$ UE

when $i \sim [m]$ for stat dist within ϵ

$m = O(|\Pi|\epsilon^{-2})$

Tasks poly-time cheating learners can do

= Tasks poly-time learners can do with UE

Not sample optimal in agnostic learning

Why? accuracy → query complexity of cheating learner

for optimal sample complexity

Extend universal prediction [MF98] to computational cases

(statistical cases)

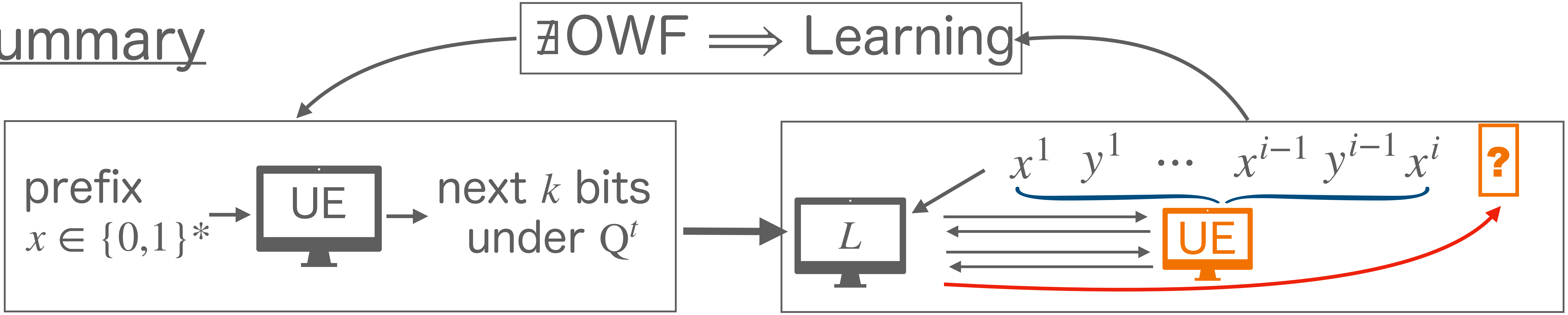
Backgrounds

Our Results

Proof Techniques

Conclusion

Summary



Thm.

The following are equivalent:

1. ∄ OWF
2. learning unknown ACDs in time $\mathrm{poly}(t, \epsilon^{-1}, 2^{\mathrm{cd}^t(s_0)})$, where s_0 is initial state
3. agnostic learning on unknown joint dists $\mathscr{D}$ over samples in time $\mathrm{poly}(t, \epsilon^{-1}, 2^{\mathrm{cd}^t(|\mathscr{D}|)})$

Q. weakest assumption for learning in time $\mathrm{poly}(\mathrm{cd})$? AIOWF or HSG or $\cdots$?