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Existence of solutions to second-order boundary value problems without growth restrictions

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Abstract. This article investigates nonlinear, second-order ordinary differential equations subject to various two-point boundary conditions. A condition is introduced that ensures *a priori* bounds on the derivatives of solutions to the problem. In particular, quadratic growth conditions on the right-hand side of the differential equation are not employed. The ideas are then applied to ensure the existence of at least one solution. The main tools involve differential inequalities and fixed-point methods.

Keywords: existence of solutions, boundary value problems, fixed-point methods, no growth condition, topological degree.

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1 Introduction

This paper considers the nonlinear, second-order differential equation

$$x'' = f(t, x, x'), \qquad t \in [0, T]; \tag{1.1}$$

coupled with any of the following boundary conditions:

$$a_1 x(0) - a_2 x'(0) = b_1, \qquad a_3 x(T) + a_4 x'(T) = b_2; \tag{1.2}$$

$$x'(0) = 0 = x'(T); \tag{1.3}$$

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{1.4}$$

Above, $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous, nonlinear function; each a_i and b_i are given constants; and $T > 0$ is also a given constant. Equations (1.1), (1.2) are collectively known as a two-point boundary value problem (BVP) with Sturm–Liouville boundary conditions. Similarly, (1.1), (1.3) are known as a Dirichlet BVP, while (1.1), (1.4) are known as a periodic BVP.

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Our motivation for studying the above problems naturally arises in the following ways. Consider the following nonlinear partial differential equations

$$\Delta u = g(t, x, u_t, u_x), \qquad (t, x) \in D, \qquad \text{(nonlinear Laplace equation)}; \tag{1.5}$$

$$u_{xx} - u_t = g(t, x, u_t, u_x), \qquad (t, x) \in D, \qquad \text{(nonlinear heat equation)}; \tag{1.6}$$

where each equation is subjected to appropriate boundary conditions. The applications of (1.5), (1.6) are well known. If stationary solutions $u(x,t) = u(x)$ to the above equations are sought then both (1.5) and (1.6) become the nonlinear, second-order ordinary differential equation

$$u'' = g(x, u, u'), \qquad x \in [0, T].$$

As a particular example to illustrate the above point, consider (1.1), (1.2) with: $a_1 = 0 = b_1 = b_2$; $a_2 = 1$, a_3 arbitrary, $a_4 = 1$ and $T = 1$. These then are models for a thermostat. Solutions of these ODEs are stationary solutions for a (nonlinear) one-dimensional heat equation, corresponding to a heated bar, with a controller at $t = 1$ adding or removing heat dependent on the temperature detected by a sensor at $t = 1$. The particular boundary conditions (1.2) correspond to the end of the bar at $t = 0$ being insulated, see [26, pp. 672–3].

A further example of steady-state temperature distribution in rods that naturally involves (1.1), (1.2) with each $a_i > 0$ can be found in [10, p. 79].

Recently, [23] presented a firm mathematical foundation for the boundary value problem associated with the nonrelativistic Thomas–Fermi equation for heavy atoms in intense magnetic fields. The analysis involved the BVP

$$x'' = \sqrt{tx}, \qquad x(0) = 1, \qquad x(t_c) = 0.$$

Another area of motivation for studying (1.1) is the appearance of ordinary differential equations in their own right, as opposed to simplifications of PDE. The reader is referred to [4, Chapter 1], [2, Chapter 1] for some nice examples, including applications of boundary value problems involving ordinary differential equations to physics, engineering and science.

These types of above applications naturally motivate a deeper theoretical study of the subject of BVPs involving ordinary differential equations.

In this work, the interest is in the existence of solutions to the BVP (1.1) subject to (1.2), (1.3) or (1.4). In particular, we are concerned with those $f(t, p, q)$ that do not satisfy the standard growth conditions in q.

The tools used in this paper involve new differential inequalities, topological degree and related fixed-point methods of integral operators. One of the useful building blocks for using topological degree theory on operator equations, is the obtention of *a priori* bounds on solutions to a certain family of equations that are related to the equation that is under consideration. In the field of second-order, nonlinear BVPs this is equivalent to obtaining conditions under which certain *a priori* bounds are guaranteed on solutions x and its first derivative x'.

The classical method of upper and lower solutions has been used to bound solutions x *a priori* where the ideas involve certain differential inequalities on the right-hand side of the differential equation and a simple maximum principle [7].

To bound x' *a priori*, authors have used a variety of conditions, including: the celebrated Bernstein–Nagumo quadratic growth conditions [5, 18, 19]; guiding functions [1, 15, 20]; and barrier strips [14], see also [9, 16] and references therein.

In this paper an alternate method for bounding x' is introduced. The ideas do not follow any of the above approaches, in particular, no growth conditions of $|f(t, p, q)|$ in $|q|$ are used.

The new results compliment and extend previous works in the literature and an example is provided to which the new results apply but the classical growth conditions do not.

The article is organised as follows.

Section 2 contains the new *a priori* bound results for first derivatives of solutions to (1.1) subject to (1.2), (1.3) or (1.4). The conditions of the theorems feature simple, wide-ranging differential inequalities that are easily verifiable in practice.

In Section 3 the ideas of Section 2 are applied, in conjunction with topological degree and fixed-point theory, to gain new theorems that ensure the existence of solutions to (1.1) subject to (1.2), (1.3) or (1.4).

Section 4 contains an example that demonstrates how to apply the new results and to clearly demonstrate the advancement made over current literature. In particular, an example is constructed so that the classical Bernstein–Nagumo growth conditions do not apply.

A solution to (1.1) is a continuously twice-differentiable function $x : [0, T] \to \mathbb{R}$, i.e., $x \in C^2([0, T])$, that satisfies (1.1) for each $t \in [0, T]$.

For more on existence of solutions to BVPs, including modern and classical approaches, see [1–16] and [18–25].

2 *A priori* bounds

In this section, some new *a priori* bound results are presented involving the first derivative of solutions to (1.1) subject to (1.2), (1.3) or (1.4). The main significance of the new theorems lies in the observation that the conditions do not involve quadratic type growth conditions on $|f(t, p, q)|$ in $|q|$.

Lemma 2.1. *Let $x \in C^2([0, T])$ and let α, K and N_1 be non-negative constants. If*

$$|x''(t)| \leq \alpha x''(t) + K, \quad \text{for all } t \in [0, T]; \tag{2.1}$$

$$\max\{|x'(0)|, |x'(T)|\} \leq N_1; \tag{2.2}$$

then there exists a non-negative constant N (depending on α, K, N_1 and T) such that $|x'(t)| \leq N$ for all $t \in [0, T]$.

Proof. If $\alpha = 0$ then $-K \leq x''(t) \leq K$ for all $t \in [0, T]$ and an integration on $[0, t]$ leads to $|x'(t)| \leq KT + N_1$, for all $t \in [0, T]$.

If $\alpha > 0$ then we have $0 \leq \alpha x''(t) + K$ for all $t \in [0, T]$ with an integration over $[0, t]$ and $[t, T]$ giving, respectively

$$x'(t) \geq Kt/\alpha - x'(0) \geq -N_1, \qquad t \in [0, T],$$
$$x'(t) \leq x'(T) + K(T - t)/\alpha \leq N_1 + KT/\alpha, \qquad t \in [0, T].$$

Thus the desired bound on x' follows by defining N as

$$N := \begin{cases} KT + N_1, & \text{for } \alpha = 0, \\ KT/\alpha + N_1, & \text{for } \alpha > 0. \end{cases}$$

□

In a similar fashion to Lemma 2.1 we have the following result.

Lemma 2.2. *Let the conditions of Lemma 2.1 hold with (2.1) replaced by:*

$$|x''(t)| \leq -\alpha x''(t) + K, \qquad t \in [0,T]. \tag{2.3}$$

Then the conclusion of Lemma 2.1 holds.

The previous two lemmas are now applied to the Sturm–Liouville BVP (1.1), (1.2).

Theorem 2.3. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous, let R be a non-negative constant and assume each $a_i > 0$. If there exist non-negative constants α and K such that*

$$|f(t,p,q)| \leq \alpha f(t,p,q) + K, \quad \text{for all } t \in [0,T],\ |p| \leq R,\ q \in \mathbb{R}; \tag{2.4}$$

then all possible solutions to (1.1), (1.2) that satisfy $|x(t)| \leq R$ for all $t \in [0,T]$, must also satisfy $|x'(t)| \leq N$ for all $t \in [0,T]$, where N is a constant involving: α, K, T, R, each a_i and each $|b_i|$.

Proof. Let $x \in C^2([0,T])$ be a solution to the BVP (1.1), (1.2) that satisfies $|x(t)| \leq R$ for all $t \in [0,T]$. If (2.4) holds then (2.1) holds for all solutions to (1.1), (1.2). In addition, since $|x(t)| \leq R$ for all $t \in [0,T]$ a rearrangement of the boundary conditions (1.2) then gives

$$\begin{aligned}\max\{|x'(0)|, |x'(T)|\} &\leq \max\left\{\frac{|b_2| + a_3 R}{a_4}, \frac{|b_1| + a_1 R}{a_2}\right\} \\ &:= N_1,\end{aligned}$$

so that (2.2) holds. Hence the desired *a priori* bound on x' follows from Lemma 2.1. □

The following *a priori* bound result may be obtained for a different class of f than those dealt with in Theorem 2.3.

Theorem 2.4. *Let the conditions of Theorem 2.3 hold with (2.4) replaced by*

$$|f(t,p,q)| \leq -\alpha f(t,p,q) + K, \quad \text{for all } t \in [0,T],\ |p| \leq R,\ q \in \mathbb{R}; \tag{2.5}$$

then the conclusion of Theorem 2.3 holds.

Proof. The proof is similar to that of Theorem 2.3 and so is omitted. □

Example 2.5. Comparing Theorems 2.3 and 2.4, we see that

$$f_1(t,p,q) := -q^4 - p^2 + t + 3, \qquad t \in [0,1];$$

satisfies (2.5) for $\alpha = 1$, $K = 8$, $R = 1$; but f_1 cannot satisfy (2.4) for any choice of non-negative α, K and positive R. Conversely, see that

$$f_2(t,p,q) := p^4 + e^q - 1 - t^2, \qquad t \in [0,1];$$

satisfies (2.4) for $\alpha = 1$, $K = 5$, $R = 1$; but f_2 cannot satisfy (2.5) for any choice of non-negative α, K and positive R.

Our attention now turns to *a priori* bounds on derivatives of solutions to the Neumann BVP (1.1), (1.3).

Theorem 2.6. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and let R be a non-negative constant. If there exist non-negative constants α and K such that (2.4) holds then all possible solutions to (1.1), (1.3) that satisfy $|x(t)| \leq R$ for all $t \in [0,T]$, also satisfy $|x'(t)| \leq N$ for all $t \in [0,T]$, where N is a constant involving: α, K and T.*

Proof. Now (2.4) implies (2.1); and (1.3) implies (2.2) for $N_1 = 0$. Thus the result follows from Lemma 2.1. □

Similarly to Theorem 2.6, the following result may be obtained.

Theorem 2.7. *If the conditions of Theorem 2.6 hold with "(2.4)" replaced with "(2.5)" then the conclusion of Theorem 2.6 holds.*

Our focus is now on *a priori* bounds on derivatives of solutions to the periodic BVP (1.1), (1.4). It is difficult to immediately verify that (1.4) implies (2.2) for this case so we adopt an alternative approach.

Consider the following BVP that is equivalent to (1.1), (1.4):

$$x'' - x = f(t, x, x') - x, \qquad t \in [0, T], \tag{2.6}$$

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{2.7}$$

We can equivalently rewrite the BVP (2.6), (2.7) in the following integral form

$$x(t) = \int_0^T G(t,s)[f(s, x(s), x'(s)) - x(s)]\, ds, \qquad t \in [0, T]. \tag{2.8}$$

Above, $G : [0, T] \times [0, T] \to \mathbb{R}$ is the unique, continuously differentiable Green's function for the following BVP

$$x'' - x = 0, \qquad t \in [0, T],$$

$$x(0) = x(T), \qquad x'(0) = x'(T).$$

Let $G_1 := \max_{(t,s)\in[0,T]\times[0,T]} \left|\frac{\partial G}{\partial t}(t,s)\right|$. We will need this constant in the theorems that follow.

Theorem 2.8. *Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and let R be a non-negative constant. If there exist non-negative constants α and K such that (2.4) holds then all possible solutions to (1.1), (1.4) that satisfy $|x(t)| \le R$ for all $t \in [0, T]$, also satisfy $|x'(t)| \le N$ for all $t \in [0, T]$, where N is a constant involving: K, T and R.*

Proof. Let x be a solution to the BVP (1.1), (1.4) that satisfies $|x(t)| \le R$ for all $t \in [0, T]$. Note that x also satisfies the BVP (2.6), (2.7) and the integral equation (2.8). If we differentiate both sides of (2.8) then x' must satisfy, for each $t \in [0, T]$ and $|x(t)| \le R$ we have

$$\begin{aligned}
|x'(t)| &\le G_1 \int_0^T |f(s, x(s), x'(s)) - x(s)|\, ds \\
&\le G_1 \int_0^T \alpha f(s, x(s), x'(s)) + K + R\, ds, \quad \text{from (2.4)} \\
&= G_1 \int_0^T \alpha \frac{d^2}{ds^2}[x(s)] + K + R\, ds \\
&\le G_1 \left[\alpha(x'(T) - x'(0)) + (K + R)T\right] \\
&= G_1(K + R)T, \quad \text{from (1.4)} \\
&:= N.
\end{aligned}$$

Hence the desired *a priori* bound on x' follows. □

Similarly, the following result may be obtained.

Theorem 2.9. *Let the conditions of Theorem 2.8 hold with "(2.4)" replaced with "(2.5)". Then the conclusion of Theorem 2.8 holds.*

Some important corollaries to the theorems in this section now follow, where we assume, respectively, that $f(t,p,q)$ is bounded below or bounded above for all $t \in [0,T]$, $|p| \leq R$, $q \in \mathbb{R}$.

Corollary 2.10. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and R be a non-negative constant. Let the conditions of Theorems 2.3, 2.4 or 2.6 hold with "(2.4)" replaced by "$f(t,p,q)$ is bounded below for all $t \in [0,T]$, $|p| \leq R$, $q \in \mathbb{R}$". Then the respective conclusions of Theorems 2.3, 2.4 and 2.6 all hold.*

Proof. We want to show that there exists non-negative constants α and K such that (2.4) holds. If $f(t,p,q)$ is bounded below for all $t \in [0,T]$, $|p| \leq R$, $q \in \mathbb{R}$ then there exists a constant C such that

$$C \leq f(t,p,q), \qquad t \in [0,T],\ |p| \leq R,\ q \in \mathbb{R}.$$

If $C \geq 0$ then $|f(t,p,q)| = f(t,p,q)$ for all $t \in [0,T]$, $|p| \leq R$, $q \in \mathbb{R}$ and so (2.4) holds with $\alpha = 1$ and $K = 0$. If $C < 0$ then $0 \leq f(t,p,q) - C$ for all $t \in [0,T]$, $|p| \leq R, q \in \mathbb{R}$. Hence

$$\begin{aligned} |f(t,p,q)| - (-C) &\leq |f(t,p,q) + (-C)| \\ &= f(t,p,q) - C \end{aligned}$$

and a rearrangement gives

$$|f(t,p,q)| \leq f(t,p,q) + 2(-C), \quad \text{for all } t \in [0,T],\ |p| \leq R,\ q \in \mathbb{R}.$$

Thus (2.4) holds with $\alpha = 1$ and $K = -2C$. □

Similarly, the following result can be obtained.

Corollary 2.11. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and R be a non-negative constant. Let the conditions of Theorems 2.3, 2.4 or 2.6 hold with "(2.5)" replaced by "$f(t,p,q)$ is bounded above for all $t \in [0,T]$, $|p| \leq R$, $q \in \mathbb{R}$". Then the respective conclusions of Theorems 2.3, 2.4 and 2.6 all hold.*

Proof. The proof is similar to that of Corollary 2.10 and thus is omitted. □

The following lemmas will be a useful tool in gaining *a priori* bounds on solutions x to our BVPs. In particular, they will be needed in our existence proofs in Section 3. The proofs of the following results are well known and use standard maximum principle techinques from the theory of lower and upper solutions [7,9]. Thus the proofs are omitted for brevity.

Lemma 2.12. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and let R_1 and R_2 be positive constants. If*

$$f(t,R_2,0) > 0, \quad \text{for all } t \in [0,T], \tag{2.9}$$
$$f(t,-R_1,0) < 0, \quad \text{for all } t \in [0,T], \tag{2.10}$$
$$\min\{R_1,R_2\} > \max\{|b_1|/a_1, |b_2|/a_2\}; \tag{2.11}$$

then all solutions x to (1.1), (1.2) that satisfy the a priori *bound $-R_1 \leq x(t) \leq R_2$ for all $t \in [0,T]$ also must satisfy $-R_1 < x(t) < R_2$ for all $t \in [0,T]$.*

Lemma 2.13. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and let R_1 and R_2 be positive constants. If (2.9) and (2.10) hold then all solutions x to (1.1), (1.3) that satisfy the* a priori *bound $-R_1 \leq x(t) \leq R_2$ for all $t \in [0,T]$ also must satisfy $-R_1 < x(t) < R_2$ for all $t \in [0,T]$.*

Lemma 2.14. *Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and let R_1 and R_2 be positive constants. If (2.9) and (2.10) hold then all solutions x to (1.1), (1.4) that satisfy the* a priori *bound $-R_1 \leq x(t) \leq R_2$ for all $t \in [0, T]$ also must satisfy $-R_1 < x(t) < R_2$ for all $t \in [0, T]$.*

For convenience, it has been assumed that each $a_i > 0$ in (1.2). This can naturally be relaxed to $a_1^2 + a_2^2 > 0$ and $a_3^2 + a_4^2 > 0$ with one of a_1 or a_3 being zero. In particular, this allows the treatment of BVPs involving, respectively, Nicoletti and Corduneanu boundary conditions

$$x'(0) = 0, \qquad a_3 x(T) + a_4 x'(T) = b_2; \tag{2.12}$$

$$a_1 x(0) - a_2 x'(0) = b_1, \qquad x'(T) = 0; \tag{2.13}$$

by suitably modifying the conditions of the theorems in this section and their associated proofs.

3 Solvability

In this section the results of Section 2 are applied, in conjunction with topological degree and fixed-point theory, to obtain some new existence theorems for solutions to (1.1) subject to (1.2), (1.3) or (1.4).

Theorem 3.1. *Let $f : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let R_1 and R_2 be positive constants such that (2.9), (2.10) and (2.11) hold. Let α and K be non-negative constants such that (2.4) holds for $R := \max\{R_1, R_2\}$. Then (1.1), (1.2) has at least one solution.*

Proof. Since each $a_i > 0$ and since f is continuous, we can equivalently rewrite the BVP (1.1), (1.2) in the following integral form

$$x(t) = \int_0^T G(t, s) f(s, x(s), x'(s))\, ds + \phi(t), \qquad t \in [0, T]. \tag{3.1}$$

Above, $G : [0, T] \times [0, T] \to \mathbb{R}$ is the unique, continuously differentiable Green's function for the following BVP

$$\begin{aligned} x'' = 0, \qquad t \in [0, T], \\ a_1 x(0) - a_2 x'(0) = 0, \\ a_3 x(T) + a_4 x'(T) = 0; \end{aligned}$$

and $\phi : [0, T] \to \mathbb{R}$ is the unique, continuously differentiable solution to the BVP

$$\begin{aligned} x'' = 0, \qquad t \in [0, T], \\ a_1 x(0) - a_2 x'(0) = b_1, \\ a_3 x(T) + a_4 x'(T) = b_2. \end{aligned}$$

In view of (3.1) and its context, we define $H : C^1([0, T]) \to C([0, T])$ by

$$(Hx)(t) := \int_0^T G(t, s) f(s, x(s), x'(s))\, ds + \phi(t), \qquad t \in [0, T]. \tag{3.2}$$

It is well known that H is a compact map. If we can show that H has at least one fixed-point, that is, $Hx = x$ for at least one x, then (3.1) will have at least one solution and so will the BVP (1.1), (1.2).

With this in mind, consider the family of equations

$$u = \lambda H u, \qquad \lambda \in [0,1]. \tag{3.3}$$

Define the set $\Omega \subset C^1([0,T])$ by

$$\Omega := \{x \in C^1([0,T]) : -R_1 < x(t) < R_2,\ |x'(t)| < N+1,\ t \in [0,T]\}$$

where N is defined in Lemma 2.1 (with N_1 defined in the proof of Theorem 2.3).

Let x be a solution to (3.3) and see that x is also a solution to the family of BVPs

$$x'' = \lambda f(t,x,x'), \qquad t \in [0,T], \tag{3.4}$$
$$a_1 x(0) - a_2 x'(0) = \lambda b_1, \tag{3.5}$$
$$a_3 x(T) + a_4 x'(T) = \lambda b_2; \tag{3.6}$$

where $\lambda \in [0,1]$.

We will show that all possible solutions to (3.3) that satisfy $x \in \overline{\Omega}$ also satisfy $x \notin \partial\Omega$. This is equivalent to showing that these solutions to the family of BVPs (3.4)–(3.6) satisfy a particular *a priori* bound, with the bound being independent of λ.

If $\lambda = 0$ then we have the zero solution to (3.4)–(3.6), so assume $\lambda \in (0,1]$ from now on.

If (2.9)–(2.11) and (2.4) hold then for $\lambda \in (0,1]$ it is not difficult to show that the family of BVPs (3.4)–(3.6) satisfy the conditions of Lemma 2.12. Thus, all solutions to (3.4)–(3.6) that satisfy $-R_1 \le x(t) \le R_2$ for all $t \in [0,T]$ must also satisfy $-R_1 < x(t) < R_2$ for all $t \in [0,T]$. In addition, (3.4)–(3.6) satisfies the conditions of Theorem 2.3 with

$$\max\{|x'(0)|, |x'(T)|\} \le \max\left\{\frac{|\lambda b_2| + a_3 R}{a_4} + \frac{|\lambda b_1| + a_1 R}{a_2}\right\} \le N_1.$$

Hence $|x'(t)| \le N$ for all $t \in [0,T]$.

Above, note that R_1, R_2 and N are all independent of λ.

Thus, if I denotes the identity, the following Leray–Schauder degrees are defined and a homotopy principle applies [17, Chap.4]

$$d(I - \lambda H, \Omega, 0) = d(I - H, \Omega, 0) = d(I, \Omega, 0) = 1.$$

By the non-zero property of Leray–Schauder degree, H has at least one fixed point in Ω and thus the BVP (1.1), (1.2) has at least one solution. □

Theorem 3.2. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let R_1 and R_2 be positive constants such that (2.9) and (2.10) hold. Let α and K be non-negative constants such that (2.4) holds for $R := \max\{R_1, R_2\}$. Then (1.1), (1.3) has at least one solution.*

Proof. Consider the following BVP that is equivalent to (1.1), (1.3):

$$x'' - x = f(t,x,x') - x, \qquad t \in [0,T], \tag{3.7}$$
$$x'(0) = x'(T). \tag{3.8}$$

Since f is continuous, we can equivalently rewrite the BVP (3.7), (3.8) in the following integral form

$$x(t) = \int_0^T G(t,s)[f(s,x(s),x'(s)) - x(s)]\, ds, \qquad t \in [0,T]. \tag{3.9}$$

Above, $G : [0,T] \times [0,T] \to \mathbb{R}$ is the unique, continuously differentiable Green's function for the following BVP

$$x'' = 0, \qquad t \in [0,T],$$
$$x'(0) = 0 = x'(T).$$

In view of (3.9) and its context, we define $H : C^1([0,T]) \to C([0,T])$ by

$$(Hx)(t) := \int_0^T G(t,s)[f(s,x(s),x'(s)) - x(s)]\, ds, \qquad t \in [0,T]. \tag{3.10}$$

It is well known that H is a compact map. If we can show that H has at least one fixed-point, that is, $Hx = x$ for at least one x, then (3.9) will have at least one solution and so will the BVP (1.1), (1.3).

With this in mind, consider the family of equations

$$u = \lambda Hu, \qquad \lambda \in [0,1]. \tag{3.11}$$

Let x be a solution to (3.11) and observe that x is also a solution to the family of BVPs

$$x'' = \lambda f(t,x,x') + (1-\lambda)x := g_\lambda(t,x,x'), \qquad t \in [0,T], \tag{3.12}$$
$$x'(0) = x'(T); \tag{3.13}$$

where $\lambda \in [0,1]$.

We show that these solutions to the family of BVPs (3.12), (3.13) satisfy a particular *a priori* bound, with the bound being independent of λ.

If $\lambda = 0$ then we have the zero solution to (3.12), (3.13), so assume $\lambda \in (0,1]$ from now on.

If (2.9), (2.10) and (2.4) hold then for $\lambda \in (0,1]$ it is not difficult to show that the family of BVPs (3.12), (3.13) satisfy the conditions of Lemma 2.13. Thus, all solutions to (3.12), (3.13) that satisfy $-R_1 \le x(t) \le R_2$ for all $t \in [0,T]$ must also satisfy $-R_1 < x(t) < R_2$ for all $t \in [0,T]$. Note that R_1 and R_2 are independent of λ.

Now, if (2.4) holds, then for all $\lambda \in [0,1]$, $t \in [0,T]$ and $|p| \le R$ we have

$$\begin{aligned}|\lambda f(t,p,q)| &\le \alpha\lambda f(t,p,q) + K\\ &\le \alpha\lambda f(t,p,q) - 2(1-\lambda)R + 2R + K,\end{aligned}$$

and so

$$|\lambda f(t,p,q)| + (1-\lambda)R \le \alpha\lambda f(t,p,q) - (1-\lambda)R + K + 2R$$

and hence

$$\begin{aligned}|\lambda f(t,p,q) + (1-\lambda)p| &\le |\lambda f(t,p,q)| + (1-\lambda)R\\ &\le \alpha\lambda f(t,p,q) - (1-\lambda)R + K + 2R\\ &\le \alpha\lambda f(t,p,q) + (1-\lambda)p + K + 2R.\end{aligned}$$

Therefore it follows

$$|g_\lambda(t,p,q)| \le \alpha g_\lambda(t,p,q) + K + 2R$$

for all $t \in [0,T]$, $|p| \le R$, $q \in \mathbb{R}$ and all $\lambda \in [0,1]$ so that Theorem 2.6 applies to the family (3.12), (3.13) with $N_1 = 0$, N being independent of λ and specifically given by

$$N := \begin{cases} (K+2R)T, & \text{for } \alpha = 0,\\ (K+2R)T/\alpha, & \text{for } \alpha > 0.\end{cases}$$

Define the set $\Omega \subset C^1([0,T])$ by

$$\Omega := \{x \in C^1([0,T]) : -R_1 < x(t) < R_2,\ |x'(t)| < N+1,\ t \in [0,T]\}$$

with N defined above.

We have shown that all possible solutions to (3.11) that satisfy $x \in \overline{\Omega}$ also satisfy $x \notin \partial\Omega$.

Thus, if I denotes the identity, then the following Leray–Schauder degrees are defined and a homotopy principle applies [17, Chapter 4]

$$d(I - \lambda H, \Omega, 0) = d(I - H, \Omega, 0) = d(I, \Omega, 0) = 1.$$

By the non-zero property of Leray–Schauder degree, H has at least one fixed point in Ω and thus the BVP (1.1), (1.3) has at least one solution. □

Theorem 3.3. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let R_1 and R_2 be positive constants such that* (2.9) *and* (2.10) *hold. Let α and K be non-negative constants such that* (2.4) *holds for $R := \max\{R_1, R_2\}$. Then* (1.1), (1.4) *has at least one solution.*

Proof. The proof is similar to that of Theorem 3.2 and so is only sketched. Consider the following BVP that is equivalent to (1.1), (1.4):

$$x'' - x = f(t, x, x') - x, \qquad t \in [0,T], \tag{3.14}$$

$$x(0) = x(T), \qquad x'(0) = x'(T). \tag{3.15}$$

Since f is continuous, we can equivalently rewrite the BVP (3.14), (3.15) in the following integral form

$$x(t) = \int_0^T G(t,s)[f(s, x(s), x'(s)) - x(s)]\, ds, \qquad t \in [0,T]. \tag{3.16}$$

Above, $G : [0,T] \times [0,T] \to \mathbb{R}$ is the unique, continuously differentiable Green's function for the following BVP

$$x'' = 0, \qquad t \in [0,T],$$

$$x(0) = x(T), \qquad x'(0) = x'(T).$$

In view of (3.16) and its context, we define $H : C^1([0,T]) \to C([0,T])$ by

$$(Hx)\,(t) := \int_0^T G(t,s)[f(s, x(s), x'(s)) - x(s)]\, ds, \qquad t \in [0,T]. \tag{3.17}$$

It is well known that H is a compact map. If we can show that H has at least one fixed-point, that is, $Hx = x$ for at least one x, then (3.16) will have at least one solution and so will the BVP (1.1), (1.4).

Apply Theorem 2.8 and Lemma 2.14 to the family of BVPs

$$x'' - x = \lambda[f(t, x, x') - x], \qquad t \in [0,T], \tag{3.18}$$

$$x(0) = x(T), \qquad x'(0) = x'(T); \tag{3.19}$$

where $\lambda \in [0,1]$ and then use degree theory in the same way as proofs of our previous results. □

An simple but useful corollary to Theorems 3.1, 3.2, and 3.3 now follows. The proof follows that of the respective theorems by respectively applying Corollary 2.10 to an appropriate family of BVPs. The proof is omitted for brevity.

Corollary 3.4. *Let the conditions of Theorems 3.1, 3.2, and 3.3 hold but with "(2.4)" replaced by "$f(t,p,q)$ be bounded below for all $t \in [0,T]$, $|p| \leq R$, $q \in \mathbb{R}$". Then the respective conclusions of Theorems 3.1, 3.2, and 3.3 hold.*

A more abstract generalization of Theorems 3.1 now follows in which we replace the assumptions involving upper and lower solutions with the *a priori* knowledge of a certain bound on solutions x to appropriate families of BVPs.

Theorem 3.5. *Let $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let R_3 and R_4 be positive constants. Let α and K be non-negative constants such that (2.4) holds for $R := \max\{R_3, R_4\}$. Suppose that all possible solutions to the family (3.4)–(3.6) that satisfy $-R_3 \leq x(t) \leq R_4$ for all $t \in [0,T]$ also satisfy $-R_3 < x(t) < R_4$ for all $t \in [0,T]$ (where the bounds are independent of λ). Then (1.1), (1.2) has at least one solution.*

Proof. The proof follows a similar line of argument to that of Theorem 3.1 by showing that Theorem 2.3 applies to the family (3.4)–(3.6). This, combined with the assumed *a priori* bound on x, gives us the sufficient knowledge to apply degree theory to obtain the existence of at least one solution, as in the proof of Theorem 3.1. □

Theorem 3.6. *Let the conditions of Theorem 3.5 hold with "(2.4)" replaced by "(2.5)". Then (1.1), (1.2) has at least one solution.*

Proof. The proof proceeds in a similar fashion to that of Theorem 3.5 by showing Theorem 2.4 applies to the family (3.4), (3.6) and so is omitted. □

Similarly, the next two abstract results follow as natural corollaries to Theorem 3.5.

Corollary 3.7. *Let the conditions of Theorem 3.5 hold with "(2.4)" replaced by "bounded below". Then (1.1), (1.2) has at least one solution.*

Corollary 3.8. *Let the conditions of Theorem 3.5 hold with "(2.4)" replaced by "bounded above". Then (1.1), (1.2) has at least one solution.*

It is clear that the above results for the Sturm–Liouville BVP may be also modified to treat the Neumann and periodic BVPs, but for brevity the statements of these results are omitted. In addition, it is possible to gain existence of solutions for the Nicoletti BVP (1.1), (2.12) and the Corduneanu BVP (1.1), (2.13) under suitably modified assumptions connected with the ideas of this section and Section 2.

4 An example

In this final section, a simple example is considered to which our new theorems are applicable. In particular, the classical Bernstein–Nagumo theory is inapplicable to what follows.

Example 4.1. Consider (1.1), (1.2) with f being given by

$$f(t,p,q) := p^2 e^q - 1 - t^2, \qquad t \in [0,1].$$

For the above f, the conditions Theorem 3.1 hold.

Proof. We see that

$$|f(t,p,q)| \leq p^2 e^q + 2, \quad \text{for all } (t,p,q) \in [0,1] \times \mathbb{R}^2;$$

and, for α and K to be chosen below

$$\begin{aligned} \alpha f(t,p,q) + K &= \alpha(p^2 e^q - 1 - t^2) + K \\ &\geq \alpha(p^2 e^q - 2) + K \\ &= p^2 e^q + 2 \quad \text{for } \alpha = 1,\ K = 4. \end{aligned}$$

Thus (2.4) holds for any choice of $R > 0$. We will choose $R = 3$.

Note that (2.9) and (2.10) hold for the choices $R_1 = 1/2$ and $R_2 = 3$. Thus, for suitable a_i and b_i, the BVP (1.1), (1.2) with the above f admits at least one solution from Theorem 3.1.

For the above f, we cannot choose a function $h : [0,\infty) \to (0,\infty)$ such that

$$|f(t,p,q)| \leq h(|q|), \quad \text{for all } t \in [0,1],\ |p| \leq R,\ q \in \mathbb{R};$$

with

$$\int^{\infty} \frac{s}{h(s)}\, ds = +\infty.$$

Thus the classical Bernstein–Nagumo quadratic growth condition does not apply to the above example. □

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