A General Free Plane Wave Ansatz to The Classcal $SU(2)$ Yang-Mills Theory

with Application to Gravity

W. Li[1,*], *A. H.* Chan[1], and *C. H.* Oh[1]

[1]Department of Physics, National University of Singapore

Abstract. In this project, the free plane wave conditions were imposed on the classical $SU(2)$ gauge field to obtain a new general Ansatz. Although afterwards it was found that this Ansatz is similar to a special case of an existing Ansatz[1], there are important differences. The idea of this Ansatz was later applied to the other nonlinear interaction of nature, namely gravity. However, this effort encountered some complications, such as the lack of an exact definition or interpretation of energy and momentum of gravitational waves.

1 Introduction

Non-Abelian Yang-Mills Theory, the cornerstone of the Standard Model, has attracted great research interest, such as that of S. Coleman[2], to search for a plane wave solution parallel to its Abelian counterpart, electromagnetism. The conventional approach is to first find a mathematical solution and then to analyze, if any, its physical properties. This could be very time consuming, as the nonlinear differential equations are highly complicated to begin with. What further compounds the problem is that a physical interpretation might not always exist. Therefore, we propose to consider proper physical conditions before writing down any specific solution, but let's first briefly review the classical $SU(2)$ Yang-Mills Theory.

2 Non-Abelian Yang-Mills Theory

Yang-Mills theory is a gauge theory. The physical system is described by a Lagrangian of fields, which remains unchanged under local gauge transformation of the fields.[1] For instance, the Lagrangian,

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi, \; where \, \Psi = (\psi_1 \;\, \psi_2)^T ,$$

should remain in the same form under an local $SU(2)$ transformation of the field,

$$\Psi' = \mathbf{S}\Psi, \; where \, \mathbf{S}(x) \equiv e^{i\theta_a(x)\cdot\sigma^a/2}.$$

*e-mail: cqtlw@nus.edu.sg

[1]Such a formulation is readily available in many sources, here we follow the notation in the well-known textbook by Peskin and Schroeder[3].

This requirement necessitates the introduction of a gauge field such that the partial derivative in the Lagrangian becomes the covariant derivative, which explicitly is

$$\partial_\mu \rightarrow \mathbf{D}_\mu \equiv \partial_\mu + i\mathbf{A}_\mu, where\, \mathbf{A}_\mu = A^a_\mu \sigma^a.$$

As a result, the new Lagrangian would retain the same form under the local gauge transformation.

Moreover, if this gauge field is promoted from a merely mathematical contrivance to a meaningful physical quantity, those terms of the derivatives of the gauge field, obeying the local gauge symmetry, should also be included in the Lagrangian, in this case, such a term does exist and is constructed as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \varepsilon_{abc} A^b_\mu A^c_\nu.$$

Finally, the interaction of the system is dictated by the complete Euler-Lagrange Equation. For the local $U(1)$ symmetry, it leads to the famous Maxwell's equations. As for $SU(2)$, it results in the following non-linear equation of motion in vacuum,

$$\partial^\mu F^a_{\mu\nu} + \varepsilon_{abc} A^{b\mu} F^c_{\mu\nu} = 0. \tag{1}$$

The $SU(2)$ classical Yang-Mills theory captures the essential non-linearity in the dynamics of the interaction while remaining mathematically tractable. Therefore, the study of wave solutions to it might shed some light on other non-linear interactions in nature, such as gravity.

3 Physical Conditions and Resultant Constraints

Having obtained Eqn. 1, we then, as proposed, first considered the physical properties of a free plane wave [2], namely a constant propagation direction, equal magnitude of the Poynting vector and energy density and energy being bounded. The last condition depends on the specific choice of solution, while the first two could be applied in general.

Since a general ansatz is the aim, it is desirable to express the energy density and components of the Poynting vector in terms of the gauge field,

$$\begin{aligned} T_{0\mu} = {} & \partial_0 A^a_\gamma \partial_\mu A^{\gamma a} - 2\partial_0 A^a_\gamma \partial^\gamma A^a_\mu + \epsilon_{abc} A^b_\mu A^{\gamma c} \partial_0 A^a_\gamma \\ & + \partial_\gamma A^a_0 \partial^\gamma A^a_\mu + A^a_0 \partial^\gamma \partial_\gamma A^a_\mu - \frac{1}{4}\eta_{0\mu} F^a_{\gamma\rho} F^{a\gamma\rho}. \end{aligned}$$

The free plane wave can be taken to be travelling in the z-direction, thereby implying $|T_{0i}| = 0$, for $i = 1, 2$. The second condition then leads to $T_{00} = |T_{03}|$.

4 A General Free Plane Wave Ansatz

Considering the forms of T_{00} and T_{0i}, it is clearer if the propagation direction is fixed, which could be either positive or negative. Here, we took $T_{00} + T_{03} = 0$, which leads to

$$\begin{aligned} & \partial_0 A^a_\gamma [(\partial_0 + \partial_3) A^{a\gamma} - 2\partial^\gamma (A^a_0 + A^a_3) + \varepsilon_{abc} A^{c\gamma} (A^b_0 + A^b_3)] \\ & + \partial_\gamma A^a_0 \partial^\gamma (A^a_0 + A^a_3) + A^a_0 \partial^\gamma \partial_\gamma (A^a_0 + A^a_3) + \frac{1}{4} F^a_{\gamma\rho} F^{a\gamma\rho} = 0. \end{aligned} \tag{2}$$

Eqn. 2 makes it very suggestive to try out the following simple guess

$$A^a_0(z + t, x, y) = -A^a_3(z + t, x, y),\ A_2 = A_3 = 0. \tag{3}$$

This Ansatz indeed satisfies the two physical constraints mentioned above, but to be a valid solution, it still has to satisfy the vacuum Yang-Mills Eqn. 1, which in this case reduces to

$$\partial^{\mu}\partial_{\mu}A^{a}_{\nu} + \varepsilon_{abc}A^{b\mu}(\partial_{\mu}A^{c}_{\nu} - \partial_{\nu}A^{c}_{\mu}) = 0. \tag{4}$$

At first glance, this new ansatz is very similar to an existing Ansatz[1],

$$A^{a}_{\mu} = \psi_{\alpha} f^{a}_{\alpha}(U)\partial_{\mu}U, \tag{5}$$

and one immediate generalization of Ansatz 3 is

$$A^{a}_{\mu} = f(U, k_{\nu}x^{\nu}, l_{\lambda}x^{\lambda})\partial_{\mu}U, \tag{6}$$

where $U = p_{\mu}x^{\mu}$, k_{ν} and l_{λ} have vanishing temporal part and spatial parts that are mutually orthogonal with the spatial part of p_{μ}. Although these two Ansatze might appear similar, the latter is more general in the sense that the function f does not need to depend only on U as the former one does, and it is narrower as it describes only free plane waves while the former can describe spherical-fronted waves as well.

Further analysis, omitted here due to length limit, also shows that the two solutions, namely one plane wave solution and one spherical fronted wave solution, based on Ansatz 5 have different colour transport properties, the former does not exchange colour with surroundings while the latter does. This difference could only be attributed to the specifically mathematical decisions made to construct the solution from Ansatz 5. Meanwhile, all possible solutions built from Ansatz 6 do not exchange colour with the environment. In fact, a closer scrutiny of the plane wave solution based on Ansatz 5 could actually be expressed in the form of the general free plane wave Ansatz 6 while the spherical fronted solution could not. This solves the mystery and illustrates the advantage of associating physical interpretation with the entire class of Ansatz as compared to the rather arbitrary search for specific mathematical solutions.

One particularly simple solution could be obtained immediately if all components of $\mathbf{A}_{\mu}$ are taken to be the same and hence the term with ε_{abc} in Eqn. 4 drops out, giving

$$\partial^{\mu}\partial_{\mu}A^{a}_{\nu} = 0.$$

Note that the resultant equation might seem to admit linear solutions as it only contains a second order derivative operator. However, this much simplified result depends on the propagation direction incorporated into Ansatz 6, namely the vector p_{μ}. Hence, two solutions with different propagation directions cannot be linearly combined to generate a new solution, thereby demonstrating the non-linearity of the described system.

5 On Gravity

Having successfully obtained a general Ansatz for the classical $SU(2)$ Yang-Mills theory, we wanted to apply the same strategy to gravity to see if we could obtain a general wave Ansatz, which might help extracting more information from the recent gravitational wave detection.

Similar to Eqn. 1, gravity is described by Einstein's Equation[4],

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu},$$

which in vacuum is explicitly,

$$R_{\beta\nu} = \frac{1}{2}g^{\alpha\mu}(\partial_{\beta}\partial_{\mu}g_{\alpha\nu} + \partial_{\alpha}\partial_{\nu}g_{\beta\mu} - \partial_{\beta}\partial_{\nu}g_{\alpha\mu} - \partial_{\alpha}\partial_{\mu}g_{\beta\nu}) + \Gamma^{\alpha}_{\lambda\alpha}\Gamma^{\lambda}_{\beta\nu} - \Gamma^{\alpha}_{\lambda\nu}\Gamma^{\lambda}_{\beta\alpha} = 0. \tag{7}$$

The inspiration drawn from the previous Ansatz regarding a free plane wave of a non-linear system is that the propagating field has a part, namely the $\partial_\mu U$ term, that describes the propagation while the magnitude can have more freedom. Hence, one analogous Ansatz for gravity is

$$g_{\mu,\nu\in\{0,3\}} = g_{\mu,\nu\in\{0,3\}}(z+t),\ g_{\mu,\nu\in\{1,2\}} = g_{\mu,\nu\in\{1,2\}}(x,y) \tag{8}$$

and other terms of $g_{\mu\nu}$ vanish. As a result of this simplifying Ansatz, the second order differential terms involving z and t indeed reduce to a much simpler form,

$$2\partial_0\partial_3 g_{03} - \partial_0\partial_0 g_{33} - \partial_3\partial_3 g_{00} = 0. \tag{9}$$

However, to study the nonlinear nature of gravity analytically, an exact solution to Einstein's equation is needed, so the $\Gamma\Gamma$ terms should also vanish. Ansatz 8 does also simplify the $\Gamma\Gamma$ terms, but when combined with Eqn. 7, it is still rather complicated. When we tried to carry over the energy momentum consideration in the Yang-Mills case to further simplify the problem, we realized that the exact energy momentum of a gravitational wave is not so straight forward. Unlike the Yang-Mills waves, the source of which is the "color charge", the source for gravitational waves is exactly the energy momentum tensor itself. Hence, it is inevitable that an exact vacuum, or "wave", solution to Einstein's equation would have a vanishing energy momentum tensor. However, a propagating wave should possess energy and momentum. This irreconcilable paradox requires deeper thinking and further studies. If this contradiction is not resolved, it would be impossible to obtain an exact vacuum solution that has a clear physical interpretation. Therefore, it is more pressing to address this contradiction than to obtain an exact mathematical solution to Eqn. 9.

6 Conclusion and Future Research Direction

Based on the above analysis, it is clearly advantageous to construct Ansatz based on physical considerations as it would be time-consuming to understand mathematical solutions whose physical interpretation is still obscure. Similar strategy has been used to study gravity. However, the fundamental difficulty with a proper definition of energy and momentum of a gravitational wave complicates this application. In the future, it could be worthwhile to look for wave properties in solutions resulted from a non-vanishing source, or perhaps to think about possible modifications to the right hand side of Einstein's equation.

References

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