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Divergence-free finite element spaces for stress tensors

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ABSTRACT

We construct finite element spaces of symmetric stress tensors that are exactly divergence-free. Moreover, their basis functions can be chosen so that they have small supports. These properties are highly desired in a number of important applications. Approximation properties of finite element spaces of divergence-free tensor functions are derived from properties of C^1 finite elements.

1. Introduction

In many problems of mathematical physics we have to deal with divergence-free functions, e.g., when solving Maxwell equations, Navier–Stokes equations, Einstein's equations, bending moments of a clamped plate, heat fluxes by a dual variational formulation of a steady-state heat conduction problem, and various continuity equations. In [1–5], we showed how to construct finite element spaces whose vector functions are exactly divergence-free in 2D and 3D. In this paper, we shall construct finite element spaces whose tensor functions are exactly divergence-free in two-dimensional domains. Such spaces are needed to calculate e.g. the stress tensor of a linear elasticity problem by means of a dual variational formulation.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We shall use the standard Sobolev space notation. The symbol $H^k(\Omega)$ stands for the Sobolev spaces of functions whose generalized derivatives up to the order k are square integrable over Ω. The norm in $(H^k(\Omega))^p$ ($p \geq 1$ is integer) is denoted by $\|\cdot\|_k$ and the scalar product in the Lebesgue space $(L^2(\Omega))^p$ is denoted by $(\cdot,\cdot)_0$. The space of symmetric 2×2 tensors

$$(L^2(\Omega))^{2\times 2}_{\mathrm{sym}} = \{\tau \in (L^2(\Omega))^{2\times 2} \mid \tau = \tau^\top\}$$

will be equipped with the scalar product

$$(\tau, \mu)_0 = \sum_{i,j=1}^{2} (\tau_{ij}, \mu_{ij})_0 \text{ for } \tau, \mu \in (L^2(\Omega))^{2\times 2}_{\mathrm{sym}}.$$

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Let us introduce the operator $\varepsilon : (H^1(\Omega))^2 \to (L^2(\Omega))^{2\times 2}_{\rm sym}$ defined by

$$\varepsilon(v) = \begin{pmatrix} \partial_1 v_1 & \frac{1}{2}(\partial_2 v_1 + \partial_1 v_2) \\ \mathrm{sym} & \partial_2 v_2 \end{pmatrix},$$

where $v = (v_1, v_2)^\top$ and $\partial_j v_i = \partial v_i/\partial x_j$ for $i, j = 1, 2$.

Denote by $C_0^\infty(\Omega)$ the space of infinitely differentiable functions with compact support in Ω. Let $d \in (L^2(\Omega))^2$ and $\tau \in (L^2(\Omega))^{2\times 2}_{\rm sym}$ be such that

$$(\tau, \varepsilon(v))_0 = -(d, v)_0 \quad \forall v \in (C_0^\infty(\Omega))^2.$$

Then we say that the *divergence of the tensor function* τ *exists in the sense of distributions in* Ω and define

$$\mathrm{Div}\ \tau = d.$$

Obviously, for any smooth tensor τ we find that

$$\mathrm{Div}\ \tau = (\partial_1 \tau_{11} + \partial_2 \tau_{12}, \partial_1 \tau_{12} + \partial_2 \tau_{22})^\top, \tag{1}$$

i.e., each component of Div τ is, in fact, the usual divergence of the corresponding row of τ. Due to the symmetry of τ we can also define the divergence of tensors by mean of columns.

2. Dual formulation of the linear elasticity problem

When solving real-life technical problems, the knowledge of the stress tensor is more important than the knowledge of displacements. We can, of course, first calculate the strain tensor by differentiating the computed displacements, and then construct the stress tensor by Hooke's law. This is the so-called primal approach. Alternatively, we can apply a dual formulation that allows us to calculate the stress tensor directly.

Let Γ_D and Γ_N be disjoint and relatively open sets of $\partial\Omega$ such that

$$\overline{\Gamma}_D \cup \overline{\Gamma}_N = \partial\Omega.$$

Let $f = (f_1, f_2)^\top \in (L^2(\Omega))^2$ be given body forces and $g = (g_1, g_2)^\top \in (L^2(\Gamma_N))^2$ given surface forces. In addition, if $\Gamma_D = \emptyset$, we assume that the following *equilibrium condition* for forces f and g and their moments is satisfied:

$$\int_\Omega f^\top v \,\mathrm{d}x + \int_{\Gamma_N} g^\top v \,\mathrm{d}s = 0 \quad \forall v \in P,$$

where $P = \{v \in (H^1(\Omega))^2 \mid \varepsilon(v) = 0\}$ is a three-dimensional space with basis $(1, 0)^\top$, $(0, 1)^\top$, and $(x_2, -x_1)^\top$, see [6, p. 95].

Define the set of statically admissible stresses as

$$T(f, g) = \left\{ \tau \in (L^2(\Omega))^{2\times 2}_{\rm sym} \,\middle|\, \int_\Omega \tau \cdot \varepsilon(v) \mathrm{d}x = \int_\Omega f^\top v \,\mathrm{d}x + \int_{\Gamma_N} g^\top v \,\mathrm{d}s \quad \forall v \in \mathcal{V} \right\}, \tag{2}$$

where

$$\mathcal{V} = V \times V, \quad V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}, \tag{3}$$

and

$$\tau \cdot \varepsilon(v) = \sum_{i,j=1}^{2} \tau_{ij} \varepsilon_{ij}(v).$$

From (1) and (2) we find that Div $\tau + f = 0$ in Ω and $\tau n = g$ on Γ_N for a sufficiently smooth $\tau \in T(f, g)$.

Now let us recall very briefly a primal formulation of the classical linear elasticity problem, see [6, p. 64]. The strain tensor $\varepsilon(v)$ is coupled with the stress tensor τ via the generalized Hooke's law

$$\tau = C : \varepsilon \qquad \Big(\tau_{ij} = \sum_{k,l=1}^{2} C_{ijkl} \varepsilon_{kl}(v)\Big),$$

where the linear elasticity coefficients $C = (C_{ijkl})_{i,j,k,l=1}^2$ satisfy $C_{ijkl} \in L^\infty(\Omega)$,

$$C_{jikl} = C_{ijkl} = C_{klij},$$

and that there exists a constant $c > 0$ such that

$$\xi \cdot (C(x) : \xi) \geq c\|\xi\|^2$$

holds a.e. in Ω for any symmetric tensor $\xi = (\xi_{ij})_{i,j=1}^2$, $\xi_{ij} \in \mathbb{R}^1$, where $\|\cdot\|$ is the standard spectral norm. Further, assume that $\bar{u} \in (H^1(\Omega))^2$ is a suitable displacement function which satisfies the Dirichlet boundary conditions on Γ_D, namely $u = \bar{u}$ on Γ_D in the sense of traces. Then the *primal problem of linear elasticity* consists of finding $u \in (H^1(\Omega))^2$ which minimizes the functional of potential energy $I : (H^1(\Omega))^2 \to \mathbb{R}^1$ defined by

$$I(v) = \frac{1}{2}\int_\Omega \varepsilon(v)\cdot(C(x):\varepsilon(v))\,\mathrm{d}x - \int_\Omega f^\top v\,\mathrm{d}x - \int_{\Gamma_N} g^\top v\,\mathrm{d}s$$

over the set $\{v \in (H^1(\Omega))^2 \mid v = \bar{u} \text{ on } \Gamma_D\}$.

In what follows, we concentrate on a dual variational formulation of the linear elasticity problem [6, p. 106]. According to [6, p. 65], the generalized Hooke's law can be inverted (see also [7]). It gives a relation between strain and stress tensor for a nonhomogeneous and anisotropic material of the elastic body:

$$\varepsilon = A:\tau \qquad \Big(\varepsilon_{ij} = \sum_{k,l=1}^2 A_{ijkl}\tau_{kl}\Big),$$

where $A = (A_{ijkl})_{i,j,k,l=1}^2$, $A_{ijkl} \in L^\infty(\Omega)$,

$$A_{jikl} = A_{ijkl} = A_{klij},$$

and that there exists a constant $c > 0$ such that

$$\eta\cdot(A(x):\eta) \ge c\|\eta\|^2$$

holds a.e. in Ω for any symmetric tensor $\eta = (\eta_{ij})_{i,j=1}^2$, $\eta_{ij} \in \mathbb{R}^1$.

Definition. The *dual problem of linear elasticity* consists of finding a 2×2 symmetric tensor σ which minimizes the functional (of the complementary energy) $J : (L^2(\Omega))^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^1$ defined by

$$J(\tau) = \frac{1}{2}a(\tau,\tau) - b(\tau) := \frac{1}{2}\int_\Omega \tau\cdot(A:\tau)\,\mathrm{d}x - \int_\Omega \tau\cdot\varepsilon(\bar{u})\,\mathrm{d}x$$

over the set $T(f,g)$.

Obviously, the bilinear form $a(\cdot,\cdot)$ is symmetric and uniformly elliptic. Further, we introduce an equivalent formulation of the dual problem. Let some particular solution $\bar{\tau} \in T(f,g)$ be given, i.e. $\operatorname{Div}\bar{\tau} + f = 0$ in Ω and $\bar{\tau}n = g$ on Γ_N in the sense of distributions, where n is the outward unit normal to $\partial\Omega$. Furthermore, we shall assume that all functions appearing below are sufficiently smooth so that the corresponding operations are correctly defined. Using the substitution

$$\tau = \tau^0 + \bar{\tau},$$

we can reformulate the dual problem as follows:

Find a 2×2 symmetric tensor σ^0 which minimizes the functional $J^0 : (L^2(\Omega))^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^1$ defined by

$$J^0(\tau^0) = \frac{1}{2}a(\tau^0,\tau^0) - b(\tau^0) + a(\bar{\tau},\tau^0) \tag{4}$$

over the *space of divergence-free tensor functions*

$$T := T(0,0) = \Big\{\tau \in (L^2(\Omega))^{2\times 2}_{\mathrm{sym}} \,\Big|\, (\tau,\varepsilon(v))_0 = 0 \quad \forall v \in \mathcal{V}\Big\}. \tag{5}$$

If $\sigma^0 \in T$ is sufficiently smooth, then $\operatorname{Div}\sigma^0 = 0$ in Ω and $\sigma^0 n = g$ on Γ_N.

Since $\bar{\tau}$ is a given function, $J(\bar{\tau})$ is a fixed constant. Then one immediately sees that

$$\begin{aligned} J(\tau^0+\bar{\tau}) - J(\bar{\tau}) &= \frac{1}{2}a(\tau^0+\bar{\tau},\tau^0+\bar{\tau}) - b(\tau^0+\bar{\tau}) - \frac{1}{2}a(\bar{\tau},\bar{\tau}) + b(\bar{\tau}) \\ &= \frac{1}{2}a(\tau^0,\tau^0) - b(\tau^0) + a(\bar{\tau},\tau^0) = J^0(\tau^0) \end{aligned}$$

and therefore,

$$\sigma = \sigma^0 + \bar{\tau}. \tag{6}$$

Remark 1. There are many possibilities how to construct a particular solution $\bar{\tau} \in T(f,g)$ appearing in (4) and (6). For instance, when $\Gamma_N = \emptyset$ we define

$$\bar{\tau}_{11}(x_1,x_2) = -\int_0^{x_1}\bar{f}_1(\xi,x_2)\mathrm{d}\xi, \quad \bar{\tau}_{22}(x_1,x_2) = -\int_0^{x_2}\bar{f}_2(x_1,\zeta)\mathrm{d}\zeta,$$

and set $\bar{\tau}_{12} = \bar{\tau}_{21} = 0$, where $(x_1,x_2) \in \mathbb{R}^2$, $\bar{f}_i = f_i$ in Ω, and $\bar{f}_i = 0$ in $\mathbb{R}^2 \setminus \Omega$ for $i = 1, 2$. Then from (1) and (2) we find that $\operatorname{Div}\bar{\tau} + f = 0$ in Ω. For $\Gamma_N \neq \emptyset$ see Appendix and [8, p. 447].

3. Construction of divergence-free tensor functions

First we introduce a method concerning how divergence-free (i.e. solenoidal) vector functions from the space

$$Q = \{q \in (L^2(\Omega))^2 \mid (q, \operatorname{grad} v)_0 = 0 \quad \forall v \in V\} \tag{7}$$

can be described. The definition of Q is based on the following Green's formula:

$$(q, \operatorname{grad} v)_0 + (\operatorname{div} q, v)_0 = \int_{\partial\Omega} v q^\top n \, \mathrm{d}s \quad \forall v \in H^1(\Omega) \tag{8}$$

for a sufficiently smooth vector function q. In this case we have div $q = 0$ in Ω and $q^\top n = 0$ on Γ_N if $q \in Q$ due to density theorems for the Sobolev spaces (3), see [9,10]. Note that the above Green's formula (8) is valid also for functions q whose divergence exists in the sense of distributions, see [11].

Further, we introduce the operator

$$\operatorname{curl} w = (\partial_2 w, -\partial_1 w)^\top, \quad w \in W,$$

where

$$W = \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_N\}.$$

Theorem 1. *Let Γ_D and Γ_N be connected. Then*

$$Q = \operatorname{curl} W.$$

Proof. Let $q \in Q$. Then by Green's formula (8) we find that

$$\operatorname{div} q = 0 \quad \text{in } \Omega$$

in the sense of distributions if all test functions v are equal to zero on whole boundary $\partial\Omega$. Taking $v \equiv 1$ in (8) we find that

$$\int_{\partial\Omega} q^\top n \, \mathrm{d}s = 0.$$

According to [11, p. 22], there exists a stream function $w \in H^1(\Omega)$ unique apart from an additive constant (which will be chosen later) such that

$$q = \operatorname{curl} w. \tag{9}$$

Let $\Gamma_N \neq \emptyset$ and let $s = (n_2, -n_1)^\top$, i.e., s is the unit tangential vector to $\partial\Omega$. Then by (7) and Green's formula (8) we obtain

$$\begin{aligned} 0 &= (\operatorname{curl} w, \operatorname{grad} \varphi)_0 = -(\operatorname{grad} w, \operatorname{curl} \varphi)_0 = -\int_{\partial\Omega} w (\operatorname{curl} \varphi)^\top n \, \mathrm{d}s \\ &= \int_{\partial\Omega} w (\operatorname{grad} \varphi)^\top s \, \mathrm{d}s = \int_{\Gamma_N} w \frac{\partial \varphi}{\partial s} \, \mathrm{d}s \end{aligned} \tag{10}$$

for all $\varphi \in C^\infty(\overline{\Omega})$ such that $\varphi = 0$ on $\overline{\Gamma}_D$. This implies that the tangential derivative of w on Γ_N is zero in the sense of distributions. Since Γ_N is connected, we can choose the stream function w in (9) to be zero on Γ_N. If $\Gamma_N = \emptyset$, then the additive constant can be chosen arbitrarily. Hence, $q \in \operatorname{curl} W$.

Conversely, let $w \in W$ and let $\varphi \in C^\infty(\overline{\Omega})$ with $\varphi = 0$ on Γ_D. Similarly to (10) we get

$$(\operatorname{curl} w, \operatorname{grad} \varphi)_0 = \int_{\Gamma_N} w \frac{\partial \varphi}{\partial s} \, \mathrm{d}s.$$

However, the last integral vanishes, since $w = 0$ on Γ_N. Hence, curl $w \in Q$ by (7). □

Note that there are analogous results based on a theorem of De Rahm, see e.g. [12].

Now recall the definition of the Hessian operator hes: $H^2(\Omega) \to (L^2(\Omega))^{2\times 2}_{\text{sym}}$,

$$\operatorname{hes} z = \begin{pmatrix} \partial_{11} z & \partial_{12} z \\ \text{sym} & \partial_{22} z \end{pmatrix}.$$

Further, we introduce the operator inv: $H^2(\Omega) \to (L^2(\Omega))^{2\times 2}_{\text{sym}}$ defined by

$$\operatorname{inv} z = \begin{pmatrix} \partial_{22} z & -\partial_{12} z \\ \text{sym} & \partial_{11} z \end{pmatrix}$$

and let

$$Z = \{z \in H^2(\Omega) \mid z = \partial_n z = 0 \text{ on } \Gamma_N\},$$

where ∂_n stands for the normal derivative. Notice that $\operatorname{inv} z$ is the inverse tensor to $\operatorname{hes} z$ up to $\det(\operatorname{hes} z)$.

Theorem 2. *Let Γ_D and Γ_N be connected. Then*

$$T = \operatorname{inv} Z,$$

where T is from (5).

Proof. Let $\tau = (\tau_{ij}) \in T$ be arbitrary. Denote its columns by q_1 and q_2. Clearly, $q_i \in Q$ for $j = 1, 2$. Since Γ_D and Γ_N are connected, we obtain by Theorem 1 that there exist two stream functions $w_1, w_2 \in W$ such that

$$q_j = \operatorname{curl} w_j, \quad j = 1, 2.$$

Due to the symmetry $\tau_{12} = \tau_{21}$ we obtain $-\partial_1 w_1 = \partial_2 w_2$. Setting $w = (w_1, w_2)$, we see that w is also divergence-free. Hence, $w \in Q$, since $w^\top n = 0$ on Γ_N. Using Theorem 1 once again, we find that there exists the third stream function $z \in W$ such that

$$w = \operatorname{curl} z.$$

Since the derivatives $\partial_1 z$ and $\partial_2 z$ also belong to W, we get that $z \in Z$. Moreover, we see that $\tau = \operatorname{inv} z$, for instance, $\partial_{11} z = -\partial_1 w_2 = \tau_{22}$.

Conversely, let $z \in Z$ be arbitrary. Since $z = \partial_n z = 0$ on Γ_N, we obtain that $\partial_1 z, \partial_2 z \in W$. Applying Theorem 1, we find that

$$\operatorname{curl}(\partial_2 z) = (\partial_{22} z, -\partial_{12} z)^\top \in Q,$$
$$\operatorname{curl}(\partial_1 z) = (-\partial_{12} z, \partial_{11} z)^\top \in Q,$$

and thus, $\operatorname{inv} z \in T$. □

Remark 2. For a given divergence-free tensor $\tau \in T$ the corresponding potential function z such that

$$\tau = \operatorname{inv} z$$

is called the *Airy function*, see [6, p. 161]. It is unique apart from a linear function. To see this we take $z_1, z_2 \in Z$ such that $\operatorname{inv} z_1 = \operatorname{inv} z_2 = \tau$. Then $\operatorname{inv}(z_1 - z_2) = 0$ implying that $z_1 - z_2$ is a linear polynomial.

4. Finite element approximation of the dual problem

Using Theorem 1, we can define finite element spaces of divergence-free vector functions by $Q_h = \operatorname{curl} W_h$, where W_h is an arbitrary finite element space of W. Similarly, Theorem 2 gives us a possibility to construct finite element spaces of divergence-free tensor functions as follows. Let $Z_h \subset Z$ be a finite element space generated by C^1-elements over some triangulation [13,14]. Then we set

$$T_h := \operatorname{inv} Z_h.$$

Remark 3. For construction of finite element spaces Z_h we refer to [14]. If basis functions $\{z^i\}$ in Z_h, $i = 1, \ldots, \dim Z_h$, have small supports, then $\{\operatorname{inv} z^i\} \subset T_h$ have also small supports. We can take for instance piecewise quadratic composite C^1-triangular elements invented by G. Heindl [15] (see also [4,16]). Applying the operator inv to any function $z_h \in Z_h$, we get a piecewise constant tensor $\operatorname{inv}(z_h) \in T_h$ which is exactly divergence-free in the sense of distributions. The basis functions $\operatorname{inv}\ z^i$ can be calculated analytically by double differentiation of C^1 piecewise quadratic basis functions z^i. Then we can easily handle, e.g., possible inequality constraints that occur in plasticity and related limit or shakedown analysis, in problems of Signorini's type or in stress hardening [6]. We could also apply the composite piecewise cubic Hsieh-Clough–Tocher C^1 triangular elements (see [17, p. 341]). Then the corresponding divergence-free basis functions $\operatorname{inv}\ z^i$ are piecewise linear and discontinuous, in general.

A conforming finite element approximation of the dual linear elasticity problem consists of finding $\sigma_h^0 \in T_h$ which minimizes the same functional (4) over T_h and the function

$$\sigma_h := \sigma_h^0 + \overline{\tau}$$

is called then an *approximate solution*. By (6), (10), and the well-known Cea's lemma there exists a constant $C > 0$ such that

$$\|\sigma - \sigma_h\|_0 = \|\sigma^0 - \sigma_h^0\|_0 \le C \inf_{\tau_h \in T_h} \|\sigma^0 - \tau_h\|_0 = C \inf_{z_h \in Z_h} \|\mathrm{inv}\, z^0 - \mathrm{inv}\, z_h\|_0$$
$$= C \inf_{z_h \in Z_h} \|\mathrm{hes}(z^0 - z_h)\|_0 \le C \inf_{z_h \in Z_h} \|z^0 - z_h\|_2,$$

where $\sigma^0 = \mathrm{inv}\, z^0$, $z^0 \in Z$, and $\tau_h = \mathrm{inv}\, z_h$ due to Theorem 2. Now we can apply standard interpolation results for C^1-elements to prove the convergence (or some rate of convergence) of σ_h to σ as $h \to 0$, see e.g. [14,17].

5. Conclusions

Divergence-free vector or tensor functions have many important application. For instance, a simultaneous use of the primal and dual formulations enables us to apply the hypercircle method which produces exact evaluation of the discretization error when the primal and dual finite element method are employed, see [6, p. 259]. We can also obtain two-sided bounds of energy and a posteriori error estimates, see [18, p. 64]. Some other applications are mentioned in Introduction.

In this paper, we show how to construct finite element stress tensor basis functions that are exactly divergence-free and have small supports which is necessary to get sparse resulting systems of linear algebraic equations. Our construction is based on a special second order differential operator inv which is applied to C^1 finite elements. The convergence and the rate of convergence of the corresponding finite element approximations follow directly from approximation properties of C^1 elements used.

Data availability

No data was used for the research described in the article.

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Appendix

Here we sketch how to construct a particular solution $\overline{\tau} \in T(f, g)$ appearing in (4) and (6) for $\Gamma_N \neq \emptyset$. For simplicity, let $\Omega = (0, 1) \times (0, 1)$ be the unit square and let $\overline{\Gamma}_2$ be the intersection of $\overline{\Omega}$ with coordinate axes x_1 and x_2. Furthermore, assume that f and g are sufficiently regular. First we shall deal with the first row of $\overline{\tau}$. Similarly to Remark 1 we define

$$F_1(x_1, x_2) = \Big(-\int_0^{x_1} f_1(\xi, x_2)\mathrm{d}\xi, 0\Big),$$

where $(x_1, x_2) \in \mathbb{R}^2$. Further we set

$$G_1 = \mathrm{curl}\, w_1 \quad \text{in } \Omega,$$

where $w_1 \in H^1(\Omega)$ is a function with the tangential component

$$(\mathrm{curl}\, w_1)^\top n = g_1 - F_1^\top n \quad \text{on } \Gamma_N.$$

Note that w_1 in not uniquely determined. Therefore, a special construction of w_1 will be given below.

Using Green's formula (8), we find that

$$\begin{aligned}(F_1 + G_1, \mathrm{grad}\, v)_0 &= (F_1, \mathrm{grad}\, v)_0 + (G_1, \mathrm{grad}\, v)_0\\ &= (-\mathrm{div}\, F_1, v)_0 + \int_{\partial\Omega} v F_1^\top n\, \mathrm{d}s + \int_{\partial\Omega} v G_1^\top n\, \mathrm{d}s\\ &= (f_1, v)_0 + \int_{\Gamma_N} v F_1^\top n\, \mathrm{d}s + \int_{\Gamma_N} v(g_1 - F_1^\top n)\mathrm{d}s\\ &= (f_1, v)_0 + \int_{\Gamma_N} g_1 v\, \mathrm{d}s\end{aligned}$$

for all $v \in V$. Therefore, $F_1 + G_1$ will be the first row of $\overline{\tau}$, since $\mathrm{div}(F_1 + G_1) + f_1 = 0$ in Ω and $(F_1 + G_1)n = g_1$ on Γ_N.

Now we show how to construct w_1 explicitly. First we divide Γ_N as follows

$$\Gamma_N = \Gamma_N^{(1)} \cup \Gamma_N^{(2)},$$

where $\Gamma_N^{(1)}$ is the horizontal segment and $\Gamma_N^{(2)}$ is the vertical segment of Γ_N. Clearly, the term $F_1^\top n$ vanishes on Γ_N and, therefore, $w_1 \in H^1(\Omega)$ can be any function satisfying

$$\begin{aligned} \partial_1 w_1 &= g_1 \quad \text{on } \Gamma_N^{(1)}, \\ -\partial_2 w_1 &= g_1 \quad \text{on } \Gamma_N^{(2)} \end{aligned}$$

and also the compatibility condition $w_1(0,0) = 0$. For instance, we can set

$$w_1(x_1, x_2) = \int_0^{x_1} g_1(\xi, 0)\mathrm{d}\xi - \int_0^{x_2} g_1(0, \zeta)\mathrm{d}\zeta .$$

Similarly we construct F_2 and G_2 so that $\operatorname{div}(F_2 + G_2) + f_2 = 0$ in Ω and $(F_2 + G_2)n = g_2$ on Γ_N, namely, let

$$F_2(x_1, x_2) = \Big(0, -\int_0^{x_2} f_2(x_1, \zeta)\mathrm{d}\zeta\Big),$$

where $(x_1, x_2) \in \mathbb{R}^2$. Then $F_2^\top n$ vanishes on Γ_N. Further, we set

$$G_2 = \operatorname{curl} w_2 \quad \text{in } \Omega,$$

so that

$$(\operatorname{curl} w_2)^\top n = g_2 - F_2^\top n \quad \text{on } \Gamma_N .$$

Here $w_2 \in H^1(\Omega)$ can be any function satisfying

$$\begin{aligned} \partial_1 w_2 &= g_2 \quad \text{on } \Gamma_N^{(1)}, \\ -\partial_2 w_2 &= g_2 \quad \text{on } \Gamma_N^{(2)} \end{aligned}$$

and also the compatibility condition $w_2(0,0) = 0$. It can be defined analogously to w_1, namely,

$$w_2(x_1, x_2) = \int_0^{x_1} g_2(\xi, 0)\mathrm{d}\xi - \int_0^{x_2} g_2(0, \zeta)\mathrm{d}\zeta .$$

Then $F_2 + G_2$ will be the second row of $\overline{\tau}$. Hence, we can put $\overline{\tau} = F + G$, where $F = (F_1^\top, F_2^\top)$ and $G = (G_1^\top, G_2^\top)$.

However, we still need to fulfill the symmetry condition $\overline{\tau}_{12} = \overline{\tau}_{21}$. Using the definition of F, we find that the both off-diagonal entries of F vanish. Therefore, from the definition of G we have to satisfy the following relations

$$\overline{\tau}_{12} = -\partial_1 w_1, \quad \overline{\tau}_{21} = \partial_2 w_2 .$$

This can be satisfied, for example, if $g_1 = g_2$ is constant and f arbitrary.

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