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Local input-output stability of recurrent networks with time-varying weights

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Abstract. We present local conditions for input-output stability of recurrent neural networks with time-varying parameters introduced for instance by noise or on-line adaptation. The conditions guarantee that a network implements a proper mapping from time-varying input to time-varying output functions using a local equilibrium as point of operation. We show how to calculate necessary bounds on the allowed inputs to keep the network in the stable range and apply the method to an example of learning an input-output map implied by the chaotic Roessler attractor.

1 Introduction

A strong motivation for investigations on recurrent neural networks (RNN) is their ability to model for given initial conditions the time-behavior of arbitrary dynamical systems [3]. This approximation capability, the possibility to incrementally adapt a network to a given task, and the intuition that a recurrent network may draw on the time-structure of its inputs render them generic candidates for learning input-output behavior. And thus a number of recent results show that RNN's can map successfully time-varying input functions into a desired time-varying trajectory for a wide range of tasks [4,7,8,11].

On the other hand, due to the non-linear nature, the usually large number of state variables (neurons) and the distributed nature of computation, even the most basic network properties as existence and number of equilibria, convergence, boundedness, or stability are different to assure. They have been subject to intensive research already in the case of zero or constant input [1,2,6], mostly with the objective to show either that the origin is globally absolute stable or to provide respective local conditions for multi-stable networks with many equilibria. However, when there is time-varying input and additional time-variance in the weights, the task to show stability becomes even more difficult and the results based on Lyapunov functions mostly rely on coarse approximations of the system by interval matrices and yield computationally intractable conditions [5,10,12].

In this contribution we extend a framework based on frequency functions introduced in [10,11] for investigating input-output stability of the origin in the presence of time-varying weights to the case of multi-stable networks. We assume that there is a

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non-trivial equilibrium with a finite basin of attraction around which we obtain a proper mapping from the given input to the desired output function. We address the problem how to avoid that the time-varying input drives the network out of this stability region into the basin of attraction of a different attractor or a region where internally driven persistent oscillations occur. Such behavior is undesirable as it contradicts the requirement of a proper map from input to output and would render generalization impossible. Further we guarantee this type of stability in the presence of time-variance in the weights for instance caused by noise or on-line adaptation, i.e. we prove that the network is locally structurally stable as defined in [5]. It is obvious that this requires to limit the allowed time-invariance as well as the inputs and we will give a constructive approach to obtain suitable bounds. In Section 2 we present the local stability framework and in Section 3 we derive the output bounds. In Section 4 we demonstrate the application of the framework for a trajectory learning task, where we expect to obtain as working point typically a non-trivial equilibrium, and finally we add some discussion.

2 Stability analysis

In this section we derive stability conditions for a network with given local equilibrium. As the resulting conditions can directly be applied based on the material presented here and without detailed knowledge of their theoretical derivation we refer the interested reader for more details to [9–11]. We assume that the network is given in the form

$$\dot{\mathbf{x}} = -\mathsf{D}\mathbf{x} + (\widetilde{\mathsf{W}} + \Delta\widetilde{\mathsf{W}})\tilde{\boldsymbol{\varphi}}(\mathbf{x}) + \mathbf{u}(t), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathsf{D} = \mathrm{diag}\{d_i\} > 0 \in \mathbb{R}^{n\times n}$ is constant, $\widetilde{\mathsf{W}} \in \mathbb{R}^{n\times n}$ is the time-stationary weight matrix, $\Delta\widetilde{\mathsf{W}}(t)$ the time-varying weight matrix, $\tilde{\boldsymbol{\varphi}}(\mathbf{x}) = (\varphi_1(x_1), \ldots, \varphi_n(x_n))^T$ the non-linear time-invariant activation function, and $\mathbf{u}(t) \in \mathbb{R}^n$ the external input function. In [11] we have shown in detail how to incorporate the time-variance $\Delta\widetilde{\mathsf{W}}$ as additional non-linear feedback, such that we can instead of (1) consider

$$\dot{\mathbf{x}} = -\mathsf{D}\mathbf{x} + \mathsf{W}\boldsymbol{\varphi}(\mathbf{y}, t) + \mathbf{u}(t), \quad \mathbf{y} = \mathsf{C}\mathbf{x} \tag{2}$$

where $\mathsf{W} \in \mathbb{R}^{n\times(n+2N)}$ is a constant matrix and N the number of time-varying weights. The output matrix $\mathsf{C} \in \mathbb{R}^{n\times(n+2N)}$ is chosen to obtain an enlarged feedback vector $\mathbf{y} \in \mathbb{R}^{n+2N}$ such that that $\boldsymbol{\varphi}(\mathbf{y})$ includes for each $\Delta\tilde{w}_{ij}(t)$ additional time-varying components $k^l_{ij}(t), k^u_{ij}(t)$. It then has the form

$$\boldsymbol{\varphi}(\mathbf{y}) = (\varphi_1(x_1), \ldots, \varphi_n(x_n), \ldots, k^l_{ij}(t)x_j, \ldots, k^u_{ij}(t)x_j, \ldots)^T \in \mathbb{R}^{n+2N},$$

such that $\Delta\tilde{w}_{ij}(t)x_j = -k^l_{ij}(t)x_j + k^u_{ij}(t)x_j$ for positive bounded parameters $0 \le k^l_{ij}(t) \le \underline{k}_{ij}, 0 \le k^u_{ij}(t) \le \bar{k}_{ij}$. The systems corresponding to (1) and (2) respectively are shown in block diagram form in Fig. 1 (a),(b). Subsequently we restrict the analysis to the system (2). Note that this inclusion of time-variance in the feedback implies that we can not assume any Lipschitz conditions on the time-varying components of $\boldsymbol{\varphi}$. The stability analysis now proceeds in four steps:

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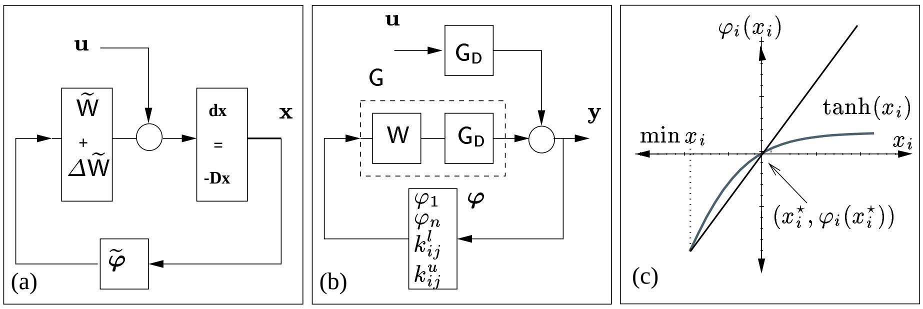

Fig. 1. (a) The original network with time-varying weights. (b) Reparametrisation of the time-variance as positive feedback using enlarged output $\mathbf{y} = \mathsf{C}\mathbf{x}$ and a modified W, $\boldsymbol{\varphi}$. $\mathsf{G_D}$ abbreviates the linear differential operator given by $\dot{\mathbf{x}} = -\mathsf{D}\mathbf{x}, \mathbf{y} = \mathsf{C}\mathbf{x}$. (c) Local sector conditions at the equilibrium $\mathbf{x}^\star$ which hold only for $x_i \geq \min x_i$.

[1] We identify in the network (2) a linear subsystem $\dot{\mathbf{x}} = -\mathsf{D}\mathbf{x} + \mathsf{W}\mathbf{e} + \mathbf{u}$ with non-linear feedback $\mathbf{e} = \boldsymbol{\varphi}(\mathbf{y}, t)$. The linear part can be Laplace transformed and is described in the complex domain by

$$\mathbf{y}(s) = \mathsf{C}(s\mathrm{I} + \mathsf{D})^{-1}\left(\mathsf{W}\mathbf{e}(s) + \mathbf{u}(s)\right) = \mathsf{G}\mathbf{e}(s) + \mathsf{G_D}\mathbf{u}(s) = \mathsf{G_D}(\mathbf{u}(s) + \mathsf{W}\mathbf{e}(s)),$$

where we define the transfer functions $\mathsf{G_D}(s) = \mathsf{C}(s\mathrm{I} + \mathsf{D})^{-1}$ and $\mathsf{G}(s) = \mathsf{G_D}\mathsf{W}$.

[2] We assume there exist local sector bounds $\bar{k}_i$ on the original non-linear activation functions φ_i, i.e. the graph of $(x_i, \varphi_i(x_i))$ is between the x_i-axis and the line $y = \bar{k}_i x_i$ if the origin is moved to the local equilibrium $(x_i^\star, \varphi_i(x_i^\star))$ as shown in Fig. 1 (c). The stability result below will be valid only if we can show that the system trajectory does not leave the region, where these local sector conditions hold. Naturally the upper bounds $\underline{k}_{ij}$ and $\bar{k}_{ij}$ for the time-varying parameters $k_{ij}^l(t), k_{ij}^u(t)$ are interpreted as sector bounds as well and therefore all components of $\boldsymbol{\varphi}(\mathbf{y})$ are in positive sectors.

[3] From the application of the multivariable Popov theorem we obtain the following frequency condition [11]: The network (1) is stable, if there exist scaling matrices $\mathsf{P} = \mathrm{diag}\{p_i, \underline{p}_{ij}, \overline{p}_{ij}\} > 0, \mathsf{Q} = \mathrm{diag}\{q_i, \mathbf{0}, \mathbf{0}\} > 0 \in \mathbb{R}^{n\times(n+2N)}$ such that

$$\sup_\omega \max_i \lambda_i \,\mathrm{Re}\left[\mathsf{P}((\mathrm{I} + \imath\omega\mathsf{Q})\mathsf{G}(\imath\omega) - \bar{\mathcal{K}}^{-1})\right] < 0, \tag{3}$$

where $\mathrm{Re}\,[\mathsf{M}] = \frac{1}{2}(\mathsf{M}^* + \mathsf{M})$ denotes the hermitian part of the complex matrix M, $\mathsf{G}(\imath\omega) = \mathsf{C}(\mathsf{D} + \imath\omega\mathrm{I})^{-1}\mathsf{W} \in \mathbb{R}^{(n+2N)^2}$ is the frequency matrix of the system (2), and $\bar{\mathcal{K}} = \mathrm{diag}\{\bar{k}_i, \ldots, \underline{k}_{ij}, \ldots, \bar{k}_{ij}, \ldots\}$ includes the upper sector bounds.

[4] Using the Kalman-Yakubovich-Lemma the condition (3) can be reformulated as a linear matrix inequality constraint. It is equivalent to the problem to find matrices $\mathsf{H} = \mathsf{H}^T > 0 \in \mathbb{R}^{n\times n}$, P, and Q such that

$$\begin{pmatrix} \mathsf{DH} + \mathsf{HD} & \mathsf{HW} + \mathsf{C}^T\mathsf{P} - \mathsf{DC}^T\mathsf{Q} \\ \widetilde{\mathsf{W}}^T\mathsf{H} + \mathsf{PC} - \mathsf{QCD} & -\mathsf{QC}\widetilde{\mathsf{W}} - \mathsf{W}^T\mathsf{C}^T\mathsf{Q} + 2\mathsf{P}\bar{\mathcal{K}}^{-1} \end{pmatrix} > 0. \tag{4}$$

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Maximization of the sector bounds $\mathcal{K}$ is now a convex optimization problem subject to the constraint (4) and can efficiently be solved by interior point algorithms available in standard software (Matlab, SCILAB) [10]. The local equilibrium leads to additional constraints on the $\bar{k}_i$, for instance for $\tanh$ we know that $\bar{k}_i \geq \varphi_i'$. As these additional constraints as well as the inequality (4) can be directly submitted to software and involve only known system data the stability condition can be applied in practice without further regard to its theoretical background. The numerical maximization then yields as well the local sector bounds on the non-linear activation functions φ_i as, if time-varying weights are included, a stable region in weight space defined component wise as $\tilde{w}_{ij} - \underline{k}_{ij} \leq \tilde{w}_{ij}(t) \leq \tilde{w}_{ij} + \bar{k}_{ij}$ in which the network may be adapted or subject to noise without the risk of instable behavior.

3 Computation of the input bounds

It remains to compute a bound on the input such that the trajectories of the system (2) do not leave the region where the local sector conditions hold. This is always necessary because for a non-trivial equilibrium the sector conditions can not hold globally. Indeed, globally valid sectors would imply, that the local equilibrium $\mathbf{x}^\star$ for the unforced system with $\Delta \mathsf{W}(t) \equiv 0$ and $\mathbf{u}(t) \equiv 0$ is also global asymptotically stable. But this contradicts the fact that the system always admits the trivial equilibrium at the origin. Thus we resort to a classical Lyapunov function approach to compute suitable bounds on $|\mathbf{u}(t)|$. Writing the dynamical equation in terms of the shifted state vector $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}^\star$ and using a Taylor series expansion of the non-linear function we obtain

$$\dot{\hat{\mathbf{x}}} = -\mathsf{D}\hat{\mathbf{x}} + \mathsf{W}\left(\boldsymbol{\varphi}(\mathbf{x}^\star) + \boldsymbol{\varphi}'(\mathbf{x}^\star)\hat{\mathbf{x}} + \frac{1}{2}\hat{\mathbf{x}}^T\boldsymbol{\varphi}''(\mathbf{x}^\star)\hat{\mathbf{x}} + O(\|\hat{\mathbf{x}}\|^3)\right) + \mathbf{u}(t) - \mathsf{D}\mathbf{x}^\star$$

Solving the Lyapunov equation for the linear parts we find $\mathsf{P} = \mathsf{P}^T > 0$ for given $\mathsf{Q} = \mathrm{diag}\{q_i\} > 0$ such that

$$(-\mathsf{D} + \mathsf{W}\boldsymbol{\varphi}'(\mathbf{x}^\star))^T \mathsf{P} + \mathsf{P}(-\mathsf{D} + \mathsf{W}\boldsymbol{\varphi}'(\mathbf{x}^\star)) = -\mathsf{Q} < 0 \tag{5}$$

and then the derivative of the corresponding quadratic Lyapunov function $\frac{1}{2}\hat{\mathbf{x}}^T\mathsf{P}\hat{\mathbf{x}}$ is

$$-\hat{\mathbf{x}}^T\mathsf{Q}\mathbf{x} + \hat{\mathbf{x}}^T\mathsf{P}\left(\mathbf{x}^T\boldsymbol{\varphi}''(\mathbf{x}^\star)\mathbf{x}^T + \mathbf{u}(t)\right), \tag{6}$$

which is smaller than zero if $|\boldsymbol{\varphi}''(\mathbf{x})|$ and $|\mathbf{u}(t)|$ are sufficiently small. In general this may result in a very small bound on $\mathbf{u}(t)$ but for the saturation functions usually employed in neural networks the first and second derivatives of $\boldsymbol{\varphi}(\mathbf{x}^\star)$ tend very rapidly to zero and substantially simplify (5), (6). We will use this in the example below.

4 Learning trajectories of the Roessler attractor

In the following we analyze stability of a network adapted to implement a mapping between functions $z_1(t), z_2(t)$ defining the input and $z_3(t)$ as reference output. The functions $z_i(t)$ are given by the chaotic Roessler dynamics

$$\dot{z_1} = -z_2 - z_3, \quad \dot{z_2} = -z_1 + 0.2z_2, \quad \dot{z_3} = 0.2 + z_1 z_3 - 5.7z_3. \tag{7}$$

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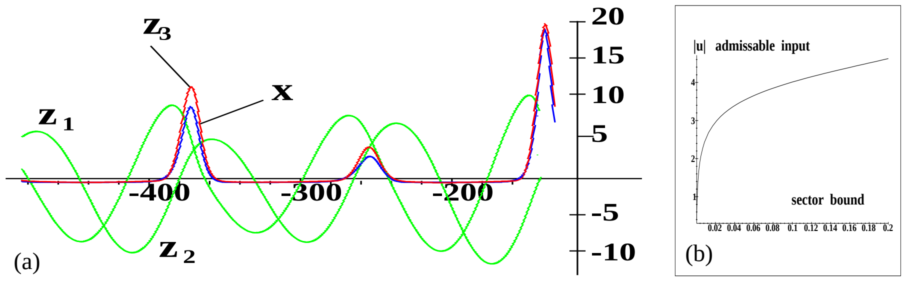

Fig. 2. (a) 500 time-steps of the input (z_1, z_2) and output (z_3) trajectories. The learned output x has very small errors only at the high peaks in z_3. (b) The bound on the admissible input at $x^\star = 4.7$ and $\varphi(x^\star) = \tanh(x^\star)$.

Note that we do not attempt to learn the vector field of the attractor but rather assume that it implicitly defines given input-output transform which can be learned. From Fig. 2 (a) it can be seen that we choose a hard task: the network has to keep the output very small most of the time and only eventually there occur fast large peaks in z_3, whose position and height have to be detected from the irregularities in the quasi-oscillations of the inputs z_1, z_2. The learned network achieves that task very well, the trajectories of x_1 and z_3 are practically coincident except for small errors at the peaks, see also Fig. 2. It turns out that the network needs a number of neurons close to saturation for implementing a kind of bias, which leads to a non-trivial equilibrium as point of operation.

We use a fully connected twenty neuron network and input weights to supply a linear combination of $z_1(t), z_2(t)$ as input to the network. As output we choose $x_1(t)$ and $z_3(t)$ is the reference function which results in the network equations

$$\dot{x}_i = -x_i + \sum_{j=1}^{20} w_{ij}\varphi_j(x_j) + u_i(t) = -x_i + \sum_{j=1}^{20} w_{ij}\varphi_j(x_j) + w_{z_1} z_1(t) + w_{z_2} z_2(t),$$

where $\varphi_j(x_j) = \tanh(x_j)$ in the experiments. For adaptation we employ fully continuous backpropagation (as in [7]) to minimize the error functional $E(t_0, t_1) = \int_{t_0}^{t_1} (x_1(t) - z_r(t))^2 dt$ taken over the first 1250 time steps of numerical integration of (7).

The adapted network has (for zero input) the equilibrium $\mathbf{x}^\star = (0, x_2^\star, \ldots, x_{20}^\star)$, where for $i > 1 : |x_i^\star| > 4.7$. Therefore $\boldsymbol{\varphi}'(\mathbf{x}) \approx (1, 0, \ldots, 0)$ and $\boldsymbol{\varphi}''(\mathbf{x}) \approx 0$ as well. The Lyapunov equation (5) now admits the simple diagonal solution $P = p\mathrm{I}$, $\mathsf{Q} = q\mathrm{I}, p, q > 0$ and (6) holds, if $-q\|\mathbf{x}(t)\|^2 + p\mathbf{x}(t)^T\mathbf{u}(t) < 0$. We can easily monitor this condition on-line and restrict inputs appropriately whenever a coordinate x_i approaches the boundary, where the local sector conditions fail. Because we can choose $p = q = 1$ in the example a global estimate for $|\mathbf{u}(t)_i|$ is given by the distance of $x_i^\star$ to the point x_{min} where the line $\bar{k}_i(x_i + x_i^\star)$ and $\tanh(x_i)$ intersect as was shown in Fig. 1 (c). As the $\tanh$ becomes very flat very quickly already small sectors yield large x_i-ranges and therefore admit reasonable large inputs $|\mathbf{u}_i(t)|$. In Fig. 2 (b) there are shown the input bounds with respect to the point $x_s^\star = 4.7$ for various sector widths, we obtained in

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the experiment $\bar{k} \approx 0.07$ and indeed all inputs were sufficiently small, i.e. we learned a locally stable network.

5 Discussion

We derived a framework to analyze stability of the local input-output behavior of a recurrent network based on frequency domain conditions and a interior point optimization scheme. Though the network first was adapted and solves the task without regard to such conditions, we gain from the a posteriori analysis that (i) we can use online-adaptation within the given bounds for the time-varying weights and (ii) can apply different inputs within the given bounds, for instance for the sake of generalization, without the risk of leaving the basin of attraction of the working equilibrium. Restricting adaptation and inputs appropriately, the proper functioning of the network is guaranteed which we regard as a step towards application of recurrent networks in more critical domains, for instance in engineering systems, where the costs of misbehavior and reset of the system can be very high.

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