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Pseudo-Riemannian Spectral Triples for the Standard Model

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Abstract. We present the importance of the pseudo-Riemannian structure in the spectral triple formalism that is used to describe the Standard Model of Particle Physics. The finite case is briefly described and its role in the context of leptoquarks is presented. The proposal for the reverse engineering program for the Standard Model is also described, together with recent results.

1 Introduction

The approach to the Standard Model of Particle Physics based on the Connes' idea of the Noncommutative Geometry (NCG) allows for the geometrical analysis of the structure of the Standard Model (SM), reveals the origin of the Higgs mechanism and, using spectral action methods and renormalization group techniques, produces *numbers* that can be compared with experimental data. The most famous example of such a procedure is related to the calculation of the mass of the Higgs boson. For the detailed discussion of that problem see [3],[9] and [10].

The main idea of NCG is based on the Connes' reconstruction theorem [8] from which it is known that the whole metric and spin structure of a closed, orientable Riemannian spin^c manifold M can be encoded in a system consisted of a commutative $*$-algebra $C^\infty(M)$ of smooth complex-valued functions on M, a Hilbert space $\mathcal{H}$ of square-integrable spinors and the Dirac operator $\mathcal{D}_M$ that acts on sections of the spinor bundle over M. In the even dimensional case in the associated Clifford algebra there exists an element γ^5 that is represented on $\mathcal{H}$ as a $\mathbb{Z}/2\mathbb{Z}$-grading and there is also a charge conjugation operator. There are several relations satisfied by these objects. Therefore, the straightforward generalization of the usual geometry is based on the imitation of that system, but the algebra $\mathcal{A}$ is not necessary commutative. This is the concept of a spectral triple - a system $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J)$ with a $*$-algebra $\mathcal{A}$ represented in a faithful way on a Hilbert space $\mathcal{H}$, $\mathbb{Z}/2\mathbb{Z}$-grading $\gamma^\dagger = \gamma$ commuting with $\mathcal{A}$, antilinear isometry J such that $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$ and (essentially) self-adjoint operator $\mathcal{D}$, called Dirac operator, with compact resolvent. They are supposed to satisfy few compatibility conditions [11],[12],[16], for example $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon'\mathrm{id}$ and $J\gamma = \epsilon\gamma J$, where the choice of signs $\epsilon, \epsilon', \epsilon'' = \pm 1$ defines the so-called KO-dimension that is an integer modulo 8.

This idea was used to describe the Euclidean version of the SM [6] and is based on the almost-commutative spectral triple, that is with an algebra of the form $C^\infty(M) \otimes \mathcal{A}_F$ for M

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being a background manifold and $\mathcal{A}_F$ a finite dimensional algebra of the form

$$\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \tag{1}$$

where $\mathbb{H}$ states for quaternions. Similarly, the Hilbert space is of the tensor product structure with the finite part

$$\mathcal{H}_F = \left(\mathcal{H}_l \oplus \mathcal{H}_q\right) \oplus \left(\mathcal{H}_{\bar{l}} \oplus \mathcal{H}_{\bar{q}}\right), \tag{2}$$

where the leptonic part $\mathcal{H}_l$ has a basis that is taken to be $\{\nu_R, e_R, (\nu_L, e_L)\}$, and for each colour in the quark sector $\mathcal{H}_q$ the basis is ordered as $\{u_R, d_R, (u_L, d_L)\}$. Note that $\dim \mathcal{H}_F = 96$, therefore the finite Dirac operator $\mathcal{D}_F$ is a 96×96-matrix.

The finite grading γ_F is just the chirality operator and the real structure J_F acts by exchanging particles with antiparticles composed with a complex conjugation. The standard choice [21] for $\mathcal{D}_F$ is of the form

$$\mathcal{D}_F = \begin{pmatrix} S & T^\dagger \\ T & \bar{S} \end{pmatrix}, \tag{3}$$

with $T\nu_R = Y_R \bar{\nu}_R$, where $Y_R \in M_N(\mathbb{C})$ with N being number of generations, and T is zero on other fermions. S is expressed in terms of Yukawa matrices.

The representation of $\mathcal{A}_F$ on $\mathcal{H}_F$ is defined on each summand separately. For $\mathcal{H}_l$ and for each color of $\mathcal{H}_q$ it is given by $\pi(\lambda, h, m) = \lambda \oplus \bar{\lambda} \oplus m$. On the antileptonic sector $\pi(\lambda, h, m) = \bar{\lambda}$ and for $\mathcal{H}_{\bar{q}}$ it has a form $1_4 \otimes m$.

Using the spectral action method, heat kernel expansion and renormalization group techniques one can reproduce [6] the effective action for the SM, in particular with the proper shape of Higgs potential. Moreover, we can express bosonic parameters by fermionic one and get numerical results that are experimentally testable predictions.

Note that the choice of $\mathcal{D}_F$ is not unique. There are other operators that satisfy all axioms for a spectral triple, therefore in principle one can construct other theories that for example allow for leptoquark fields [17]. There were several approaches based on K-theoretic arguments [7],[15] or the introduction of additional conditions for a spectral triple like second order condition [11] or Hodge duality [13] that would protect from such situation. Unfortunately, they were too sophisticated or not enough for that purpose.

In [5] we proposed a new point of view on the lepton-quark symmetry based on the existence of an additional $\mathbb{Z}/2\mathbb{Z}$-grading that distinguishes between these sectors and is a shadow of a pseudo-Riemannian structure on the finite spectral triple for the SM. This approach is briefly summarized in the next section. Formulation of these finite pseudo-Riemannian spectral triples was also a first step in the project of the reverse engineering for the SM that we discuss in section 3. The goal is to characterize possible pseudo-Riemannian spectral triples or their modifications that can describe the SM together with all its hidden structures or symmetries.

2 Finite Pseudo-Riemannian Spectral Triples and Leptoquarks

There are a lot of different approaches to the incorporation of pseudo-Riemannian structures in the NCG. We have to mention the ground-breaking papers [19] and [1]. Also recently, there appear many interesting results, especially in [2] and [20]. We highly recommend the overview [14] of this topic and references therein.

Motivated by properties of the Clifford algebra associated to the indefinite metric of signature (p, q) we proposed in [5] a notion of the finite (real & even) pseudo-Riemannian spectral triple of signature (p, q). We introduced such a spectral triple as a system $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \gamma, J, \beta)$

with $\mathcal{A}$ being a $*$-algebra represented on a Hilbert space $\mathcal{H}$. For even $p+q$ $\gamma = \gamma^\dagger$ is a $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{H}$ that commutes with $\mathcal{A}$. J is an antilinear isometry satisfying 0^{th}-order condition, i.e such that $[Ja^*J^{-1}, b] = 0$ for all $a, b \in \mathcal{A}$. The pseudo-Riemannian structure is determined by an operator $\beta = \beta^\dagger = \beta^{-1}$ commuting with $\mathcal{A}$.

Instead of having the Dirac operator $\mathcal{D}$ selfadjoint we demand so-called β-selfadjointness, that is the condition

$$\mathcal{D}^\dagger = (-1)^p \beta \mathcal{D} \beta. \tag{4}$$

Moreover, commutators of $\mathcal{D}$ with elements of the algebra have to be bounded and furthermore we demand that $\mathcal{D}\gamma = -\gamma\mathcal{D}$. As in the the Riemannian case we have relations between $\mathcal{D}, \gamma$ and J : $\mathcal{D}J = \epsilon J\mathcal{D}$, $J^2 = \epsilon' \mathrm{id}$ and $J\gamma = \epsilon'' \gamma J$. The KO-dimensions defined by $\epsilon, \epsilon', \epsilon'' = \pm 1$ are collected in the table 1.

Table 1. *KO*-dimensions

$p-q$ mod 8	0	1	2	3	4	5	6	7
ϵ	+	−	+	+	+	−	+	+
ϵ'	+	+	−	−	−	−	+	+
ϵ''	+		−		+		−	

Furthermore, we demand the following relations

$$\beta\gamma = (-1)^p \gamma\beta, \quad \beta J = (-1)^{\frac{p(p-1)}{2}} \epsilon^p J\beta, \tag{5}$$

and the 1^{st}-order condition: $[Ja^*J^{-1}, [\mathcal{D}, b]] = 0$ for all $a, b \in \mathcal{A}$. There are other technical conditions [5], but we will not discuss them here. Moreover, we say that the triple is orientable if γ is an image of a certain Hochschild cycle, and is time-orientable if β is an image of such p-cycle.

We observed that for such a spectral triple we can construct a Riemannian spectral triple with additional grading β, a Dirac operator

$$\mathcal{D}_E = \frac{1}{2}(\mathcal{D} + \mathcal{D}^\dagger) + \frac{i}{2}(\mathcal{D} - \mathcal{D}^\dagger) \tag{6}$$

and the real structure changed into $J_E = J\beta$ or $J'_E = J_E\gamma$ depending on the value of p. This procedure is an analogue of the Wick rotation for the Clifford structure.

To illustrate this procedure we consider the Lorentzian noncommutative torus. Denote by $\{|n, m, \pm\rangle\}_{n,m\in\mathbb{Z}}$ the orthonormal basis of the Hilbert space $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2$ and for unitary $\lambda \in \mathbb{C}$ define operators

$$U|n, m, \pm\rangle = |n+1, m, \pm\rangle, \qquad V|n, m, \pm\rangle = \lambda^{-n}|n, m+1, \pm\rangle. \tag{7}$$

They generates the polynomial algebra $\mathcal{A}\left(\mathbb{T}^2_\lambda\right)$ over the noncommutative torus. There exists a time-orientable pseudo-Riemannian spectral triple of signature $(1, 1)$

$$\left(\mathcal{A}\left(\mathbb{T}^2_\lambda\right), \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^2, D, J, \gamma, \beta\right) \tag{8}$$

with

$$D|n, m, \pm\rangle = (n \pm m)|n, m, \mp\rangle, \quad \gamma|n, m, \pm\rangle = \pm|n, m, \pm\rangle, \tag{9}$$

$$J|n, m, \pm\rangle = \mp\lambda^{nm}|-n, -m, \pm\rangle, \quad \beta|n, m, \pm\rangle = \pm i|n, m, \mp\rangle. \tag{10}$$

In that case

$$D_E|n, m, \pm\rangle = (n \pm m)|n, m, \mp\rangle, \quad J'_E|n, m, \pm\rangle = \pm i\lambda^{mn}|-n, -m, \mp\rangle \tag{11}$$

and we end up with the well-known equivariant spectral triple for $\mathcal{A}\left(\mathbb{T}^2_\lambda\right)$. For detailed discussion we refer to [4] and [18].

Now, for the spectral triple for the SM discussed in the previous section we introduced an operator β that is equal 1 on leptonic sector and -1 on quark sector. In [5] we observed that it is consistent with the pseudo-Riemannian spectral triple of signature $(4k, 4k+2) \pmod 8$ with $k \in \mathbb{Z}$, e.g. $(0, 2)$. As a result we can treat $(\mathcal{A}_F, \mathcal{H}_F, \mathcal{D}_F, \gamma_F, J_F, \beta)$ as a Riemannian restriction of that pseudo-Riemannian spectral triple.

Moreover, using the representation of the SM spectral triple from [11] and the form of a general Dirac operator presented therein we found all possible β that makes the SM spectral triple time-orientable and such that this β is consistent with the KO-dimension 6. It turns out that the only possible choice that is not unphysical is the one discussed above and it restricts the class of possible Dirac operators to that which do not allow for leptoquarks and therefore explains the origin of the lepton number conservation in the SM as a shadow of the pseudo-Riemannian structure.

3 Reverse Engineering for the Standard Model

Here we briefly describe the proposal for future research and present recent results for the reverse engineering for the structure of the SM. This is still work in progress.

We propose to analyse the Lagrangian of the SM without the assumption that it follows from the almost-commutative spectral triple with the usual axioms presented in previous sections. Conversely, we start with the reading of the operator D such that the fermionic part can be presented in the form $\Psi^\dagger D \Psi$, where Ψ states for fermions in the model, and try to relate this operator with an operator that can be connected with a Dirac operator $\mathcal{D}$ for some generalized pseudo-Riemannian spectral triple through $D = \beta\mathcal{D}$, where β determines the pseudo-Riemannian structure and is an analogue for the finite pseudo-Riemannian structure discussed in the previous section, but now we need to deal with non-finite triples. We do not demand that all conditions for a spectral triple have to be satisfied, but rather we would like to find all conditions that are really satisfied in that model. For example we do not assume that the model has to be described by an almost-commutative geometry, but it can be a more general spectral triple or some its modification. Moreover, we concentrate only on algebraic conditions for these triples. We would like to also avoid the fermion doubling problem from the very beginning.

In [4] we try to find all possible (possibly slightly modified) pseudo-Riemannian spectral triples that can describe the Standard Model. We can for example fix the gradation and determine all compatible real structures, β operators etc., or conversely, fix some other ingredient and search for the rest that are compatible with that one.

Recent results [4] show that the first order condition is not satisfied in the full model in that formalism, but for some specific cases it is fulfilled, e.g. in the so-called *locally constant version*, i.e. when $f \in C^\infty(M)$ are locally constant.

We also noted that the structure of the full model can be related to the KO-dimension zero, but for the almost-commutative geometry with Lorentzian background, i.e. of signature $(1, 3)$, and the finite part of signature $(0, 2)$ that was discussed in the previous section, we expected to have the KO-dimension 4 for the full model. The recent results suggest [4] that the spectral triple of the SM in that formulation is more general than the almost-commutative structure, but some ingredients are of the product-like type, for example the gradation and the operator β that gives the pseudo-Riemannian structure, but the finite parts of these product are different than in the usual Connes' spectral triple. The real structure is determined by the charge conjugation operator in the full SM.

Additional questions that appear in that consideration are related to the analytical properties of these pseudo-Riemannian spectral triples, but most of them are still open problems in that formulation. We postpone them for the future research and at that moment concentrate on algebraic conditions, like first order condition etc.

4 Summary

We presented the approach to the Standard Model based on the Noncommutative Geometry methods. The main ideas was briefly described and the problems related to the existence of pseudo-Riemannian structures in the spectral triple formalism was presented. We described one role of that structure in the case of the finite spectral triple for the Standard Model. The shadow of the existence of that structure in that triple allows for the exclusion of leptoquarks and, as a result, we infer the lepton number conservation.

Moreover, we briefly described the reverse engineering program for the Standard Model. The recent results were presented and the proposals for future research were mentioned. This is still work in progress and we hope that new results will appear in the nearest future.

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