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Synchronization in a Neural Network of Phase Oscillators with Time Delayed Coupling

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Abstract

We investigate a neural network model designed as a system of the central oscillator and peripheral oscillators interacting with a time delay according to the phase-locking scheme. The delay corresponds to the finite velocity of signal propagation along the nervous fibers. We study the synchronization under various values of delay. It is shown under some conditions, that for a finite delay time there exists a multitude of synchronization frequencies in contrast to the case without delay where the system has at most one solution. The criteria for multiple solutions existence and their stability are found. The asymptotic behaviour under increasing connection strengths is analyzed.

1. Introduction

Recent experimental observations show the significant role of oscillatory processes in the functioning of the nervous system. There is a hypothesis that information processing in the brain can be described in terms of synchronization of activity of various neuron ensembles [1,2]. This causes a great number of oscillatory neural network models of olfactory, visual and motor systems as well as models of memory and attention [3]. The investigation of these models is also stimulated by the fact, that a system of interacting oscillators exhibits complex dynamics, which is suitable for mathematical analysis. The architecture of the oscillator network model considered here was developed by V.I.Kryukov for attention modeling [4]. The network consists of the central oscillator (CO) and N peripheral oscillators (PO). The CO has forward and backward connections with all POs. The POs are not coupled with each other and interact via the CO. An oscillator behaviour is described by one variable, phase of oscillations. The CO interacts with all the POs according to the phase-locking scheme. It is supposed that the septo-hippocampal region can play the role of the CO and cortex columns can play the role of POs. Synchronization of the CO with all POs or with some part of them is considered as synchronization of oscillations between septo-hippocampal region and some parts of the cortex. Attention has been interpreted as the result of such synchronization. The finite velocity of signal propagation along the nervous fibers causes a finite delay in transmission of information between neurons. Therefore the time delay should be taken into account when one models biological neural networks. The network dynamics is described by the following system of equations:

$$\begin{aligned} \frac{dx_0}{dt} &= \omega_0 + K\sum_{i=1}^{N} \sin(x_i(t-\tau) - x_0(t)), \\ \frac{dx_i}{dt} &= \omega_i + K\sin(x_0(t-\tau) - x_i(t)), \\ & i = 1,2,\ldots,N, \end{aligned} \tag{1}$$

where x_0 and ω_0 are the phase and the natural frequency of the CO, respectively; x_i and ω_i, $i = 1,2,\ldots,N$, are the phases and the natural frequencies of POs; K is the strength of coupling between oscillators; τ is the time delay.

Our aim is to derive the criteria for all the oscillators synchronization at a common constant frequency Ω and to analyse the asymptotic behaviour of the network when connection strengths are increasing. To study the synchronization mode we consider the solutions of system (1) in the form:

$$x_0(t) = \Omega t, \quad x_i(t) = \Omega t + \phi_i, \tag{2}$$

where $\phi_i := x_i(t) - x_0(t), \; i = 1,2,\ldots,N$, are constants.

Note, that for $\tau = 0$ and $|\omega_i - \Omega| \leq K, \; i = 1,2,\ldots,N$, system (1) has the single stable solution (2) with $\Omega = \sum_{i=0}^{N} w_i/(N+1)$ and $\phi_i = \arcsin(\omega_i - \Omega)/K$ [5].

We will show that the nonzero delay significantly changes the behavior of system (1). In this case the system admits several stable solutions of the form (2) for the fixed values of the system parameters and, hence, the neural network model admits several synchronization frequencies. Investigating the model of N oscillators interacting via the central oscillator as a model of attention, we have got the following new results concerning collective synchronization in large assemblies of coupled neural oscillators interacting with a time delay:

1.For any values of N and τ we derive the necessary and sufficient conditions of the existence of solutions of system (1) in the case of full synchronization.

2.We formulate a simple criterion of stability of solutions.

3.We also derive rigorous results concerning the asymptotic behaviour of the network when K is increasing. The results allow us to obtain some estimates of the network behaviour for finite values of K.

Thus, we can compute the number of the synchronization frequencies as a function of K and τ and the values of frequencies and phase differences of synchronized oscillators.

2. Synchronization of oscillators at a common constant frequency.

Substituting the solution (2) into (1), we obtain the function $f(\Omega)$ to determine the values of the synchronization frequency Ω and the formulas for the values ϕ_i:

$$f(\Omega) = \Omega - \omega_0 - K\sum_{i=1}^{N} \sin(\arcsin((-\Omega)/K) - 2\Omega\tau) = 0 \tag{3}$$

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and

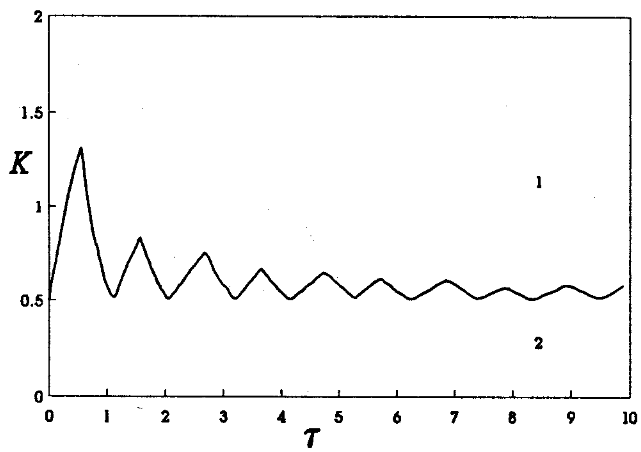
$$\phi_i = \arcsin(\omega_i - \Omega)/K - \Omega\tau, \quad i = 1, 2, \ldots, N.$$

This allows us to determine all the solutions of (1). The necessary condition of the existence of the solutions $K \geq K_c = \Delta\omega/2$, where $\Delta\omega := \max_i \omega_i - \min_i \omega_i$, is given by (3). The sufficient condition is formulated in the following statement.

Statement 1. For fixed ω_0 and ω_i there exists a value K^*, $K^* \geq K_c$, such that (3) has a solution for any values τ if and only if $K \geq K^*$ For any fixed $K, K_c \leq K < K^*$, there exists p_k ($p_k < \infty$) intervals $I_s = (\tau_1(s), \tau_2(s+1))$ with $0 \leq s \leq p_k - 1$ of values τ such, that (3) has no solution for all $\tau \in \bigcup I_s$.

In the Figure 1. the solid line separates the region of the existence of solutions from the region of the values K and τ, for which the system has no solution.

Figure 1. The region 1 is the region of the existence of solutions. The region 2 is the region, where the system (1) has no solution. $N = 9$, $\omega_0 = 1$, and ω_i are uniformly distributed on [1,2].



Thus, if the coupling strength K between the oscillators is sufficiently small, the synchronization of all the oscillators at a common frequency is possible for some values of the delay only. The following statement determine the number of zeros of the function $f(\Omega)$ depending on the influence of the parameters K, τ, ω_i.

Statement 2. Suppose, that Ω_n is the n-th solution of (3) (Ω_n are arranged in the increasing order in n) and $D_n = \Omega_n - \Omega_n$. Then for any $n \geq 1$

$$\lim_{K \to \infty} D_n = \pi/2\tau.$$

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Thus all the solutions of the equation (3) are spaced asymptotically (for $K \to \infty$) at the same distance. Consequently, when $K \to \infty$, the number of solutions (q) of (3) over the whole region of the existence of the function $f(\Omega)$ can be determined as:

$$q \approx [2(2K - \Delta\omega)\tau/\pi], \qquad (4)$$

where [ ] denotes the rounding down a real number. It follows from (4) that the number of solutions of equations (3) increases with K and τ. A decrease in the scatter of natural frequencies $\Delta\omega$ also increases q. Thus, we conclude that the important feature of system (1) for $\tau \neq 0$ and fixed K, τ, ω_i is the existence of multiple solutions and, consequently, of multiple synchronization frequencies $\Omega(K, \tau, \omega_i)$.

3. Stability of solutions.

Criterion. A solution of system (1) in form (2) is stable iff

$$\sum_{i=1}^{N} \cos(\phi_i - \Omega\tau) > 0,$$

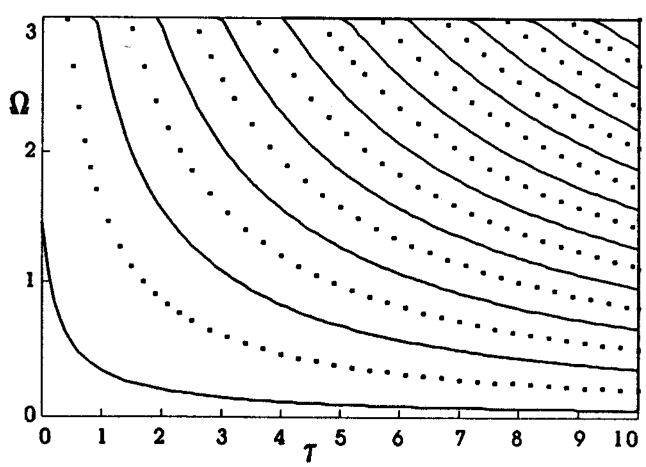
where

$$\phi_i = \arcsin(\omega_i - \Omega)/K) - \Omega\tau, \quad i = 1, 2, \ldots, N.$$

To derive this statement we apply the amplitude-phase method [6] to a quasipolynomial of the linear system, obtained by the linearization of (1) near the solution.

Figure 2 shows the behavior of all the solutions (2) of system (1) for $K = 2$.

Figure 2. The solid lines correspond to the stable solutions, and the dotted lines correspond to the unstable ones.



4. Asymptotic behavior of phase differences

Numerical experiments have shown that either $\phi_i \approx 0(mod2\pi)$ or $\phi_i \approx \pi(mod2\pi)$ for all stable solutions, and either $\phi_i \approx \pi/2(mod2\pi)$ or $\phi_i \approx 3\pi/2(mod2\pi)$ $(i = 1, 2, ..., N)$ for unstable solutions.

Figure 3. The average phase difference ($K = 2$).

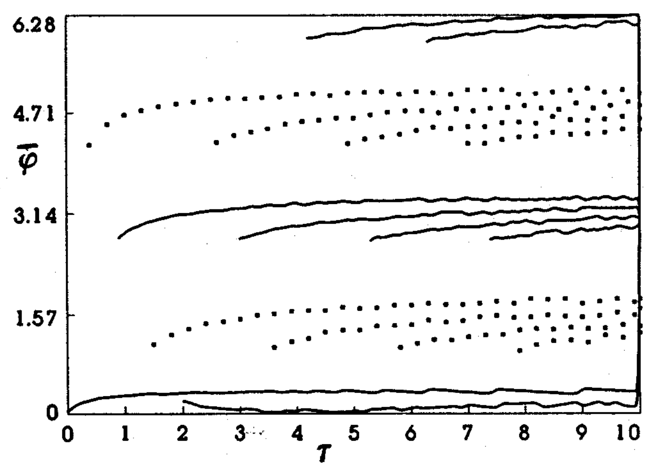


Fig.3 shows the average phase difference $\bar{\phi} = \sum_{i=1}^{N} \phi_i/N$ versus τ for all solutions of system (1) for $K = 2$. The solid lines indicate the stable solutions and the dotted lines indicate the unstable ones. It is clear that all POs are synchronized with the CO either in phase or in antiphase. Moreover the following statement is valid:

Statement 3.

$\lim_{K\to\infty} \phi_i = 0(mod2\pi)$ or $\quad \lim_{K\to\infty} \phi_i = \pi(mod2\pi)$

for stable solutions;

$\lim_{K\to\infty} \phi_i = \pi/2(mod2\pi)$ or $\quad \lim_{K\to\infty} \phi_i = 3\pi/2(mod2\pi)$

for unstable solutions;

$i = 1, 2, \ldots, N$.

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5. Summary

The oscillator network behaviour described by system (1) with time delay coupling differs significantly from the one without delay. The main features of this behaviour are the following. For fixed values of the parameters several stable solutions of the system (and consequently, multiple synchronization frequencies) exist. The number of frequencies increases if the strength of coupling and the delay increase and the scatter of the oscillator natural frequencies decreases. Introduction of a delay τ into system (1) change substantially the existence conditions of its solutions: for small values of the coupling strength K the solutions exist for certain intervals of values τ only. Consequently, for small values of the coupling strength there are intervals of τ where the synchronization is impossible. At the same time, for τ lying out of these intervals the synchronization occurs at one or more frequencies.

The oscillator network exhibits two types of stable synchronous regimes: the first one takes place when all POs and the CO are synchronized in phase, and the second one occurs when the synchronization takes place in antiphase.

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