Efficient supersingularity testing over $\mathbb{F}_p$ and CSIDH key validation

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Problem: how should we check if $\mathcal{E}_A$ is supersingular?

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For CSIDH-512, where $n = 74$, this results in ≈ 33 scalar multiplications.

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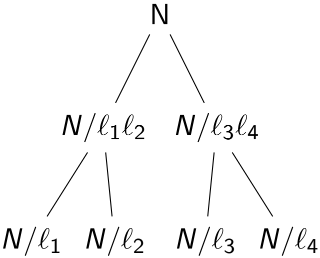

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An ℓ*-isogeny graph*:

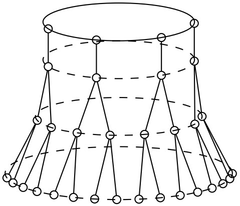
- vertices = ℓ-isogenous curves (up to isomorphism)
- edges = ℓ-isogenies (up to isomorphism)

Sutherland Test

2-isogeny graphs over $\mathbb{F}_p$ form a forest of same-sized trees, where the roots are connected by a cycle. We call them *volcanoes*.

The cycle of roots is the *crater*.

The leaves of the trees form the *floor*.



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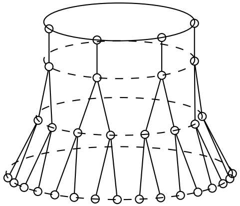
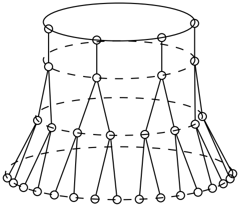


Supersingular 2-isogeny graphs over $\mathbb{F}_{p^2}$ will form a 3-regular, connected graph.

Ordinary (not supersingular) graphs over $\mathbb{F}_{p^2}$ form a larger volcano.

Sutherland Test



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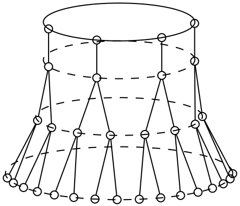

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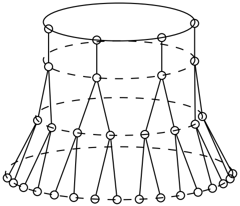

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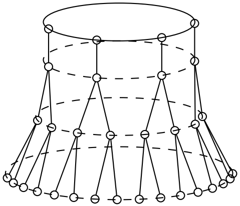

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- if none of the paths hit the floor, then the graph is not a volcano.

Sutherland Test

We will traverse the graph using *modular polynomials*.
The (classical) modular polynomial of level 2 is

$$\Phi_2(j_1, j_2) = j_1^3 + j_2^3 - j_1^2 j_2^2 + 1488(j_1^2 j_2 + j_1 j_2^2) - 162000(j_1^2 + j_2^2) + 40773375 j_1 j_2 + 8748000000(j_1 + j_2) - 157464000000000\,.$$

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It has the property that

$$\text{there exists a 2-isogeny } \mathcal{E}_1 \to \mathcal{E}_2 \iff \Phi_2(j(\mathcal{E}_1), j(\mathcal{E}_2)) = 0\,.$$

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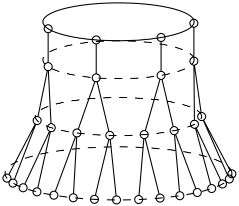

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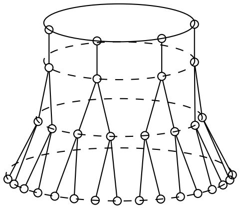

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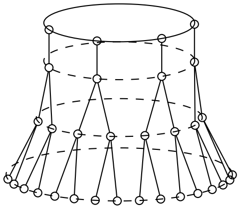

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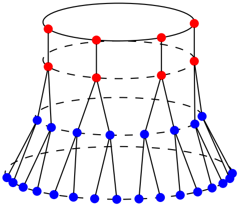
Repeat $\log_2 p + 1$ times.

- after this many steps, we have walked the maximal height of the volcano, so we must be supersingular

Modified Sutherland

Sutherland's modification in the $\mathbb{F}_p$ case:

- Assuming we are supersingular, the $\mathbb{F}_p$ volcano will be very short—only two levels tall;
- Within two steps, one neighbour will be defined over $\mathbb{F}_{p^2}$. We take this path down;
- Gives an approximate $3\times$ speedup.



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- Improved bound on the maximum length of the descending path (if ordinary) of the volcano

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- We replaced the modular polynomial computations by computing explicit 2-isogenies as follows:

$$\varphi : \mathcal{E} \longrightarrow \mathcal{E}/\langle(\alpha, 0)\rangle \cong \mathcal{E}' : y^2 = x(x^2 + a_2'x + a_4').$$

Doliskani Test

For $m \geq 0$, the m-th division polynomial $\psi_{\mathcal{E},m}$ of an elliptic curve $\mathcal{E}$ satisfies

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For a random $u \in \mathbb{F}_{p^2}$,

$$\psi_{\mathcal{E},p}(u)^2 = 1 \underset{\text{prob. 1-1/2p}}{\Longrightarrow} \psi^2_{\mathcal{E},p} = 1 \iff \mathcal{E} \text{ s.s.}$$

Modified Doliskani

Our modification:

Scalar multiplication can be defined in terms of division polynomials as

$$[m](x, y) = \left(\frac{\phi_{\mathcal{E},m}(x)}{\psi_{\mathcal{E},m}(x)^2}, \frac{\omega_{\mathcal{E},m}(x, y)}{\psi_{\mathcal{E},m}(x)^3} \right)$$

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In our version of Doliskani's test, we use (projective) differential addition to compute

$$[p](u:1) = \left(\lambda\phi_{\mathcal{E},p}(u) : \lambda\psi_{\mathcal{E},p}^2(u) \right)$$

where λ is determined by the ladder algorithm.

Comparison

Test algorithm	Asymptotics		Supersingular input			Non-Supersingular input		
	Time ($\mathbb{F}_p$-ops)	Space ($\mathbb{F}_p$-elts)	MCycles: Avg.	Med.	Stack (B)	MCycles: Avg.	Med.	Stack (B)
Random point	$O(n \log p)$	$O(1)$	63.4	62.2	2890	65.3	62.9	2890
Product tree	$O((\log n)(\log p))$	$O(\log n)$	6.7	6.1	4344	1.7	1.6	3896
Sutherland	$O(\log^2 p)$	$O(1)$	35.4	35.1	**2696**	**0.8**	**0.4**	**2696**
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- Using an Intel i7-10610U processor running at 4.90 GHz (see paper for details);
- Cycles were measured using the `bench` utility provided in the CSIDH code package;
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Conclusion: we suggest using Doliskani for CSIDH key validation.
See `https://ia.cr/2022/880` for details.