Signed k-domatic numbers of digraphs

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Abstract

Let D be a finite and simple digraph with vertex set $V(D)$, and let $f : V(D) \to \{-1, 1\}$ be a two-valued function. If $k \geq 1$ is an integer and $\sum_{x \in N^-[v]} f(x) \geq k$ for each $v \in V(D)$, where $N^-[v]$ consists of v and all vertices of D from which arcs go into v, then f is a signed k-dominating function on D. A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions of D with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed k-dominating family* (of functions) of D. The maximum number of functions in a signed k-dominating family of D is the *signed k-domatic number* of D, denoted by $d_{kS}(D)$. In this note we initiate the study of the signed k-domatic numbers of digraphs and present some sharp upper bounds for this parameter.

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1 Introduction

In this paper, D is a finite and simple digraph with vertex set $V = V(D)$ and arc set $A = A(D)$. Its underlying graph is denoted $G(D)$. We write $\deg_D^+(v) = \deg^+(v)$ for the *outdegree* of a vertex v and $\deg_D^-(v) = \deg^-(v)$ for its *indegree*. The *minimum* and *maximum indegree* are $\delta^-(D)$ and $\Delta^-(D)$. The sets $N^+(v) = \{x \mid (v, x) \in A(D)\}$ and $N^-(v) = \{x \mid (x, v) \in A(D)\}$ are called the *outset* and *inset* of the vertex v. Likewise, $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. For an arc $(x, y) \in A(D)$, the vertex y is an *outer neighbor* of x and x is an *inner neighbor* of y. Note that for any digraph D with m arcs,

$$\sum_{u \in V(D)} \deg^-(u) = \sum_{u \in V(D)} \deg^+(u) = m. \tag{1}$$

Consult [6] and [7] for notation and terminology which are not defined here.

For a real-valued function $f : V(D) \longrightarrow \mathbf{R}$ the weight of f is $w(f) = \sum_{v \in V(D)} f(v)$, and for $S \subseteq V(D)$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V(D))$. If $k \geq 1$ is an integer, then the

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signed k-dominating function is defined as a function $f : V(D) \longrightarrow \{-1, 1\}$ such that $f(N^-[v]) = \sum_{x \in N^-[v]} f(x) \geq k$ for every $v \in V(D)$. The *signed k-domination number* for a digraph D is

$$\gamma_{kS}(D) = \min\{w(f) \mid f \text{ is a signed } k\text{-dominating function of } D\}.$$

A $\gamma_{kS}(D)$-function is a signed k-dominating function on D of weight $\gamma_{kS}(D)$. As the assumption $\delta^-(D) \geq k-1$ is necessary, we always assume that when we discuss $\gamma_{kS}(D)$, all digraphs involved satisfy $\delta^-(D) \geq k-1$ and thus $n(D) \geq k$. Then the function assigning $+1$ to every vertex of D is a SkD function, called the function ϵ, of weight n. Thus $\gamma_{kS}(D) \leq n$ for every digraph of order n with $\delta^- \geq k-1$. Moreover, the weight of every SkD function different from ϵ is at most $n-2$ and more generally, $\gamma_{kS}(D) \equiv n \pmod 2$. Hence $\gamma_{kS}(D) = n$ if and only if ϵ is the unique SkD function of D.

The signed k-domination number of digraphs was introduced by Atapour, Hajypory, Sheikholeslami and Volkmann [1]. When $k = 1$, the signed k-domination number $\gamma_{kS}(D)$ is the usual *signed domination number* $\gamma_S(D)$, which was introduced by Zelinka in [16] and has been studied by several authors (see for example [8]).

Observation 1. ([1]) Let D be a digraph of order n. Then $\gamma_{kS}(D) = n$ if and only if $k-1 \leq \delta^-(D) \leq k$ and for each $v \in V(D)$ there exists a vertex $u \in N^+[v]$ such that $\deg^-(u) = k-1$ or $\deg^-(u) = k$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed k-dominating functions on D with the property that $\sum_{i=1}^d f_i(v) \leq 1$ for each $v \in V(D)$, is called a *signed k-dominating family* on D. The maximum number of functions in a signed k-dominating family on D is the *signed k-domatic number* of D, denoted by $d_{kS}(D)$. The signed k-domatic number is well-defined and $d_{kS}(D) \geq 1$ for all digraphs D in which $d_D^-(v) \geq k-1$ for all $v \in V$, since the set consisting of any one SkD function, for instance the function ϵ, forms a SkD family of D. A *$d_{kS}(D)$-family* of a digraph D is a SkD family containing $d_{kS}(D)$ SkD functions. When $k = 1$, the signed k-domatic number of a digraph D is the usual *signed domatic number* $d_S(D)$, which was introduced by Sheikholeslami and L. Volkmann [9] and has been studied in [13].

Observation 2. Let D be a digraph of order n. If $\gamma_{kS}(D) = n$, then ϵ is the unique SkD function of D and so $d_{kS}(D) = 1$.

In this paper we initiate the study of the signed k-domatic number of digraphs, and we present different bounds on $d_{kS}(D)$. Some of our results are extensions of well-known properties of the signed domatic number $d_S(D) = d_{1S}(D)$ of digraphs (see for example [9]) as well as the signed k-domatic number of graphs G (see for example [5, 14]).

We make use of the following results and observations in this paper.

Observation 3. Let $k \geq 1$ be an integer, and let D be a digraph with $\delta^-(D) \geq k-1$. If for every vertex $v \in V(D)$ the set $N^+[v]$ contains a vertex x such that $\deg^-(x) \leq k$, then $d_{kS}(D) = 1$.

Proof. Assume that $N^+[v]$ contains a vertex x_v such that $\deg^-(x_v) \leq k$ for every vertex $v \in V(D)$, and let f be a signed k-dominating function on D. Since $\deg^-(x_v) \leq k$, we deduce that $f(v) = 1$. Hence $f(v) = 1$ for each $v \in V(D)$ and thus $d_{kS}(D) = 1$. □

A digraph is *r-inregular* if each vertex has indegree r.

Corollary 4. If D is an r-inregular digraph and $k = r-1$ or r, then $\gamma_{kS}(D) = n$ and $d_{kS}(D) = 1$.

Observation 5. The signed k-domatic number of a digraph is an odd integer.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed k-dominating family on D such that $d = d_{kS}(D)$. Suppose to the contrary that $d_{kS}(D)$ is an even integer. If $x \in V(D)$ is an arbitrary vertex, then $\sum_{i=1}^d f_i(x) \leq 1$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number and we obtain $\sum_{i=1}^{d} f_i(x) \le 0$ for each $x \in V(G)$. If v is an arbitrary vertex, then it follows that

$$d \cdot k = \sum_{i=1}^{d} k \le \sum_{i=1}^{d} \sum_{x \in N^-[v]} f_i(x) = \sum_{x \in N^-[v]} \sum_{i=1}^{d} f_i(x) \le 0.$$

which is a contradiction, and the proof is complete. □

2 Properties and upper bounds

In this section we present basic properties of the signed k-domatic number, and we find some sharp upper bounds for this parameter.

Proposition 6. If $k > p \ge 1$ are integers, then $d_{pS}(D) \ge d_{kS}(D)$ for any digraph D.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family on D such that $d = d_{kS}(D)$. Then $\{f_1, f_2, \ldots, f_d\}$ is also a SpD family on D and thus $d_{pS}(D) \ge d_{kS}(D)$. □

Theorem 7. Let D be a digraph and $v \in V(D)$. Then

$$d_{kS}(D) \le \begin{cases} \dfrac{\deg^-(v)+1}{k+1} & \text{if} \quad \deg^-(v) \equiv k \pmod 2 \\ \dfrac{\deg^-(v)+1}{k} & \text{if} \quad \deg^-(v) \equiv k+1 \pmod 2. \end{cases}$$

Moreover, if the equality holds, then for each function f_i of a SkD family $\{f_1, f_2, \cdots, f_d\}$ and for every $u \in N^-[v]$, $\sum_{u \in N^-[v]} f_i(u) = k+1$ if $\deg^-(v) \equiv k \pmod 2$, $\sum_{u \in N^-[v]} f_i(u) = k$ if $\deg^-(v) \equiv k+1 \pmod 2$ and $\sum_{i=1}^{d} f_i(u) = 1$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family of D such that $d = d_{kS}(D)$. If $\deg^-(v) \equiv k \pmod 2$, then

$$\begin{aligned} d &= \textstyle\sum_{i=1}^{d} 1 \le \sum_{i=1}^{d} \frac{1}{k+1} \sum_{u \in N^-[v]} f_i(u) \\ &= \textstyle\frac{1}{k+1} \sum_{u \in N^-[v]} \sum_{i=1}^{d} f_i(u) \le \frac{1}{k+1} \sum_{u \in N^-[v]} 1 \\ &= \textstyle\frac{\deg^-(v)+1}{k+1}. \end{aligned}$$

Similarly, if $\deg^-(v) \equiv k+1 \pmod 2$, then

$$\begin{aligned} d &= \textstyle\sum_{i=1}^{d} 1 \le \sum_{i=1}^{d} \frac{1}{k} \sum_{u \in N^-[v]} f_i(u) \\ &= \textstyle\frac{1}{k} \sum_{u \in N^-[v]} \sum_{i=1}^{d} f_i(u) \le \frac{1}{k} \sum_{u \in N^-[v]} 1 \\ &= \textstyle\frac{\deg^-(v)+1}{k}. \end{aligned}$$

If $d_{kS}(D) = \frac{\deg^-(v)+1}{k+1}$ when $\deg^-(v) \equiv k \pmod 2$ or $d_{kS}(D) = \frac{\deg^-(v)+1}{k}$ when $\deg^-(v) \equiv k+1$ (mod 2), then the two inequalities occurring in the proof of each corresponding case become equalities, which gives the properties given in the statement. □

Corollary 8. Let D be a digraph and $1 \le k \le \delta^-(D) + 1$. Then

$$d_{kS}(D) \le \begin{cases} \dfrac{\delta^-(D)+1}{k+1} \le \dfrac{n}{k+1} & \text{if} \quad \delta^-(D) \equiv k \pmod 2 \\ \dfrac{\delta^-(D)+1}{k} \le \dfrac{n}{k} & \text{if} \quad \delta^-(D) \equiv k+1 \pmod 2. \end{cases}$$

The next corollary is a consequence of Observation 5 and Corollary 8.

Corollary 9. If D is a digraph of minimum degree δ^-, then $d_{kS}(D) = 1$ for every integer k such that $\frac{\delta^-+1}{3} < k \le \delta^- + 1$.

Corollary 10. Let $k \geq 1$ be an integer, and let D be a $(k+1)$-inregular digraph of order n. If $k \geq 2$ or $k = 1$ and $n \not\equiv 0 \pmod 3$, then $d_{kS}(D) = 1$.

Proof. By Corollary 8, $d_{kS}(D) \leq \frac{k+2}{k}$. If $k \geq 2$, then it follows from Observation 5 that $d_{kS}(D) = 1$. Now let $k = 1$. Then $d_{kS}(D) = 1$ or $d_{kS}(D) = 3$ by Observation 5. Suppose to the contrary that $d_{kS}(D) = 3$. Let f belong to a signed k-dominating family on D of order 3. By Theorem 7, we have $\sum_{x \in N^-[v]} f(x) = 1$ for every $v \in V(D)$. This implies that

$$n = \sum_{v \in V(D)} \sum_{x \in N^-[v]} f(x) = \sum_{x \in N^-[v]} \sum_{v \in V(D)} f(x) = 3w(f).$$

Since $w(f)$ is an integer, 3 is a divisor of n which contradicts the hypothesis $n \not\equiv 0 \pmod 3$, and the proof is complete. $\square$

Corollary 11. Let $k \geq 1$ be an integer, and let D be a $(k+4)$-inregular digraph of order n. If $k \geq 2$ or $k = 1$ and $n \not\equiv 0 \pmod 5$, then $d_{kS}(D) = 1$.

Proof. According to Corollary 8, $d_{kS}(D) \leq \frac{k+5}{k+1}$. If $k \geq 2$, then we deduce from Observation 5 that $d_{kS}(D) = 1$. Now let $k = 1$. In view of Observation 5, $d_{kS}(D) = 1$ or $d_{kS}(D) = 3$. Suppose to the contrary that $d_{kS}(D) = 3$. Let f belong to a signed k-dominating family on D of order 3. By Theorem 7, we have $\sum_{x \in N^-[v]} f(x) = 2$ for every $v \in V(D)$. This implies that

$$2n = \sum_{v \in V(D)} \sum_{x \in N^-[v]} f(x) = \sum_{x \in N^-[v]} \sum_{v \in V(D)} f(x) = 5w(f).$$

Thus 5 is a divisor of n, a contradiction to the hypothesis $n \not\equiv 0 \pmod 5$. $\square$

Corollary 12. Let $k \geq 1$ be an integer, and let D be a $(k+2)$-inregular digraph of order n. Then $d_{kS}(D) = 1$.

Proof. By Corollary 8, $d_{kS}(D) \leq \frac{k+3}{k+1}$. Therefore Observation 5 implies that $d_{kS}(D) = 1$. $\square$

Theorem 13. Let $k \geq 1$ be an integer, and let D be an r-inregular digraph of order n such that $r \geq k-1$. If $r < 3k-1$, then $d_{kS}(D) = 1$, and if $r \geq 3k-1$ and $(n, r+1) = 1$, then

$$d_{kS}(D) < \begin{cases} \dfrac{\delta^-(D)+1}{k+1} & \text{if} \quad \delta^-(D) \equiv k \pmod 2 \\ \dfrac{\delta^-(D)+1}{k} & \text{if} \quad \delta^-(D) \equiv k+1 \pmod 2. \end{cases}$$

Proof. If $r < 3k-1$, then it follows from Corollary 8 that $d_{kS}(D) \leq \frac{r+1}{k} < 3$. Therefore Observation 5 implies that $d_{kS}(D) = 1$.

Now assume that $r \geq 3k-1$ and $(n, r+1) = 1$. First let $r = \delta^-(D) \equiv k \pmod 2$ (if $\delta^-(D) \equiv k+1 \pmod 2$, then the proof is similar). Suppose to the contrary that $d_{kS}(D) \geq \frac{\delta^-(D)+1}{k+1}$. Then by Corollary 8, $d_{kS}(D) = \frac{\delta^-(D)+1}{k+1}$. Let f belong to a signed k-dominating family on D of order $\frac{\delta^-(D)+1}{k+1}$. By Theorem 7, we have $\sum_{x \in N^-[v]} f(x) = k+1$ for every $v \in V(D)$. This implies that

$$n(k+1) = \sum_{v \in V(D)} \sum_{x \in N^-[v]} f(x) = \sum_{x \in N^-[v]} \sum_{v \in V(D)} f(x) = (r+1)w(f).$$

Since $w(f)$ is an integer and $(n, r+1) = 1$, the number $r+1$ is a divisor of $k+1$. It follows from $k-1 \leq \delta^-(D) = r$ that $r = k-1$ or $k = r$, a contradiction to the hypothesis that $r \geq 3k-1$. $\square$

Theorem 14. Let D be a digraph with $\delta^-(D) \geq k-1$, and let $\Delta = \Delta(G(D))$ be the maximum degree of $G(D)$. Then

$$d_{kS}(D) \leq \begin{cases} \dfrac{\Delta(G(D))+2}{2(k+1)} & \text{if} \quad \delta^-(D) \equiv k \pmod 2 \\[2ex] \dfrac{\Delta(G(D))+2}{2k} & \text{if} \quad \delta^-(D) \equiv k+1 \pmod 2. \end{cases}$$

Proof. First of all, we show that $\delta^-(D) \leq \Delta/2$. Suppose to the contrary that $\delta^-(D) > \Delta/2$. Then $\Delta^+(D) \leq \Delta - \delta^-(D) < \Delta/2$, and (1) leads to the contradiction

$$\frac{\Delta \cdot |V(D)|}{2} < \sum_{u \in V(D)} \deg^-(u) = \sum_{u \in V(D)} \deg^+(u) < \frac{\Delta \cdot |V(D)|}{2}.$$

Applying Corollary 8, we deduce the desired result. $\square$

Let D be a digraph. By D^{-1} we denote the digraph obtained by reversing all the arcs of D. A digraph without directed cycles of length 2 is called an *oriented graph*. An oriented graph D is a *tournament* when either $(x, y) \in A(D)$ or $(y, x) \in A(D)$ for each pair of distinct vertices $x, y \in V(D)$.

Theorem 15. For every oriented graph D of order n and $1 \leq k \leq \min\{\delta^-(D)+1, \delta^-(D^{-1})+1\}$,

$$d_{kS}(D) + d_{kS}(D^{-1}) \leq \frac{n+1}{k} \tag{2}$$

with equality if and only if D is an r-regular tournament of order $n = 2r+1$ and $r = k-1$.

Proof. Since $\delta^-(D) + \delta^-(D^{-1}) \leq n-1$, Corollary 8 implies that

$$d_{kS}(D) + d_{kS}(D^{-1}) \leq \frac{\delta^-(D)+1}{k} + \frac{\delta^-(D^{-1})+1}{k} \leq \frac{n+1}{k}.$$

If D is an r-regular tournament of order $n = 2r+1$ and $r = k-1$, then D^{-1} is also an r-regular tournament, and it follows from Observation 3 that

$$d_{kS}(D) + d_{kS}(D^{-1}) = 2 = \frac{2(r+1)}{k} = \frac{n+1}{k}.$$

If D is not a tournament or D is a non-regular tournament, then $\delta^-(D) + \delta^-(D^{-1}) \leq n-2$ and hence we deduce from Corollary 8 that

$$d_{kS}(D) + d_{kS}(D^{-1}) \leq \frac{n}{k}.$$

If D is an r-regular tournament, then $n = 2r+1$. If $k-1 < r < 3k-1$, then Theorem 13 leads to

$$2 = d_{kS}(D) + d_{kS}(D^{-1}) < \frac{n+1}{k}.$$

Finally, assume that $r \geq 3k-1$. We observe that $(n, r+1) = (2r+1, r+1) = 1$. Using Theorem 13, we deduce that

$$d_{kS}(D) + d_{kS}(D^{-1}) < \frac{\delta^-(D)+1}{k} + \frac{\delta^-(D^{-1})+1}{k} = \frac{n+1}{k},$$

and the proof is complete. $\square$

Theorem 16. Let D be a digraph of order n and $\delta^-(D) \geq k-1 \geq 0$. Then $\gamma_{kS}(D) \cdot d_{kS}(D) \leq n$. Moreover if $\gamma_{kS}(D) \cdot d_{kS}(D) = n$, then for each $d = d_{kS}(D)$-family $\{f_1, f_2, \cdots, f_d\}$ of D each function f_i is a $\gamma_{kS}(D)$-function and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a SkD family of D such that $d = d_{kS}(D)$ and let $v \in V$. Then

$$\begin{array}{rcl} d \cdot \gamma_{kS}(D) & = & \sum_{i=1}^{d} \gamma_{kS}(D) \\ & \leq & \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ & = & \sum_{v \in V} \sum_{i=1}^{d} f_i(v) \\ & \leq & \sum_{v \in V} 1 \\ & = & n. \end{array}$$

If $\gamma_{kS}(D) \cdot d_{kS}(D) = n$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{kS}(D)$-family $\{f_1, f_2, \cdots, f_d\}$ of D and for each i, $\sum_{v \in V} f_i(v) = \gamma_{kS}(D)$, thus each function f_i is a $\gamma_{kS}(D)$-function, and $\sum_{i=1}^{d} f_i(v) = 1$ for all v. □

Corollary 17 is a consequence of Theorem 16 and Observation 5 and improves Observation 2.

Corollary 17. If $\gamma_{kS}(D) > \frac{n}{3}$, then $d_{kS}(D) = 1$.

Corollary 18. If D is a digraph of order n, then $\gamma_{kS}(D) + d_{kS}(D) \leq n + 1$.

Proof. By Theorem 16,

$$\gamma_{kS}(D) + d_{kS}(D) \leq d_{kS}(D) + \frac{n}{d_{kS}(D)}. \tag{3}$$

Using the fact that the function $g(x) = x + n/x$ is decreasing for $1 \leq x \leq \sqrt{n}$ and increasing for $\sqrt{n} \leq x \leq n$, this inequality leads to the desired bound immediately. □

Corollary 19. Let D be a digraph of order $n \geq 3$. If $2 \leq \gamma_{kS}(D) \leq n - 1$, then

$$\gamma_{kS}(D) + d_{kS}(D) \leq n.$$

Proof. Theorem 16 implies that

$$\gamma_{kS}(D) + d_{kS}(D) \leq \gamma_{kS}(D) + \frac{n}{\gamma_{kS}(D)}. \tag{4}$$

If we define $x = \gamma_{kS}(D)$ and $g(x) = x + n/x$ for $x > 0$, then because $2 \leq \gamma_{kS}(D) \leq n - 1$, we have to determine the maximum of the function g on the interval $I : 2 \leq x \leq n - 1$. It is easy to see that

$$\begin{array}{rcl} \max\limits_{x \in I}\{g(x)\} & = & \max\{g(2), g(n-1)\} \\ & = & \max\{2 + \frac{n}{2}, n - 1 + \frac{n}{n-1}\} \\ & = & n - 1 + \frac{n}{n-1} < n + 1, \end{array}$$

and we obtain $\gamma_{kS}(D) + d_{kS}(D) \leq n$. This completes the proof. □

Corollary 20. Let D be a digraph of order n, and let $k \geq 1$ be an integer. If $\min\{\gamma_{kS}(D), d_{kS}(D)\} \geq a \geq 2$, then

$$\gamma_{kS}(D) + d_{kS}(D) \leq a + \frac{n}{a}.$$

Proof. Since $\min\{\gamma_{kS}(D), d_{kS}(D)\} \geq a \geq 2$, it follows from Theorem 16 that $a \leq d_{ks}(D) \leq \dfrac{n}{a}$. If we define $x = d_{kS}(D)$ and $g(x) = x + n/x$ for $x > 0$, then we deduce from inequality (3) that

$$\begin{array}{rcl} \gamma_{kS}(D) + d_{kS}(D) & \leq & d_{kS}(D) + \dfrac{n}{d_{kS}(D)} \\ & \leq & \max\{g(a), g(n/a)\} = a + \dfrac{n}{a}\ . \end{array}$$

□

3 Signed k-domatic number of graphs

The *signed k-dominating function* of a graph G is defined in [15] as a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $\sum_{x \in N_G[v]} f(x) \geq k$ for all $v \in V(G)$. The sum $\sum_{x \in V(G)} f(x)$ is the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed k-dominating functions f on G is called the *signed k-domination number* of G, denoted by $\gamma_{kS}(G)$. In the special case when $k = 1$, $\gamma_{kS}(G)$ is the signed domination number investigated in [3] and has been studied by several authors (see for example [2, 4]).

A set $\{f_1, f_2, \dots, f_d\}$ of distinct signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a *signed k-dominating family* on G. The maximum number of functions in a signed k-dominating family on G is the *signed k-domatic number* of G, denoted by $d_{kS}(G)$. This parameter was introduced by Favaron, Sheikholeslami and Volkmann in [5]. In the case $k = 1$, we write $d_S(G)$ instead of $d_{1S}(G)$ which was introduced by Volkmann and Zelinka [14] and has been studied in [10, 11, 12] .

The *associated digraph* $D(G)$ of a graph G is the digraph obtained from G when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}(v) = N_G(v)$ for each vertex $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 21. If $D(G)$ is the associated digraph of a graph G, then $\gamma_{kS}(D(G)) = \gamma_{kS}(G)$ and $d_{kS}(D(G)) = d_{kS}(D)$.

There are a lot of interesting applications of Observation 21, as for example the following results. Using Observation 5, we obtain the first one.

Corollary 22. (Volkmann and Zelinka [14]) The signed domatic number $d_S(G)$ of a graph G is an odd integer.

Since $\delta^-(D(G)) = \delta(G)$, the next result follows from Observation 21 and Corollary 8.

Corollary 23. (Favaron, Sheikholeslami and Volkmann [5]) If G is a graph with minimum degree $\delta(G) \geq k - 1$, then

$$d_{kS}(G) \leq \begin{cases} \dfrac{\delta(G)+1}{k+1} & \text{if} \quad \delta(G) \equiv k \pmod 2 \\ \dfrac{\delta(G)+1}{k} & \text{if} \quad \delta(G) \equiv k+1 \pmod 2. \end{cases}$$

The case $k = 1$ in Corollary 23 can be found in [14].

In view of Observation 21 and Corollary 18, we obtain the next result immediately.

Corollary 24. (Favaron, Sheikholeslami and Volkmann [5]) If G is a graph of order n and minimum degree $\delta(G) \geq k - 1$, then

$$\gamma_{kS}(G) + d_{kS}(G) \leq n + 1.$$

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