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# STABILITY ANALYSIS OF DIAGONAL RECURRENT NEURAL NETWORKS

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**Abstract:** In this paper, we investigate the local stability of the diagonal recurrent neural network (DRNN), a particular case of recurrent neural networks, in which a recurrent neuron only possesses a self-feedback connection. The local stability conditions for a DRNN are presented. Subsequently it is shown how the extent of stability can be investigated in terms of the equilibrium state's region of attraction in state space. Then, some examples are given to illustrate the stability problem of diagonal recurrent neural networks.

## 1. Introduction

One of the most attractive properties of neural networks is their input/output mapping capability. In identification and control, we usually have to deal with the problem of building input/output nonlinear dynamic models. In these cases, the application of recurrent neural networks (RNN) to construct a system model is more appropriate than using feedforward neural networks (FNN) which are widely applied to investigate the static properties of the systems.

Usually, the architecture of a fully connected recurrent neural network is very complicated. Recently, Ku et. al. (1992) have proposed a rather simple structure of RNN called the diagonal recurrent neural network (DRNN), which only contains a hidden layer of dynamic nodes with self-feedback connection, without any interconnection with the other nodes in the same layer. This kind of local recurrent network has been proved to be rather effective for real-time control (Ku et. al., 1992) and dynamic identification (Tan and De Keyser, 1993).

Since the DRNN includes feedback loops, the stability of the network is an important issue which certainly affects the learning capability of the DRNN. In experiments, we found that, in some cases, the neural models failed due to instability. Thus, the stability analysis is the aim of our paper. In section 4, some examples are given to demonstrate the stability problem of diagonal recurrent neural networks.

## 2. Diagonal Recurrent Neural Network

The architecture of the diagonal recurrent neural network is described as follows:

$$x_i(t)=s[n_i(t)]=s[I^T(t)V_i+v_{i,n+1}x_i(t-1)] \quad (1)$$

$$\hat{y}(t)=X^T(t)W \quad (2)$$

where $\hat{y}$ is the output of the neural network; $X(t)\in R^h$ is a state vector; $I(t)\in R^n$ is an input vector; $W\in R^h$ as well as $V_i\in R^n$ are weight vectors; $v_{i,n+1}$ is the recurrent weight of the *ith*

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hidden node; and s(.) is a sigmoid function,i.e.

$$s(x) \triangleq \frac{1-e^{-x}}{1+e^{-x}} \tag{3}$$

In identification and control, we usually use two kinds of neural network models which are respectively called the series-parallel model (SPM) where $I(t)=[y(t-1),...,y(t-n_y),u(t-1),...,u(t-n_u)]^T$, and y and u respectively denote the system output and input, and the parallel model (PM) where $I(t)=[\hat{y}(t-1),...,\hat{y}(t-n_y),u(t-1),...,u(t-n_u)]^T$ (Narendra and Parthasarathy,1990).

## 3. Local Stability of the DRNN

Since the hidden layer of the DRNN has feedback connections, we may meet the stability problem in training or in model validation. If the network is unstable, it will lose its learning capability.

First we consider the PM network. Let {u(t)} be zero. Then the autonomous dynamic equation of the network becomes

$$x_i(t)=s[v_{i,n+1}x_i(t-1)+\sum_{j=1}^{n_y}v_{ij}\hat{y}(t-j)]=s[w_1\sum_{j=1}^{n_y}v_{ij}x_1(t-j)+...+w_h\sum_{j=1}^{n_y}v_{ij}x_h(t-j)+v_{i,n+1}x_i(t-1)] \tag{4}$$

It is obvious that in this case the network has the structure of a fully connected recurrent network . Define

$$z_{11}(t)\triangleq x_1(t-n_y),\ ...,\ z_{1n_y}(t)\triangleq x_1(t-1);\ ...;\ z_{hl}(t)\triangleq x_h(t-n_y),\ ...,\ z_{hn_y}(t)\triangleq x_h(t-1)$$

and

$$Z_1(t)\triangleq[z_{11}(t),\ ...,\ z_{1n_y}]^T;\ ...;\ Z_h(t)\triangleq[z_{hl}(t),\ ...,\ z_{hn_y}(t)]^T.$$

Then we have

$$z_{11}(t+1)=z_{12}(t),\ ...,\ z_{1n_y-1}(t+1)=z_{1n_y}(t),\ z_{1n_y}(t+1)=s[Z_1(t),\ ...,\ Z_h(t),\beta_1]$$

$$z_{21}(t+1)=z_{22}(t),\ ...,\ z_{2n_y-1}(t+1)=z_{2n_y}(t),\ z_{2n_y}(t+1)=s[Z_1(t),\ ...,\ Z_h(t),\beta_2]$$

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$$z_{hl}(t+1)=z_{h2}(t),\ ...,\ z_{hn_y-1}(t+1)=z_{hn_y(t)},\ z_{hn_y}(t+1)=s[Z_1(t),\ ...,\ Z_h(t),\beta_h]$$

where $\beta_i$ (i=1,...,h) are a new set of weight vectors composed from the components of $V_i$ (i=1,...,h), and from W, and where with a slight abuse of notation $s[Z_1(t),...,Z_h(t),\beta_i]$ means that the argument of s(.) is a linear combination of $Z_1(t)...Z_h(t)$ with coefficients taken from the vector $\beta_i$. It is assumed that $s(.)\in C^r(E)$, $(r\geq1)$, where E is an open subset of $R^h$ which contains the equilibrium point $Z_e$. The linearized system around $Z_e$ is

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$$Z^*(t+1) = D \cdot Z^*(t) \tag{5}$$

where $Z^*(t)=[Z^*_1(t),\dots, Z^*_h(t)]^T$ and $D \triangleq [\partial Z(t+1)/\partial Z^T(t)]_{z=z_e}$ is a Jacobian matrix. According to the linearization principle (Vidyasagar, 1993), the system (4) is asymptotically stable in the neighbourhood of $Z_e$ if the system given in (5) is asymptotically stable, i.e. the matrix D has all its eigenvalues in the open unit circle $\{|z|<1\}$ on the z plane. The system (4) is unstable around the equilibrium point if the matrix D has at least one eigenvalue located outside the unit circle $\{|z|>1\}$.

For the SPM network, if we let I(t)=0, each state equation of the system (1) only contains a single self-recurrent state variable. In this specific case, the Jacobian matrix D becomes

$$D = diag[s'(n_1)v_{1,n+1},\dots,s'(n_h)v_{h,n+1}] \tag{6}$$

where $[n_1,\dots,n_h]^T$ represents the network's equilibrium state. Therefore the stability condition for the SPM network within the neighbourhood of the equilibrium point is

$$|s'(n_i)v_{i,n+1}|<1 \; ; i=1,\dots,h \tag{7}$$

For the sigmoid function (3), the maximum value of $s'(n_i)$ is 1/2. So a sufficient stability condition for the SPM network is given by

$$|v_{i,n+1}| \le \frac{1}{\max[s'(n_i)]} \le 2 \; ; i=1,\dots,h \tag{8}$$

When the Jacobian matrix D has its eigenvalues inside the unit circle then the extent of stability can be studied in terms of the equilibrium state's region of attraction in state space. To do this and rewrite the state equation (4) as

$$Z^*(t+1) = D \cdot Z^*(t) + h(Z^*(t))$$

where h(.) represents the higher order terms around the equilibrium point. Suppose the higher order terms of the above equation satisfy

$$\|h(Z^*(t))\|_2 \le \alpha \|Z^*(t)\|_2 \quad ; \quad \|Z^*(t)\|_2 \le b \tag{10}$$

where $\alpha$ and b are positive constants. Consider a quadratic Lyapunov function

$$V(t) \triangleq Z^{*T}(t) \cdot P \cdot Z^*(t) \tag{11}$$

where P is a positive definite and symmetric matrix. The difference of the Lyapunov function is

$$\Delta V(t+1) = V(t+1) - V(t) = [DZ^* + h(Z^*)]^T P [DZ^* + h(Z^*)] - Z^{*T} P Z^* \tag{12}$$
$$= Z^{*T} D^T P D Z^* + h^T(Z^*) P h(Z^*) + Z^{*T} D^T P h(Z^*) + h^T(Z^*) P D Z^* - Z^{*T} P Z^*$$

where for simplification we denote $Z^*(t)$ as $Z^*$. From the relation

$$(DZ^* - h(Z^*))^T P (DZ^* - h(Z^*)) \ge 0 \tag{13}$$

we derive

$$Z^{*T}D^TPDZ^{*}+h^T(Z^{*})Ph(Z^{*})\geq Z^{*T}D^TPh(Z^{*})+h^T(Z^{*})PDZ^{*} \qquad (14)$$

Considering the assumption (10) and the fact that P is positive definite and symmetric, we obtain

$$h^T(Z^{*})Ph(Z^{*})\leq\lambda_{max}(P)h^T(Z^{*})h(Z^{*})\leq\alpha^2\lambda_{max}(P)Z^{*T}Z^{*} \quad ;\|Z^{*}(t)\|\leq b \qquad (15)$$

where $\lambda_{max}(P)$ is the maximum eigenvalue of P. Substituting the inequalities. (14) and (15) into eqn. (12) yields

$$\Delta V(t+1)\leq Z^{*T}[(D^TPD-P)+D^TPD+2\alpha^2\lambda_{max}(P)\cdot I]Z^{*} \qquad (16)$$

Since all eigenvalues of D lie inside the unit circle, the Lyapunov equation

$$2D^TPD-P=-Q \qquad (17)$$

possesses a unique positive definite solution P for any chosen symmetric positive definite Q (Vidyasagar, 1993) . Now

$$\Delta V(t+1)\leq Z^{*T}[-Q+2\alpha^2\lambda_{max}(P)\cdot I]Z^{*} \quad ;\|Z^{*}(t)\|\leq b \qquad (18)$$

To ensure the matrix on the right hand side of the above formula is negative definite, we should have

$$\alpha\leq\sqrt{\frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}} \qquad (19)$$

Let

$$V_{min}\triangleq\min\{Z^{*T}(t)PZ^{*}(t)\} \quad ;\|Z^{*}(t)\|=b \qquad (20)$$

Then the set $G\triangleq\{Z^{*T}(t)PZ^{*}(t)<V_{min}\}$ is a region of attraction of the network's equilibrium state $Z_e$. Finding $V_{min}$ is a standard problem of constrained minimization. So G can be found by combining (10), (17), (19) and (20).

## 4. Examples

In the following, we shall discuss some examples illustrating the stability problems which may arise in recurrent networks.

**Example 1:** Consider an autonomous PM diagonal recurrent network with state equations

$$\begin{aligned} x_1(t)&=s[-0.0176x_1(t-1)+0.2353\hat{y}(t-1)] \\ x_2(t)&=s[-3.95x_2(t-1)+2\hat{y}(t-1)] \\ \hat{y}(t)&=0.5x_1(t)+0.85x_2(t) \end{aligned} \qquad (21)$$

The equilibrium state is $x_1(t)\equiv x_2(t)\equiv 0$. At this equilibrium state, $s'(n_i)$ reaches its maximum value $s'(n_i)_{max}=0.5$. The corresponding eigenvalues of the Jacobian matrix are 0.0911 and -1.1661. So the neural network is unstable. Fig. 1 shows that persistent oscillations occur in the response.

**Example 2:** The neural network of this example is an autonomous SPM diagonal recurrent network given by eqn. (22). Suppose k=-2.15. In this case, it is obvious that the stability condition (8) is not satisfied. One of the eigenvalues of the Jacobian matrix at the system's equilibrium state equals -1.075. Thus the neural network is unstable. Fig. 2a) displays the persistent oscillation of the network's response.

$$\begin{aligned} x_1(t) &= s[0.25x_1(t-1)] \\ x_2(t) &= s[k\cdot x_2(t-1)] \\ \hat{y}(t) &= 0.65x_1(t)+0.45x_2(t) \end{aligned} \qquad (22)$$

For k=2.15, instead of converging to zero which is the output of the network at the equilibrium point, the response of the neural network shows a monotonic increase at the beginning, then it remains stuck in the saturation region. This behaviour is illustrated in Fig. 2b). The reason of this phenomenon is that the Jacobian matrix has an eigenvalue at 1.075.

Both examples show that the diagonal recurrent neural network becomes unstable if its architecture parameters violate the stability conditions of the previous section. We notice that the instability of the network does not lead to limitless divergence but to saturation or persistent oscillation due to the boundedness of the sigmoid function. The result of instability of a network is that its learning capability is lost.

## 5. Conclusion

Though the diagonal recurrent neural network is more useful for studying the dynamics of nonlinear systems than the feedforward neural network, the feedback connections of this neural network may lead to instability. This paper has discussed this phenomenon and has given some rules to check the stability of the diagonal recurrent neural network. The examples have shown that the character of the sigmoid function causes saturation or persistent oscillations in the output of the unstable neural network.. The stability analysis of the DRNN may provide some insight in the problem of the local feedback stabilization of diagonal recurrent neural networks.

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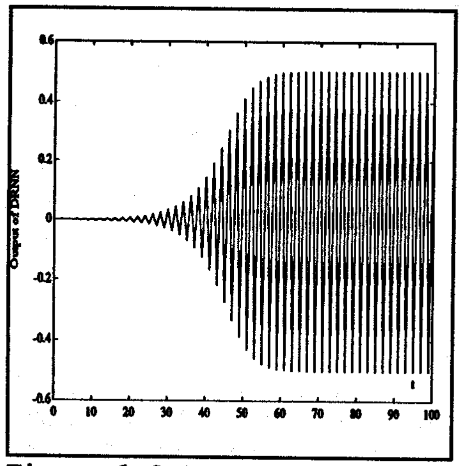


**Figure 1** Autonomous response of a DRNN (Example 1)

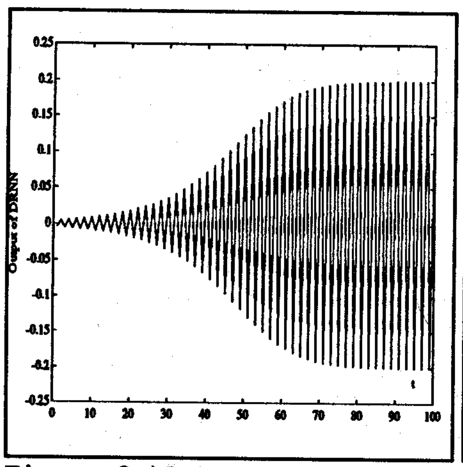


**Figure 2**a) Autonomous response of a DRNN (Example 2)

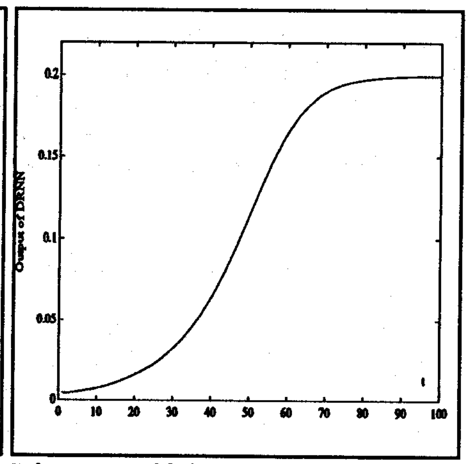


**Figure 2**b) Autonomous response of a DRNN (Example 2)